Improved bounds for cross-Sperner systems

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Abstract

A collection of non-empty families $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \in \mathcal{P}([n])^k$ is *cross-Sperner* if there is no pair $i \neq j$ for which some $F_i \in \mathcal{F}_i$ is comparable to some $F_j \in \mathcal{F}_j$. Two natural measures of the 'size' of such a family are the sum $\sum_{i=1}^k |\mathcal{F}_i|$ and the product $\prod_{i=1}^k |\mathcal{F}_i|$. We prove new upper and lower bounds on the maximum size of such a family under both of these measures for general n and $k \geqslant 2$ which improve considerably on the previous best bounds. In particular, we construct a rich family of counterexamples to a conjecture of Gerbner, Lemons, Palmer, Patkós, and Szécsi from 2011.

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1 Introduction

A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is an *antichain* (also known as a *Sperner* family) if for all distinct $F, G \in \mathcal{F}$, neither $F \subseteq G$ nor $G \subseteq F$ (i.e. F and G are *incomparable*). One of the principal results in extremal combinatorics is Sperner's theorem [21], which states that the largest size of an antichain in $\mathcal{P}([n])$ is $\binom{n}{\lfloor n/2 \rfloor}$. This can be seen to be tight by taking a 'middle layer', that is $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.

It is natural to consider a generalisation of Sperner's theorem to multiple families of sets. For $k \geqslant 2$, say that a collection of non-empty families $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \in \mathcal{P}([n])^k$, is *cross-Sperner* if for all $i \neq j$, the sets F_i and F_j are incomparable for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$. (We may also write that $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is *cross-Sperner in* $\mathcal{P}([n])$.) The study of such objects goes back to the 1970s when Seymour [20] deduced from a result of Kleitman [13] that a cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ in $\mathcal{P}([n])$ satisfies

$$|\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2} \leqslant 2^{n/2}, \tag{1.1}$$

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hence resolving a related conjecture of Hilton (see [3]). Equality is obtained in Seymour's bound precisely when the minimal sets of $\mathcal{F}$ are pairwise disjoint from the minimal sets intersecting each set of $\mathcal{G}$. A broad spectrum of research concerning discrete objects with 'Sperner-like' properties have since emerged (see, for example, [1, 2, 4, 5, 7, 10, 11, 12, 18, 22]). Many related results concern families satisfying both Sperner-type properties, and additional properties such as conditions on intersections (see, for example [6, 14, 16, 17, 19]).

Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ be cross-Sperner in $\mathcal{P}([n])$. There are several natural measures of the 'size' of such a family. These include the sum $\sum_{i=1}^k |\mathcal{F}_i|$ and the product $\prod_{i=1}^k |\mathcal{F}_i|$. The general study of these quantities was initiated by Gerbner, Lemons, Palmer, Patkós, and Szécsi [8], who essentially proved best possible bounds on cross-Sperner *pairs* of families.

Concerning the product, they gave a direct proof that a cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ in $\mathcal{P}([n])$ satisfies

$$|\mathcal{F}| \cdot |\mathcal{G}| \leqslant 2^{2n-4}. \tag{1.2}$$

To see that this bound is tight, consider $\mathcal{F} = \{F \subseteq [n] : 1 \in F, n \notin F\}$ and $\mathcal{G} = \{G \subseteq [n] : 1 \notin G, n \in G\}$. It is straightforward to see that the bound in (1.2) can also be obtained as a direct consequence of (1.1) via the AM–GM inequality[1].

First, let us focus on product bounds for $k \geqslant 3$. It is convenient to define

$$\pi(n,k) := \max\left\{\prod_{i=1}^{k} |\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n])\right\}.$$

In [8], it was observed that (1.1) can be used to obtain the upper bound $\pi(n,k) \leqslant 2^{k(n-2)}$. For $k > 4$, an improved bound of $\pi(n,k) \leqslant \left(\frac{2^n}{k}\right)^k$ can be obtained by a simple application of the AM-GM inequality.[2] Gerbner, Lemons Palmer, Patkós, and Szécsi [8] conjectured that $\pi(n,k) \leqslant 2^{k(n-\ell^*)}$, where $\ell^* = \ell^*(k)$ is the least positive integer such that $\binom{\ell^*}{\lfloor \ell^*/2 \rfloor} \geqslant k$. They described a construction which provides a matching lower bound to their conjecture: let $A_1, \ldots, A_k$ be an antichain in $\mathcal{P}([\ell])$ and let $(\mathcal{F}_1, \ldots, \mathcal{F}_k) \in \mathcal{P}([n])$ be defined by $\mathcal{F}_i := \{F \in [n] : F \cap [\ell] = A_i\}$.

Our first theorem strongly disproves this conjecture.

Theorem 1.1. *Let n and $k \geqslant 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leqslant \pi(n,k).$$

A crude application of Stirling's approximation yields that $\ell^*(k) = \omega(\log k)$. So in particular, there is a function $g(k)$ tending to infinity with k such that $2^{k(n-\ell^*)} = O\left(2^{kn}(k \cdot g(k))^{-k}\right)$. Therefore our lower bound is exponentially larger than the conjectured $2^{k(n-\ell^*)}$.

We also improve the previous best known upper bound by a factor of 2^k.

[1] Observe that $2\left(|\mathcal{F}||\mathcal{G}|\right)^{1/4} \leqslant |\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2} \leqslant 2^{n/2}$.

[2] Similarly to above, we have $\left(\prod_{i=1}^k |\mathcal{F}_i|\right)^{\frac{1}{k}} \leqslant \frac{\sum_{i=1}^k |\mathcal{F}_i|}{k} \leqslant \frac{2^n}{k}$.

Theorem 1.2. *Let n and $k \geqslant 2$ be integers. Then*

$$\pi(n,k) \leqslant \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil}$$

Regarding bounds on the sum, in [8] it is shown that for n sufficiently large, a cross-Sperner pair in $\mathcal{P}([n])$ satisfies

$$|\mathcal{F}| + |\mathcal{G}| \leqslant 2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 2. \tag{1.3}$$

This is tight, which can be seen by taking $\mathcal{F} = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and letting $\mathcal{G}$ be all subsets of $[n]$ that are not comparable to F. Gerbner, Lemons Palmer, Patkós, and Szécsi [8] also asked about bounds for the sum for general k. Analogously to in the product case, define

$$\sigma(n,k) := \max \left\{ \sum_{i=1}^{k} |\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n]) \right\}.$$

In our next theorem, we determine upper and lower bounds on $\sigma(n,k)$. Recall that each family is non-empty in a cross-sperner system.

Theorem 1.3. *Let n, k be integers with $n \geqslant 2k$ and $k \geqslant 2$. Then*

$$2^n - \frac{3}{\sqrt{2}}\sqrt{2^n k} + 2(k-1) \;\leqslant\; \sigma(n,k) \;\leqslant\; 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

When k is a power of 2 and $n - \log_2 k$ is even, we can further improve the lower bound to $2^n - 2\sqrt{2^n k} + 2(k-1)$, which is extremely close to the upper bound.

In order to prove Theorem 1.2 and the upper bound in Theorem 1.3, we exploit a connection between $\sigma(n,k)$ and the *comparability number* of a set (given in Section 2). In doing so, we recover a simple proof of (1.3) (see Theorem 2.4) that holds for *all* n (recall the result of [8] holds for large n).

The article is structured as follows. We introduce the comparability number in Section 2 and provide a lower bound (Theorem 2.3) that will be used in the proofs of Theorems 1.2 and 1.3. In Section 3 we prove Theorems 1.1 and 1.2 bounding the product. In Section 4 we prove Theorem 1.3 bounding the sum. We conclude in section 5 with some discussion and open questions.

2 Minimizing Comparability

Given a family $\mathcal{F} \subseteq \mathcal{P}([n])$ define the *comparability number of* $\mathcal{F}$ to be

$$c(n,\mathcal{F}) := |\{X \subseteq [n] : X \text{ is comparable to some } A \in \mathcal{F}\}|.$$

When the setting is clear from context, we may write $c(\mathcal{F})$ for $c(n,\mathcal{F})$. Define

$$c(n,m) = \min\{c(n,\mathcal{F}) : \mathcal{F} \subseteq \mathcal{P}([n]), |\mathcal{F}| = m\}.$$

As noted in [8], there is a direct relationship between $\sigma(n,2)$ and $c(n,m)$. Observe that if $(\mathcal{F},\mathcal{G})$ is cross-Sperner in $\mathcal{P}([n])$, we have

$$|\mathcal{F}|+|\mathcal{G}|\leqslant|\mathcal{F}|+2^n-c(n,|\mathcal{F}|),$$

as only sets incomparable to every member of $\mathcal{F}$ can be added to $\mathcal{G}$. We will use analogous ideas in Section 4 to provide upper bounds on $\sigma(n,k)$ for $k\geqslant 3$.

Our goal in this section is to find a lower bound on $c(n,m)$. We begin by showing that families that minimize comparability are 'convex'.

Lemma 2.1. *Let $\mathcal{F}\subseteq\mathcal{P}([n])$. Let $\mathcal{F}':=\mathcal{F}\cup\{Z\in\mathcal{P}([n]):X\subseteq Z\subseteq Y,$ where $X,Y\in\mathcal{F}\}$. Then $c(\mathcal{F}')=c(\mathcal{F})$.*

Proof. Let Z be a set such that $X\subseteq Z\subseteq Y$, for some $X,Y\in\mathcal{F}$. Observe that any set in $\mathcal{P}([n])$ that is comparable to Z is either comparable to X or to Y. So $c(\mathcal{F}\cup Z)=c(\mathcal{F})$. Repeatedly applying this observation gives the result. □

Theorem 2.3 can now be deduced from the Harris-Kleitman inequality. Recall that a family $\mathcal{U}\subseteq\mathcal{P}([n])$ is an *upset* if for all $X\in\mathcal{U}$, if $X\subseteq Y$, then $Y\in\mathcal{U}$. A family $\mathcal{D}\subseteq\mathcal{P}([n])$ is a *downset* if for all $X\in\mathcal{D}$, if $Y\subseteq X$, then $Y\in\mathcal{D}$.

Lemma 2.2 (Harris-Kleitman Inequality [13])**.** *Let $\mathcal{U}\subseteq\mathcal{P}([n])$ be an upset and $\mathcal{D}\subseteq\mathcal{P}([n])$ be a downset. Then*

$$\frac{|\mathcal{U}\cap\mathcal{D}|}{2^n}\leqslant\frac{|\mathcal{U}|}{2^n}\cdot\frac{|\mathcal{D}|}{2^n}.$$

We will apply Lemma 2.2 to prove a lower bound on $c(n,m)$. For convenience, for a family $\mathcal{F}\subseteq\mathcal{P}([n])$, define

$$\mathcal{U}_{\mathcal{F}}=\{X\in\mathcal{P}([n]):F\subseteq X\text{ for some }F\in\mathcal{F}\}$$

and

$$\mathcal{D}_{\mathcal{F}}=\{X\in\mathcal{P}([n]):X\subseteq F\text{ for some }F\in\mathcal{F}\}.$$

Theorem 2.3. *For $1\leqslant m\leqslant 2^n$,*

$$c(n,m)\geqslant 2^{n/2+1}\sqrt{m}-m.$$

Proof. Let $\mathcal{F}\subseteq\mathcal{P}([n])$ be such that $|\mathcal{F}|=m$ and $c(\mathcal{F})=c(n,m)$. We may assume $\mathcal{F}$ is convex. If not, by Lemma 2.1 we may add sets to make it convex and then remove minimal or maximal elements to obtain $\mathcal{F}'$ such that $|\mathcal{F}'|=|\mathcal{F}|$ and $c(\mathcal{F}')\leqslant c(\mathcal{F})$. Note that $c(\mathcal{F})=|\mathcal{U}_{\mathcal{F}}|+|\mathcal{D}_{\mathcal{F}}|-|\mathcal{U}_{\mathcal{F}}\cap\mathcal{D}_{\mathcal{F}}|$. Since $\mathcal{F}$ is convex, $|\mathcal{F}|=|\mathcal{U}_{\mathcal{F}}\cap\mathcal{D}_{\mathcal{F}}|=m$. Using the AM-GM inequality we get

$$c(\mathcal{F})\geqslant 2\sqrt{|\mathcal{U}_{\mathcal{F}}||\mathcal{D}_{\mathcal{F}}|}-m.$$

Since $\mathcal{U}_{\mathcal{F}}$ is an upset and $\mathcal{D}_{\mathcal{F}}$ is a downset, we apply Lemma 2.2 to get

$$c(\mathcal{F})\geqslant 2\sqrt{2^n m}-m=2^{\frac{n}{2}+1}\sqrt{m}-m,$$

as required. □

It is now a simple consequence of Theorem 2.3 to see that (1.3) holds for all n.

Theorem 2.4. *Let* $(\mathcal{F},\mathcal{G})$ *be cross-Sperner in* $\mathcal{P}([n])$*. Then*

$$|\mathcal{F}|+|\mathcal{G}| \leqslant 2^n - 2^{\lfloor n/2\rfloor} - 2^{\lceil n/2\rceil} + 2.$$

Proof. Let $(\mathcal{F},\mathcal{G}) \in \mathcal{P}([n])^2$ be a cross-Sperner pair. Suppose $|\mathcal{F}| = m$. Since $\mathcal{F},\mathcal{G} \neq \emptyset$, $1 \leqslant m \leqslant 2^n - 1$. Moreover, we may assume without loss of generality that $|\mathcal{F}| \leqslant |\mathcal{G}|$. We know $|\mathcal{F}||\mathcal{G}| \leqslant 2^{2n-4}$ by (1.2), which implies that $m \leqslant 2^{n-2}$.

Then, $c(\mathcal{F}) \geqslant |\mathcal{U}_{\mathcal{F}}| + |\mathcal{D}_{\mathcal{F}}| - m$ and $|\mathcal{G}| \leqslant 2^n - c(\mathcal{F}) \leqslant 2^n - |\mathcal{U}_{\mathcal{F}}| - |\mathcal{D}_{\mathcal{F}}| + m$. Thus

$$|\mathcal{F}|+|\mathcal{G}| \leqslant 2^n - |\mathcal{U}_{\mathcal{F}}| - |\mathcal{D}_{\mathcal{F}}| + 2m. \tag{2.1}$$

We have the following two cases.

Case 1: Suppose $m = 1$. Since $\mathcal{F}$ only consists of one set, say F, we have $|\mathcal{U}_{\mathcal{F}}| = 2^{n-|F|}$ and $|\mathcal{D}_{\mathcal{F}}| = 2^{|F|}$. Observe that $2^{|F|} + 2^{n-|F|}$ is minimized when $|F| \in \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$. So (2.1) yields

$$|\mathcal{F}|+|\mathcal{G}| \leqslant 2^n - 2^{\lfloor n/2\rfloor} - 2^{\lceil n/2\rceil} + 2,$$

as required. This completes the case $m = 1$.

Case 2: Now suppose $m \geqslant 2$. By Theorem 2.3, $|\mathcal{U}_{\mathcal{F}}| + |\mathcal{D}_{\mathcal{F}}| \geqslant 2^{\frac{n}{2}+1}\sqrt{m}$, so Equation (2.1) gives

$$|\mathcal{F}|+|\mathcal{G}| \leqslant 2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m.$$

By differentiation with respect to m we see that the expression on the right-hand side is decreasing in the range $2 \leqslant m \leqslant 2^{n-2}$. It is therefore maximized at $m = 2$, where we have

$$2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m = 2^n - 2^{\frac{n+3}{2}} + 4.$$

Note that for all $n \geqslant 2$,

$$2^{\frac{n+3}{2}} - 4 \geqslant 2^{\lfloor n/2\rfloor} + 2^{\lceil n/2\rceil} - 2.$$

This implies that $2^n - 2^{\lfloor n/2\rfloor} - 2^{\lceil n/2\rceil} + 2 \geqslant 2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m$ for all $2 \leqslant m \leqslant 2^{n-2}$ and $n \geqslant 2$. This completes the case $m \geqslant 2$.

We conclude that $|\mathcal{F}|+|\mathcal{G}| \leqslant 2^n - 2^{\lfloor n/2\rfloor} - 2^{\lceil n/2\rceil} + 2$, as desired. □

3 Bounding $\pi(n,k)$

The goal of this section is to prove Theorems 1.1 and 1.2.

3.1 Lower Bound on $\pi(n, k)$

Theorem 1.1 follows directly from the following (slightly stronger) statement.

Lemma 3.1. *Let n, k be integers with $k \geqslant 2$ and $n > k \log_2 k + k$. Then*

$$\pi(n,k) \geqslant \left(\left(\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}} \right) \left(1 - \frac{1}{k} \right)^{k-1} \right)^k 2^{kn}$$

Proof. Partition $[n]$ into k parts $A_1, A_2, \cdots, A_k$ each of size $\left\lfloor \frac{n}{k} \right\rfloor$ or $\left\lceil \frac{n}{k} \right\rceil$. For each $1 \leqslant i \leqslant k$, take $\mathcal{X}_i$ to be an initial segment of colex in $\mathcal{P}(A_i)$ such that $|\mathcal{X}_i| = \lambda_i 2^{|A_i|}$ for some $0 < \lambda_i < 1$, which will be chosen to be optimal at the end. Set $\mathcal{Y}_i := \mathcal{P}(A_i) \setminus \mathcal{X}_i$. Now we construct a cross-Sperner system $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$. Define

$$\mathcal{F}_i := \{F \in \mathcal{P}([n]) : F \cap A_i \in \mathcal{X}_i, F \cap A_j \in \mathcal{Y}_j \text{ for all } j \neq i\}.$$

Refer to Example 3.3 for an example of this construction.

To see that $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$, consider $S \in \mathcal{F}_i$ and $T \in \mathcal{F}_j$. We must show that S and T are incomparable. If $S \subseteq T$, then $S \cap A_j \subseteq T \cap A_j$, so there is some $Y \in \mathcal{Y}_j$ and $X \in \mathcal{X}_j$ such that $Y \subseteq X$, a contradiction. Analogously, we see that T cannot be a subset of S. Hence $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner as required.

Observe that

$$|\mathcal{F}_i| = |\mathcal{X}_i| \prod_{j \neq i} |\mathcal{Y}_j|,$$

and so

$$\pi(n,k) \geqslant \prod_{i=1}^{k} |\mathcal{F}_i| = \prod_{i=1}^{k} \left(|\mathcal{X}_i| \prod_{j \neq i} |\mathcal{Y}_j| \right).$$

To complete the proof of Lemma 3.1 it remains to optimise the sizes of the λ_i. We have

$$|\mathcal{F}_i| = \lambda_i 2^{|A_i|} \prod_{j \neq i} (1 - \lambda_j) 2^{|A_j|} = \lambda_i 2^{|A_1| + |A_2| + \cdots |A_k|} \prod_{j \neq i} (1 - \lambda_j) = \lambda_i 2^n \prod_{j \neq i} (1 - \lambda_j).$$

So

$$\prod_{i=1}^{k} |\mathcal{F}_i| = \left(\prod_{i=1}^{k} \lambda_i (1 - \lambda_i)^{k-1} \right) 2^{kn} \tag{3.1}$$

For each $1 \leqslant i \leqslant k$, set $\lambda_i = \frac{1}{2^{|A_i|}} \left\lfloor \frac{2^{|A_i|}}{k} \right\rfloor$. We have

$$\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}} \leqslant \frac{1}{k} - \frac{1}{2^{|A_i|}} \leqslant \lambda_i \leqslant \frac{1}{k}.$$

For $n > k \log_2 k + k$ we have $2^{-\lfloor n/k \rfloor} \leqslant 2^{-(n/k - 1)} < \frac{1}{k}$ and so λ_i is not zero. Therefore, with this choice of λ_i we get

$$\prod_{i=1}^{k} |\mathcal{F}_i| \geqslant \left(\left(\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}} \right) \left(1 - \frac{1}{k} \right)^{k-1} \right)^k 2^{kn},$$

as required. □

Remark 3.2. Note that if k is a power of 2, in the proof of Lemma 3.1 we have $\lambda_i = \frac{1}{k}$ for all $1 \leqslant i \leqslant k$. Therefore in this case we can eliminate the $-\frac{1}{2^{\lfloor n/k \rfloor}}$ term.

For clarity, we provide an example of the construction given in Lemma 3.1.

Example 3.3. Let $n = 6$ and $k = 3$. Partition $[6]$ into

$$A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5,6\}.$$

To provide a more illustrative example, we choose $\lambda_i = \frac{1}{2}$ rather than $\frac{1}{4}$ as the proof of Lemma 3.1 stipulates. Let

$$\mathcal{X}_1 = \{\emptyset, \{1\}\}$$
$$\mathcal{X}_2 = \{\emptyset, \{3\}\}$$
$$\mathcal{X}_3 = \{\emptyset, \{5\}\}.$$

So

$$\mathcal{Y}_1 = \{\{2\}, \{1,2\}\}$$
$$\mathcal{Y}_2 = \{\{4\}, \{3,4\}\}$$
$$\mathcal{Y}_3 = \{\{6\}, \{5,6\}\}.$$

Then we construct our cross-Sperner system to be

$$\mathcal{F}_1 = \{\{4,6\}, \{4,5,6\}, \{3,4,6\}, \{3,4,5,6\}, \{1,4,6\}, \{1,4,5,6\}, \{1,3,4,6\}, \{1,3,4,5,6\}\}$$
$$\mathcal{F}_2 = \{\{2,6\}, \{2,5,6\}, \{2,3,6\}, \{2,3,5,6\}, \{1,2,6\}, \{1,2,5,6\}, \{1,2,3,6\}, \{1,2,3,5,6\}\}$$
$$\mathcal{F}_3 = \{\{2,4\}, \{2,4,5\}, \{2,3,4\}, \{2,3,4,5\}, \{1,2,4\}, \{1,2,4,5\}, \{1,2,3,4\}, \{1,2,3,4,5\}\}.$$

We now deduce Theorem 1.1 (restated below for convenience) from Lemma 3.1.

Theorem 1.1. *Let n and $k \geqslant 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leqslant \pi(n,k).$$

Proof. Take n sufficiently large so that

$$\frac{1}{2^{\lfloor n/k \rfloor}} \leqslant \frac{1}{k} - \frac{1}{ek}\left(1 + \frac{1}{k-1}\right)^{k-1} = \frac{1}{ek}\left(e - \left(1 + \frac{1}{k-1}\right)^{k-1}\right).$$

This is possible as $\left(1 + \frac{1}{k-1}\right)^{k-1} < e$ for all k. Substituting this into Lemma 3.1, we see that

$$\pi(n,k) \geqslant \left(\frac{1}{ek}\left(1 + \frac{1}{k-1}\right)^{k-1}\left(1 - \frac{1}{k}\right)^{k-1}\right)^k 2^{kn} = \left(\frac{1}{ek}\right)^k 2^{kn}.$$ □

3.2 Upper Bound on $\pi(n,k)$

The goal of this subsection is to prove Theorem 1.2, restated below for convenience.

Theorem 1.2. *Let n and $k \geqslant 2$ be integers. Then*

$$\pi(n,k) \leqslant \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil}$$

We will use the following observation.

Lemma 3.4. *Let $1 \leqslant j < k$ and let $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ be cross-Sperner. Then $\left(\bigcup_{i=1}^{j} \mathcal{F}_i, \bigcup_{i=j+1}^{k} \mathcal{F}_i\right)$ is cross-Sperner in $\mathcal{P}([n])$.*

Proof. Suppose for contradiction that $\left(\bigcup_{i=1}^{j} \mathcal{F}_i, \bigcup_{i=j+1}^{k} \mathcal{F}_i\right)$ is not cross-Sperner. Then there exists some $X \in \bigcup_{i=1}^{j} \mathcal{F}_i$ and $Y \in \bigcup_{i=j+1}^{k} \mathcal{F}_i$ such that $X \subseteq Y$ or $Y \subseteq X$. Since $X \in \mathcal{F}_i$ for some $1 \leqslant i \leqslant j$, and $Y \in \mathcal{F}_t$ for some $j+1 \leqslant t \leqslant k$ we deduce that $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ is not cross-Sperner, a contradiction. ☐

We now use Lemma 3.4, along with Theorem 2.3, to give an upper bound on $\pi(n,k)$.

Proof of Theorem 1.2. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. Let $a = \lfloor k/2 \rfloor$ and $b = \lceil k/2 \rceil$. Observe that $a + b = k$. Let $\mathcal{G} = \cup_{i=1}^{a} \mathcal{F}_i$ and $\mathcal{H} = \cup_{a+1}^{k} \mathcal{F}_i$. Notice that $(\mathcal{G}, \mathcal{H}) \subseteq \mathcal{P}([n])^2$ is cross-Sperner by Lemma 3.4, so if $|\mathcal{G}| = m$, then $|\mathcal{H}| \leqslant 2^n - c(n,m)$. Moreover, $\prod_{i=1}^{a} |\mathcal{F}_i| \leqslant \left(\frac{m}{a}\right)^a$ and $\prod_{j=a+1}^{k} |\mathcal{F}_j| \leqslant \left(\frac{2^n - 2^{n/2+1}\sqrt{m} + m}{b}\right)^b$ since each product is maximized when the families are of equal sizes and $c(n,m)$ is bounded below by Theorem 2.3. Thus,

$$\prod_{i=1}^{k} |\mathcal{F}_i| = \prod_{i=1}^{a} |\mathcal{F}_i| \prod_{j=a+1}^{k} |\mathcal{F}_j| \leqslant \left(\frac{m}{a}\right)^a \left(\frac{2^n - 2^{n/2+1}\sqrt{m} + m}{b}\right)^b := h(m). \tag{3.2}$$

To find an upper bound on the left hand side of (3.2), we differentiate with respect to m to find the value of m that maximises the right hand side.

$$\frac{d}{dm} h(m) = \left(\frac{m}{a}\right)^a \left(\frac{(2^{n/2} - \sqrt{m})^2}{b}\right)^b (a(\sqrt{m} - 2^{n/2}) + b\sqrt{m})(m^{3/2} - m2^{n/2})^{-1}.$$

Setting this equal to zero yields $m \in \{0, 2^n, \frac{a^2 2^n}{k^2}\}$. A simple calculation shows that (3.2) is maximized when $m = \frac{a^2 2^n}{k^2}$. As $2^n - 2^{n/1+1}\sqrt{m} + m = (2^{n/2} - \sqrt{m})^2 = \frac{2^n}{k^2} b^2$ when $m = \frac{a^2 2^n}{k^2}$,

$$\prod_{i=1}^{k} |\mathcal{F}_i| \leqslant \left(\frac{2^n}{k^2}\right)^k a^a b^b = \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil},$$

as required. ☐

Note that for k even, the upper bound given by Theorem 1.2 is $\left(\frac{2^n}{2k}\right)^k$. For k odd, it is not hard to check that the upper bound is less than $\left(1 + \frac{1}{k}\right)\left(\frac{2^n}{2k}\right)^k$.

4 Bounding $\sigma(n,k)$

The goal of this section is to prove Theorem 1.3.

4.1 Lower Bound on $\sigma(n,k)$

For our proof of the lower bound in Theorem 1.3 we need the following counting lemma.

Lemma 4.1. *Let $\mathcal{A} := \{F_1, F_2, \dots F_{k-1}\}$ be an antichain in $\mathcal{P}([n])$ where $F_i := \{i\} \cup \{n-\ell+1, \dots, n\}$. Then $c(\mathcal{A}) = k2^\ell + 2^{n-\ell}\left(1 - \frac{1}{2^{k-1}}\right) - (k-1)$.*

Proof. First note that the existence of the antichain $\mathcal{A}$ implies that $k-1 < n-\ell+1$. For each i, let $\mathcal{S}_i$ be the collection of sets comparable to F_i. For ease of notation, let $G := \{n-\ell+1, \dots, n\}$ Observe that

$$|\mathcal{S}_i| = |\mathcal{U}_{F_i} \cup \mathcal{D}_{F_i}| = 2^{\ell+1} + 2^{n-\ell-1} - 1, \tag{4.1}$$

since $|F_i| = \ell + 1$ and $\mathcal{U}_{F_i} \cap \mathcal{D}_{F_i} = \{F_i\}$.

Note that for each $i > 1$, we have

$$\mathcal{D}_{F_i} \setminus \bigcup_{j<i} \mathcal{D}_{F_j} = \mathcal{D}_{F_i} \setminus \mathcal{D}_{F_1} = \{\{i\} \cup Y : Y \subseteq G\}. \tag{4.2}$$

Similarly, observe that for each $i > 1$, we have

$$\mathcal{U}_{F_i} \setminus \bigcup_{j<i} \mathcal{U}_{F_i} = \{Z \subseteq [n] : Z \supseteq F_i, Z \cap \{1, \dots, i-1\} = \emptyset\}. \tag{4.3}$$

So now putting together (4.1) (to bound $|\mathcal{S}_1|$), (4.2), and (4.3), we obtain

$$\begin{aligned} \left|\bigcup_{i=1}^{k-1} \mathcal{S}_i\right| &= |\mathcal{S}_1| + \sum_{i=2}^{k-1} \left|\mathcal{D}_{F_i} \setminus \bigcup_{j<i} \mathcal{D}_{F_j}\right| + \sum_{i=2}^{k-1} \left|\mathcal{U}_{F_i} \setminus \bigcup_{j<i} \mathcal{U}_{F_j}\right| - (k-2) \\ &= 2^{\ell+1} + 2^{n-\ell-1} - 1 + (k-2)2^\ell + \left(\sum_{i=2}^{k-1} 2^{n-\ell-i}\right) - (k-2). \end{aligned}$$

The final term occurs as the sets F_i are counted both in their downset and their upset. Simplifying we get

$$c(\mathcal{A}) = k2^\ell + 2^{n-\ell}\left(1 - \frac{1}{2^{k-1}}\right) - (k-1).$$

□

We now prove the lower bound given in Theorem 1.3. We actually prove a slightly stronger statement.

Lemma 4.2. *Let $n, k \in \mathbb{N}$ where $n \geqslant 2k - 1 - \log_2 k \geqslant 1$. Then*

$$\sigma(n,k) \geqslant 2^n - \frac{3}{\sqrt{2}}\left(1 - \frac{1}{2^{k-1}}\right)^{\frac{1}{2}} \sqrt{2^n k} + 2(k-1).$$

Proof. Let a be an integer with the same partity as n to be specified later. Let $G := \{n - \frac{n-a}{2} + 1, \dots, n\}$. Let $\mathcal{A} = \{F_1, F_2, \dots, F_{k-1}\}$ be an antichain in $\mathcal{P}([n])$, where $F_i = G \cup \{i\}$. This is possible as long as $n - \frac{n-a}{2} \geqslant k-1$, that is, $n \geqslant 2(k-1) - a$.

By Lemma 4.1 (setting $\ell = \frac{n-a}{2}$), we obtain

$$c(\mathcal{A}) = k2^{\frac{n-a}{2}} + 2^{\frac{n+a}{2}}\left(1 - \frac{1}{2^{k-1}}\right) - (k-1).$$

Define $\mathcal{F}_i := \{F_i\}$ for $1 \leqslant i \leqslant k-1$ and $\mathcal{F}_k := \{Z \subseteq [n] : Z$ is incomparable to F_i for all $1 \leqslant i \leqslant k-1\}$. By construction, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. We have

$$\begin{aligned}\sum_{i=1}^{k} |\mathcal{F}_i| &= (k-1) + 2^n - c(\mathcal{A}) \\ &= 2^n - \sqrt{2^n}\left(\frac{k}{\sqrt{2^a}} + \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right) + 2(k-1). \qquad (4.4)\end{aligned}$$

Differentiating this expression with respect to a gives

$$\frac{\ln 2}{2}\sqrt{2^n}\left(\frac{k}{\sqrt{2^a}} - \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right).$$

Thus we can see that if there were no restrictions on a the maximum value of (4.4) would be achieved when $2^a = k\frac{2^{k-1}}{2^{k-1}-1}$; that is, $a = \log_2(k) + \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right)$. However, we require a to be an integer with the same parity as n. Set a to be the unique such integer such that

$$-1 < a - \log_2(k) - \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right) \leqslant 1$$

and let $c = a - \log_2(k) - \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right)$. Note that $n \geqslant 2k - 1 - \log_2 k \geqslant 1$ by hypothesis. This ensures that $n \geqslant 2(k-1) - a$ for any such value of a. From (4.4) we have

$$\begin{aligned}\sum_{i=1}^{k} |\mathcal{F}_i| &= 2^n - \sqrt{2^n}\left(\frac{k}{\sqrt{2^a}} + \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right) + 2(k-1) \\ &= 2^n - \sqrt{2^n k}\left(1 - \frac{1}{2^{k-1}}\right)^{1/2}\left(\frac{1}{\sqrt{2^c}} + \sqrt{2^c}\right) + 2(k-1) \qquad (4.5) \\ &\leqslant 2^n - \sqrt{2^n k}\left(1 - \frac{1}{2^{k-1}}\right)^{1/2}\left(\frac{3}{\sqrt{2}}\right) + 2(k-1)\end{aligned}$$

where the last inequality follows from the fact that the bracketed expression in (4.5) is maximised when $c = 1$ for c in the range $-1 < c \leqslant 1$. □

For certain values of k we can prove a stronger lower bound which essentially matches the upper bound of Theorem 1.3.

Corollary 4.6. *Let $n, k \in \mathbb{N}$ and suppose that $k = 2^a$ where a has the same parity as n and $n \geqslant 2(k-1) - a$. Then*

$$\sigma(n,k) \geqslant 2^n - 2\sqrt{2^n k}\left(1 - \frac{1}{2^k}\right) + 2(k-1).$$

Proof. If we apply the proof of Lemma 4.2 with $a = \log_2 k$, then the result follows from (4.4). □

4.2 Upper Bound on $\sigma(n, k)$

Lemma 4.3. *For $k \geqslant 2$ and n such that $2^n \geqslant (k-1)(1+\sqrt{k-1})^2$,*

$$\sigma(n,k) \leqslant 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

Proof. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. We may and will assume that $|\mathcal{F}_1| \leqslant |\mathcal{F}_2| \leqslant \cdots \leqslant |\mathcal{F}_k|$. Define $\mathcal{G} := \cup_{i=1}^{k-1} \mathcal{F}_i$. Let $m = |\mathcal{G}|$ and observe that, as each family is non-empty, we have $m \geqslant k-1$.

By Theorem 2.3, $|\mathcal{F}_k| \leqslant 2^n - c(n,m) \leqslant 2^n - 2^{n/2+1}\sqrt{m} + m = \left(\sqrt{2^n} - \sqrt{m}\right)^2$. Since the families are ordered by increasing size, $|\mathcal{F}_k| \geqslant \frac{m}{k-1}$. Putting this together gives

$$\frac{m}{k-1} \leqslant |\mathcal{F}_k| \leqslant \left(\sqrt{2^n} - \sqrt{m}\right)^2.$$

Rearranging, we obtain

$$\sqrt{m} \leqslant \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right). \tag{4.7}$$

Now consider the sum

$$\sum_{i=1}^{k} |\mathcal{F}_i| = |\mathcal{G}| + |\mathcal{F}_k| \leqslant m + \left(\sqrt{2^n} - \sqrt{m}\right)^2. \tag{4.8}$$

Let $x = \frac{1}{2}\sqrt{2^n} - \sqrt{m}$. Substituting this into the right hand side of (4.8) gives

$$\left(\tfrac{1}{2}\sqrt{2^n} - x\right)^2 + \left(\tfrac{1}{2}\sqrt{2^n} + x\right)^2 = 2^{n-1} + 2x^2$$

and it is clear that the right hand side of (4.8) is maximised when $|x| = \left|\frac{1}{2}\sqrt{2^n} - \sqrt{m}\right|$ is as large as possible. Combining $m \geqslant k-1$ with (4.7) gives $\sqrt{k-1} \leqslant \sqrt{m} \leqslant \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right)$, we need only find which of these end values is further from $\frac{1}{2}\sqrt{2^n}$.

If we have $2^n \geqslant (k-1)(1+\sqrt{k-1})^2$ then

$$\tfrac{1}{2}\sqrt{2^n} - \sqrt{k-1} \geqslant \tfrac{1}{2}\sqrt{2^n} - \frac{\sqrt{2^n}}{1+\sqrt{k-1}} = \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right) - \tfrac{1}{2}\sqrt{2^n}$$

and thus expression (4.8) is maximised when $m = k-1$. Substituting $m = k-1$ into (4.8) gives

$$\sum_{i=1}^{k} |\mathcal{F}_i| \leqslant (k-1) + \left(\sqrt{2^n} - \sqrt{k-1}\right)^2$$
$$= 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

□

Proof of Theorem 1.3. Lemmas 4.2 and 4.3 together give Theorem 1.3. Observe that $2^{2k} \geqslant (k-1)(1+\sqrt{k-1})^2$ for $k \geqslant 2$ so the conditions of Lemma 4.3 hold. □

5 Closing remarks

In Section 3 we provided upper and lower bounds on $\pi(n,k)$ in Theorems 1.1 and 1.2. Comparing these bounds shows that they differ by a factor of $\left(\frac{e}{2}\right)^k$ for k even and less than $\left(1+\frac{1}{k}\right)\left(\frac{e}{2}\right)^k$ for k odd. It would be interesting to tighten this gap. We believe that (for large n) the bound given in Lemma 3.1 ought to be essentially best possible.

Conjecture 5.1. *Let $k \geqslant 2$ be fixed and n be sufficiently large with respect to k. Then*

$$\pi(n,k) = (1+o(1))\left(\frac{(k-1)^{k-1}}{k^k}2^n\right)^k.$$

Our lower bound on $\pi(n,k)$ holds in the case $n > k\log_2 k + k$. For small fixed values of n and k, we also have some bounds for $\pi(n,k)$, see [15]. In particular, we have $f(4,3) = 9$, $f(5,3) \geqslant 81$ $f(6,3) \geqslant 810$ and $f(5,4) \geqslant 108$.

Note added before submission: In the final stages of preparation of this article, we noticed a recent paper of Gowty, Horsley, and Mammoliti [9], concerning the comparability number. They give a very different proof of Theorem 2.3 (see Corollary 1.2 of [9]) and use it as we do to deduce Theorem 2.4. They also provide some very interesting further analysis of the comparability number and sets that minimise $c(n,m)$.

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