A Note on Lenses in Arrangements of Pairwise Intersecting Circles in the Plane

Rom Pinchasi[a]

Submitted: May 8, 2023; Accepted: May 14, 2024; Published: Jun 14, 2024

Abstract

Let $\mathcal{F}$ be a family of n pairwise intersecting circles in the plane. We show that the number of lenses, that is convex digons, in the arrangement induced by $\mathcal{F}$ is at most $2n-2$. This bound is tight. Furthermore, if no two circles in $\mathcal{F}$ touch, then the geometric graph G on the set of centers of the circles in $\mathcal{F}$ whose edges correspond to the lenses generated by $\mathcal{F}$ does not contain pairs of avoiding edges. That is, G does not contain pairs of edges that are opposite edges in a convex quadrilateral. Such graphs are known to have at most $2n-2$ edges.

1 Introduction

Given a family $\mathcal{F}$ of circles in the plane we consider the planar arrangement that is induced by the circles in $\mathcal{F}$ and denote it by $\mathcal{A}(\mathcal{F})$. The arrangement $\mathcal{A}(\mathcal{F})$ consists of *vertices* that are intersection points of circles in $\mathcal{F}$ and also of *edges* that are arcs of circles in $\mathcal{F}$ delimited by two consecutive vertices. Finally, $\mathcal{A}(\mathcal{F})$ consists also of *faces* that are the connected components of the plane after removing from it the union of all circles in $\mathcal{F}$. A *digon* in $\mathcal{A}(\mathcal{F})$ is a face in the arrangement $\mathcal{A}(\mathcal{F})$ that has two edges. We distinguish between two types of digons in arrangements of circles. A *lens* in $\mathcal{A}(\mathcal{F})$ is a face with two edges in $\mathcal{A}(\mathcal{F})$ that is equal to the intersection of two discs bounded by circles in $\mathcal{F}$. Each of the two circles in $\mathcal{F}$ corresponding to a lens is said to *support* the lens. The lens is said to be *created* by these two circles supporting it. Lenses are in fact just the convex digons in $\mathcal{A}(\mathcal{F})$. The arrangement $\mathcal{A}(\mathcal{F})$ may contain also digons that are not convex. These are digons that are equal to the difference of two discs bounded by circles in $\mathcal{F}$. They are called *lunes*.

Grünbaum[6] conjectured that the number of digons in arrangements of n pairwise intersecting pseudo-circles is at most $2n-2$. This conjecture of Grünbaum was verified in by Agarwal et al.[1] for arrangements of pseudo-circles surrounding a common point. In a

[a] Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. room@technion.ac.il. Visiting professor at EPFL, Lausanne, Switzerland.

recent work by Felsner, Roch, and Scheucher[4], Grünbaum's conjecture was verified for any arrangement of pairwise intersecting pseudo-circles under an additional assumption that the family of pseudo-circles contains three pseudo-circles every two of which create a digon in the arrangement.

In this paper we will be concerned with digons that are lenses in a family of pairwise intersecting circles. We will show that the number of lenses in an arrangement of n pairwise intersecting circles without any further assumption is at most $2n-2$.

Theorem 1. *Let $\mathcal{F}$ be a family of n pairwise intersecting circles in the plane. Then $\mathcal{A}(\mathcal{F})$ has at most $2n-2$ lenses. This bound is tight for $n \geqslant 4$.*

The simple construction in Figure 1 shows that the bound in Theorem 1 is best possible for $n \geqslant 4$. There are 5 circles in this construction and 8 lenses. One can generalize the construction for any number of circles by suitably adding more circles to the three smaller circles in the figure.

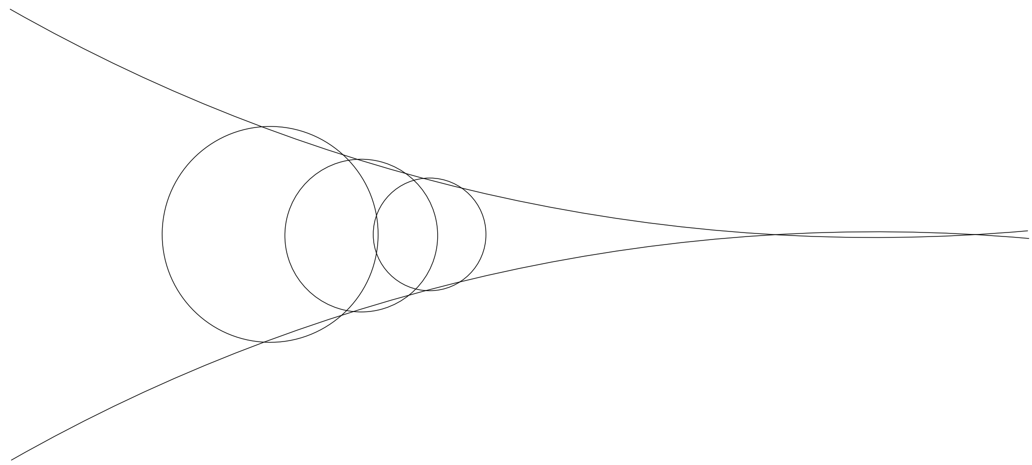

Figure 1: A family of 5 pairwise intersecting circles with 8 lenses.

It is interesting to note that the number of lenses in arrangements of pairwise intersecting unit circles is different. It is shown in[10] that there are at most n lenses in any arrangement of pairwise intersecting unit circles in the plane and this bound is best possible.

There has been a lot of research about lenses in arrangements of circles and pseudo-circles that are not necessarily pairwise intersecting. We will not survey here the vast literature about digons in arrangements of circles and pseudo-circles and on related situations where we allow curves to intersect more than twice and only refer the reader to[5] and the many references therein. The case where circles need not be pairwise intersecting is of completely different nature. We remark that in such a case the best constructions show that it is possible that n circles will determine $\Omega(n^{4/3})$ many lenses. The best known upper bound is $O(n^{3/2}\log n)$ given in[8], that is following the footsteps of[11]. The case of unit circles is of particular interest because of its relation to the celebrated unit distance problem posed by Paul Erdős[3].

Going back to families of pairwise intersecting circles, the number of lunes in these arrangements was studied in[2].

Theorem 2 ([2])**.** *Let $\mathcal{F}$ be a family of n pairwise intersecting circles in the plane. Let G be the geometric graph on the set of centers of the circles in $\mathcal{F}$ whose edges correspond to pairs of discs bounded by circles in $\mathcal{F}$ whose difference is a lune in $\mathcal{A}(\mathcal{F})$. Then G is a bipartite planar embedding. Consequently, $\mathcal{A}(\mathcal{F})$ has at most $2n-4$ lunes.*

Theorem 2 is used in[2] to derive a linear upper bound (that is not tight) for the number of lenses in arrangements of pairwise intersecting circles in the plane. Theorem 1, that we prove here, provides the tight upper bound for the number of lenses in a family of pairwise intersecting circles in the plane.

Similar to the proof of Theorem 2, the proof of Theorem 1 relies too on studying the corresponding geometric graph $G = G(\mathcal{F})$ whose vertices are the centers of circles in $\mathcal{F}$ and two centers are connected by an edge in $G(\mathcal{F})$ iff the corresponding circles in $\mathcal{F}$ create a lens in $\mathcal{A}(\mathcal{F})$. We will show that unless we allow two circles in $\mathcal{F}$ to touch, then G does not contain a pair of *avoiding* edges. Two straight line segments (or edges in a geometric graph G) are called *avoiding* if they are opposite edges in a convex quadrilateral. Handling the case of $\mathcal{F}$ having pairs of touching circles will require only a bit more effort. This is because if we allow touching circles in $\mathcal{F}$, the geometric graph $G(\mathcal{F})$ may contain pairs of avoiding edges. Luckily, we will be able to compensate for this.

The important property of geometric graphs that do not contain pair of avoiding edges is given in[7] and[12] (see also[9] for a different and shorter proof of the same result, based on graph drawing).

Theorem 3 ([7, 12])**.** *Let G be a geometric on n vertices. If G does not have a pair of avoiding edges, then it has at most $2n-2$ edges. This bound is tight for $n \geqslant 4$.*

Our main goal is to prove Theorem 1. In order to prove Theorem 1 we start by making some assumptions without loss of generality that will help to simplify the presentation.

We first observe that we may assume that no two discs bounded by the circles in $\mathcal{F}$ may contain each other. Assume that C_1 and C_2 are two circles in $\mathcal{F}$ such that the disc bounded by C_1 fully contains the disc bounded by C_2. Because every two circles in $\mathcal{F}$ intersect, it must be that C_2 touches C_1 internally. We claim that in such a case C_1 cannot support any lens. The reason is that if C_1 create a lens together with a circle C_3 in $\mathcal{F}$, then necessarily C_3 and C_2 cannot intersect (see Figure 2).

We can therefore remove C_1 from $\mathcal{F}$ and conclude Theorem 1 by induction on the number of circles in $\mathcal{F}$. We henceforth assume that no two discs bounded by circles in $\mathcal{F}$ may contain each other. We conclude that if two circles in $\mathcal{F}$ touch, then they must touch externally and consequently no more than two circles in $\mathcal{F}$ may be pairwise touching at a common point.

For reasons that will become clear later, it will be more convenient for us to assume that if C_1 and C_2 are two circles in $\mathcal{F}$ that support a lens, then the segment connecting the center of C_1 to the center of C_2 is not collinear with any center of circle in $\mathcal{F}$ that is not

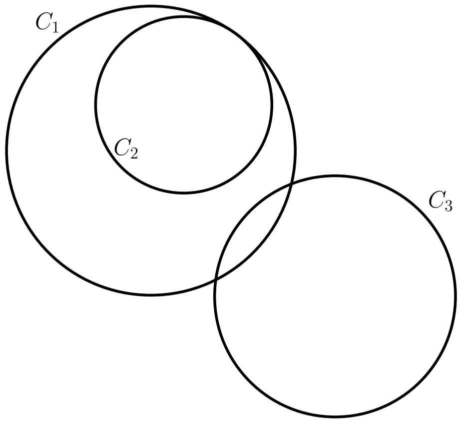

Figure 2: C_2 cannot be contained in the disc bounded by C_1.

on that segment. We can indeed assume this without loss of generality. This is naturally the case if we assume that no three centers of circles in $\mathcal{F}$ are collinear. If we want to avoid this assumption, then we can just apply a generic inversion to the plane. In such a case three centers of circles in $\mathcal{F}$ will remain collinear only if they form a *pencil*, that is only if they pass through two common points (or mutually touch at a point, which is impossible in our case). However, in such a case where we have several circles in $\mathcal{F}$ passing through two common points, then only the two extreme circles may create together a lens (see Figure 3). In such a case the segment connecting the centers of these two extreme circles will not be collinear with any other center of a circle in $\mathcal{F}$ not on that segment. In

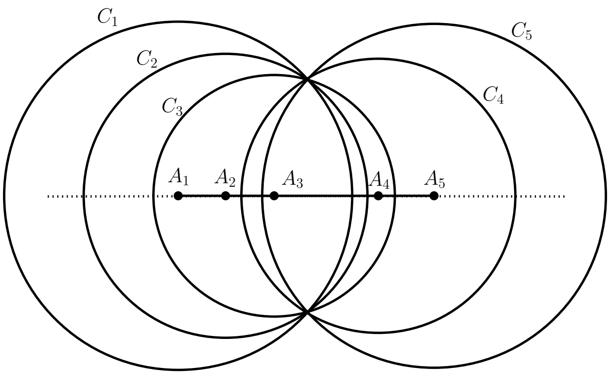

Figure 3: A pencil of circles. Only the two extreme circles, C_1 and C_5 in this figure, may create a lens.

order to prove Theorem 1 and in order to allow also touching circles in $\mathcal{F}$, we define the geometric graph $G = G(\mathcal{F})$ on the set of centers of the pairwise intersecting circles in $\mathcal{F}$ in the following way. The edges of G will be either red, or blue. We connect two centers of circles in $\mathcal{F}$ by a red edge if the corresponding circles create a lens. We connect two centers of circles in $\mathcal{F}$ by a blue edge if the corresponding circles touch.

The following theorem is of independent interest and is also an intermediate step before proving Theorem 1.

Theorem 4. *The geometric graph $G(\mathcal{F})$ does not contain pairs of avoiding edge unless there are four circles C_1, C_2, C_3 and C_4 in $\mathcal{F}$ passing through a common point at which C_1 and C_3 touch each other and also C_2 and C_4 touch each other. Only in such a case it is possible to have avoiding edges in $G(\mathcal{F})$. The avoiding edges in such a case can only be a pair of two opposite red edges in the quadrilateral whose vertices are the centers of such four circles C_1, C_2, C_3, and C_4.*

We present the proof of Theorem 4 in Section 2. Then we bring the proof of Theorem 1 in Section 3.

2 Proof of Theorem 4.

Assume that e and f are two avoiding edges in $G(\mathcal{F})$. Then e and f remain two avoiding edges in the graph G that corresponds only to the four circles in $\mathcal{F}$ whose centers are the four endpoints of e and f. This is because by removing, or ignoring, all other circles in $\mathcal{F}$ we cannot destroy the digons, or pairs of touching circles corresponding to the edges e and f. For the proof we will indeed assume that $\mathcal{F}$ is a family of only four circles. These are the four circles in the original family $\mathcal{F}$ that are centered at the endpoints of the avoiding segments e and f. Consequently, $G = G(\mathcal{F})$ is a geometric graph with only four vertices. Denote by A_1 and A_2 the vertices of e and let A_3 and A_4 be the vertices of f. Because e and f are avoiding, we assume without loss of generality that $A_1A_2A_3A_4$ is a convex quadrilateral. For $i = 1, 2, 3, 4$ denote by C_i the circle in $\mathcal{F}$ centered at A_i. Denote by D_i the closed circular disc bounded by C_i.

In order to simplify the presentation of the proof we would like to assume that there is a point M common to three of the circles C_1, C_2, C_3, and C_4. We can assume this without loss of generality. This is because if this is not the case we inflate the circle C_1 keeping its center A_1 fixed until the first time C_1 passes through an intersection point of two other circles from $\mathcal{F}$, namely an intersection point of two of the circles C_2, C_3, and C_4.

More precisely, assume that no three of C_1, C_2, C_3, and C_4 pass through a common intersection point. Because C_1 and C_2 create a lens or externally touch, it must be that the intersection points on C_2 with the circles C_3 and C_4 lie outside of D_1. We start inflating C_1 until the first time it meets an intersection points of two of C_2, C_3, and C_4. When this happens C_1 still intersects with each of C_2, C_3, and C_4 because each of C_2, C_3, and C_4 contains at least one intersection point that is not surrounded by C_1, while each of C_2, C_3, and C_4 contains also points in D_1 because before we inflated C_1 it intersected each of C_1, C_2, and C_3, while D_1 only increases through the inflation of C_1. The only thing that is left two show is that e and f are still edges in $G(\mathcal{F})$, that is, we need to show that even after the inflation of C_1 it is still true that C_3 and C_4 create a lens or touch and the same is true for C_1 and C_2. The reason this is true is that the intersection points of C_1 with any of C_2, C_3, and C_4 move continuously with the inflation of C_1. Therefore, if C_3 and C_4 create a lens, then the edges $C_3 \cap D_4$ and $C_4 \cap D_3$, remain edges in $\mathcal{A}(\mathcal{F})$ also after inflating C_1. It is also clear that if C_3 and C_4 touch, then they remain touching regardless of the inflation of C_1. If C_1 and C_2 create a lens, then the edges $C_1 \cap D_2$ and $C_2 \cap D_1$

remain edges in $\mathcal{A}(\mathcal{F})$ also after inflating C_1 and therefore C_1 and C_2 create a lens also after we inflate C_1.

The only case where we need to be careful is if C_1 and C_2 touch each other before we inflate C_1. In this case as we start inflating C_1 it creates a lens with C_2 and by the argument above it will create a lens with C_2 also when the inflation of C_1 is stopped.

We conclude that by inflating C_1, keeping its center fixed, we may assume that three of the circles in $\mathcal{F}$ pass through a common point while e and f remain two avoiding edges in $G(\mathcal{F})$. Therefore, we assume without loss of generality that C_1, C_2, and C_3 pass through a common intersection point that we denote by M.

Denote by s_1, s_3, and s_4 the arcs $C_2 \cap D_1, C_2 \cap D_3$, and $C_2 \cap D_4$, respectively. We note that each s_i may be degenerate and equal to a single point in case C_i and C_2 touch. Notice that M is an endpoint of both s_1 and s_3. For $i = 1, 3, 4$ let S_i denote the center of the arc s_i. We notice that S_i is the point of intersection of the ray $\overrightarrow{A_2A_i}$ with C_2. For this reason and because $A_1A_2A_3A_4$ is a convex quadrilateral, the point S_4 must lie in the shorter arc of C_2 delimited by S_1 and S_3 (see Figures 4 and 5).

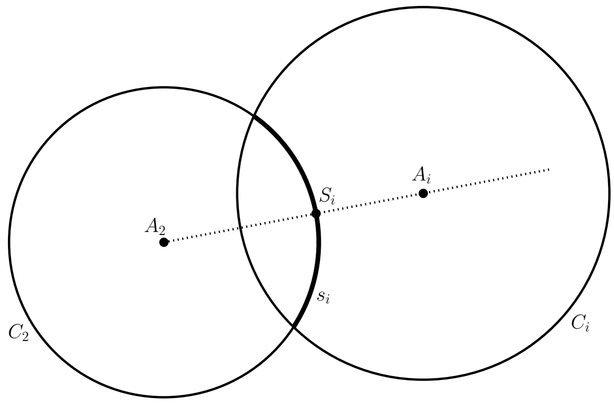

Figure 4: The arc s_i with its center S_i.

We claim that the intersection of s_1 and s_3 is equal to the point M. Indeed, otherwise s_1 and s_3 are both nondegenerate arcs and the relative interiors of s_1 and s_3 overlap. Then arc s_1 is an edge of the lens $D_1 \cap D_2$ and cannot contain any intersection points in its relative interior. The arc s_1 can also not be equal to s_3, or else A_1, A_2, and A_3 are collinear, contrary to the assumption that they are three vertices of a convex quadrilateral. We conlude from here that it must be that s_3 strictly contains s_1 (see Figure 5).

Because S_4 lies in the shorter arc of C_2 delimited by S_1 and S_3, it follows that S_4 lies in the relative interior of s_3. Consequently, S_4 lies in the interior of D_3. At the same time S_4 lies also in D_4. This shows that S_4, that is a point on C_2, belongs to the lens or is equal to the touching point $D_3 \cap D_4$. This is possible only if S_4 is an intersection point of C_3 and C_4. However, the latter case is impossible because S_4 lies in the interior of D_3.

Having shown that the intersection of s_1 and s_3 is equal to the point M, we observe that M cannot belong to the interior of D_4, or else $D_3 \cap D_4$ cannot be a lens nor a touching point. Consequently, M cannot belong to the relative interior of the arc s_4. Combining

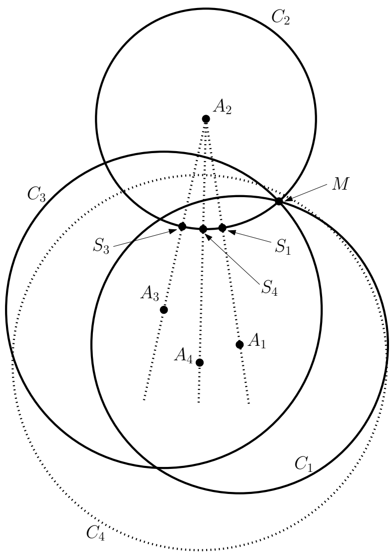

Figure 5: The impossible case where s_3 contains s_1.

this with the fact that S_4 belongs to the shorter arc of C_2 delimited by S_1 and S_3, we conclude that s_4 is contained in s_1, or it is contained in s_3 (see Figure 6).

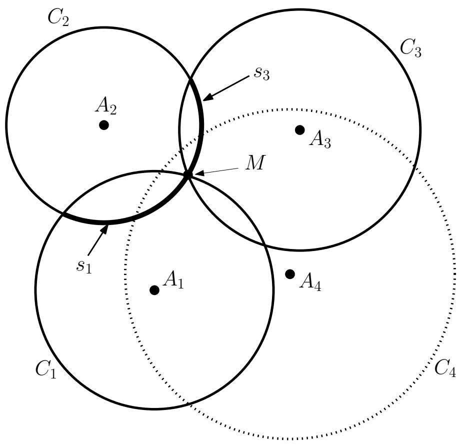

Figure 6: M cannot belong to the relative interior of s_4.

We claim that s_4 must be equal to the point M and hence C_2 and C_4 touch each other at M. In order to prove this we consider the two possible cases $s_4 \subset s_1$, and $s_4 \subset s_3$.

Case 1. $s_4 \subset s_1$. In this case s_1 must be a nondegenerate arc, or else both C_1 and C_4 touch C_2 at the same point M which is impossible. Therefore, s_1 is an edge of the lens $D_1 \cap D_2$ and it cannot contain any intersection points in its relative interior. Hence either s_4 is equal to an endpoint of s_1, or s_4 is equal to s_1. The latter case is impossible as it would imply that A_1, A_2, and A_4 are collinear, contrary to the assumption that they are vertices of a convex quadrilateral. In the former case s_4 must be equal to the point M because S_4, that is equal to s_4 in this case, lies in the shorter arc of C_2 delimited by S_1

and S_3 (see Figure 7).

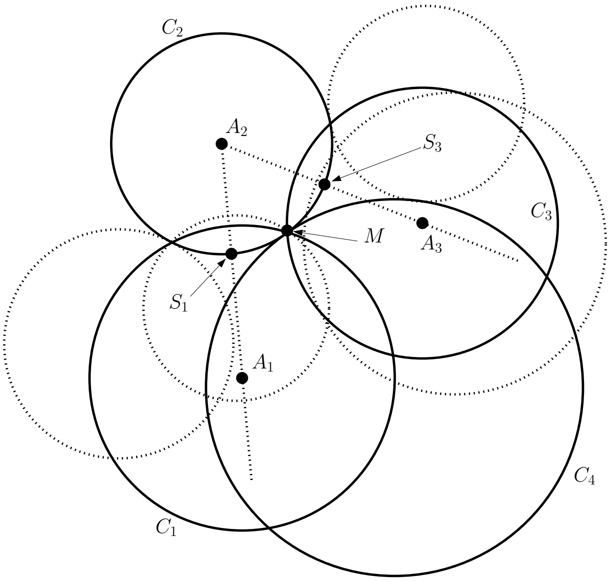

Figure 7: C_4 must touch C_2 at M. The dotted circles show impossible positions of C_4.

Case 2. $s_4 \subset s_3$. In this case s_3 must be a nondegenerate arc, or else both C_1 and C_3 touch C_2 at the same point M which is impossible. We notice that s_4 is contained in both D_4 and D_3 and therefore s_4 is contained in the lens $D_4 \cap D_3$. This is impossible unless s_4 is a single point that is equal to an intersection point of C_3 and C_4 and in particular belongs to C_3. Then s_4 must be equal to one of the two endpoints of s_3. It must be equal to the point M because S_4, that is equal to s_4 in this case, lies in the shorter arc of C_2 delimited by S_1 and S_3 (see Figure 7).

Having shown that that C_1, C_2, and C_4 pass through M we can argue symmetrically that C_3 touches C_1 at M, as illustrated in Figure 8.

We observe that in such a case the circle C_1 could not be inflated by a factor greater than 1, or else it would be disjoint from C_3 in the original configuration of the circles. To summarize, we have shown that if e and f are two avoiding edges in $G(\mathcal{F})$, then necessarily C_1 touches C_3 at a point M and C_3 touches C_4 at the same point M. This concludes the proof of Theorem 4. □

3 Proof of Theorem 1.

We need to show that $\mathcal{F}$ determines at most $2n-2$ lenses. We consider the geometric graph $G = G(\mathcal{F})$ defined just before the statement of Theorem 4. Proving Theorem 1 is equivalent to showing that the number of red edges in $G(\mathcal{F})$ is at most $2n-2$.

If no two circles in $\mathcal{F}$ touch each other, then it follows from Theorem 4 that the graph $G(\mathcal{F})$ consists of red edges only. Moreover, no two edges in $G(\mathcal{F})$ are avoiding. If the set of vertices of $G(\mathcal{F})$, that is the set of centers of the circles in $\mathcal{F}$, is in general position in

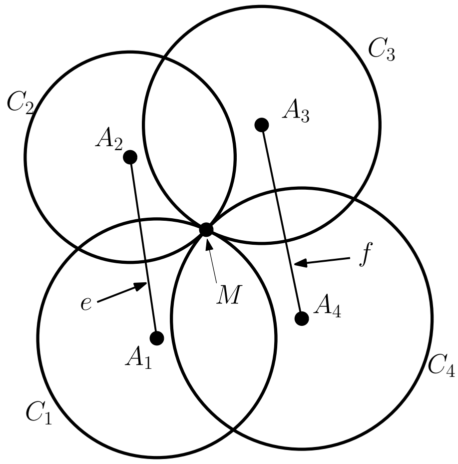

Figure 8: The only configuration in which e and f may be avoiding. Both pairs of circles C_2 and C_4 as well as C_1 and C_3 mutually touch at M.

the sense that no three of them are collinear, then we can apply Theorem 3 and conclude that $G(\mathcal{F})$ has at most $2n-2$ edges. Consequently, $\mathcal{F}$ determines at most $2n-2$ lenses. If the set of vertices of $G(\mathcal{F})$ is not in general position, then strictly speaking we cannot apply Theorem 3 because Theorem 3, as most other theorems about geometric graphs, is stated and proved for geometric graphs whose set of vertices is in general position. This is a standard assumption in most results concerning geometric graphs. In order to be able to apply Theorem 3, we perturb a bit the vertices of $G(\mathcal{F})$ to make them lie in general position. We notice that by perturbing the vertices of $G(\mathcal{F})$ we may not create pairs of avoiding edges, unless there is an edge in $G(\mathcal{F})$ that is collinear with another vertex in $G(\mathcal{F})$ not on this edge (see Figure 9, where in the perturbed picture on the right e and f_2 are avoiding as well as e and f_3).

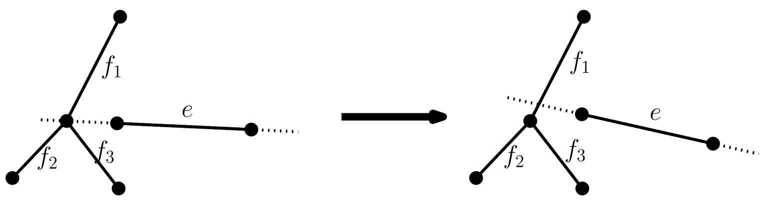

Figure 9: The edge e is collinear with a vertex not on e.

As we assumed without loss of generality, this cannot not happen. No edge e in $G(\mathcal{F})$ is collinear with a vertex not on on e. We may therefore apply Theorem 3 to the perturbed geometric graph and conclude that $G(\mathcal{F})$ has at most $2n-2$ edges as before. This proves Theorem 1 in the case where no pair of circles in $\mathcal{F}$ touch each other.

In the more general case where we allow circles in $\mathcal{F}$ to touch each other, the graph $G(\mathcal{F})$ may contain pairs of avoiding red edges. However, in this case we can use the blue edges in $G(\mathcal{F})$ as follows. By Theorem 4, whenever the graph $G(\mathcal{F})$ contains pairs of avoiding edges it must be because of the very special structure as described in the statement of Theorem 4 and shown in Figure 8. If e and f are two avoiding edges in $G(\mathcal{F})$, then they are necessarily red edges that correspond to two lenses in $\mathcal{A}(\mathcal{F})$. Let $A_1A_2A_3A_4$ be the convex quadrilateral such that $e = A_1A_2$ and $f = A_3A_4$. We assume without loss of generality that A_1, A_2, A_3, and A_4 is the clockwise cyclic order of these points as vertices of the convex quadrilateral $A_1A_2A_3A_4$ (see Figure 8). For $i = 1, 2, 3, 4$ denote by C_i the circle in $\mathcal{F}$ centered at A_i. By Theorem 4, A_1 and A_3 are connected by a blue edge in $G(\mathcal{F})$ and so are A_2 and A_4. This corresponds to that C_1 and C_3 must touch, at a point that we denote by M, and so are C_2 and C_4.

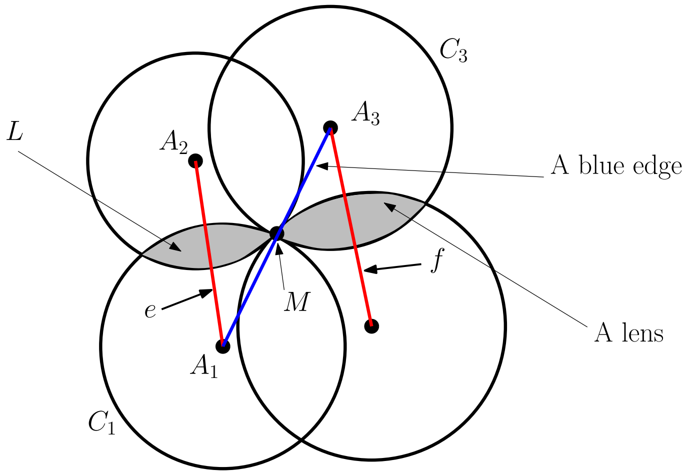

Figure 10: The blue edge A_1A_3 is uniquely charged.

We remove either the red edge e, or the red edge f from $G(\mathcal{F})$ and charge the avoiding pair e and f to the blue edge A_1A_3. We claim that the blue edge A_1A_3 cannot be charged for another red edge in this way. This is because once we fix A_1 and A_3 and therefore also C_1 and C_3, then we determine also the touching point M of C_1 and C_3. We claim that the edges e and f are determined as well. Indeed, e is the edge that corresponds to the unique lens L supported by C_1 such that M is a vertex of L in $\mathcal{A}(\mathcal{F})$ and A_1, A_2, M is the clockwise order of these three points, where A_2 is the center of the other circle in $\mathcal{F}$ supporting L (see Figure 10). Any other such lens L' would overlap with L, which is impossible.

By a symmetric argument we show that the edge f is determined by the blue edge A_1A_3. We conclude from this that the blue edge A_1A_3 in $G(\mathcal{F})$ can be charged to at most one pair of avoiding edges in $G(\mathcal{F})$.

After repeating this procedure for every remaining pair of avoiding edges we are left with a subgraph G' of $G(\mathcal{F})$ in which no two edges are avoiding. The number of edges, red and blue, in G' is greater than or equal to the number of red edges in G. We apply Theorem 3 to G', after possibly perturbing the vertices of G', as we did already before, and conclude that G' has at most $2n-2$ edges. Therefore, there are at most $2n-2$ red edges in $G(\mathcal{F})$. Consequently, there are at most $2n-2$ lenses in $\mathcal{A}(\mathcal{F})$, as desired. $\square$

Acknowledgements

We thank Eyal Ackerman for helpful comments about the case where the centers of the circles in $\mathcal{F}$ are not in general position. Supported by ISF grant (grant No. 1091/21).

References

[1] P. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky, Lenses in Arrangements of Pseudocircles and their Applications, *J. ACM*, **51**, (2004), 139–186.

[2] N. Alon, H. Last, R. Pinchasi, and M. Sharir, On the Complexity of Arrangements of Circles in the Plane, *Discrete Comput. Geom.*, **26** (2001), 465–492.

[3] P. Erdős, On sets of distances of n points *Amer. Math. Monthly.* **53** (1946), 248–250.

[4] S. Felsner, S. Roch, and M. Scheucher, Arrangements of Pseudocircles: On Digons and Triangles. In: Angelini, P., von Hanxleden, R. (eds) Graph Drawing and Network Visualization. GD 2022. Lecture Notes in Computer Science, vol **13764**. Springer, Cham.

[5] Handbook of discrete and computational geometry. Third edition. Edited by Jacob E. Goodman, Joseph O'Rourke and Csaba D. Tóth. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018.

[6] B. Grünbaum, Arrangements and Spreads, CBMS Regional Conference Series in Mathematics, vol. 10. AMS (1972).

[7] M. Katchalski, H. Last, On geometric graphs with no two edges in convex position. Dedicated to the memory of Paul Erdős. Discrete Comput. Geom. **19** (1998), no. 3, Special Issue, 399–404.

[8] A. Marcus, G. Tardos, Intersection reverse sequences and geometric applications. *J. Combin. Theory Ser. A* **113** (2006), no. 4, 675–691.

[9] R. Pinchasi, Geometric graphs with no two parallel edges, *Combinatorica*, **28** (2008) no. 1, 127–130.

[10] R. Pinchasi, Gallai-Sylvester Theorem for Pairwise Intersecting Unit Circles, *Discrete Comput. Geom.*, **28** (2002), 607–624.

[11] R. Pinchasi and R. Radoičić, On the Number of Edges in a Topological Graph with no Self-intersecting Cycle of Length 4, *Towards a Theory of Geometric Graphs*, 233–243, Contemp. Math., 342, Amer. Math. Soc. Providence, RI, 2004.

[12] P. Valtr, On geometric graphs with no k pairwise parallel edges, Discrete Comput. Geom. **19** (1998), no. 3, Special Issue, 461–469.