On Triangle-Free Graphs Maximizing Embeddings of Bipartite Graphs

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Abstract

In 1991 Győri, Pach, and Simonovits proved that for any bipartite graph H containing a matching avoiding at most 1 vertex, the maximum number of copies of H in any large enough triangle-free graph is achieved in a balanced complete bipartite graph. In this paper we improve their result by showing that if H is a bipartite graph containing a matching of size x and at most $\frac{1}{2}\sqrt{x-1}$ unmatched vertices, then the maximum number of copies of H in any large enough triangle-free graph is achieved in a complete bipartite graph. We also prove that such a statement cannot hold if the number of unmatched vertices is $\Omega(x)$.

Mathematics Subject Classifications: 05C35

1 Introduction

A classical theorem of Turán [8] states that the unique K_r-free graph on n vertices with the maximum number of edges is the balanced complete $(r-1)$-partite graph, denoted by $T_{r-1}(n)$. This result was further generalized by Zykov [9] (and independently by Erdős [2]), who proved that among K_r-free n-vertex graphs, also $T_{r-1}(n)$ maximizes the number of copies of any complete graph K_s for $s < r$. In general, the maximum number of copies of a given graph H among all K_r-free n-vertex graphs (for $r > \chi(H)$) is not always achieved in $T_{r-1}(n)$. For example if H is a star on 4 vertices and $r = 3$. Nevertheless, recently, Morrison, Nir, Norin, Rzążewski and Wesolek [7], answering a conjecture of Gerbner and Palmer [4], showed that for any graph H the maximum number of copies of H in a large enough K_r-free n-vertex graph is obtained in $T_{r-1}(n)$ as long as r is large enough.

A natural generalization of the previously mentioned results is to search for sufficient conditions for the maximum number of copies of a given graph H in a K_r-free G to be maximized when G is some (not necessarily balanced) complete $(r-1)$-partite graph. Note that an easy application of the graph removal lemma (see e.g. [3, Theorem 2])

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implies that if $\chi(H) < \chi(F)$ then the maximum number of copies of H in F-free graphs is asymptotically the same as in $K_{\chi(F)}$-free graphs, so solving the problem when a complete graph is forbidden is essentially solving it in the more general case as well.

One necessary condition to have the maximum number of copies of H achieved in a complete $(r-1)$-partite graph is $\chi(H) < r$. However, this is not sufficient as shown by the following example.

Example 1 (Győri, Pach, Simonovits [6])**.** Let H be a bipartite graph on $2k$ vertices formed by two disjoint stars $K_{1,k-2}$ with centers connected by a path of length 3. The number of copies of H in any n-vertex bipartite graph is maximized in $T_2(n)$, since H has the same number of vertices in both color classes. However, for sufficiently large k and n, the number of copies of H in $T_2(n)$ is significantly smaller than in a blow-up of a five-cycle with blobs of sizes $\frac{n}{2k}, \frac{n}{2k}, \frac{n}{2k}, \frac{n}{2k}$, and $n - \frac{2n}{k}$.

For a similar example for higher values of r, see [5].

In the most interesting case $r = 3$, asking when the maximum number of copies of a given bipartite graph in triangle-free graphs is achieved in a complete bipartite graph, Győri, Pach, and Simonovits proved the following sufficient condition.

Theorem 2 (Győri, Pach, and Simonovits [6])**.** *Let H be a bipartite graph on m vertices containing a matching of size $\lfloor \frac{m}{2} \rfloor$. Then, for $n > m$, $T_2(n)$ is the unique n-vertex triangle-free graph maximizing the number of copies of H.*

Intuitively, if H contains a large matching, then $T_2(n)$ is the best choice for a triangle-free maximizer, because it contains the largest number of copies of the matching itself. The condition that H must have a perfect matching (or an almost perfect matching if $2 \nmid m$) cannot be relaxed to a matching on $m-2$ vertices if we want to have $T_2(n)$ as the maximizer. For example the maximum number of copies of a star $K_{1,3}$ in triangle-free graphs is not achieved in $T_2(n)$, but in a non-balanced complete bipartite graph.

We show that the maximizer is a complete bipartite graph even using a far weaker condition on the size of a matching in H.

Theorem 3. *Let H be a bipartite graph containing a matching of size x and at most $\frac{1}{2}\sqrt{x-1}$ unmatched vertices. Then, for n sufficiently large, a complete bipartite graph maximizes the number of copies of H among all triangle-free n-vertex graphs.*

We do not know whether the bound on the number of unmatched vertices $O(\sqrt{x})$ is optimal, but we show that it needs to be sublinear.

Theorem 4. *For any constant $\lambda > 0$ and integer $n_0 > 0$, there exist an integer x, a bipartite graph H containing a matching of size x and at most λx unmatched vertices, and a triangle-free non-bipartite graph G on more than n_0 vertices, such that the number of copies of H in G is larger than in any bipartite graph on the same number of vertices.*

2 Proofs

We start with introducing the needed notation and some preliminary results.

By the number of copies of H in G, denoted by $H(G)$, we mean the number of subgraphs of G isomorphic to H. For easier calculations we consider injective embeddings of H in G, i.e., injective functions $\varphi : V(H) \to V(G)$ such that $\varphi(v_1)\varphi(v_2) \in E(G)$ if $v_1v_2 \in E(H)$. We denote the number of different injective embeddings of H in G by $\overline{H(G)}$. Observe that $\overline{H(G)}$ differs from $H(G)$ just by a factor equal to the number of automorphisms of H, so maximizing $\overline{H(G)}$ is equivalent to maximizing $H(G)$.

For two graphs H and G we define the *H-degree* of a vertex $v \in V(G)$, denoted $h(v)$, as the number of injective embeddings of H in G whose image contains v. Analogously, for $u, v \in V(G)$ we define $h(u, v)$ as the number of injective embeddings whose image contains vertices u and v, and $h(u, \bar{v})$ as the number of injective embeddings whose image contains vertex u and does not contain vertex v.

Lemma 5. *For an m-vertex graph H let G be a triangle-free n-vertex graph that maximizes the number of injective embeddings of H. Then for any two vertices $u, v \in V(G)$ it holds $h(v) \leqslant h(u) + O\left(n^{m-2}\right)$.*

Proof. Modify the graph G by deleting u and adding instead a copy of v (not adjacent to v). The obtained graph remains triangle-free after such modification. In this process we lose $h(u)$ injective embeddings and gain $h(v, \bar{u})$ new ones. Since $h(v) = h(v, u) + h(v, \bar{u})$, we get $h(v, \bar{u}) - h(u) = h(v) - h(u, v) - h(u) \leqslant 0$, so $h(v) \leqslant h(u) + h(u, v) = h(u) + O\left(n^{m-2}\right)$. ☐

Lemma 6. *Let G be a triangle-free graph on n vertices with maximum degree Δ. Then $|E(G)| \leqslant \Delta(n - \Delta)$.*

Proof. Consider a vertex of degree Δ and let A be the set of its neighbors. The set $V(G) \setminus A$ contains $n - \Delta$ vertices of degree at most Δ. Moreover, from triangle-freeness of G there are no edges between vertices in A. Thus, $|E(G)| \leqslant \Delta(n - \Delta)$.

Note that the equality holds only for the complete bipartite graph $K_{\Delta, n-\Delta}$. ☐

We are ready to prove the main theorem.

Proof of Theorem 3. If H contains an isolated vertex, then the graph H' obtained by removing it satisfies the assumptions of Theorem 3 (with a smaller number of unmatched vertices). Moreover, for every n-vertex graph G we have $\overline{H(G)} = \overline{H'(G)}(n - |V(H')|)$. Thus, if the theorem holds for H' then it also holds for H. Therefore, we may assume that H does not contain isolated vertices. We may also assume that the matching of size x is a maximal matching in H and $x \geqslant 2$.

Let m be the number of vertices in H and c be the number of connected components of H. For a sufficiently large n let G be a triangle-free graph on n vertices that has the largest number of copies of H. By summing up the H-degrees of all vertices in G we count each injective embedding exactly m times, so

$$\sum_{v\in V(G)} h(v) = m\cdot \overline{H(G)} \geqslant m\cdot \overline{H(T_2(n))} = m\cdot 2^c\left(\frac{n}{2}\right)^m + O\left(n^{m-1}\right).$$

The last equality holds because we can count the number of injective embeddings of H in $T_2(n)$ by embedding each vertex connected to already embedded vertices in $n/2$ ways and a vertex in a new component of H in n ways. The lower order error term comes from the possibility of selecting the same vertex multiple times.

For any vertex $u\in V(G)$ by summing up the inequality from Lemma 5 for each vertex $v\in V(G)$, and combining it with the above bound, we obtain

$$n\cdot h(u) \geqslant \sum_{v\in V(G)} h(v) + O(n^{m-1}) \geqslant m\cdot 2^c\left(\frac{n}{2}\right)^m + O\left(n^{m-1}\right).$$

Therefore

$$h(u) \geqslant m2^{c-1}\left(\frac{n}{2}\right)^{m-1} + O\left(n^{m-2}\right). \tag{1}$$

Our plan now is to upper-bound $h(u)$ for a vertex u of the minimum degree δ in G. Consider an arbitrary vertex $w\in V(H)$. We estimate the number of injective embeddings that map w to u in the following way. As H does not contain isolated vertices and the matching of size x is maximal, w is adjacent to some matched vertex. This implies that there exists a matching M in H of size x, which contains w. Thus, we have δ ways to embed in G the neighbor of w in the matching M. Then, we have at most $|E(G)|$ ways to embed in G any edge of M from the same component as already embedded vertices, and at most $2|E(G)|$ ways for each edge in a new component. Finally, we have at most Δ^{m-2x} ways to embed all vertices of H not belonging to M, where Δ is the maximum degree of G. Therefore, taking into account that we can choose w in $V(H)$ in m ways and applying Lemma 6 to bound the number of edges of G, we obtain

$$h(u) \leqslant m\delta 2^{c-1}|E(G)|^{x-1}\Delta^{m-2x} \leqslant m\delta 2^{c-1}\Delta^{m-x-1}(n-\Delta)^{x-1}. \tag{2}$$

By combining inequalities (1) and (2) we conclude the following bound for δ

$$\delta \geqslant \frac{\left(\frac{n}{2}\right)^{m-1}}{\Delta^{m-x-1}(n-\Delta)^{x-1}} + O(1). \tag{3}$$

Our goal is to show that under the assumptions of Theorem 3 from inequality (3) we derive that $\delta > \frac{2}{5}n$. Then, since the Andrásfai-Erdős-Sós [1] theorem gives that every n-vertex triangle-free graph with minimum degree greater than $\frac{2}{5}n$ is bipartite, the graph G will be bipartite.

For convenience, we replace $m-2x$ with $2d$. Since a function $f(z)=z^a(1-z)^b$ attains its maximum in $[0,1]$ for $z=\frac{a}{a+b}$, the denominator $\Delta^{m-x-1}(n-\Delta)^{x-1} = \Delta^{2d+x-1}(n-\Delta)^{x-1}$ is maximized for

$$\Delta = \frac{2d+x-1}{2d+2x-2}n.$$

Thus, from (3) we obtain

$$\delta \geqslant \frac{(d+x-1)^{2d+2x-2}}{2(2d+x-1)^{2d+x-1}(x-1)^{x-1}} n + O(1).$$

Note that it is enough to show that

$$\frac{(d+x-1)^{2d+2x-2}}{2(2d+x-1)^{2d+x-1}(x-1)^{x-1}} > \frac{2}{5}$$

as then, for large enough n, we have the wanted inequality $\delta > \frac{2}{5}n$.

Rearranging the terms and using $2d \geqslant 0$, Bernoulli's inequality and $d^2 \leqslant \frac{1}{16}(x-1)$ implied by assumptions of the theorem we obtain

$$\begin{aligned}
&\frac{(d+x-1)^{2d+2x-2}}{2(2d+x-1)^{2d+x-1}(x-1)^{x-1}} \\
&= \frac{1}{2}\left(1-\frac{d}{2d+x-1}\right)^{2d+x-1}\left(1+\frac{d}{x-1}\right)^{x-1} \\
&\geqslant \frac{1}{2}\left(1-\frac{d}{x-1}\right)^{2d+x-1}\left(1+\frac{d}{x-1}\right)^{x-1} \\
&= \frac{1}{2}\left(1-\frac{d}{x-1}\right)^{2d}\left(\left(1-\frac{d}{x-1}\right)\left(1+\frac{d}{x-1}\right)\right)^{x-1} \\
&= \frac{1}{2}\left(1-\frac{d}{x-1}\right)^{2d}\left(1-\frac{d^2}{(x-1)^2}\right)^{x-1} \\
&\geqslant \frac{1}{2}\left(1-\frac{2d^2}{x-1}\right)\left(1-\frac{d^2}{x-1}\right) \\
&\geqslant \frac{1}{2}\left(1-\frac{2}{16}\right)\left(1-\frac{1}{16}\right) \\
&> \frac{2}{5}
\end{aligned}$$

as needed. □

Proof of Theorem 4. It is enough to consider rational λ. Let $d \geqslant 1$ and $x \geqslant 3$ be large enough integers such that $\lambda = \frac{2d}{x}$. We modify Example 1. Consider a graph H consisting of two stars having $d+1$ leaves with centers connected by a path of length 3 and an additional $x-3$ paths of length 2 starting in a central vertex from the aforementioned path, see Figure 1. Note that H has a matching of size x and $2d = \lambda x$ unmatched vertices.

We will show that if x is large enough then $\overline{H(T_2(n))} < \overline{H(G)}$, where G is an unbalanced blow-up of a C_5 with parts of sizes an, bn, cn, cn, cn for large enough n and some values of a, b, c with $a+b+3c=1$ to be established later. Since H is balanced, $T_2(n)$ maximizes the number of injective embeddings of H among all bipartite graphs, and so the inequality $\overline{H(T_2(n))} < \overline{H(G)}$ implies that no bipartite graph on n vertices contains more copies of H than G.

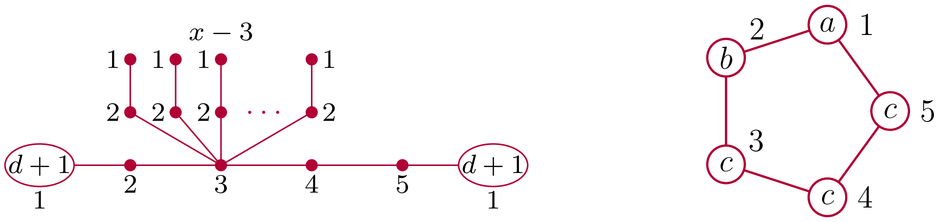

Figure 1: Graphs H and G.

The intuition behind this example is as follows. Every embedding of H in a bipartite graph needs to have the same number of vertices in each part. Therefore, there is a relatively small number of copies of H in such a graph if x and d are large enough, because then the probability that random $2x + 2d$ vertices in a bipartite graph are distributed equally between the parts is quite low. On the other hand, in a random choice of $2x+2d$ vertices in G with a large a and a tiny c, we have a high probability that random $2d + 2$ vertices in G are in the set of size an and a still fairly high probability of having additional $x - 3$ edges between parts of sizes an and bn. This gives many copies of H in G. In the rest of the proof we will count this precisely.

Since H is a connected balanced bipartite graph, $\overline{H(T_2(n))} \leqslant 2(\frac{1}{2})^{2x+\lambda x} n^{2x+\lambda x}$.

On the other hand, to lower bound $\overline{H(G)}$ we count the number of embeddings where each vertex of H is embedded to a specific blob of G marked in Figure 1

$$\begin{aligned}\overline{H(G)} &\geqslant a^{x+\lambda x-1} b^{x-2} c^3 n^{2x+\lambda x} + O(n^{2x+\lambda x-1})\\ &= a^{x+\lambda x-1}(1-a-3c)^{x-2} c^3 n^{2x+\lambda x} + O(n^{2x+\lambda x-1}).\end{aligned}$$

Note that for arbitrary $\lambda > 0$ it is possible to choose $a > \frac{1}{2}$ such that

$$a^{\lambda+1}(1-a) - \left(\frac{1}{2}\right)^{\lambda+2} > 0.$$

This is due to the fact that the function $f(z) = z^{\lambda+1}(1-z) - \left(\frac{1}{2}\right)^{2+\lambda}$ has a root in $\frac{1}{2}$ and its derivative at $\frac{1}{2}$ is greater than zero.

Now consider the function $g(z) = a^{\lambda+1}(1-a-3z) - \left(\frac{1}{2}\right)^{2+\lambda}$. It is continuous at $z = 0$ and $g(0) = f(a) > 0$. Thus, we can choose $c > 0$ small enough so that $g(c) > 0$. For such c define

$$p = \frac{a^{\lambda+1}(1-a-3c)}{\left(\frac{1}{2}\right)^{\lambda+2}} > 1$$

and take $x > \log_p\left(\frac{2a(1-a-3c)^2}{c^3}\right)$. Then

$$\frac{a^{x+\lambda x}(1-a-3c)^x}{\left(\frac{1}{2}\right)^{2x+2\lambda}} > \frac{2a(1-a-3c)^2}{c^3},$$

which implies

$$a^{x+\lambda x-1}(1-a-3c)^{x-2}c^3 > 2\left(\frac{1}{2}\right)^{2x+2\lambda}.$$

This means that for large enough n we have $\overline{H(G)} \geqslant \overline{H(T_2(n))}$ as needed. □

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