

# Two Graph Games

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Submitted: Jul 25, 2023; Accepted: Apr 24, 2024; Published: Jun 14, 2024

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## Abstract

We isolate two simple infinite determined games on analytic graphs. With their help, we prove that certain large classes of analytic graphs have countable chromatic number.

**Mathematics Subject Classifications:** 05C15, 05C63, 05C57, 03E15

## 1 Introduction

Chromatic number of graphs is a notoriously difficult topic. In this paper, we isolate two graph games which can be used to give conceptual and brief proofs of two strengthenings of countable chromatic number for analytic graphs on Polish spaces: the countable coloring number, and left-separation.

**Definition 1.** Let  $\Gamma$  be a graph on a set  $X$ .

1. [2]  $\Gamma$  has *countable coloring number* if there is a well-ordering  $\leq$  on  $X$  such that for every  $x \in X$ , the set  $\{y \in X : y \leq x \text{ and } y \Gamma x\}$  is finite.
2. If in addition  $X$  is a Polish space,  $\Gamma$  is *left-separated* if there is a well-ordering  $\leq$  on  $X$  such that for every  $x \in X$ ,  $x$  is not an accumulation point of the set  $\{y \in X : y \leq x \text{ and } y \Gamma x\}$ .

It is easy to see that countable coloring number implies left-separation, which in turn implies countable chromatic number (Corollary 19). Our first game provides a brief proof of a previously known dichotomy for countable coloring number.

**Definition 2.** The topological graph  $\Delta_0$  is the graph on the space  $X_0$  which is the disjoint union of  $2^\omega$  and  $2^{<\omega}$  (the former with the usual topology and the latter with discrete topology) connecting points  $t \in 2^{<\omega}$  and  $x \in 2^\omega$  if  $t \subset x$ .

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**Theorem 3.** [1] *Let  $\Gamma$  be an analytic graph on a Polish space  $X$ . Exactly one of the following occurs:*

1. *there is a continuous injective homomorphism from  $\Delta_0$  to  $\Gamma$ ;*
2.  *$\Gamma$  has countable coloring number.*

The second game provides a quick way to verify left-separation for many graphs, even though in this case we do not get a dichotomy.

**Definition 4.** The topological graph  $\Delta_1$  is the graph of the space  $X_1$  which is the disjoint union of  $2^\omega$  and  $2^{<\omega}$  connecting points  $t \in 2^{<\omega}$  and  $x \in 2^\omega$  if  $t \subset x$ . Unlike for the graph  $\Delta_0$  of Definition 2, the topology on  $X_1$  is induced by the metric  $d(x, y) = 2^{-n}$  where  $n$  is the length of the longest common initial segment of  $x$  and  $y$ , so the initial segments of any point  $x \in 2^\omega$  converge to  $x$ .

**Theorem 5.** *Let  $\Gamma$  be an analytic graph on a Polish space  $X$ . One of the following occurs:*

1. *there is a continuous injective homomorphism from  $\Delta_1$  to  $\Gamma$ ;*
2.  *$\Gamma$  is left-separated.*

As a fairly immediate corollary, one can reprove a result of Komjáth citekomjath:rn: given a dimension  $d > 0$  and a countable set  $a \subset \mathbb{R}$  of positive reals, the graph  $\Gamma$  on  $\mathbb{R}^d$  connecting points whose Euclidean distance belongs to the set  $C$  is countably chromatic (Example 29). There are many analytic graphs which satisfy both (1) and (2) of the above theorem— $\Delta_1$  is a trivial example. To partially address this issue, we provide a game-free argument to show that many commonly occurring  $\sigma$ -algebraic graphs are left-separated.

**Definition 6.** Let  $X$  be a Polish field.

1. If  $d \geq 1$  is a number, a set  $A \subset X^d$  is *algebraic* if there is a finite collection  $\{p_i : i \in j\}$  of polynomials with parameters in  $X$  and  $d$  many free variables such that  $A = \{\bar{x} \in X^d : \forall i \in j \ p_i(\bar{x}) = 0\}$ ;
2. If  $Y \subset X$  is a subfield, then the set  $A \subset X^d$  is  *$Y$ -algebraic* if the defining polynomials can be selected so that all their coefficients belong to  $Y$ ;
3. A graph  $\Gamma$  on  $X^d$  is *algebraic* if the set  $\{\langle \bar{x}, \bar{y} \rangle \in (X^d)^2 : \bar{x} \Gamma \bar{y} \text{ or } \bar{x} = \bar{y}\}$  is an algebraic subset of  $X^{2d}$ ;
4. A graph  $\Gamma$  on  $X^d$  is *irreflexive-algebraic* if the set  $\{\langle \bar{x}, \bar{y} \rangle \in (X^d)^2 : \bar{x} \Gamma \bar{y}\}$  is an algebraic subset of  $X^{2d}$ ;
5.  $\Gamma$  is  *$\sigma$ -algebraic* if it is a union of countably many algebraic graphs.

**Theorem 7.** *If  $X$  is a Polish field,  $d > 0$  is a number, and  $\Gamma$  is the union of countably many pairwise disjoint irreflexive-algebraic graphs on  $X^d$ , then  $\Gamma$  is left-separated.*

There are many open questions. In particular, we do not know if there is an exact characterization of left-separation of analytic graphs by a determined game and a minimal problematic graph.

For the notation, a graph  $\Gamma$  on a set  $X$  is a symmetric irreflexive relation on  $X$ . For a set  $A \subseteq X$  we write  $\Gamma(A) = \{x \in X : \forall y \in A \ y = x \vee y \Gamma x\}$ . An amalgamation operation on well-orderings is repeated in several proofs of this paper. If  $b$  is a set,  $b = \bigcup_{\alpha \in \kappa} c_\alpha$  is an increasing union for some ordinal  $\kappa$ , and for each  $\alpha \in \kappa$  a relation  $\leq_\alpha$  is a well-ordering on the set  $c_\alpha$ , the *amalgamation* of the orderings  $\leq_\alpha$  is a relation on  $b$  defined as follows. If  $x, y \in b$  are elements and  $\alpha_x, \alpha_y \in \kappa$  are the smallest ordinals  $\alpha$  such that  $x \in c_\alpha$  and  $y \in c_\alpha$  respectively, we put  $x \leq y$  if either  $\alpha_x \in \alpha_y$  or  $(\alpha_x = \alpha_y \text{ and } x \leq_{\alpha_y} y)$  holds. It is immediate that an amalgamation of well-orderings is a well-ordering again. The only fact about algebraic sets used in this paper is the Hilbert Basis Theorem: an intersection of an arbitrary collection of algebraic sets is equal to the intersection of a finite subcollection.

## 2 Coloring number

In this section we provide a characterization of countable coloring number of analytic graphs via a simple infinite game. This provides an efficient, forcing-free proof of a dichotomy result of [1]. For the sake of exposition and parallel of the treatment in Section 3, we maintain the proof of the following well-known proposition.

**Proposition 8.** [2] *Let  $\Gamma$  be a graph on a Polish space  $X$ . The following are equivalent:*

1.  $\Gamma$  has countable coloring number;
2.  $\Gamma$  has an orientation in which every point has finite outflow.

*Proof.* To show that (1) implies (2), choose a well-ordering  $\leq$  on  $X$  witnessing countable coloring number of  $\Gamma$ . Orient every edge of  $\Gamma$  from the larger to the smaller vertex in the ordering  $\leq$  and observe that the outflow of every vertex is finite. To show that (2) implies (1), let  $\vec{\Gamma}$  be an orientation of  $\Gamma$  in which every vertex has finite outflow. Say that a set  $b \subset X$  is  $\vec{\Gamma}$ -closed if with every vertex  $b$  contains all vertices in its outflow. It is clear that every infinite subset of  $X$  is a subset of a  $\vec{\Gamma}$ -closed set of the same cardinality.

By transfinite induction on the cardinality of an infinite  $\vec{\Gamma}$ -closed set  $b$ , prove that  $\Gamma \upharpoonright b$  has countable coloring number. In the end, applying this for  $b = X$ , (1) will be proved. The case in which  $b$  is countable is easy: the well-ordering can be chosen as one of ordertype  $\omega$ . Now, suppose that the statement has been proved for all infinite  $\vec{\Gamma}$ -closed sets of cardinality smaller than  $b$ . Stratify  $b$  as  $b = \bigcup_{\alpha \in \kappa} c_\alpha$  as a continuous increasing union of  $\vec{\Gamma}$ -closed sets of cardinality strictly smaller than  $b$ . Use the inductive assumption to find a well-ordering  $\leq_\alpha$  on  $c_\alpha$  witnessing the fact that  $\Gamma \upharpoonright c_\alpha$  has countable coloring number. Let  $\leq$  be the well-ordering of  $b$  which is the amalgamation of orderings  $\leq_\alpha$ ; we claim that  $\leq$  is a witness to  $b$  having countable coloring number.

To see this, suppose that  $x \in b$  is any vertex. Find the smallest ordinal  $\alpha$  such that  $x \in c_\alpha$  holds; note that  $\alpha$  must be a successor ordinal,  $\alpha = \beta + 1$  for some  $\beta$ . Now, consider

the set  $d = \{y \in b : y \leq x \text{ and } y \Gamma x\}$ . The set  $d$  is the union of  $d_0 = \{y \in c_\beta : y \leq x \text{ and } y \Gamma x\}$  and  $d_1 = \{y \in c_\alpha \setminus c_\beta : y \leq x \text{ and } y \Gamma x\}$ . If  $y \in d_0$  then  $y \vec{\Gamma} x$  cannot hold by the closure of the set  $c_\beta$ , therefore  $x \vec{\Gamma} y$  must hold. Thus, the set  $d_0$  must be finite, since it is a subset of the finite  $\vec{\Gamma}$ -outflow of the vertex  $x$ . The set  $d_1$  is finite by the choice of the well-ordering  $\leq_\alpha$ . In sum, the set  $d$  is finite as required.  $\square$

It is now time to define the first game of the paper.

**Definition 9.** Let  $\Gamma$  be an analytic graph on a Polish space  $X$ . Choose a complete compatible metric on the space  $X$ . The *coloring game*  $G_0(\Gamma)$  is played between Players I and II for infinitely many rounds as follows. In round  $n$  Player I plays a point  $x_n \in X$  and a basic open ball  $B_n$  of radius smaller than  $2^{-n}$ . Player II answers with a point  $y_n \in X$ . Player I must make sure that his points are all distinct and  $y_n \notin \bar{B}_{n+1} \subset B_n$  holds. In the end, let  $z \in X$  be the unique point in  $\bigcap_n B_n$ —the *outcome* of the play. Player I wins if  $\forall n \in \omega \ z \Gamma x_n$  holds.

In principle, the game depends on the choice of the complete compatible metric on  $X$ . However, the existence of a winning strategy is independent of that choice. This is one of the consequences of the following central theorem.

**Theorem 10.** *Let  $\Gamma$  be an analytic graph on a Polish space  $X$ . The game  $G_0(\Gamma)$  is determined. In addition,*

1. *Player I has a winning strategy in  $G_0(\Gamma)$  if and only if there is an injective continuous homomorphism from  $\Delta_0$  to  $\Gamma$ ;*
2. *Player II has a winning strategy in  $G_0(\Gamma)$  if and only if  $\Gamma$  has countable coloring number.*

*Proof.* Determinacy of the game  $G_0(\Gamma)$  follows from a straightforward unraveling argument. First, use the assumption that the graph  $\Gamma$  is analytic to find a continuous function  $k : \omega^\omega \rightarrow X^{\omega+1}$  such that its range is the set of all tuples  $\langle x_n : n \in \omega, z \rangle$  such that for all  $n \in \omega$ ,  $x_n \Gamma z$ . Then, consider the game  $G_0^u(\Gamma)$  played in the same way as  $G_0(\Gamma)$  except Player I in addition plays strings  $t_n \in \omega^n$  such that  $t_0 \subset t_1 \subset \dots$ . In the end, Player I wins if he wins the play of  $G_0(\Gamma)$  and in addition  $k(\bigcup_n t_n) = \langle x_n : n \in \omega, z \rangle$ . It is not difficult to see that the unraveled game is closed for Player I and therefore determined [4]. Thus, the following two claims prove the determinacy of the game  $G_0(\Gamma)$ :

**Claim 11.** *If Player I has a winning strategy in  $G_0^u(\Gamma)$ , then he has a winning strategy in  $G_0(\Gamma)$ .*

*Proof.* Game  $G_0^u(\Gamma)$  is clearly more difficult for Player I to win than  $G_0(\Gamma)$ .  $\square$

**Claim 12.** *If Player II has a winning strategy in  $G_0^u(\Gamma)$ , then he has a winning strategy in  $G_0(\Gamma)$ .*

*Proof.* Let  $\sigma$  be a winning strategy for Player II in the unraveled game  $G_0^u(\Gamma)$ . Let  $\kappa$  be an uncountable cardinal large enough such that  $X$  belongs to  $H_\kappa$ , the set of all sets whose transitive closure has cardinality less than  $\kappa$ . To describe the winning strategy for Player II in  $G_0(\Gamma)$ , as the game progresses, let Player II on the side construct an increasing elementary chain  $\langle M_n : n \in \omega \rangle$  of countable elementary submodels of  $H_\kappa$  such that  $\sigma \in M_0$  and  $x_n \in M_n$  for every  $n \in \omega$ , and play the points  $y_n$  in such a way that  $\{y_n : n \in \omega\} = \bigcup_n M_n \cap X$ . We claim that this is a winning strategy for Player II.

Suppose towards a contradiction that Player I has a winning counterplay against this strategy, consisting of points  $x_n$  for  $n \in \omega$ , basic open sets  $B_n$ , and an outcome point  $z \in X$ . Let  $u \in \omega^\omega$  be a point such that  $k(u) = \langle x_n : n \in \omega, z \rangle$ , and  $\{y_n : n \in \omega\}$  be the sequence of responses the strategy of Player II produced. Player I will now beat the strategy  $\sigma$  in the game  $G_0^u(\Gamma)$  in the following way. His first move is  $(x_0, \emptyset, B_0)$ , Player II responds with  $y'_0$ , following  $\sigma$ . In a subsequent round  $n$  Player I chooses  $i \in \omega$  large enough so that  $(x_n, u \upharpoonright n, B_i)$  is a legal move, Player II follows  $\sigma$  and responds  $y'_n$ . Such  $i \in \omega$  exists since the model  $M = \bigcup_n M_n$  is closed under the strategy  $\sigma$  and it must be the case that  $\{y'_k : k < n\} \subset \{y_k : k \in \omega\}$ . Since Player II won in the play of the game  $G_0^u(\Gamma)$ , the two plays produced the same outcome  $z$ , and  $\{y'_k : k \in \omega\} \subseteq \{y_k : k \in \omega\}$ , he must have won the play of the game  $G_0(\Gamma)$ , contradicting the initial assumption.  $\square$

For the right-to-left implication of item (1), let  $h$  be the injective continuous homomorphism. Let Player I play the game  $G_0(\Gamma)$  in such a way that  $x_n = h(t_n)$  for some binary string  $t_n \in 2^{<\omega}$  and basic open sets  $B_n$  such that  $t_0 \subset t_1 \subset \dots$ ,  $y_n \notin B_{n+1}$ , and  $\{h(z) : z \in 2^\omega \text{ and } t_n \subset z\} \subset \bar{B}_n$ . This is not difficult to do. In the end, the unique point in  $\bigcap_n B_n$  is equal to  $h(\bigcup_n t_n)$ , as such is  $\Gamma$ -connected to  $x_n$  for all  $n \in \omega$ , and it is not equal to any point  $y_n$  for  $n \in \omega$ .

For the left-to-right implication of item (1), let  $\sigma$  be a winning strategy for Player I. Two claims will be helpful.

**Claim 13.** *Let  $p$  be a finite play of the game  $G_0(\Gamma)$  played according to the strategy  $\sigma$ , ended with a move of Player I. Let  $a \subset X$  be a finite set. There is an extension played according to the strategy  $\sigma$  such that*

1. *the last point played by I does not belong to  $a$ ;*
2. *the last open set is disjoint from  $a$ .*

*Proof.* The first item follows from the demand that the points played by I are pairwise distinct. The second follows from the fact that Player II is free to produce the points of the set  $a$  among his answers.  $\square$

**Claim 14.** *Let  $p$  be a finite play of the game  $G_0(\Gamma)$  played according to the strategy  $\sigma$ , ended with a move of Player I. There are extensions  $p_0, p_1$  played according to the strategy  $\sigma$  in which the closures of the last open sets played by Player I are disjoint.*

*Proof.* Produce any infinite extension of the play  $p$  using the strategy  $\sigma$ . Let  $z_0 \in X$  be the final point of the play. Consider any infinite play of the strategy  $\sigma$  extending  $p$  in which Player II plays  $z_0$  as one of his moves. Let  $z_1$  be the final point of this second play. Since  $z_0 \neq z_1$  and the diameters of open sets played by the strategy  $\sigma$  must converge to zero, there must be initial segments  $p_0, p_1$  of the two plays in which the last open sets played by the strategy  $\sigma$  are disjoint.  $\square$

Now, by tree induction on  $t \in 2^{<\omega}$  build finite plays  $p_t$  of the game  $G_0(\Gamma)$  in which Player I follows his strategy  $\sigma$ , produces final move  $x_t, B_t$ , the points  $x_t$  for  $t \in 2^{<\omega}$  are pairwise disjoint, and for each  $t \in 2^{<\omega}$  the sets  $B_{t \smallfrown 0}$  and  $B_{t \smallfrown 1}$  are pairwise disjoint, and they do not contain any of the points  $x_s$  for binary strings  $s$  shorter than  $t$ . This is easily possible using the claims. In the end, let  $h$  be the map defined by  $h(t) = x_t$  and  $h(z) =$  the unique point in  $\bigcap_{n \in \omega} B_{z \upharpoonright n}$ . It is not difficult to check that this is a continuous injective homomorphism from  $\Delta_0$  to  $\Gamma$ .

For the left-to-right direction of item (2), suppose that Player II has a winning strategy  $\sigma$  in the game  $G_0(\Gamma)$ . The following claim is key. Call a set  $b \subset X$  *closed* under the strategy  $\sigma$  if for every partial finite play  $p$  using only points in  $b$  and ending with a move of Player II, the answer  $\sigma(p)$  of the strategy belongs to  $b$  as well.

**Claim 15.** *If  $b \subset X$  is a set closed under the strategy  $\sigma$  and  $z \in X \setminus b$  is a point, then the set  $\{x \in b: x \Gamma z\}$  is finite.*

*Proof.* Otherwise, Player I can play points  $x_n$  in the infinite set  $\{x \in b: x \Gamma z\}$  and the basic open neighborhoods  $B_n$  so that they all contain the point  $z$ . In the end of such a play against the strategy  $\sigma$ , he wins, since the set  $b$  is closed under the strategy  $\sigma$ , so all points the strategy  $\sigma$  plays are in  $b$  and therefore not equal to  $z$ . This contradicts the choice of  $\sigma$ .  $\square$

Now, by transfinite induction on the infinite cardinality of a set  $b \subset X$  closed under the strategy  $\sigma$ , we argue that  $G \upharpoonright b$  has countable coloring number. This is clear for countable cardinality of  $b$  since then the witnessing well-ordering of  $b$  can be chosen of ordertype  $\omega$ . Now suppose that  $b \subset X$  is an infinite set closed under the strategy  $\sigma$  and for all sets of cardinality smaller than  $|b|$  the statement has been proved. Stratify  $b = \bigcup_{\alpha \in \kappa} c_\alpha$  as a well-ordered continuous increasing union of sets of smaller cardinality, all of them of closed under the strategy  $\sigma$ . For each  $\alpha \in \kappa$  pick a well-ordering  $\leq_\alpha$  on  $c_\alpha$  witnessing the countable coloring number of  $\Gamma \upharpoonright c_\alpha$ . Now, let  $\leq$  be the amalgamation well-ordering on  $b$  and argue that it witnesses the left-separation of the graph  $\Gamma \upharpoonright b$ .

Suppose that  $x \in b$  is an arbitrary point and argue for the finiteness of the set  $a = \{y \in b: y \leq x \text{ and } y \Gamma x\}$ . Let  $\alpha \in \kappa$  be the first ordinal such that  $x \in c_\alpha$  holds. By the continuity of the stratification, the ordinal  $\alpha$  is a successor of some  $\beta$ . The set  $a \cap c_\beta$  is finite by Claim 15 applied to  $c_\beta$  and  $x$ . The set  $a \setminus c_\beta$  is a subset of  $\{y \in c_\alpha: y \leq_\alpha x \text{ and } y \Gamma x\}$  which is finite by the choice of the well-ordering  $\leq_\alpha$ . It follows that the set  $a = (a \cap c_\beta) \cup (a \setminus c_\beta)$  is finite as required.

The right-to-left direction of item (2) is proved in the contrapositive. If Player II has no winning strategy, then Player I must have it, and by (1) there must be a continuous

homomorphic injection of  $\Delta_0$  to  $\Gamma$ . To conclude that  $\Gamma$  does not have countable coloring number, it is then enough to show that  $\Delta_0$  does not have it. Indeed, suppose towards a contradiction that  $\tilde{\Delta}_0$  is an orientation of  $\Delta_0$  in which every vertex has finite outflow. By a counting argument, there must be a point  $x \in 2^\omega$  which does not belong to the outflow of any point in  $2^{<\omega}$ . By another counting argument, there must be an initial segment  $t \subset x$  which does not belong to the outflow of  $x$ . Since  $\{t, x\}$  is an edge in  $\Delta_0$ , this is a contradiction.  $\square$

Theorem 3 is now an immediate corollary.

**Example 16.** [3] If  $a$  is a countable set of positive reals and  $\Gamma$  is the graph on  $\mathbb{R}^2$  connecting points whose distance belongs to  $a$ , then  $\Gamma$  has countable coloring number. To construct a winning strategy for Player II, just make sure as Player I produces points  $x_n$  for  $n \in \omega$ , if  $n_0, n_1$  are distinct numbers and  $r_0, r_1$  are elements of  $a$ , the (at most two) points in  $\mathbb{R}^2$  which have distance  $r_0$  from  $x_{n_0}$  and distance  $r_1$  from  $x_{n_1}$  are on the sequence Player II plays. This is easy to do, and it constitutes a winning strategy for Player II.

### 3 Left-separation

Left-separation of graphs on Polish spaces is characterized by a tool parallel to orientation with finite outflows. This tool has the advantage of abstracting away from well-orderings.

**Definition 17.** Let  $\Gamma$  be a graph on a Polish space  $X$ . A  $\Gamma$ -neighborhood assignment is a function  $\pi$  which to every point  $x \in X$  assigns an open neighborhood  $\pi(x) \subset X$  of  $x$  so that if  $x, y$  are  $\Gamma$ -related points then either  $x \notin \pi(y)$  or  $y \notin \pi(x)$ .

**Proposition 18.** Let  $\Gamma$  be a graph on a Polish space  $X$ . The following are equivalent:

1.  $\Gamma$  is left-separated;
2. there is a  $\Gamma$ -neighborhood assignment.

*Proof.* To show that (1) implies (2), choose a well-ordering  $\leq$  on  $X$  witnessing left-separation of  $\Gamma$ . For every point  $x \in X$ , let  $\pi(x) \subset X$  be any open neighborhood of  $x$  containing no points of the set  $\{y \in X : y \leq x \text{ and } y \Gamma x\}$ . Clearly, this is a  $\Gamma$ -neighborhood assignment. To show that (2) implies (1), let  $\pi$  be a  $\Gamma$ -neighborhood assignment. Let  $d$  be a compatible metric on the space  $X$ . For every number  $n \in \omega$ , let  $A_n = \{x \in X : \pi(x) \text{ contains an open ball of radius } 2^{-n} \text{ around } x\}$ . Clearly, the sets  $A_n$  increase with respect to inclusion as  $n$  grows, and  $X = \bigcup_n A_n$ . Let  $\leq$  be any well-ordering on  $X$  in which  $A_n$  is an initial segment for every  $n \in \omega$ . We claim that  $\leq$  witnesses left-separation of the graph  $\Gamma$ .

Indeed, suppose that  $y \leq x$  are  $\Gamma$ -connected points in the space  $X$  and  $n$  is the smallest number such that  $x \in A_n$  holds. Since  $y \in A_n$  and  $\pi$  is a  $\Gamma$ -neighborhood assignment,  $d(x, y) \geq 2^{-n}$  must hold. Thus, the open  $2^{-n}$ -ball around  $x$  contains no points of the set  $\{y \in X : y \leq x \text{ and } y \Gamma x\}$  as desired.  $\square$

**Corollary 19.** *Let  $\Gamma$  be a graph on a Polish space  $X$ . If  $\Gamma$  is left-separated, then it is countably chromatic.*

*Proof.* Let  $\pi$  be a  $\Gamma$ -neighborhood assignment. Shrinking the values of  $\pi$  if necessary, we may assume that they are all basic open sets. Then  $\pi$  is in fact a  $\Gamma$ -coloring with a countable range.  $\square$

It is important to understand that, unlike the coloring number, the definition of left-separation and  $\Gamma$ -neighborhood assignments depends on the topology of the underlying space.

**Example 20.** Let  $\Delta_2$  be a graph on  $2^\omega$  connecting points  $x, y$  if  $x$  contains only finitely many 1's,  $y$  contains infinitely many 1's, and all 1's of  $x$  appear in  $\text{lcis}(x, y)$ , the longest common initial segment of  $x$  and  $y$ . Then  $\Delta_2$  is not left-separated.

It is obvious that if  $2^\omega$  is equipped with a larger topology in which the set of its rational points is clopen, the graph  $\Delta_2$  becomes left-separated.

*Proof.* Suppose that  $\pi$  is a neighborhood assignment on  $2^\omega$ ; shrinking  $\pi$  if necessary, we assume that its values are in fact clopen sets. It will be enough to find a point  $y$  and points  $x_n$  for  $n \in \omega$  such that  $\lim_n x_n = y$  and for all  $n \in \omega$   $x_n \Gamma y$  and  $y \in \pi(x)$  both hold. Then  $\pi(y)$  must contain all but finitely many points  $x_n$ , violating the definitory property of a  $\Gamma$ -neighborhood assignment.

To find the points  $x_n$ , just construct them so that each of them has exactly  $n$  many unit entries, all unit entries of  $x_n$  appear in  $\text{lcis}(x_n, x_{n+1})$ , and  $x_n \in \bigcap_{m \in n} \pi(x_m)$ . This is easy to accomplish. In the end, let  $y = \lim_n x_n$ . It is clear from the construction that  $y$  is  $\Gamma$ -connected with every point  $x_n$ , and belongs to  $\pi(x_n)$  as well. This concludes the proof.  $\square$

There is a natural graph game associated with existence of a  $\Gamma$ -neighborhood assignment. Even though this time the game does not provide a full characterization, it still provides interesting information difficult to obtain otherwise.

**Definition 21.** Let  $\Gamma$  be a graph on a Polish space  $X$ . The game  $G_1(\Gamma)$  proceeds as follows. Players I and II alternate for  $\omega$  many rounds, playing points  $x_n \in X$  and  $y_n \in X$  for  $n \in \omega$ . Player I wins if the points  $x_n$  converge to some point  $z \in X$  (this point is referred to as the *outcome* of the play), and for each  $n \in \omega$ ,  $z \Gamma x_n$  and  $z \neq y_n$  both hold.

**Theorem 22.** *Let  $\Gamma$  be an analytic graph on a Polish space  $X$ . The game  $G_1(\Gamma)$  is determined. Moreover,*

1. *Player I has a winning strategy if and only if there is a continuous homomorphic injection from  $\Delta_1$  to  $\Gamma$ ;*
2. *if Player II has a winning strategy then  $\Gamma$  is left-separated.*

*Proof.* The determinacy proof is very close to that of Theorem 10. We only indicate the unraveled version of the game and leave the rest to the reader. Fix a continuous function  $k: \omega^\omega \rightarrow X^{\omega+1}$  such that its range consists exactly of points  $\langle x_n: n \in \omega, z \rangle$  such that  $z = \lim_n x_n$  and for every  $n \in \omega$ ,  $x_n \Gamma z$ . The unraveled game  $G_1^u(\Gamma)$  is played just as  $G_1(\Gamma)$ , except Player I in addition plays finite sequences  $t_n$  of natural numbers of length at least  $n$  such that  $y_n$  does not belong to the closure of the set  $\{k(u)(\omega): t_{n+1} \subset u\} \subset X$ ; in addition, the sequences  $t_n$  are required to form an inclusion increasing sequence. Player I wins if  $\langle x_n: n \in \omega \rangle = k(\bigcup_n t_n) \upharpoonright \omega$ . This is a closed game for Player I, therefore determined. It is more difficult for Player I than the original game  $G_1(\Gamma)$ , and Player II has a winning strategy in the original game if and only if he has a winning strategy in the unraveled game.

The right-to-left implication of item (1) is easy. Let  $\pi: X_1 \rightarrow X$  be the injective continuous homomorphism of  $\Delta_1$  to  $\Gamma$ . Let Player I play points  $x_n = \pi(t_n)$  where  $t_n \in 2^n$  form an inclusion increasing sequence such that  $y_n \notin \pi''\{z \in 2^\omega: t_{n+1} \subset z\}$ . This is clearly a winning strategy for Player I.

The left-to-right implication of item (1) is notationally more difficult. Let  $\sigma$  be a winning strategy for Player I. Let  $d$  be a compatible complete metric on  $X$ . For a finite play  $p$  of the game let  $z(p)$  be the set of all points that Player I can produce as outcomes in plays extending  $p$  and observing the strategy  $\sigma$ . The following claims will streamline the construction.

**Claim 23.** *Let  $a \subset X$  be a finite set and  $p$  a finite play observing the strategy  $\sigma$ . There is an extension  $q$  of  $p$  observing  $\sigma$  such that  $z(q)$  is disjoint from  $a$ .*

*Proof.* If this failed for some set  $a$  and a play  $p$ , Player II could win against the strategy  $\sigma$  from  $p$  by listing all points in  $a$  among his points and finding an infinite play in which the strategy  $\sigma$  produces infinitely many points arbitrarily close to points in  $a$ . Note that the outcome of the play would have to belong to the set  $a$ .  $\square$

**Claim 24.** *Let  $\varepsilon > 0$  be a positive real and  $p$  a finite play of the game observing the strategy  $\sigma$ . There is an extension  $q$  of  $p$  observing  $\sigma$  such that  $z(q)$  has metric diameter smaller than  $\varepsilon$ .*

*Proof.* If this failed for some  $\varepsilon > 0$  and  $p$ , then Player II could find a counterplay starting with  $p$  in which the strategy  $\sigma$  produces points  $x_n$  for  $n \in \omega$  in such a way that for every  $n$  there is  $m \geq n$  such that  $d(x_m, x_n) > \varepsilon/2$ . Such a sequence of points cannot converge, and Player II would win the play, contradicting the choice of the strategy  $\sigma$ .  $\square$

**Claim 25.** *Let  $p$  be a finite play of the game observing the strategy  $\sigma$ . There are extensions  $q_0, q_1$  of  $p$  observing the strategy  $\sigma$  such that the closures of the sets  $z(q_0)$  and  $z(q_1)$  are disjoint.*

*Proof.* Find two infinite extensions  $r_0, r_1$  of  $p$  such that for every  $\varepsilon > 0$  there is an initial segment  $q_{\varepsilon 0} \subset r_0$  such that  $z(q_{\varepsilon 0}) < \varepsilon$ , similarly for  $r_1$ , and in addition, the outcome  $z_0$  of  $r_0$  is one of the points Player II played in the play  $r_1$ . This is possible by the previous claim. The outcome  $z_1$  of  $r_1$  is distinct from  $z_0$ , and they are at a distance  $\delta > 0$ . Find

finite intermediate stages  $q_0, q_1$  or  $r_0, r_1$  such that the sets  $z(q_0), z(q_1)$  both have diameters smaller than  $\delta/4$ . It is not difficult to see that  $q_0, q_1$  work as desired.  $\square$

Now, by a tree induction on  $t \in 2^{<\omega}$  produce points  $x_t$  and plays  $p_t$  observing the strategy  $\sigma$  so that  $s \subset t$  implies  $p_s \subset p_t$ ,  $x_t$  is among the points produced by the strategy  $\sigma$  in  $p_t$ , the points  $x_t$  for  $t \in 2^{<\omega}$  are pairwise distinct, and for every  $n \in \omega$  the sets  $z(p_t)$  for  $t \in 2^n$  are pairwise disjoint and contain none of the points  $x_s$  for  $s \in 2^n$ . This is easily possible by the three above claims. In the end, let  $h: 2^{<\omega} \cup 2^\omega \rightarrow X$  be the map defined as follows.  $h(t) = x_t$  for  $t \in 2^{<\omega}$ , and  $h(y) =$  the outcome of the play  $\bigcup_n p_{y \upharpoonright n}$  for  $y \in 2^\omega$ . It is not difficult to see that the map  $h$  is a continuous injective homomorphism of  $\Delta_1$  to  $\Gamma$ .

For the second item, suppose that  $\sigma$  is a winning strategy for Player II. The following is immediate.

**Claim 26.** *Let  $b \subset X$  be a set closed under the strategy  $\sigma$  and  $z \in X \setminus b$  be a point. Then  $z$  is not an accumulation point of the set  $\{y \in b: z \Gamma y\}$ .*

*Proof.* Otherwise, Player I can construct a winning counterplay using a sequence of points in  $b$  whose limit is  $z$ .  $\square$

Now, by induction on cardinality of an infinite set  $b \subseteq X$  closed under the strategy  $\sigma$  prove that  $\Gamma \upharpoonright b$  is left-separated. Applied to the set  $b = X$ , this provides item (2). Now, the induction statement is immediate for a countable set  $b$ , since the well-ordering  $\leq$  can have ordertype  $\omega$  in this case. Now suppose that  $b \subseteq X$  is a set closed under  $\sigma$  such that left-separation has been proved for all infinite sets closed under the strategy  $\sigma$  of smaller cardinality.

Stratify  $b = \bigcup_{\alpha \in \kappa} c_\alpha$  as a continuous increasing union of sets closed under  $\sigma$  of strictly smaller cardinality than  $b$ . For each ordinal  $\alpha \in \kappa$  use the induction assumption to select a well-ordering  $\leq_\alpha$  on  $c_\alpha$  witnessing the fact that  $\Gamma \upharpoonright c_\alpha$  is left-separated. Amalgamate the well-orderings  $\leq_\alpha$  for  $\alpha \in \kappa$  to obtain a well-ordering  $\leq$  on  $b$ . Now, argue that the resulting well-ordering  $\leq$  on the set  $b$  witnesses left-separation of  $\Gamma \upharpoonright b$ .

To see this, let  $x \in b$  be an element and  $\alpha_x$  be the smallest ordinal  $\alpha \in \kappa$  such that  $x \in c_{\alpha+1}$ . Then, the set  $\{y \in b: y \leq x \text{ and } y \Gamma x\}$  is a subset of the union of the sets  $\{y \in c_\alpha: x \Gamma y\}$  and  $\{y \in c_{\alpha+1}: y \leq_{\alpha+1} x \text{ and } x \Gamma y\}$ . The former set does not have  $x$  as an accumulation point by Claim 26 applied to the set  $c_\alpha$ ; the latter set does not have  $x$  as an accumulation point by the choice of  $\leq_{\alpha+1}$ . This concludes the induction step.  $\square$

Theorem 5 is now an immediate corollary. There are other colorful corollaries.

**Corollary 27.** *Let  $X$  be a Polish field. Let  $\Gamma$  be a closed  $\sigma$ -algebraic graph on  $X^d$  for some  $d \geq 1$ . Exactly one of the following occurs:*

1.  $\Gamma$  contains a perfect clique;
2. there is a  $\Gamma$ -neighborhood assignment.

*Proof.* It is immediate that (1) implies the negation of (2); we need to show that the negation of (1) implies (2). For every closed set  $A \subset X^d$ , let  $h(A) = \{x \in A : \forall y \in A \ x = y \vee x\Gamma y\}$ . Since  $\Gamma$  is a closed graph and  $A$  is closed,  $h(A)$  is closed. The set  $h(A)$  is a  $\Gamma$ -clique by its definition. If there is a closed set  $A$  for which  $h(A)$  is uncountable, then  $h(A)$  contains a perfect clique and item (1) prevails. Thus, assume that for every closed set  $A$ ,  $h(A)$  is countable and work to find a winning strategy for Player II in the game  $G_1(\Gamma)$ , verifying item (2).

Let  $\Gamma = \bigcup_n \Gamma_n$  be a presentation of  $\Gamma$  as a union of countably many algebraic graphs on  $X^d$ . Consider the strategy for Player II in which, if Player I produces points  $x_i$  for  $i \in \omega$ , Player II enumerates all points in sets  $h(A)$ , where  $A$  runs through all possible sets of the form  $\bigcap_{i \in a} \Gamma_{n_i}(x_i)$  where  $a \subset \omega$  is a finite set and  $\langle n_i : i \in a \rangle$  is a tuple of natural numbers. Since the sets  $h(A)$  are all countable, this is easy for Player II to do. We will argue that this is a winning strategy for Player II.

To this end, suppose that points  $x_i$  for  $i \in \omega$  converge to  $z$ , and  $\forall i \ x_i \Gamma z$  holds. For each  $i \in \omega$  let  $n_i \in \omega$  be a number such that  $x_i \Gamma_{n_i} z$  holds. Use the Hilbert Basis Theorem to find a finite set  $a \subset \omega$  such that  $A = \bigcap_{i \in \omega} \Gamma_{n_i}(x_i)$  is equal to the set  $\bigcap_{i \in a} \Gamma_{n_i}(x_i)$ . The set  $h(A)$  is countable, and all of its elements are among the points the strategy for Player II produced when confronted with the points  $x_i$  for  $i \in \omega$ . Thus, it will be enough to show that  $z \in h(A)$  holds. And indeed, if  $y \in A$  is any point distinct from  $z$ , then  $y$  is  $\Gamma$ -connected to every point  $x_i \in \omega$ , and by the closedness of the graph  $\Gamma$ , it is also connected to  $z$  as well.  $\square$

**Corollary 28.** *Let  $X$  be a Polish field. Let  $\Gamma$  be a closed  $\sigma$ -algebraic graph on  $X^d$  for some  $d \geq 1$ . Suppose that  $\Gamma = \bigcup_m \Gamma_m$  is a union of algebraic graphs such that*

*(\*) for any distinct points  $x, y \in X$  there is an open neighborhood  $O$  of  $x$  such that for every  $z \in O$  and every  $m \in \omega$ , at least one of  $x \Gamma_m z$ ,  $y \Gamma_m z$  fails.*

*Then there is a  $\Gamma$ -neighborhood assignment.*

**Example 29.** Let  $d \geq 1$  be a number, let  $a$  be a countable set of positive reals, and let  $\Gamma$  be the graph on  $\mathbb{R}^d$  connecting points  $x, y$  if their Euclidean distance belongs to the set  $a$ . Clearly  $\Gamma = \bigcup_m \Gamma_m$  where  $\{r_m : m \in \omega\}$  is any listing of the set  $a$  and  $\Gamma_m$  connects points of Euclidean distance  $r_m$ . Then (\*) is satisfied: for distinct points  $x, y \in \mathbb{R}^n$  choose  $O$  to be any open ball around  $x$  whose radius is smaller than half of the distance between  $x$  and  $y$ . By Corollary 28, the graph  $\Gamma$  is left-separated. This reproves a result of Schmerl [5, Theorem 4].

*Proof of corollary.* By Theorem 10, it is enough to produce a winning strategy for Player II in the game  $G_1(\Gamma)$ . To do this, let Player I play a sequence of points  $x_n \in X$  for  $n \in \omega$ . Player II plays in such a way that if  $n_0 \in \omega$  is a number and  $m_n$  for  $n \in n_0$  are numbers and the set  $\bigcap_{n \in n_0} \Gamma_{m_n}(x_n)$  is a singleton, then the unique point in the intersection appears on the sequence played by Player II. This is easily possible as there are only countably many finite strings of natural numbers.

In the end, Player II must have won. To see this, suppose that  $z = \lim_n x_n$  and the point  $z$  is  $\Gamma$ -connected to every point  $x_n$  for  $n \in \omega$ . It follows that there must be numbers  $m_n$  for  $n \in \omega$  such that  $z \Gamma_{m_n} x_n$  holds. By the Hilbert Basis Theorem, there must be a number  $n_0$  such that  $C = \bigcap_{n \in \omega} \Gamma_{m_n}(x_n) = \bigcap_{n \in n_0} \Gamma_{m_n}(x_n)$ . The set  $C$  must be the singleton  $\{z\}$ : if it contained another point  $w$  different from  $z$ , there would be an open neighborhood  $O \subset X$  of  $z$  witnessing (\*) of the assumptions for  $\Gamma$ , there would have to be a number  $n \in \omega$  such that  $x_n \in O$ , and then the point  $x_n$  and the number  $m_n$  would violate the choice of the set  $O$ . It follows that the point  $z$  has appeared on the sequence of points that Player II produced, resulting in a victory for Player II.  $\square$

In the last theorem of this paper, we provide a proof of left-separation for  $\sigma$ -algebraic graphs which works even in some cases where Player I has a winning strategy in the game  $G_1$ .

**Theorem 30.** *Let  $d > 0$  be a natural number. Let  $X$  be a Polish field and  $\Gamma$  be a  $\sigma$ -algebraic graph on  $X^d$ . Suppose that  $\Gamma$  is a countable disjoint union of irreflexive-algebraic graphs. Then  $\Gamma$  is left-separated.*

*Proof.* Fix a partition  $\Gamma = \bigcup_n \Gamma_n$  into pairwise disjoint irreflexive-algebraic graphs. Let  $Y \subset X$  be a subfield. Say that  $\pi$  is a  $Y$ -suitable map if

- $\text{dom}(\pi)$  consists of all  $Y$ -algebraic subsets of  $X^d$ ;
- for every  $Y$ -algebraic set  $A$ ,  $\pi(A) \subset X^d$  is an open neighborhood of  $A$ ;
- $A \subseteq B$  implies  $\pi(A) \subseteq \pi(B)$ ;
- for every  $A \in \text{dom}(\pi)$ , for all but finitely many  $n \in \omega$ ,  $\pi(\Gamma_n(A)) \cap A = \emptyset$ .

By induction on cardinality of a subfield  $Y \subset X$  we now show that there is a  $Y$ -suitable map. To conclude the proof of the theorem from this, let  $\pi$  be an  $X$ -suitable map. Let  $\chi$  be any neighborhood assignment on  $X^d$  such that  $\chi(x) \subset \pi(\{x\})$  and  $\chi(x)$  is disjoint from the finitely many sets  $\Gamma_n(\{x\})$  for which  $\pi(\Gamma_n(\{x\}))$  contains the point  $x$ . It is not difficult to see that this will be the desired  $\Gamma$ -neighborhood assignment.

Suppose first that  $Y$  is countable, and enumerate nonempty  $Y$ -algebraic sets as  $\{A_i : i \in \omega\}$ . By recursion on  $i \in \omega$  build the values  $\pi(A_i)$  so that (a) for  $i, j \in \omega$ ,  $A_j \subseteq A_i$  implies  $\pi(A_j) \subseteq \pi(A_i)$ ; and (b) for every  $j < i$ , either  $A_j \cap A_i \neq \emptyset$ , or  $\pi(A_i) \cap A_j = \emptyset$ , or there is  $k < j$  such that  $A_k \subseteq A_i$  and  $\pi(A_k) \cap A_j \neq \emptyset$ . To see how this is done, suppose the values of  $\pi(A_j)$  for  $j < i$  have been constructed. Let  $O \subset X^d$  be an open neighborhood of the set  $A_i$  which is disjoint from  $A_j$  for all  $j < i$  such that  $A_j \cap A_i = \emptyset$ , and let  $\pi(A_i) = (O \cup \bigcup \{\pi(A_j) : j < i \text{ and } A_j \subseteq A_i\}) \cap \bigcap \{\pi(A_j) : j < i \text{ and } A_i \subseteq A_j\}$ .

First note that this choice of  $\pi(A_i)$  preserves the recursion hypothesis. Item (a) is satisfied at  $i$  by the definition of  $\pi(A_i)$  and item (a) below  $i$ . To verify item (b), suppose that  $j < i$  is a number such that  $A_i \cap A_j = \emptyset$  and  $\pi(A_i) \cap A_j \neq \emptyset$ . Then, by the definition of  $\pi(A_i)$ , there must be some  $l < i$  such that  $A_l \subset A_i$  and  $\pi(A_l) \cap A_j \neq \emptyset$ . There are two

cases. If  $l < j$  then let  $k = l$  and observe that  $k$  witnesses (b) for  $j$  and  $i$ . If  $l > j$ , find  $k < j$  witnessing (b) for  $j$  and  $l$  and observe that  $k$  witnesses (b) for  $j$  and  $i$ .

Second, observe that the map  $\pi$  is then  $Y$ -suitable. To see this, let  $A \subset X^d$  be a nonempty  $Y$ -algebraic set. Find  $j \in \omega$  such that  $A = A_j$ . The sets  $\Gamma_n(A)$  are all  $Y$ -algebraic and disjoint from  $A_j$  by the irreflexivity assumptions. Since the graphs  $\Gamma_n$  are pairwise disjoint, for all but finitely many  $n \in \omega$  there is no  $k < j$  such that  $A_k \subseteq \Gamma_n(A)$ . Item (b) then shows that for all such  $n$ ,  $\pi(\Gamma_n(A))$  must be disjoint from  $A$  as desired.

Now suppose that  $Y \subset X$  is a subfield of uncountable cardinality  $\kappa$  and for all subfields of smaller cardinality suitable maps have been produced. Express  $Y = \bigcup_{\beta \in \alpha} Y_\beta$  as a continuous increasing union of subfields of smaller cardinality, for some ordinal  $\alpha$ . By recursion on  $\beta \in \alpha$  build  $Y_\beta$ -suitable maps  $\pi_\beta$  in such a way that  $\gg \in \beta$  implies  $\pi_\gg \subset \pi_\beta$ . At limit ordinals  $\beta$ , take a union and observe that all items above are preserved. Now, suppose that  $\pi_\beta$  has been constructed, and work to find  $\pi_{\beta+1}$ .

First, use the induction hypothesis to find a suitable map  $\pi$  on  $Y_{\beta+1}$ . The difficulty now is that  $\pi$  may not extend  $\pi_\beta$ . Thus, define  $\pi_{\beta+1}$  in the following way. Let  $\pi_{\beta+1}(A) = (\pi(A) \cap \pi_\beta(B)) \cup \bigcup \{\pi_\beta(C) : C \subseteq A \text{ is a } Y_\beta\text{-algebraic set}\}$ , where  $B$  is the smallest  $Y_\beta$ -algebraic set containing  $A$  as a subset.

It is immediate that  $\pi_{\beta+1}$  is a map which preserves inclusion, and it extends  $\pi_\beta$ . Now, let  $A \subset X^d$  be an  $Y_{\beta+1}$ -algebraic set, and work to show that for all but finitely many  $n \in \omega$ , the set  $\pi_{\beta+1}(\Gamma_n(A))$  is disjoint from  $A$ . Let  $B \subset X^d$  be the smallest  $Y_\beta$ -algebraic set containing  $A$  as a subset.

**Claim 31.** *For every number  $n \in \omega$ ,  $\Gamma_n(B)$  is the largest  $Y_\beta$ -algebraic subset of  $\Gamma_n(A)$ .*

*Proof.* Suppose that  $C \subseteq \Gamma_n(A)$  is an  $Y_\beta$ -algebraic set. Consider the set  $D = \Gamma_n(C)$ .  $D$  is  $Y_\beta$ -algebraic and it is a superset of  $A$ . Therefore,  $B \subseteq D$ , which implies that  $C \subseteq \Gamma_n(D) \subseteq \Gamma_n(B)$  holds as desired.  $\square$

Thus, for every number  $n \in \omega$ ,  $\pi_{\beta+1}(\Gamma_n(A)) \subseteq \pi(\Gamma_n(A)) \cup \pi_\beta(\Gamma_n(B))$ . For all but finitely many  $n \in \omega$ , the first entry of this union is disjoint from  $A$  and the other from  $B$  by the choice of  $\pi$  and  $\pi_\beta$ . Since  $A \subseteq B$  holds, for all but finitely many  $n$  the union is disjoint from  $A$  as desired. This concludes the transfinite recursion step and the proof of the theorem.  $\square$

## 4 Questions

Many questions regarding  $\Gamma$ -neighborhood assignments remain unresolved. In the following,  $\Gamma$  stands for an analytic graph on a Polish space. However, the questions are already interesting for smaller classes of graphs such as  $\sigma$ -algebraic graphs on Euclidean spaces.

**Question 32.** Are the following equivalent:

1.  $\Gamma$  has countable chromatic number;
2.  $\Gamma$  is left-separated in a suitable Polish topology on  $X$ .

**Question 33.** Are the following equivalent:

1.  $\Gamma$  is left-separated;
2. there is no continuous homomorphic injection from  $\Delta_2$  to  $\Gamma$ .

**Question 34.** Is the statement “ $\Gamma$  is left-separated” absolute between models of set theory?

**Question 35.** For an analytic set  $A \subset X$ , is the statement “ $\Gamma \cap (A \times A)$  is left-separated”  $\Pi_1^1$ ?

## Acknowledgements

The first author was supported by the Czech Academy of Sciences (RVO 67985840). The second author was partially supported by the Czech Academy of Sciences (RVO 67985840) and grant EXPRO 20-31529X of GA ĆR, and grant NSF 2348371.

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