BE-Diperfect Digraphs with Stability Number Two

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Submitted: Feb 23, 2024; Accepted: May 19, 2024; Published: Jun 14, 2024

Abstract

In 1982, Berge defined the class of α-diperfect digraphs. A digraph D is α-diperfect if every induced subdigraph of H of D satisfies the following property: for every maximum stable set S of H there is a path partition $\mathcal{P}$ of H in which every $P \in \mathcal{P}$ contains exactly one vertex of S. Berge conjectured a characterization of α-diperfect digraphs by forbidding induced orientations of odd cycles. In 2018, Sambinelli, Nunes da Silva and Lee proposed a similar class of digraphs. A digraph D is BE-diperfect if every induced subdigraph H of D satisfies the following property: for every maximum stable set S of H there is a path partition $\mathcal{P}$ of H in which (i) every $P \in \mathcal{P}$ contains exactly one vertex of S and (ii) P either begins or ends at a vertex of S. They also conjectured that the BE-diperfect digraphs can be characterized by forbidding induced orientations of odd cycles; we refer to this as the Begin-End Conjecture. In 2023, de Paula Silva, Nunes da Silva and Lee presented an infinite family of counterexamples with stability number two to Berge's Conjecture. On the other hand, these digraphs are not counterexamples to the Begin-End Conjecture. In this paper, we prove that the latter conjecture holds for digraphs with stability number two.

Mathematics Subject Classifications: 05C20, 05C75

1 Introduction

We assume that the reader is familiar with common terminology in Graph Theory [1, 7]. In what follows we specify some notation. Let $D = (V, A)$ denote a digraph. The *underlying graph* of D, denoted by $U(D)$, is the simple graph with vertex set $V(D)$ such that u and v are adjacent in $U(D)$ if and only if $(u, v) \in A(D)$ or $(v, u) \in A(D)$.

The *converse* of a digraph D is a digraph D' obtained from D by reversing each arc of D, i.e., $V(D') = V(D)$ and $(u, v) \in A(D')$ if and only if $(v, u) \in A(D)$. Note that every statement that holds for a digraph D has an analogue that holds for its converse

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digraph. This property is called *Principle of Directional Duality* and we use it in several places throughout this text.

Henceforth, when we say a path of a digraph, we mean a directed path. A path P of a digraph D is *hamiltonian* if $V(P) = V(D)$. By a *cycle* in a digraph D we mean a digon (a directed cycle of length two) or a subdigraph C of D such that $U(C)$ is a cycle. A *path partition* of D is a collection of pairwise disjoint paths of D that cover $V(D)$. Let $\pi(D)$ denote the smallest size of path partition of D. A *stable set* of a digraph D is a stable set of its underlying graph $U(D)$. The *stability number* of a digraph is the size of its maximum stable set, denoted by $\alpha(D)$. In 1960, Gallai and Milgram [15] showed that, for every digraph, the size of a minimum path partition $\pi(D)$ is less than or equal to its stability number.

Theorem 1 (Gallai-Milgram [15])**.** *For every digraph D, $\alpha(D) \geqslant \pi(D)$.*

Actually, it is possible to prove a stronger statement using the concept of *orthogonality*, defined next. Let $\mathcal{P}$ be a path partition and let S be a stable set of D. We say that $\mathcal{P}$ and S are *orthogonal* if $|S \cap P| = 1$ for every $P \in \mathcal{P}$; we also say that S is orthogonal to $\mathcal{P}$ or vice versa. Theorem 2 implies that there is a stable set orthogonal to every minimum path partition of D and so, inequality $\pi(D) \leqslant \alpha(D)$ immediately holds. The first proof of this theorem is due to Linial [17]. Other proofs of this result can be found in [7, 11].

Theorem 2. *Let D be a digraph and let $\mathcal{P}$ be a path partition of D. Then,*

(i) there exists a path partition $\mathcal{Q}$ of D such that ini($\mathcal{Q}$) $\subset$ ini($\mathcal{P}$), ter($\mathcal{Q}$) $\subset$ ter($\mathcal{P}$) and $|\mathcal{Q}| = |\mathcal{P}| - 1$, or

(ii) there exists a stable set S which is orthogonal to $\mathcal{P}$.

An *arborescence* is a connected digraph in which every vertex has in-degree one except for the *root*, which has in-degree zero. The *leaves* of an arborescence are the vertices with out-degree zero. An *arborescence forest* F is a disjoint union of arborescences of a digraph D. Let $R(F)$ and $L(F)$ denote the set of roots and leaves, respectively, of the arborescences of F. Let H be an arborescence. Let y be a leaf of H; a maximal path of H which ends at y and does not contain any vertex with out-degree at least two, is called a *terminal branch*. A *strong component* of a digraph D is a maximal strongly connected subdigraph of D. A strong component C of D is a *source-component* (respectively, *sink-component*) if no vertex of C has an in-neighbour (respectively, out-neighbour) in $D - V(C)$.

Let D be a digraph. We say that $B \subseteq V(D)$ is a *basis* if every vertex of D can be reached by a vertex of B and no two distinct vertices in B are connected by a path in D. Berge showed [4] that any basis of D is the root set of a spanning arborescence forest of D with at most $\alpha(D)$ leaves. Note that Berge's result implies $\alpha(D) \geqslant \pi(D)$ (Gallai-Milgram Theorem).

Theorem 3 (Berge [4])**.** *Let D be a digraph and let B be a basis of D. Then, there are a spanning arborescence forest F in D with $R(F) = B$ and a stable set meeting every terminal branch of F. In particular, F has at most $\alpha(D)$ leaves.*

In 1982, Berge [3] introduced a new class of digraphs which he called α-diperfect digraphs. A digraph D is *α-diperfect* if every induced subdigraph H of D has the following property: for every maximum stable set S of H, there exists a path partition $\mathcal{P}$ orthogonal to S. Berge was interested in obtaining a characterization of α-diperfect in terms of forbidden induced subdigraphs, similar to his conjecture on perfect graphs [2]. This conjecture was proved in 2006 by Chudnovsky, Robertson, Seymour and Thomas and it is known as the Strong Perfect Graph Theorem (Theorem 4).

Theorem 4 (Chudnovsky *et al* [9])**.** *A graph G is perfect if and only if G does not contain an odd cycle with five or more vertices or its complement as an induced subgraph.*

Berge [3] proved that symmetric digraphs (defined next) and digraphs whose underlying graph is perfect are α-diperfect. A *super-orientation* of a graph G is a digraph D obtained from G by replacing each edge uv of G by an arc (u, v), or an arc (v, u), or both. A digraph D is *symmetric* if D is a super-orientation of a graph G in which every edge uv of G is replaced by both arcs (u, v) and (v, u). On the other hand, Berge also showed that there are super-orientations of odd cycles that are not α-diperfect. We say that a super-orientation D of an odd cycle is an *anti-directed odd cycle* if $U(D) = (y_0, \ldots, y_{2k}, y_0)$ with $k \geqslant 2$ and each of $y_0, y_1, y_2, y_3, y_5, y_7, \ldots, y_{2k-1}$ is either a source or a sink in D. Berge [3] proved that a super-orientation D of an odd cycle with at least five vertices is α-diperfect if and only if D is not an anti-directed odd cycle. Based on that, he proposed the following conjecture that aims to characterize α-diperfect digraphs.

Conjecture 5 (Berge [3])**.** A digraph D is α-diperfect if and only if D does not contain an anti-directed odd cycle as an induced subdigraph.

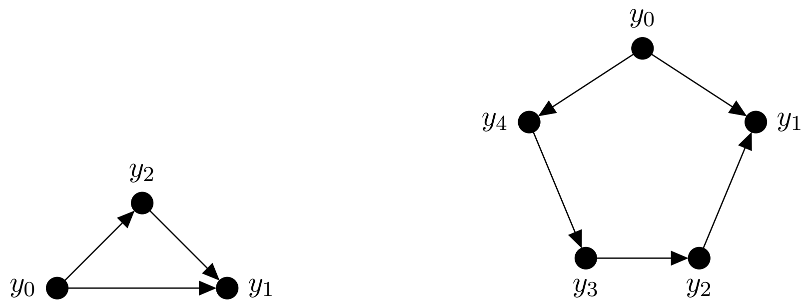
Motivated by Berge's Conjecture, Sambinelli, Nunes da Silva and Lee [19] proposed in 2018 a similar conjecture. Before we state it, we need some definitions.

A digraph D is *BE-diperfect* if every induced subdigraph H of D satisfies the following property: for every maximum stable set S of H there is a path partition $\mathcal{P}$ of H in which (i) $\mathcal{P}$ is orthogonal to S and (ii) P either begins or ends at a vertex of S. We also say that $\mathcal{P}$ is an *S_{BE}-path partition of* D. A super-orientation D of an odd cycle is a *blocking odd cycle* if $U(D) = (y_0, \ldots, y_{2k}, y_0)$ with $k \geqslant 1$ and each of y_0 and y_1 is either a source or a sink in D. We also say that y_0, y_1 forms a *blocking pair* of D.

Blocking odd cycles are not BE-diperfect. Figure 1 shows examples of blocking odd cycles and maximum stable sets that do not admit an S_{BE}-path partition. Let $\mathfrak{D}$ be the family of digraphs such that $D \in \mathfrak{D}$ if and only if D does not contain an induced blocking odd cycle.

Conjecture 6 (Sambinelli, Nunes da Silva and Lee [20])**.** A digraph D is BE-diperfect if and only if $D \in \mathfrak{D}$.

In 2023, de Paula Silva, Nunes da Silva and Lee [10] showed that Berge's Conjecture is false by presenting an infinite family of non-α-diperfect super-orientations of complements of odd cycles with at least seven vertices. On the other hand, these digraphs are not counterexamples to Conjecture 6. In fact, it can be shown that a super-orientation D of



(a) The set $\{y_2\}$ is a maximum stable set, but there is no hamiltonian path which begins or ends at y_2.

(b) The set $S = \{y_2, y_4\}$ is a maximum stable set, but there is no path partition $\mathcal{P}$ orthogonal to S in which every path $P \in \mathcal{P}$ begins or ends at a vertex of S.

Figure 1: Examples of blocking odd cycles.

the complement of an odd cycle with at least seven vertices is BE-diperfect if and only if D does not contain a blocking odd cycle with three vertices as an induced subdigraph [12].

Sambinelli, Nunes da Silva and Lee [20] and Freitas and Lee [13], [14] proved that Conjectures 5 and 6 are true for some generalizations of tournaments. In his PhD thesis, Sambinelli [19] proved that Conjecture 6 holds for every digraph with stability number two and at least four strong components. Also, Freitas [12] in his PhD thesis proved that Conjecture 6 holds for every digraph with stability number two that does not contain digons. In this paper, we prove that Conjecture 6 holds for digraphs with stability number two. In other words,

Theorem 7. *Let $D \in \mathfrak{D}$. If D has stability number two, then D is BE-diperfect.*

Corollary 8. *Let D be a digraph with stability number two. Then, D is BE-diperfect if and only if D does not contain a blocking odd cycle with three or five vertices as an induced subdigraph.*

We also note that there are many other results closely related to this topic. In 2001, Thomassé [21] proved that every strong digraph D with $\alpha(D) > 1$ has a spanning arborescence with at most $\alpha(D) - 1$ leaves (Las Vergnas Conjecture [16]), i.e., D has a path partition with at most $\alpha(D) - 1$ paths. Such a result extends Chen-Manalastas's Theorem [8] that states that every strong digraph with stability number two has a hamiltonian path. In 2007, Bessy and Thomassé [6] proved that every strong digraph D has $\alpha(D)$ directed cycles that cover $V(D)$. A digraph is D *k-strong* if D has at least $k+1$ vertices and for every set $S \subset V(D)$ such that $|S| = k$, it follows that $D - S$ is strong. The maximum k for which D is k-strong is denoted by $\kappa(D)$. In 2008, Bessy [5] conjectured that for every digraph D with $\kappa(D) \geqslant \alpha(D)$ and for every $S \subset V(D)$ with $\alpha(D)$ vertices, there is a path partition $\mathcal{P}$ in which every path begins at a vertex of S. He also proved this conjecture when $\alpha(D) = 2$.

2 Arborescences in digraphs

We start this section by relaxing the concept of a basis of a digraph. We say that $X \subseteq V(D)$ is a *quasi-basis* if every vertex of D can be reached by a vertex of X. Thus, every subset $X \subseteq V(D)$ that contains at least one vertex of each source-component of D is a quasi-basis of D. We denote by $\text{ter}(P)$ (respectively, $\text{ini}(P)$) the terminal (respectively, initial) vertex of a path P. Similarly, if $\mathcal{P}$ is a collection of paths, we denote by $\text{ter}(\mathcal{P})$ ($\text{ini}(\mathcal{P})$) the set of terminal (respectively, initial) vertices of the paths in $\mathcal{P}$. In the following lemmas, we present properties about paths, arborescences and quasi-basis of a digraph.

Lemma 9. *If $S \subset V(D)$ is a quasi-basis of a digraph D such that $|S| \geqslant \alpha(D)$, then*

(i) there is a path partition $\mathcal{P}$ of D such that $\text{ini}(\mathcal{P}) = S$; or

(ii) there is a non-empty $X \subset S$ such that X is a quasi-basis of $D - (S - X)$.

Proof. Let D' be the digraph obtained from D by identifying the vertices of S into a vertex u. So, $\{u\}$ is a basis in D'. By Theorem 3, there is a spanning arborescence H of D' with $R(H) = \{u\}$ and a stable set Y meeting every terminal branch of H. Since u is the root of H and $D' - u = D - S$ is a subdigraph of D, we may deduce that H has at most $|Y| \leqslant \alpha(D)$ leaves. Thus, the out-degree of u is at most $\alpha(D)$. It is easy to see that H corresponds to a spanning arborescence forest F of D such that $R(F) = S$ and $A(F) = A(H)$. Let $X \subseteq S$ be the set of roots of non-trivial arborescences of F. Since $|S| \geqslant \alpha(D)$, if $X = S$, then every arborescence of F is a path and assertion (i) holds. Otherwise, X is a quasi-basis of $D - (S - X)$ and assertion (ii) holds. □

If w is a vertex of an arborescence F with out-degree at least two and w is not the root of F, then we say that w is a *branch vertex* of F.

Lemma 10. *Let D be a digraph with stability number two. If $S = \{u, v\}$ is a quasi-basis of D, then D has a path partition $\mathcal{P}$ such that every $P \in \mathcal{P}$ has exactly one vertex of S.*

Proof. By Lemma 9, we may assume that (ii) holds, i.e., there is a non-empty $X \subset S$ such that X is quasi-basis of $D - (S - X)$. Without loss of generality, we may assume that $X = \{u\}$ is quasi-basis of $D - v$. By Theorem 3, there is a spanning arborescence H in $D - v$ such that $R(H) = \{u\}$ and $|L(H)| \leqslant 2$. If $|L(H)| = 1$, then H is a path beginning at u. Hence, $\{H, v\}$ is the desired path partition of D. So, assume that $|L(H)| = 2$. Let Q_1 and Q_2 be the terminal branches of H. Note that $Q = H - V(Q_1) - V(Q_2)$ is a path. Let $\mathcal{Q} = \{Q_1, Q_2, (v)\}$ be a path partition of $D - V(Q)$. By Theorem 2, there is a path partition $\mathcal{P} = \{P_1, P_2\}$ of $D - V(Q)$ such that $\text{ini}(\mathcal{P}) \subseteq \text{ini}(\mathcal{Q})$ and $\text{ter}(\mathcal{P}) \subseteq \text{ter}(\mathcal{Q})$. Assume without loss of generality that $v \in V(P_1)$. Note that $\text{ini}(P_2) \in \{\text{ini}(Q_1), \text{ini}(Q_2)\}$. Then, $\{P_1, QP_2\}$ is a path partition of D satisfying the required properties. □

Lemma 10 implies the following.

Corollary 11. *Let D be a digraph with stability number two. Suppose that D has a maximum stable set S that does not admit an S-path partition, i.e., D is a counterexample to Conjecture 5. Then, S is not a quasi-basis of D.*

All the counterexamples of Conjecture 5 given by de Paula Silva, Nunes da Silva and Lee [10] have stability number two and are important examples that illustrate Corollary 11. So far, there is no characterization of α-diperfect digraphs with stability number two. On the other hand, the situation is different when we deal with BE-diperfect digraphs with stability number two, as we discuss next.

Let D be a digraph with stability number two and let S be a maximum stable set that is a quasi-basis of D. Although Lemma 10 ensures the existence of an S-path partition of D, its proof guarantees that only that one of such paths begins at a vertex of S. To be able to prove that both paths begin or end at a vertex of S, that is, to be able to obtain an S_{BE}-path partition of D, we have to assume that D does not contain blocking odd cycles as induced subdigraphs. Observe that, since $\alpha(D) = 2$, the only possible blocking odd cycles that D may contain as induced subdigraphs are blocking odd cycles with three or five vertices. Henceforth, we will refer to a blocking odd cycle with three vertices as a *transitive triangle*.

Let $\mathfrak{D}_2$ be the family of digraphs such that $\alpha(D) = 2$ and $D \in \mathfrak{D}$. The following lemmas are technical results that will help us to find an S_{BE}-path partition when S is a quasi-basis of D and $D \in \mathfrak{D}_2$. We use the notation $X \mapsto Y$ to denote that every vertex of X dominates every vertex of Y in D and no vertex of Y dominates a vertex of X in D. If $X = \{u\}$ (respectively, $Y = \{v\}$), we may write directly $u \mapsto Y$ (respectively, $X \mapsto v$).

Lemma 12. *Let $D \in \mathfrak{D}_2$ and let $\mathcal{P} = \{P_1, P_2, (v)\}$ be a path partition of D. Then,*

(i) there is a path partition $\mathcal{R} = \{R_1, R_2\}$ of D such that $v \in ini(\mathcal{R}) \cup ter(\mathcal{R})$, $ter(\mathcal{R}) \subset ter(\mathcal{P})$ and $ini(R_i) \in \{ini(P_1), ini(P_2)\}$ when $v \notin V(R_i)$, or

(ii) there is a spanning arborescence H of $D - v$ with at most two leaves such that $R(H) \subset \{ini(P_1), ini(P_2)\}$ and $L(H) \cap \{ter(P_1) \cup ter(P_2)\}$ is not empty.

Proof. Let $P_1 = (x_1, \ldots, x_k)$ and let $P_2 = (y_1, \ldots, y_\ell)$. The following assertions are straightforward.

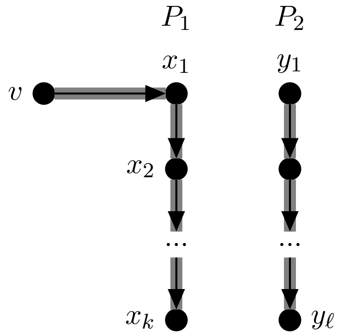
(1) If v dominates x_1 or y_1, then (i) holds. Similarly, if x_k or y_ℓ dominates v, then (i) holds. See Figure 2a.

(2) If x_1 dominates v and x_k dominates x_1, then (i) holds. Similarly, If y_1 dominates v and y_ℓ dominates y_1, then (i) holds. See Figure 2b.

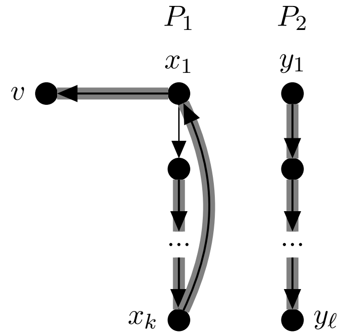
(3) If x_1 and y_1 are adjacent, then (ii) holds. See Figure 2c.

(4) If x_k dominates y_1, then both (i) and (ii) hold. Similarly, if y_ℓ dominates x_1, then both (i) and (ii) holds. See Figure 2d.

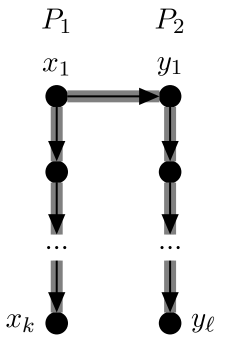
(5) If x_1 dominates y_ℓ and y_ℓ dominates y_1, then (ii) holds. Similarly, if y_1 dominates x_k and x_k dominates x_1, then (ii) holds. See Figure 2e.



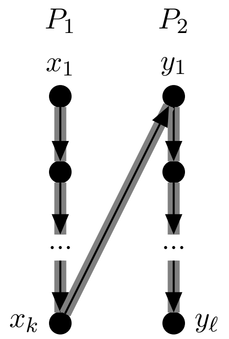
(a) Extending P_1 when v dominates x_1.



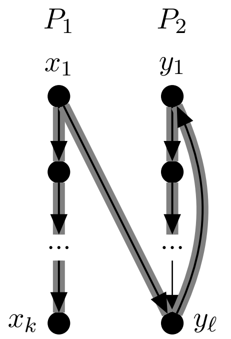
(b) Extending P_1 when x_1 dominates v and x_k dominates x_1.



(c) Building an arborescence rooted at x_1 when x_1 dominates y_1.



(d) Building an arborescence rooted at x_1 when x_k dominates y_1.



(e) Building an arborescence rooted at x_1 when x_1 dominates y_ℓ and y_ℓ dominates y_1.

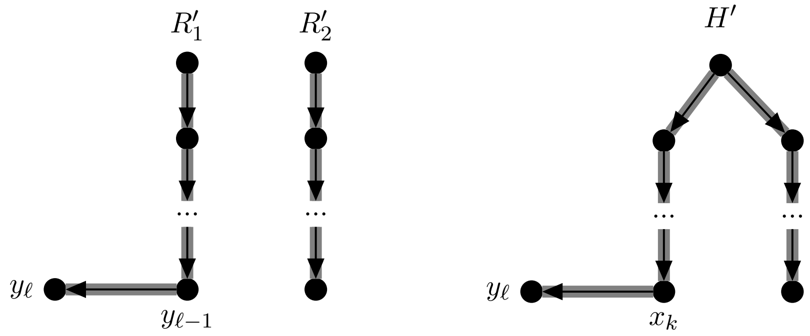
Figure 2: Auxiliary illustration for the proof of Lemma 12.

The proof follows by induction on the number of vertices. Suppose first that x_k and y_ℓ are adjacent. Without loss of generality, we may assume that x_k dominates y_ℓ. If $\ell = 1$, then the result follows by (4). Otherwise, let $D' = D - y_\ell$, let $P_2' = P_2 - y_\ell$ and let $\mathcal{P}' = \{P_1, P_2', (v)\}$. By induction hypothesis applied to D' and $\mathcal{P}'$, it follows that

(a) there is a path partition $\mathcal{R}' = \{R_1', R_2'\}$ of D' such that $v \in \text{ini}(\mathcal{R}') \cup \text{ter}(\mathcal{R}')$, $\text{ter}(\mathcal{R}') \subset \text{ter}(\mathcal{P})$ and $\text{ini}(R_i') \in \{\text{ini}(P_1), \text{ini}(P_2')\}$ when $v \notin V(R_i')$, or

(b) there is a spanning arborescence H' of $D[V(P_1) \cup V(P_2')]$ with at most two leaves such that $R(H) \subset \{x_1, y_1\}$ and $L(H) \cap \{x_k, y_{\ell-1}\}$ is not empty.

Suppose first that (a) holds. Then, at least one of x_k or $y_{\ell-1}$ must belong to $\text{ter}(\mathcal{R}')$. Without loss of generality, we may assume that $\text{ter}(R_1') \in \{x_k, y_{\ell-1}\}$. If $\text{ter}(R_1') = y_{\ell-1}$, then $R = \{R_1' y_\ell, R_2'\}$ is a path partition for which (i) holds (see Figure 3a). The argument is analogous when $\text{ter}(R_1') = x_k$. Now suppose that (b) holds. Then, x_k or $y_{\ell-1}$ is a leaf of H'. If $y_{\ell-1} \in L(H')$, then $H = H' + (y_{\ell-1}, y_\ell)$ is a spanning arborescence of $D[V(P_1) \cup V(P_2)]$ for which (ii) holds (see Figure 3b). Otherwise, $x_k \in L(H')$ and $H = H' + (x_k, y_\ell)$ is a spanning arborescence of $D[V(P_1) \cup V(P_2)]$ for which (ii) holds.



(a) Extending R_1' with y_ℓ.

(b) Extending H' by adding y_ℓ as a leaf.

Figure 3: Auxiliary illustration for the proof of Lemma 12.

Thus, we may assume that x_k and y_ℓ are non-adjacent. By (3), we may assume that $\{x_1, y_1\}$ is a maximum stable set of D. Thus, v must be adjacent to x_1 or y_1. In particular, if $k = \ell = 1$, then the result follows by (1). Assume then that $k > 1$ or $\ell > 1$. Suppose first that $\ell = 1$ (so $k > 1$). Towards a contradiction, assume that neither (i) nor (ii) holds. By (1) and (3), we may assume that both v and x_1 are non-neighbors of y_ℓ. Since x_k and y_ℓ are also non-adjacent, and $\alpha(D) = 2$, it follows that $U(D[\{v, x_1, x_k\}])$ must induce a triangle C. By (1), we may assume that $x_1 \mapsto v$ and $v \mapsto x_k$. By (2), $x_1 \mapsto x_k$. Thus, C is a transitive triangle, a contradiction. Hence, we may assume that $\ell > 1$. By a similar argument, we may assume that $k > 1$.

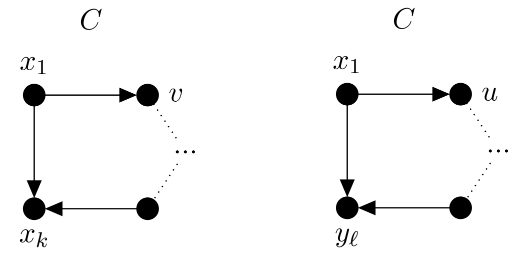
Let $S_1 = \{x_1, y_1\}$, $S_2 = \{x_k, y_\ell\}$ and $S_3 = \{v\}$. Since $\alpha(D) = 2$, $D[S_1 \cup S_2 \cup S_3]$ cannot be bipartite. Hence, there is an induced odd cycle C in $U(D[S_1 \cup S_2 \cup S_3])$ which contains v. Without loss of generality, we may assume that the neighbor of v in C that belongs to S_1 is x_1. By (1), $x_1 \mapsto v$. We consider the following cases.

Case 1. The neighbors of x_1 in C are v and x_k.

By (2), $x_1 \mapsto x_k$. Note that the other neighbor of x_k in C is either v or y_1. If v and x_k are adjacent, then by (1), $v \mapsto x_k$. Similarly, if y_1 and x_k are adjacent, then by (4), $y_1 \mapsto x_k$. In either case, x_1 and x_k are respectively, a source and a sink in C; hence, x_1, x_k is a blocking pair of C, a contradiction (see Figure 4a).

Case 2. The neighbors of x_1 in C are v and y_ℓ.

By (4), $x_1 \mapsto y_\ell$. Note that the other neighbor of y_ℓ in C is either v or y_1. If y_ℓ and v are adjacent, then by (1), $v \mapsto y_\ell$. Similarly, if y_ℓ and y_1 are adjacent, then by (5), $y_1 \mapsto y_\ell$. In either case, x_1 and y_ℓ are respectively, a source and a sink in C; hence, x_1, y_ℓ is a blocking pair of C, a contradiction (see Figure 4b).



(a) x_1, x_k is a blocking pair of C.

(b) x_1, y_ℓ is a blocking pair of C.

Figure 4: Auxiliary illustration for the proof of Lemma 12.

This completes the proof of the lemma. □

Lemma 13. *Let $D \in \mathfrak{D}_2$. If $S = \{u, v\}$ is a quasi-basis of D, then D has a path partition $\mathcal{P}$ with $|\mathcal{P}| = 2$ in which every $P \in \mathcal{P}$ begins or ends at a vertex of S.*

Proof. We may assume that Lemma 9 (ii) holds, otherwise the result follows. So, there is a non-empty $X \subset S$ such that X is a quasi-basis of $D - (S - X)$. Without loss of generality, we may assume that $X = \{u\}$ is a (quasi-)basis of $D - v$. By Theorem 3, there is a spanning arborescence F in $D - v$ such that $R(F) = \{u\}$ and $|L(F)| \leqslant 2$. Choose F that minimizes $|L(F)|$ and, subject to that, F maximizes the distance between the root u and its branch vertex w^*, if $|L(F)| = 2$. If $|L(F)| = 1$, then F is a path beginning at u. Hence, $\mathcal{R} = \{F, (v)\}$ is a path partition of D with the required properties. Otherwise, $|L(F)| = 2$. Let P_1 and P_2 be the terminal branches of F. Note that $Q = F - V(P_1) - V(P_2)$ is a path. Let $\mathcal{P} = \{P_1, P_2, (v)\}$. Note that $\mathcal{P}$ is a path partition of $D - V(Q)$. By Lemma 12, it follows that

(i) there is a path partition $\mathcal{R} = \{R_1, R_2\}$ of $D - V(Q)$ such that $v \in \text{ini}(\mathcal{R}) \cup \text{ter}(\mathcal{R})$, $\text{ter}(\mathcal{R}) \subset \text{ter}(\mathcal{P})$ and $\text{ini}(R_i) \in \{\text{ini}(P_1), \text{ini}(P_2)\}$ when $v \notin V(R_i)$, or

(ii) there is a spanning arborescence H of $D[V(P_1) \cup V(P_2)]$ with at most two leaves such that $R(H) \subset \{\text{ini}(P_1), \text{ini}(P_2)\}$ and $L(H) \cap \{\text{ter}(P_1) \cup \text{ter}(P_2)\}$ is not empty.

Suppose that (i) holds. Without loss of generality, we may assume that $v \in V(R_1)$. Hence, $\text{ini}(R_2) \in \{\text{ini}(P_1), \text{ini}(P_2)\}$. Thus, $\{R_1, QR_2\}$ is a path partition of D with the required properties. Suppose then that (ii) holds. Since $R(H) = \{r\} \subset \{\text{ini}(P_1), \text{ini}(P_2)\}$, then $F' = (Q \cup H) + (w^*, r)$ is a spanning arborescence of $D - v$ with at most two leaves and $R(F') = \{u\}$. However, either $|L(F')| < |L(F)|$ or the distance between u and the branch vertex of F' is greater than the distance between u and the branch vertex of F, a contradiction to the choice of F. □

Lemma 13 and the Principle of Directional Duality immediately imply the following.

Corollary 14. *Let $D \in \mathfrak{D}_2$ be a digraph and let S be a maximum stable set of D. If $S \cap V(H)$ is not empty for every source-component H of D, then D has an S_{BE}-path partition. Similarly, if $S \cap V(H)$ is not empty for every sink-component H of D, then D has an S_{BE}-path partition.*

In particular, since every non-empty subset of vertices of a strong digraph D is a quasi-basis of D, the following holds.

Corollary 15. *Let $D \in \mathfrak{D}_2$ be a digraph and let S be a maximum stable set of D. If D is strong, then D has an S_{BE}-path partition.*

3 Main Result

In this section, we prove that Conjecture 6 holds for every digraph with stability number two.

Lemma 16. *Let D be a digraph and let S_1, S_2 and S_3 be disjoint stable sets of D such that there is no arc from S_j to S_i for $1 \leqslant i < j \leqslant 3$. Then, $D[S_1 \cup S_2 \cup S_3]$ is either bipartite or contains a blocking odd cycle as an induced subdigraph. Moreover, if $D \in \mathfrak{D}$, then $|S_1| + |S_2| + |S_3| \leqslant 2\alpha(D)$.*

Proof. Let $D' = D[S_1 \cup S_2 \cup S_3]$. Suppose that D' is not bipartite. Then, there is an induced odd cycle C in D'. Clearly, C must contain vertices of all the three sets S_1, S_2 and S_3. Note that all the vertices of S_1 are sources in D'. Similarly, all the vertices of S_3 are sinks in D'. Moreover, there is at least one pair of adjacent vertices in C such that $u \in S_1$ and $v \in S_3$. Thus, u, v is a blocking pair of C and C is a blocking odd cycle of D'. If $D \in \mathfrak{D}$, then D' must be bipartite with each part having at most $\alpha(D)$ vertices. Therefore, $|S_1| + |S_2| + |S_3| \leqslant 2\alpha(D)$. □

We say that a digraph D is *semicomplete* if D is a super-orientation of a complete graph and it is *complete* if D is a symmetric super-orientation of a complete graph. In 1934, Rédei proved that every semicomplete digraph has a hamiltonian path.

Theorem 17 (Rédei [18]). *Every semicomplete digraph has a hamiltonian path.*

Sambinelli, Nunes da Silva and Lee [20] proved that a semicomplete digraph $D \in \mathfrak{D}$ has a hamiltonian path P beginning or ending at any $v \in V(D)$, that is, D is BE-diperfect.

Theorem 18 (Sambinelli, Nunes da Silva and Lee [20]). *If $D \in \mathfrak{D}$ is a semicomplete digraph, then D is BE-diperfect.*

The same authors showed that any minimum counterexample for Conjecture 6 cannot be partitioned into certain subdigraphs, as we state in the following lemma.

Lemma 19 (Sambinelli, Nunes da Silva and Lee [20]). *If a digraph D can be partitioned into k vertex-disjoint induced subdigraphs, say $H_1, H_2, \ldots, H_k$, such that every H_i is BE-diperfect and $\alpha(D) = \sum_{i=1}^{k} \alpha(H_i)$, then D is BE-diperfect.*

For digraphs with stability number two, the previous lemma and Theorem 18, immediately imply the following.

Corollary 20. *Let $D \in \mathfrak{D}_2$ and let S be a maximum stable set of D. If D can be partitioned into two vertex-disjoint semicomplete digraphs, then D admits an S_{BE}-path partition.*

We say that a path $P = (w_1, \dots, w_\ell)$ of D is *symmetric* if $P^{-1} = (w_\ell, w_{\ell-1}, \dots, w_1)$ is also a path of D, i.e., each pair w_i, w_{i+1} is joined by a digon.

Lemma 21. *Let $D \in \mathfrak{D}_2$ such that every proper induced subdigraph of D is BE-diperfect. Let S be a maximum stable set of D such that there is a pair of vertex-disjoint symmetric paths P_1 and P_2 satisfying the following properties:*

- *at least one of P_1 and P_2 is non-trivial,*
- *$S = \{ini(P_1), ini(P_2)\}$ and*
- *$\{ter(P_1), ter(P_2)\}$ is a stable set.*

Then, D admits an S_{BE}-path partition.

Proof. Let P_1' be the path obtained from P_1 by deleting its terminal vertex t_1. Similarly, let P_2' be the path obtained from P_2 by deleting its terminal vertex t_2. Let $D' = D - (V(P_1') \cup V(P_2'))$. By hypothesis $S' = \{t_1, t_2\}$ is a maximum stable set of D' and D' admits an S'_{BE}-path partition $\{R_1, R_2\}$. We may assume without loss of generality and by the Principle of Directional Duality that $\text{ini}(R_1) = t_1$. If $\text{ini}(R_2) = t_2$, then $\{P_1'R_1, P_2'R_2\}$ is an S_{BE}-path partition of D. Otherwise, $\text{ter}(R_2) = t_2$ and $\{P_1'R_1, R_2(P_2')^{-1}\}$ is an S_{BE}-path partition of D. □

The following two lemmas ensure the existence of an S_{BE}-path partition when $D - S$ has a source and a sink of D.

Lemma 22. *Let $D \in \mathfrak{D}_2$ and let $S = \{u, v\}$ be a maximum stable set of D. If there is a pair of adjacent vertices $x, y \in V(D) \setminus S$ that are, respectively, a source and a sink of D, then D admits an S_{BE}-path partition.*

Proof. We claim that each vertex of $D - \{x, y\}$ is adjacent to precisely one of x or y. Towards a contradiction, suppose first that there is $w \in V(D) \setminus \{x, y\}$ that is adjacent to both x and y. Then, $D[\{w, x, y\}]$ induces a transitive triangle, a contradiction. Thus, since $\alpha(D) = 2$, we may assume without loss of generality that (a) $x \mapsto u$ and x is non-adjacent to v, and (b) $v \mapsto y$ and y is non-adjacent to u. Suppose now that there is $w \in V(D) \setminus \{x, y\}$ that is non-adjacent to x and neither to y. Note that $w \notin \{u, v\}$. Hence, by (a) and (b), $U(D[\{x, y, u, v, w\}])$ must be an induced odd cycle C; however, x, y is a blocking pair of C, a contradiction. Thus, each vertex of $D - \{x, y\}$ is adjacent to precisely one of x or y. This implies that we may partition $V(D) \setminus \{x, y\}$ into two sets X and Y such that every vertex of X is adjacent to x and non-adjacent to y, and every vertex of Y is adjacent to y and non-adjacent to x. Note that, since $\alpha(D) = 2$, both $X \cup \{x\}$ and $Y \cup \{y\}$ are semicomplete digraphs. So $V(D)$ can be partitioned into two semicomplete digraphs. By Corollary 20, D admits an S_{BE}-path partition. □

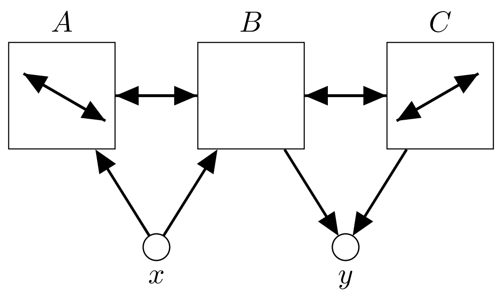


Figure 5: Auxiliary illustration for the proof of Lemma 23. Both A and C induce complete digraphs. Moreover, for every pair of adjacent vertices $w \in B$ and $z \in V(D) \setminus \{x, y\}$, there is a digon joining w and z.

Lemma 23. *Let $D \in \mathfrak{D}_2$ and let $S = \{u, v\}$ be a maximum stable set of D. If there is a pair of non-adjacent vertices $x, y \in V(D) \setminus S$ that are, respectively, a source and a sink of D, then D admits an S_{BE}-path partition.*

Proof. Towards a contradiction, suppose that the result does not hold and let D be a counterexample with a minimum number of vertices. Since $\alpha(D) = 2$, every vertex of $D - \{x, y\}$ is adjacent to x or y. Let A be the set of vertices that are adjacent to x and non-adjacent to y, let B be the set of vertices that are adjacent to both x and y and let C be the set of vertices that are adjacent to y and non-adjacent to x. Note that $V(D) = A \cup B \cup C \cup \{x, y\}$ and both $A \cup \{x\}$ and $C \cup \{y\}$ induce semicomplete digraphs. Moreover, since D does not contain transitive triangles as induced subdigraphs, $x \mapsto A \cup B$ and $B \cup C \mapsto y$, it follows that

(i) $D[A]$ and $D[C]$ are complete digraphs,

(ii) any pair of adjacent vertices in $A \cup B$ must be joined by a digon, and

(iii) any pair of adjacent vertices in $B \cup C$ must be joined by a digon.

Note that (ii) and (iii) implies that there is a digon joining any pair of adjacent vertices $w \in B$ and $z \in V(D) \setminus \{x, y\}$. See Figure 5.

Claim 24. *Every vertex of $B \setminus \{u, v\}$ is adjacent to both u and v.*

Proof. Let $w \in B \setminus \{u, v\}$. Since $\alpha(D) = 2$, w must be adjacent to u or v. Towards a contradiction, suppose that w is adjacent to u and non-adjacent to v. Then, by (ii) and (iii), there is a digon joining u and w, a contradiction to Lemma 21 with $P_1 = (u, w)$ and $P_2 = (v)$ (see Figure 6a). The argument is analogous when w is adjacent to v and non-adjacent to u. ☐

Claim 25. *Set $B \setminus \{u, v\}$ induces a complete digraph.*

Proof. Towards a contradiction, suppose that there is a stable set $\{w_1, w_2\} \subseteq B \setminus \{u, v\}$. By Claim 24, (ii) and (iii), there is a digon joining u and w_1 and a digon joining v and w_2, a contradiction to Lemma 21 with $P_1 = (u, w_1)$ and $P_2 = (v, w_2)$ (see Figure 6b). ☐

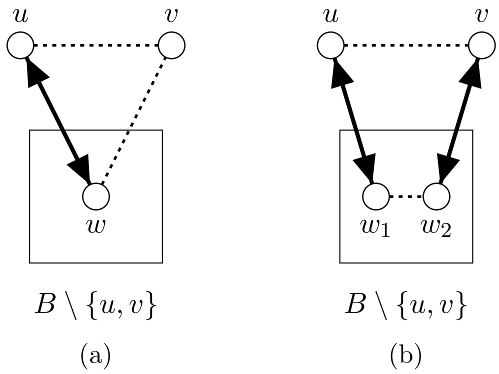


Figure 6: Auxiliary illustration for the proof of Claims 24 and 25

.

Claim 26. *If $w \in B \setminus \{u, v\}$, then w is adjacent to every vertex of A or w is adjacent to every vertex of C.*

Proof. Towards a contradiction, suppose that w is non-adjacent to a vertex $z_1 \in A$ and to a vertex $z_2 \in C$. By Claim 24, $\{z_1, z_2\} \cap \{u, v\} = \emptyset$. Moreover, by (ii) and (iii), there is a digon joining w and u and a digon joining w and v. Since $\alpha(D) = 2$, we may assume without loss of generality that u is adjacent to z_1. If $u \in A \cup B$, then by (ii), there is a digon joining u and z_1, a contradiction to Lemma 21 with $P_1 = (u, z_1)$ and $P_2 = (v, w)$ (see Figure 7a). Then, $u \in C$. By (i), it follows that there is a digon joining u and z_2, a contradiction to Lemma 21 with $P_1 = (u, z_2)$ and $P_2 = (v, w)$ (see Figure 7b). □

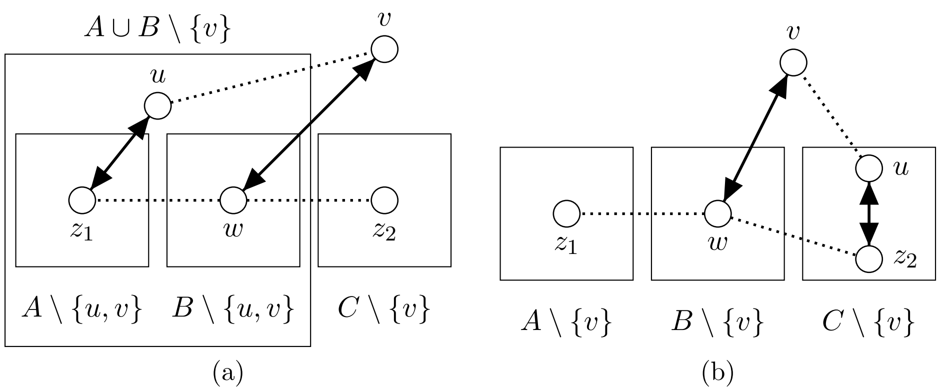


Figure 7: Auxiliary illustration for the proof of Claim 26.

Claim 27. *If $u \in B$, then v is adjacent to every vertex of $(A \cup B \cup C) \setminus \{u, v\}$. Similarly, if $v \in B$, then u is adjacent to every vertex of $(A \cup B \cup C) \setminus \{u, v\}$.*

Proof. Suppose that $u \in B$. By Claim 24, every vertex of $B \setminus \{u, v\}$ is adjacent to both u and v. So, towards a contradiction, suppose that there is a vertex $w \in (A \cup C) \setminus \{v\}$ that is non-adjacent to v. Since $\alpha(D) = 2$, it follows that w is adjacent to u. By (ii) and (iii), there must be a digon joining u and w. Hence, there is a contradiction to Lemma 21 with $P_1 = (u, w)$ and $P_2 = (v)$ (see Figure 6a exchanging the roles of u and w). The argument is analogous when $v \in B$. □

Let $B_A \subseteq B \setminus \{u, v\}$ be the set of vertices that are adjacent to every vertex of A and let $B_C = B \setminus (\{u, v\} \cup B_A)$. Note that, $B \setminus \{u, v\} = B_A \cup B_C$ and, by Claim 26, every vertex of B_C is adjacent to every vertex of C. By Claim 25, (i), (ii) and (iii), both $D[A \cup B_A]$ and $D[C \cup B_C]$ are complete digraphs (see Figure 8a). Suppose that $B \cap \{u, v\}$ is empty. Then D can be partitioned into two semicomplete digraphs, $D[A \cup B_A \cup \{x\}]$ and $D[C \cup B_C \cup \{y\}]$ a contradiction to Lemma 20. Thus, we may assume that $u \in B$ or $v \in B$. Consider first the case in which $\{u, v\} \subseteq B$. By Claim 27, both u and v are adjacent to every vertex of $(A \cup B \cup C) \setminus \{u, v\}$. Then, $D[A \cup B_A \cup \{u, x\}]$ and $D[C \cup B_C \cup \{v, y\}]$ is a partition of D into two semicomplete digraphs, a contradiction to Lemma 20. Then, we may assume without loss of generality that $u \in A$ and $v \in B$.

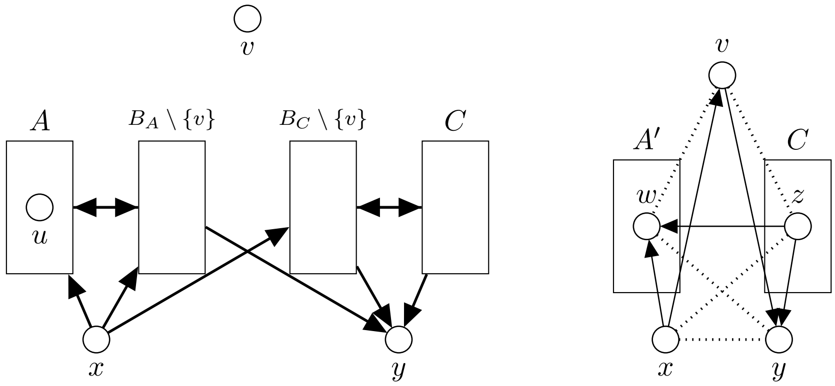
If $C \cup B_C$ is empty, then $D[A \cup B_A \cup \{x\}]$ and $D[\{v, y\}]$ is a partition of D into two semicomplete digraphs, a contradiction to Lemma 20. Thus, we may assume that $C \cup B_C$ is non-empty. Suppose that v is adjacent to some vertex w of $C \cup B_C$. Recall that $v \in B$ and, by (iii), v and w are joined by a digon. Since $D[A \cup B_A]$ is a complete digraph, there is a hamiltonian path P_1 of $D[A \cup B_A]$ ending at u. Similarly, since $D[C \cup B_C]$ is a complete digraph, there is a hamiltonian path P_2 of $D[C \cup B_C]$ beginning at w. Then, $\{xP_1, vP_2y\}$ is an S_{BE}-path partition of D, a contradiction. Thus, we may assume that there is no vertex of $C \cup B_C$ that is adjacent to v. Note that, by Claim 24, this implies that $B_C = \emptyset$ (so $C \neq \emptyset$). Let A' be the subset of vertices of A that are non-adjacent to v. Note that $u \in A'$ and, since $\alpha(D) = 2$, every vertex of A' is adjacent to every vertex of C. We claim that every vertex of A' dominates every vertex of C. Towards a contradiction, suppose that there is $w \in A'$ and $z \in C$ such that $(z, w) \in A(D)$ and $(w, z) \notin A(D)$. Then, $D[\{w, x, v, y, z\}]$ is a blocking odd cycle, a contradiction (see Figure 8b). So every vertex of A' dominates every vertex of C.

Since $u \in A'$, $D[A']$ and $D[C]$ are complete digraphs and every vertex of A' dominates every vertex of C, there is a hamiltonian path P_1 of $D[A' \cup C]$ beginning at u and ending at a vertex of C. Similarly, since $D[(A \setminus A') \cup B_A \cup \{v\}]$ is a complete digraph, there is a hamiltonian path P_2 of $D[A \setminus A' \cup B_A \cup \{v\}]$ ending at v. Then, $\{P_1y, xP_2\}$ is an S_{BE}-path partition of D, a contradiction. □

Corollary 28. *Let $D \in \mathfrak{D}_2$ and let S be a maximum stable set of D. If $D - S$ has a source and a sink of D, then D admits an S_{BE}-path partition.*

We are finally ready to prove that every digraph $D \in \mathfrak{D}_2$ is BE-diperfect.

Proof of Theorem 7. Towards a contradiction, suppose that the result does not hold and let D be a counterexample with a minimum number of vertices. Let S be a maximum stable set of D such that D does not admit an S_{BE}-path partition. By Corollary 14, we may assume that there is a source-component D_1 of D for which $S \cap V(D_1) = \emptyset$ and a sink-component D_2 of D for which $S \cap V(D_2) = \emptyset$.



(a) Both $A \cup B_A$ and $C \cup B_C$ induce complete digraphs.

(b) The set pf vertices $\{w, x, v, y, z\}$ induce a blocking odd cycle, with blocking pairs x, w, z, w and z, y.

Figure 8: Auxiliary illustration for the proof of Lemma 23.

Claim 29. *Digraphs D_1 and D_2 are semicomplete.*

Proof. Towards a contradiction, let S_1 be a stable set of size two of D_1, let $S_2 = S$ and let $S_3 = \{y\}$ where y is any vertex of D_2. Since $|S_1| + |S_2| + |S_3| = 5$, there is a contradiction to Lemma 16. □

Claim 30. *Digraphs D_1 and D_2 are trivial.*

Proof. Towards a contradiction, suppose that D_1 is not trivial. By Claim 29, D_1 is semicomplete. By Redéi's Theorem, D_1 has a hamiltonian path P_1. Let P_1' be the path obtained from P_1 by deleting its terminal vertex t. Let $D' = D - V(P_1')$. Clearly, $D' \in \mathfrak{D}_2$ and S is also a maximum stable set of D'. Since D is a minimum counterexample to the statement, D' admits an S_{BE}-path partition $\{R_1, R_2\}$. Since t is a source of D', we may assume without loss of generality that $t = \text{ini}(R_1)$. Note that this implies that $\text{ter}(R_1) \in S$. Thus, $\{P_1'R_1, R_2\}$ is an S_{BE}-path partition of D, a contradiction. Thus, D_1 must be a trivial component of D. By the Principle of Directional Duality, D_2 must also be a trivial component of D. □

Let $V(D_1) = \{x\}$ and let $V(D_2) = \{y\}$. Note that x and y are, respectively, a source and a sink of D in $D - S$. By Corollary 28, D admits an S_{BE}-path partition, a contradiction. □

4 Final remarks

In this paper, we proved that a digraph D with stability number two is BE-diperfect if and only if D does not contain a blocking odd cycle with three or five vertices as an induced subdigraph. This provides support for Conjecture 6. In contrast, Berge's original conjecture regarding α-diperfect digraphs (Conjecture 5) is false for digraphs with stability number two. As we mentioned in the introduction, de Paula Silva *et al* [10] exhibited orientations of complements of odd cycles with at least seven vertices which are not α-diperfect. So anti-directed odd cycles and those orientations of complement of odd cycles are all minimal non-α-diperfect digraphs. It is natural to ask whether a digraph which does not contain any of those digraphs as an induced subdigraph is α-diperfect. This seems to be a very hard question. On the other hand, the only known minimal non-BE-diperfect digraphs are blocking odd cycles which have a somewhat simpler structure.

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