Bounded-Degree Planar Graphs Do Not Have Bounded-Degree Product Structure

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Abstract

Product structure theorems are a collection of recent results that have been used to resolve a number of longstanding open problems on planar graphs and related graph classes. One particularly useful version states that every planar graph G is contained in the strong product of a 3-tree H, a path P, and a 3-cycle K_3; written as $G \subseteq H \boxtimes P \boxtimes K_3$. A number of researchers have asked if this theorem can be strengthened so that the maximum degree in H can be bounded by a function of the maximum degree in G. We show that no such strengthening is possible. Specifically, we describe an infinite family $\mathcal{G}$ of planar graphs of maximum degree 5 such that, if an n-vertex member G of $\mathcal{G}$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_c$ where P is a path and H is a graph of maximum degree Δ and treewidth t, then $t\Delta c \geqslant 2^{\Omega(\sqrt{\log\log n})}$.

Mathematics Subject Classifications: 05C76

1 Introduction

Recently, product structure theorems have been a key tool in resolving a number of longstanding open problems on planar graphs. Roughly, a *product structure theorem* for a graph family $\mathcal{G}$ states that every graph in $\mathcal{G}$ is isomorphic to a subgraph of the product

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of two or more "simple" graphs. As an example, there are a number of graph classes $\mathcal{G}$ for which there exists integers t and c such that, for each $G \in \mathcal{G}$ there is a graph H of treewidth[1] t and a path P such that G is isomorphic to a subgraph of the strong product[2] of H, P, and a clique K of order c. This is typically written as $G \subseteq H \boxtimes P \boxtimes K_c$, where the notation $G_1 \subseteq G_2$ is used to mean that G_1 is isomorphic to some subgraph of G_2. See references [2, 3, 5, 8–13, 15, 17] for examples.

In some applications of product structure theorems it is helpful if, in addition to having treewidth t, the graph H has additional properties, possibly inherited from G. For example, one very useful version of the planar graph product structure theorem states that for every planar graph G there exists a *planar* graph H of treewidth 3 and a path P such that $G \subseteq H \boxtimes P \boxtimes K_3$ [8, Theorem 36(b)]. The planarity of H in this result has been leveraged to obtain better constants and even asymptotic improvements for graph colouring and layout problems, including queue number [8], p-centered colouring [7], and ℓ-vertex ranking [1].

In this vein, the authors have been repeatedly asked if H can have bounded degree whenever G does; that is:

> For each $\Delta \in \mathbb{N}$, let $\mathcal{G}_\Delta$ be the family of planar graphs of maximum degree Δ. Do there exist functions $t : \mathbb{N} \to \mathbb{N}$, $d : \mathbb{N} \to \mathbb{N}$, and $c : \mathbb{N} \to \mathbb{N}$ such that, for each $\Delta \in \mathbb{N}$ and each $G \in \mathcal{G}_\Delta$ there exists a graph H of treewidth at most $t(\Delta)$ and maximum degree $d(\Delta)$ and a path P such that $G \subseteq H \boxtimes P \boxtimes K_{c(\Delta)}$?

In the current paper we show that the answer to this question is no, even when $\Delta = 5$.

Theorem 1. For infinitely many integers $n \geqslant 1$, there exists an n-vertex planar graph G of maximum degree 5 such that, for every graph H of treewidth t and maximum degree Δ, every path P, and every integer c, if $G \subseteq H \boxtimes P \boxtimes K_c$ then $t\Delta c \geqslant 2^{\Omega(\sqrt{\log\log n})}$.

The graph family $\mathcal{G} := \{G_h : h \in \mathbb{N}\}$ that establishes Theorem 1 consists of complete binary trees of height h augmented with edges to form, for each $i \in \{1, \dots, h\}$, a path D_i that contains all vertices of depth i. See Figure 1.

2 Proof of Theorem 1

Throughout this paper, all graphs G are simple and undirected with vertex-set $V(G)$ and edge-set $E(G)$. For a set S, $G[S]$ denotes the subgraph of G induced by $S \cap V(G)$ and

[1] A *tree decomposition* of a graph H is a collection $\mathcal{T} := (B_x : x \in V(T))$ of subsets of $V(H)$ indexed by the nodes of some tree T such that (i) for each $v \in V(H)$, the induced subgraph $T[x \in V(T) : v \in B_x]$ is connected; and (ii) for each edge $vw \in E(H)$, there exists some $x \in V(T)$ with $\{v, w\} \subseteq B_x$. The *width* of such a tree decomposition is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of H is the minimum width of any tree decomposition of H.

[2] The *strong product* $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is the graph with vertex-set $V(G_1 \boxtimes G_2) := V(G_1) \times V(G_2)$ and that includes the edge with endpoints (v, x) and (w, y) if and only if (i) $vw \in E(G_1)$ and $x = y$; (ii) $v = w$ and $xy \in E(G_2)$; or (iii) $vw \in E(G_1)$ and $xy \in E(G_2)$.

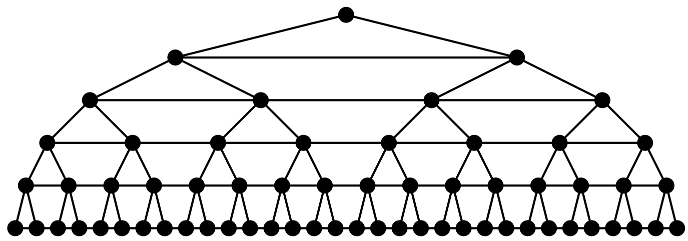

Figure 1: The graph G_5 from the graph family $\{G_h : h \in \mathbb{N}\}$ that establishes Theorem 1.

$G - S := G[V(G) \setminus S]$. For every $v \in V(G)$, let $N_G(v) := \{w : vw \in E(G)\}$ and for every $S \subseteq V(G)$, let $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$. We write $G_1 \cong G_2$ if G_1 and G_2 are isomorphic and $G_1 \subseteq G_2$ if G_1 is isomorphic to some subgraph of G_2. Inside of asymptotic notation, $\log n := \max\{1, \log_2 n\}$.

2.1 Partitions

Let G and H be graphs. An *H-partition* $\mathcal{H} := \{B_x : x \in V(H)\}$ of G is a partition of $V(G)$ whose parts are indexed by the vertices of H with the property that, if vw is an edge of G with $v \in B_x$ and $w \in B_y$ then $x = y$ or $xy \in E(H)$. The *width* of $\mathcal{H}$ is the size of its largest part; that is, $\max\{|B_x| : x \in V(H)\}$. If H is in a class $\mathcal{G}$ of graphs then we may call $\mathcal{H}$ a $\mathcal{G}$-partition of G. Specifically, if H is a tree, then $\mathcal{H}$ is a *tree-partition* of G and if H is a path, then $\mathcal{H}$ is a *path-partition* of G. A path-partition $\mathcal{P} := \{P_x : x \in V(P)\}$ of G is also referred to as a *layering* of G and the parts of $\mathcal{P}$ are referred to as *layers*. A set of layers $\{P_{x_1}, \dots, P_{x_q}\} \subseteq \mathcal{P}$ is *consecutive* if $P[\{x_1, \dots, x_q\}]$ is connected.

As in previous works, we make use of the following relationship between H-partitions and strong products, which follows immediately from the preceding definitions.

Observation 2. For every integer $c \geqslant 1$, and all graphs G, H, and J, $G \subseteq H \boxtimes J \boxtimes K_c$ if and only if G has an H-partition $\mathcal{H} := \{B_x : x \in V(H)\}$ and a J-partition $\mathcal{J} := \{C_y : y \in V(J)\}$ such that $|B_x \cap C_y| \leqslant c$, for each $(x, y) \in V(H) \times V(J)$.

The following important result of Ding and Oporowski [4] (also see [6, 16]) allows us to focus on the case where the first factor in our product is a tree.

Theorem 3 (Ding and Oporowski [4]). If H is a graph with maximum degree Δ and treewidth t, then H has a tree-partition of width at most $24\Delta(t + 1)$.

Corollary 4. If $G \subseteq H \boxtimes P \boxtimes K_c$ where H has treewidth t and maximum degree Δ then there exists a tree T such that $G \subseteq T \boxtimes P \boxtimes K_{24c\Delta(t+1)}$.

Proof. By Theorem 3, H has a tree-partition $\mathcal{T} := \{B_x : x \in V(T)\}$ of width at most $24\Delta(t+1)$. By Observation 2, $H \subseteq T \boxtimes K_{24\Delta(t+1)}$. Therefore, $G \subseteq T \boxtimes K_{24\Delta(t+1)} \boxtimes P \boxtimes K_c \cong T \boxtimes P \boxtimes K_{24c\Delta(t+1)}$. □

The *length* of a path is the number of edges in it. Given two vertices $v, w \in V(G)$, $\mathrm{dist}_G(v, w)$ denotes the minimum length of a path in G that contains v and w, or $\mathrm{dist}_G(v, w)$ is infinite if v and w are in different connected components of G. For any $R \subseteq V(G)$, the *diameter* of R in G is $\mathrm{diam}_G(R) := \max\{\mathrm{dist}_G(v, w) : v, w \in R\}$.

Observation 5. Let G be a graph, let $R \subseteq V(G)$, and let $\mathcal{L}$ be a layering of G. Then there exists a layer $L \in \mathcal{L}$ such that $|R \cap L| \geqslant |R|/(\mathrm{diam}_G(R) + 1)$.

Proof. By the definition of layering, the vertices in R are contained in a set of at most $\mathrm{diam}_G(R) + 1$ consecutive layers of $\mathcal{L}$. The result then follows from the Pigeonhole Principle. □

We also make use of the following basic fact about tree-partitions:

Observation 6. Let G be a graph, let $\mathcal{T} := \{B_x : x \in V(T)\}$ be a tree-partition of G, let $x \in V(T)$, and let $v, w \in N_G(B_x)$ be in the same component of $G - B_x$. Then T contains an edge xy with $v, w \in B_y$.

Proof. Suppose that $v \in B_y$ and $w \in B_z$ for some $y, z \in V(T)$. Since $v, w \in N_G(B_x)$, T contains the edges xy and xz. All that remains is to show that $y = z$. For the purpose of contradiction, assume $y \neq z$. Since v and w are in the same component of $G - B_x$, G contains a path from v to w that avoids all vertices in B_x, which implies that T contains a path P_{yz} from y to z that does not include x. This is a contradiction since then P_{yz} and the edges xy and yz form a cycle in T, but T is a tree. □

2.2 Percolation in Binary Trees

The *depth* of a vertex v in a rooted tree T is the length of the path $A_T(v)$ from v to the root of T. Each vertex $a \in V(A_T(v))$ is an *ancestor* of v, and v is a *descendant* of each vertex in $V(A_T(v))$. We say that a set $B \subseteq V(T)$ is *unrelated* if no vertex of B is an ancestor of any other vertex in B.

For each $h \in \mathbb{N}$, let T_h denote the complete binary tree of height h; that is, the rooted ordered tree with 2^h leaves, each having depth h and in which each non-leaf vertex has exactly two children, one *left child* and one *right child*. Note that the ordering of T_h induces an ordering on every unrelated set $B \subseteq V(T_h)$, which we refer to as the *left-to-right ordering*. Specifically, $v \in B$ appears before $w \in B$ in the left-to-right ordering of B if and only if there exists a common ancestor a of both v and w such that the path from a to v contains the left child of a and the path from a to w contains the right child of a.

We use the following two percolation-type results for T_h.

Lemma 7. Let $h \geqslant 1$, let r be the root of T_h, and let $S \subseteq V(T_h)$ with $1 \leqslant |S| < 2^h$. Then there exists a vertex v of T_h such that

(i) the depth of v is at most $\log_2 |S| + 1$;
(ii) $v \neq r$ and the parent of v is in $S \cup \{r\}$; and

(iii) $T_h - S$ contains a path from v to a leaf of T_h.

Proof. The proof is by induction on h. When $h = 1$, $|S| \leqslant 1$. In particular, at least one child v of r is not in S. The depth of v is $1 \leqslant \log_2 |S| + 1$, so v satisfies (i). The parent of v is $r \in S \cup \{r\}$, so v satisfies (ii). $T_1 - S$ contains a length-0 path from v to itself (a leaf of T_1), so v satisfies (iii).

For $h \geqslant 2$, let ℓ be the maximum integer such that $S \cup \{r\}$ contains all 2^ℓ vertices of depth ℓ. Observe that $2^\ell \leqslant |S|$, so $\ell \leqslant \log_2 |S| < h$. Let L be the set of $2^{\ell+1}$ depth-$(\ell+1)$ vertices in T_h. By the Pigeonhole Principle some vertex $r' \in L$ is the root of a complete binary tree T' with root r' of height $h - \ell - 1$ with $|S \cap V(T')| \leqslant |S|/2^{\ell+1} < 2^{h-\ell-1}$.

If $V(T') \cap S = \emptyset$ then choosing $v := r'$ satisfies the requirements of the lemma. Otherwise, by applying induction on T' and $S' := S \cap V(T')$ we obtain a vertex v' of depth at most $\ell + 1 + \log_2(|S'|) + 1 \leqslant \log_2 |S| + 1$ whose parent is in $S \cup \{r'\}$, and such that $T_h - S$ contains a path from v' to a leaf of T_h. Thus v' satisfies requirements (i) and (iii). If the parent of v' is in S then v' also satisfies requirement (ii) and the lemma is proven, with $v := v'$. Otherwise, the parent of v' is r', in which case r' satisfies requirements (i)–(iii) and we are done, with $v := r'$. □

Lemma 8. Let $h \geqslant 1$, let r be the root of T_h, and and let $S \subseteq V(T_h)$ with $1 \leqslant |S| < 2^{h-1}$. Then there exist two unrelated vertices v_1 and v_2 of T_h such that, for each $i \in \{1, 2\}$:

(i) the depth of v_i is at most $\log_2 |S| + 2$;
(ii) $v_i \neq r$ and the parent of v_i is in $S \cup \{r\}$; and
(iii) $T_h - S$ contains a path from v_i to a leaf of T_h.

Proof. Let T_1 and T_2 be the two maximal subtrees of T_h rooted at the children r_1 and r_2, respectively of r. (Each of T_1 and T_2 is a complete binary tree of height $h - 1$.) For each $i \in \{1, 2\}$, let $S_i := S \cap V(T_i)$. If $S_i = \emptyset$ then we choose $v_i = r_i$ and this satisfies requirements (i)–(iii). If $S_i \neq \emptyset$ then, since $|S_i| \leqslant |S| < 2^{h-1}$, we can apply Lemma 7 to T_i and S_i to obtain a vertex $v_i' \in V(T_i)$ of depth at most $1 + \log_2 |S_i| + 1 \leqslant \log_2 |S| + 2$ and such that $T_h - S$ contains a path from v_i' to a leaf of T_h. Therefore, v_i' satisfies (i) and (iii). Furthermore, the parent of v_i' is in $S \cup \{r_i\}$. If the parent of v_i' is in S, then v_i' also satisfies (ii), so we set $v_i := v_i'$. If the parent of v_i' is not in S, then the parent of v_i' is $r_i \notin S$ and r_i satisfies (i)–(iii), so we set $v_i := r_i$. Finally, since $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$, v_1 and v_2 are unrelated. □

2.3 A Connectivity Lemma

The $x \times y$ *grid* $G_{x\times y}$ is the graph with vertex-set $V(G_{x\times y}) := \{1, \ldots, x\} \times \{1, \ldots, y\}$ and that contains an edge with endpoints (x_1, y_1) and (x_2, y_2) if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. An edge of $G_{x\times y}$ is *horizontal* if its two endpoints agree in the second (y) coordinate. For each $i \in \{1, \ldots, x\}$, the vertex-set $\{i\} \times \{1, \ldots, y\}$ is called *column* i of $G_{x\times y}$. A set C of columns is *consecutive* if $G_{x\times y}[\cup C]$ is connected.

Lemma 9. Let $x, y, p \geqslant 1$ be integers, let G be a graph obtained by subdividing horizontal edges of $G_{x\times y}$, and let $S \subseteq V(G)\setminus V(G_{x\times y})$ be a set of subdivision vertices of size $|S| < py$. Then some component of $G - S$ contains at least x/p consecutive columns of $G_{x\times y}$.

Proof. For each $i \in \{1, \dots, x-1\}$, in order to separate column i from column $i+1$, S must contain at least y subdivision vertices on the horizontal edges between columns i and $i+1$. Since $|S| < py$, this implies that there are at most $p-1$ values of $i \in \{1, \dots, x-1\}$ for which columns i and $i+1$ are in different components of $G - S$. These at most $p-1$ values of i partition $\{1, \dots, x\}$ into at most p intervals, at least one of which contains at least x/p consecutive columns that are contained in a single component of $G - S$. □

2.4 The Proof

Recall that, for each $h \in \mathbb{N}$, G_h is the planar supergraph of the complete binary tree T_h of height h obtained by adding the edges of a path D_i that contains all vertices of depth i, in left-to-right order, for each $i \in \{1, \dots, h\}$. Since T_h is a spanning subgraph of G_h, the *depth* of a vertex v in G_h refers to the depth of v in T_h. The *height* of a depth-d vertex of T_h is $h - d$. We are now ready to prove the following result that, combined with Corollary 4 is sufficient to prove Theorem 1:

Theorem 10. For every $h \in \mathbb{N}$, every tree T, and every path P, if $G_h \subseteq T \boxtimes P \boxtimes K_c$ then $c \geqslant 2^{\Omega(\sqrt{\log h})}$.

It is worth noting that, unlike Theorem 1, there is no restriction on the maximum degree of the tree T.

Before diving into technical details, we first sketch our strategy for proving Theorem 10. We may assume that $c \leqslant 2^{\sqrt{\log_2 h}}$ since, otherwise there is nothing to prove. Recall Observation 2, which states that if $G_h \subseteq T \boxtimes P \boxtimes K_c$ then G_h has a tree-partition $\mathcal{T} := \{B_x : x \in V(T)\}$ and a path-partition (that is, layering) $\mathcal{P} := \{P_y : y \in V(P)\}$ with $|B_x \cap P_y| \leqslant c$ for each $(x, y) \in V(T) \times V(P)$. However, since G_h has diameter $2h$, Observation 5 (with $R = B_x$) implies that $|B_x| \leqslant c(2h+1)$ for each $x \in V(T)$. This upper bound on $|B_x|$ is used to establish all of the results described in the following paragraph.

Refer to Figure 2. We will construct a sequence of sets $\mathcal{R}_1, \dots, \mathcal{R}_{t+1}$ and a sequence of nodes $x_1, \dots, x_{t+1}$ of T, where each $\mathcal{R}_i$ is a family of unrelated sets in T_h such that $\cup\mathcal{R}_i \subseteq B_{x_i}$. The first family $\mathcal{R}_1$ consists of $q_1 \geqslant h/(25c)$ singleton sets whose union is an unrelated set in T_h. For each $i \in \{2, \dots, t+1\}$, $\mathcal{R}_i$ has size $q_i \geqslant q_1/(10c)^{i-1} - 3$. For each $\mathcal{R}_i := \{R_{i,1}, \dots, R_{i,q_i}\}$, each $R_{i,j} \subseteq V(T)$ is an unrelated set in T_h of size 2^{i-1} that has a common ancestor $a_{i,j}$ of height at least $h/5$ that is at distance at most $(i-1)(\log_2(ch)+2)$ from every element in $R_{i,j}$. Furthermore, $\{a_{i,1}, \dots, a_{i,q_i}\}$ is an unrelated set. These properties imply that $\cup\mathcal{R}_i$ is also an unrelated set.

We do this for some appropriately chosen integer $t \in \Theta(\sqrt{\log h})$ in order to ensure that $q_{t+1} \geqslant 1$, so $\mathcal{R}_{t+1}$ contains at least one part R of size 2^t. By Observation 5, there

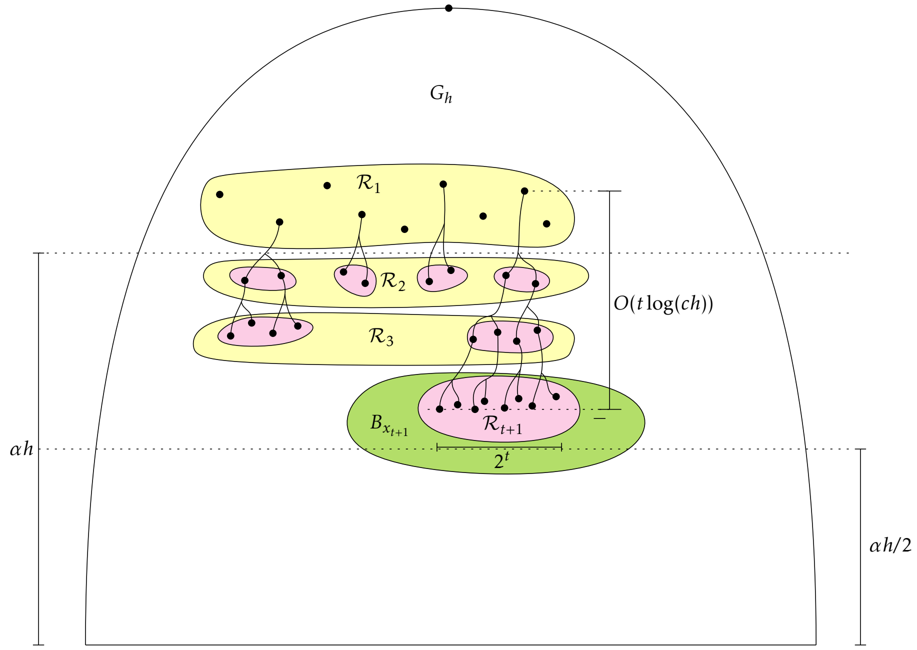

Figure 2: The proof of Theorem 10.

exists some $y \in V(P)$ such that

$$|R \cap P_y| \geqslant \frac{|R|}{\mathrm{diam}_{G_h}(R)+1} \geqslant \frac{2^t}{2t(\log_2(ch)+2)+1} = 2^{t-\log_2(t\log_2(ch))-O(1)} = 2^{\Omega(t)} = 2^{\Omega(\sqrt{\log_2 h})} .$$

Since $R \subseteq B_{x_{t+1}}$, $|B_{x_{t+1}} \cap P_y| \geqslant 2^{\Omega(\sqrt{\log h})}$. Since $c \geqslant \max\{|B_x \cap P_y| : (x,y) \in V(T)\times V(P)\}$, the assumption that $c \leqslant 2^{\sqrt{\log h}}$ therefore leads to the conclusion that $c \geqslant 2^{\Omega(\sqrt{\log h})}$, which establishes Theorem 1.

We now proceed with the details of the proof outlined above. The next two lemmas will be used to obtain the set $\mathcal{R}_1$ that allows us to start the argument. Informally, the first lemma says that every balanced separator S of G_h must contain a vertex of depth i for each $i \in \{i_0, \ldots, h\}$, where $i_0 \in O(\log|S|)$.

Lemma 11. Let $h \in \mathbb{N}$ with $h \geqslant 1$, let $S \subseteq V(G_h)$, $S \neq \emptyset$. If $G_h - S$ has no component with more than $|V(G_h)|/2$ vertices then $S \cap V(D_i) \neq \emptyset$ for each $i \in \{i_0, \ldots, h\}$, where $i_0 := \lceil \max\{2\log_2|S|+2, \log_2(1+(h+2)|S|)-1\}\rceil$.

Proof. Let C be the vertex set of a component of $G_h - S$ that maximizes $C \cap V(D_h)$. For each $i \in \{0,\ldots,h\}$, let $C_i := C \cap V(D_i)$ and let $S_i := S \cap V(D_i)$. We will show that, for each $i \geqslant i_0$, C_i is non-empty but does not contain all 2^i vertices in D_i. Therefore $S_i \supseteq N_{G_h}(C_i) \cap V(D_i) \neq \emptyset$ for each $i \in \{i_0,\ldots,h\}$.

For each $i \in \{0,\ldots,h-1\}$, the vertices in C_{i+1} are adjacent to at least $|C_{i+1}|/2$ vertices of D_i, so $|C_i| \geqslant |N_{G_h}(C_{i+1}) \cap V(D_i) \setminus S_i| \geqslant |C_{i+1}|/2 - |S_i|$. Iterating this inequality $h-i$

times gives $|C_i| \geqslant |C_h|/2^{h-i} - \sum_{j=i}^{h-1} |S_j|/2^{h-i-1} \geqslant |C_h|/2^{h-i} - |S|$. The vertices in S partition $V(D_h) \setminus S$ into at most $|S|+1$ connected components. Since C is chosen to maximize $|C_h|$, $|C_h| \geqslant (2^h - |S|)/(|S|+1) > 2^h/(|S|+1) - 1$. Therefore,

$$|C_i| \geqslant \frac{|C_h|}{2^{h-i}} - |S| > \frac{2^h/(|S|+1)-1}{2^{h-i}} - |S| \geqslant 2^{i-\log_2(|S|+1)} - |S| - 1 \geqslant 0 \tag{1}$$

for $i \geqslant 2\log_2|S| + 2$. Since $i_0 \geqslant 2\log_2|S| + 2$, this establishes that C_i is non-empty for each $i \in \{i_0, \dots, h\}$.

For each $i \in \{0, \dots, h-1\}$, the vertices in C_i are adjacent to at least $2|C_i|$ vertices of D_{i+1}, so $|C_{i+1}| \geqslant 2|C_i| - |S_{i+1}|$. Iterating this $h-i$ times gives:

$$|C_h| \geqslant 2^{h-i}|C_i| - \sum_{j=i+1}^{h} 2^{h-j}|S_j| \geqslant 2^{h-i}|C_i| - 2^{h-i-1}|S| \ . \tag{2}$$

Suppose that $|C_{i^*}| = 2^{i^*}$ for some $i^* \in \{0, \dots, h\}$. Then Equation (2) implies that $|C_h| \geqslant 2^h - 2^{h-i^*-1}|S|$. Therefore, by Equation (1),

$$|C| = \sum_{i=0}^{h} |C_i| \geqslant \sum_{i=0}^{h} \left(\frac{|C_h|}{2^{h-i}} - |S| \right) > 2|C_h| - 1 - (h+1)|S| \geqslant 2^{h+1} - 2^{h-i^*}|S| - 1 - (h+1)|S| \ . \tag{3}$$

However, $2^h > |V(G_h)|/2 \geqslant |C|$, and combining this with Equation (3) gives $2^h > 2^{h+1} - 2^{h-i^*}|S| - 1 - (h+1)|S|$. Rewriting this inequality, we get

$$2^h < 2^{h-i^*}|S| + 1 + (h+1)|S| \ . \tag{4}$$

Multiplying each side of Equation (4) by 2^{i^*-h} then gives:

$$\begin{aligned} 2^{i^*} &< |S| + 2^{i^*-h}(1 + (h+1)|S|) \\ &\leqslant |S| + 1 + (h+1)|S| && \text{(since } i^* \leqslant h \text{, so } 2^{i^*-h} \leqslant 1) \\ &= 1 + (h+2)|S| \ . \end{aligned}$$

Taking the logarithm of each side then gives $i^* < \log_2(1+(h+2)|S|) \leqslant i_0$. This establishes that $|C_i| < 2^i$ for each $i \in \{i_0, \dots, h\}$ and completes the proof. □

The following lemma shows that every tree-partition of G_h must have a part with a large unrelated set that is far from the leaves of T_h and will be used to obtain our first set $\mathcal{R}_1$.

Lemma 12. For every $\alpha \in (0, 1/4)$, there exists h_0 such that the following is true, for all integers $h \geqslant h_0$ and all $c \in [1, h]$. If $\mathcal{T} := \{B_x : x \in V(T)\}$ is a tree-partition of G_h of width less than ch then there exists a node $x \in V(T)$ and a subset $R \subseteq B_x$ such that

(i) R is unrelated;
(ii) $|R| \geqslant \alpha^2 h/c$; and

(iii) Each vertex in R has height at least αh.

Proof. It is well-known and easy to show that there exists a node x of T such that $G - B_x$ has no component with more than $|V(G_h)|/2$ vertices [14, (2.6)]. Let Y be the set of vertices in B_x that have height at least $h/4$. By Lemma 11, $|Y| \geqslant 3h/4 - O(\log(ch+1))$.

Let T_Y be the minimal (connected) subtree of T_h that spans Y, and let L be the set of leaves of T_Y (excluding the root of T_Y if this happens to be contained in Y). Observe that $L \subseteq Y$ is an unrelated set. Therefore, L satisfies (i) and, by definition, each vertex in L has height at least $h/4 > \alpha h$, so L satisfies (iii). If $|L| \geqslant \alpha h \geqslant \alpha^2 h/c$ then L also satisfies (ii). In this case, we can take $R := L$ and we are done. We now assume that $|L| < \alpha h$.

Let Z consist of all vertices in $V(T_h) \setminus V(T_Y)$ whose parents are in $Y \setminus L$. Observe that Z is an unrelated set of vertices each having height at least $h/4$. For each v of T_Y, let d_v denote the number of children of v in T_Y. Then,

$$\sum_{v \in Y \setminus L} (d_v - 1) \leqslant \sum_{v \in V(T_Y) \setminus L} (d_v - 1) = |L| - 1 \ ,$$

where the second equality is a standard fact about rooted trees. Rewriting this, we get $\sum_{v \in Y \setminus L} d_v < |Y \setminus L| + |L| = |Y|$. On the other hand, each $v \in Y \setminus L$ contributes $2 - d_v$ vertices to Z, so

$$|Z| = \sum_{v \in Y \setminus L} (2 - d_v) \ .$$

Combining these two formulas, we obtain

$$|Z| \geqslant 2|Y \setminus L| - |Y| = |Y| - 2|L| \geqslant 3h/4 - O(\log(ch+1)) - 2\alpha h \geqslant h/4 - O(\log(ch+1)) \ .$$

Refer to Figure 3. For each $r \in Z$, Lemma 7 applied to the subtree of T_h rooted at r with $S = B_x$ implies that r has a descendant v such that (a) the parent of v is in $B_x \cup \{r\}$; (b) the height of v is at least $h/4 - O(\log(ch+1))$; and (c) $T_h - B_x$ contains a path Q_v from v to a leaf of T_h.

Form the set Z' using the following rule for each $r \in Z$: If the vertex v described in the preceding paragraph is a child of r then place r in Z', otherwise place v in Z'. Since each $r \in Z$ is a child of some vertex in $Y \subseteq B_x$, this ensures that the parent of v is in B_x for each $v \in Z'$. Since Z is an unrelated set and Z' is obtained by replacing each vertex in Z with one of its descendants, Z' is an unrelated set. Since $\alpha < 1/4$, for sufficiently large h, $|Z'| \geqslant h/4 - O(\log(ch+1)) \geqslant \alpha h$ and each vertex in Z' has height at least $h/4 - O(\log(ch+1)) \geqslant \alpha h$.

Now observe that the union of the paths Q_v for each $v \in Z'$ and the paths $D_{h-\lceil \alpha h \rceil+1}, \dots, D_h$ contains a subgraph G' isomorphic to a graph that can be obtained from the grid $G_{\lceil \alpha h \rceil \times \lceil \alpha h \rceil}$ by subdividing horizontal edges. Since B_x does not contain any vertex of Q_v for each $v \in Z'$, $B_x \cap V(G')$ contains only vertices corresponding to subdivision vertices. Therefore, by Lemma 9, some component of $G - B_x$ contains a subset

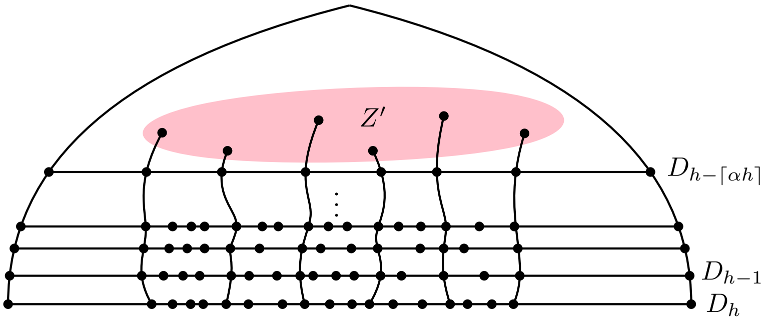

Figure 3: A step in the proof of Lemma 12.

$R \subseteq Z'$ of size at least $\alpha^2 h/c$. Each element in R has a parent in B_x. By Observation 6 some neighbour y of x in T has a bag B_y that contains all of R. This completes the proof. □

A set $\mathcal{R} := \{R_1, \ldots, R_q\}$ of subsets of $V(T_h)$ is (k, ℓ, m)-*compact* if it has the following properties:

1. For each $i \in \{1, \ldots, q\}$, R_i is unrelated and $|R_i| \geqslant k$.
2. For each $i \in \{1, \ldots, q\}$ there exists a common ancestor a_i of R_i such that $\text{dist}_{T_h}(v, a_i) \leqslant \ell$ for each $v \in R_i$.
3. $a_1, \ldots, a_q$ are unrelated and each has height at least m.

This definition has the following implications: (i) $\cup\mathcal{R}$ is an unrelated set; and (ii) If a_i precedes a_j in the left-to-right ordering of $\{a_1, \ldots, a_q\}$ then every element of R_i precedes every element of R_j in the left-to-right order of $\cup\mathcal{R}$. We say that a vertex v of T_h is *compatible* with $S \subseteq V(T_h)$ if the parent of v is in S and $T_h - S$ contains a path from v to a leaf of T_h. A (k, ℓ, m)-compact set $\mathcal{R}$ is *compatible* with S if each vertex in $\cup\mathcal{R}$ is compatible with S.

Lemma 13. Let $\mathcal{R} := \{R_1, \ldots, R_q\}$ be a (k, ℓ, m)-compact set, and let $S \supseteq \cup\mathcal{R}$ have size $1 \leqslant |S| < 2^{m-\ell-2}$. Then, there exists a $(2k, \ell + \log_2 |S| + 2, m)$-compact set $\mathcal{R}' := \{R'_1, \ldots, R'_q\}$ that is compatible with S.

Proof. For each $i \in \{1, \ldots, q\}$ and each $r \in R_i$, replace r with the descendants v_1 and v_2 of r described in Lemma 8 and call the resulting set R'_i. Then $|R'_i| = 2|R_i| \geqslant 2k$ and $\text{dist}_{T_h}(v, a_i) \leqslant \ell + \log_2 |S| + 2$ for each $v \in R'_i$, where a_i is the common ancestor of R_i in the definition of (k, ℓ, m)-compact. Therefore $\mathcal{R}' := \{R'_1, \ldots, R'_q\}$ is a $(2k, \ell + \log_2(|R|) + 2, m)$-compact set. □

The next lemma is the last ingredient in the proof of Theorem 1.

Lemma 14. Let $\mathcal{T} := \{B_x : x \in V(T)\}$ be a tree-partition of G_h and let $\mathcal{P} := \{P_y : y \in V(P)\}$ be a path-partition of G_h. Then there exists $(x, y) \in V(T) \times V(P)$ such that $|B_x \cap P_y| \geqslant 2^{\Omega(\sqrt{\log h})}$.

Proof. Let $x \in V(T)$ be a node that maximizes $|B_x|$. Then $\operatorname{diam}_{G_h}(B_x) \leqslant \operatorname{diam}_{T_h}(B_x) \leqslant 2h$ so, by Observation 5, $|B_x \cap P_y| \geqslant |B_x|/(2h+1)$ for some $y \in V(P)$. If $|B_x| \geqslant h2^{\sqrt{\log_2 h}}$ then there is nothing more to prove, so we may assume that $|B_x| < ch$ where $c := 2^{\sqrt{\log_2 h}}$. Note that $c \geqslant 1$ for every $h \geqslant 1$.

By Lemma 12, with $\alpha := 1/5$, T contains a node x_1 such that B_{x_1} contains an unrelated set R of size $q_1 := |R| \geqslant h/(25c)$ where each vertex in R has height at least $m := m_1 := h/5$. Let $\mathcal{R}_1 := \{\{v\} : v \in R\}$. By definition $\mathcal{R}_1$ is a $(1, 0, m)$-compact set. $\mathcal{R}_1$ will be the first in a sequence of sets $\mathcal{R}_1, \dots, \mathcal{R}_{t+1}$, where t will be fixed below. For each $i \in \{1, \dots, t+1\}$, $\mathcal{R}_i$ will satisfy the following properties:

(a) $\mathcal{R}_i$ is a $(2^{i-1}, (i-1)(\log_2(ch)+2), m_i)$-compact set, where $m_i \geqslant h/5-(i-1)(\lfloor\log_2(ch)\rfloor+2)$.

(b) $q_i := |\mathcal{R}_i|$, with $q_i > q_{i-1}/(10c) - 2$ if $i \geqslant 2$.

(c) There exists $x_i \in V(T)$ such that $\cup\mathcal{R}_i \subseteq B_{x_i}$.

Note that, by a simple inductive argument, one can show that

$$q_i > q_1/(10c)^{i-1} - 3 \ .$$

Indeed, the base case $i = 1$ holds trivially, and for the inductive case ($i \geqslant 2$) we have $q_i > q_{i-1}/(10c) - 2 > (q_1/(10c)^{i-2} - 3)/(10c) - 2 > q_1/(10c)^{i-1} - 3$.

It is straightforward to verify that $\mathcal{R}_1$ satisfies (a)–(c). Let $t := \min\{t_1, t_2\}$ where $t_1 := \lfloor\log_{10c}(q_1/3)\rfloor$ and $t_2 := \lfloor h/(10(\sqrt{\log_2 h} + \log_2 h) + 2)\rfloor$. Observe that, since $c = 2^{\sqrt{\log_2 h}}$, $t_1 \geqslant \log_{10c}(h/75c) \in \Omega(\log_c h) \subseteq \Omega(\sqrt{\log h})$ and that $t_2 \geqslant h/(10(\sqrt{\log_2 h} + \log_2 h) + 2) - 1 \in \Omega(h/\log h)$. Therefore $t \in \Omega(\sqrt{\log h})$. These specific values of t_1 and t_2 are chosen for the following reasons:

(i) Since $t \leqslant t_1$, $q_{t+1} > q_1/(10c)^{t_1} - 3 \geqslant 0$, so $q_{t+1} \geqslant 1$.
(ii) Since $t \leqslant t_2$, $m_i \geqslant h/5 - t_2(\lfloor\log_2(ch)\rfloor + 2) \geqslant h/10$ for each $i \in \{2, \dots, t+1\}$.

We now describe how to obtain $\mathcal{R}_{i+1}$ from $\mathcal{R}_i$ for each $i \in \{1, \dots, t\}$. By Lemma 13 (applied to $\mathcal{R} := \mathcal{R}_i$ and $S := B_{x_i}$), T_h contains a $(2^i, i\log_2(ch) + 2, m)$-compact set $\mathcal{R}^+_{i+1}$ of size q_i that is compatible with B_{x_i}. For each $v \in \cup\mathcal{R}^+_{i+1}$, v has height at least $m_{i+1} := m_i - (\lfloor\log_2(ch)\rfloor + 2) \geqslant h/5 - i(\lfloor\log_2(ch)\rfloor + 2)$. Therefore $\mathcal{R}^+_{i+1}$ satisfies (a), but does not necessarily satisfy (c). Next we show how to extract $\mathcal{R}_{i+1} \subseteq \mathcal{R}^+_{i+1}$ that also satisfies (b) and (c).

For each $v \in \cup\mathcal{R}^+_{i+1}$, $T_h - B_{x_i}$ contains a path Q_v from v to a leaf of T_h. The union of the paths in $D_{h-m_{i+1}}, \dots, D_h$ and the paths in $\mathcal{C}_i := \{Q_v : v \in \cup\mathcal{R}^+_{i+1}\}$ contains a subgraph G' isomorphic to a graph that can be obtained from $G_{2^i q_i \times m_{i+1}}$ by subdividing horizontal edges. By Lemma 9 applied to $G := G'$ with $S := B_{x_i}$ and $p := ch/m_{i+1}$, some component X' of $G' - B_{x_i}$ contains $q'_i \geqslant 2^i q_i m_{i+1}/(ch) \geqslant 2^i q_i/(10c)$ consecutive columns $C_1, \dots, C_{q'_i}$ of G'. The component X' is contained in some component X of $G_h - B_{x_i}$.

Since $\cup\mathcal{R}_i$ is unrelated, it has a left to-right-order. This order defines a total order $\prec$ on the paths in $\mathcal{C}_i$, in which $Q_v \prec Q_w$ if and only if v precedes w in left-to-right order. The resulting total order $(\prec, \mathcal{C}_i)$ corresponds to the order of the columns in G' and each part in $\mathcal{R}^+_{i+1}$ corresponds to 2^i consecutive columns of G'. There are at most two parts $R \in \mathcal{R}_i$ such that $0 < |R \cap (C_1 \cup \cdots \cup C_{q'_i})| < |R|$. These two parts account for at most $2(2^i - 1)$ of the columns in $C_1, \ldots, C_{q'_i}$. Therefore, the number of parts of $\mathcal{R}^+_{i+1}$ completely contained in $C_1 \cup \cdots \cup C_{q'_i}$ is at least

$$(q'_i - (2^{i+1} - 2))/2^i > q_i/(10c) - 2 \ .$$

We define $\mathcal{R}_{i+1} \subseteq \mathcal{R}^+_{i+1}$ as the set of parts in $\mathcal{R}^+_{i+1}$ that are completely contained in $C_1 \cup \cdots \cup C_{q'_i}$. The preceding calculation shows that $\mathcal{R}_{i+1}$ satisfies (b). Since $\cup\mathcal{R}_{i+1}$ is contained in a single component X of $G_x - B_{x_i}$ and each vertex in $\cup\mathcal{R}_{i+1}$ has a neighbour (its parent in T) in B_{x_i}, Observation 6 implies that T contains an edge $x_i x_{i+1}$ with $\cup\mathcal{R}_{i+1} \subseteq B_{x_{i+1}}$. Therefore $\mathcal{R}_{i+1}$ satisfies (c).

This completes the definition of $\mathcal{R}_1, \ldots, \mathcal{R}_{t+1}$. Properties (a)–(c) imply that, $\mathcal{R}_{t+1}$ is a $(2^t, t(\log_2(ch) + 2), m)$-compact set of size $q_{t+1} \geqslant 1$ and $\cup\mathcal{R}_{t+1} \subseteq B_{x_{t+1}}$ for some $x_{t+1} \in V(T)$. Let R be one of the sets in $\mathcal{R}_{t+1}$. Since $\mathcal{R}_{t+1}$ is $(2^t, t(\log_2(ch) + 2), m)$-compact, $|R| \geqslant 2^t$ and all vertices in R have a common ancestor a whose distance to each element of R is at most $t(\log_2(ch) + 2)$. Therefore, $\operatorname{diam}_{G_h}(R) \leqslant \operatorname{diam}_{T_h}(R) \leqslant 2t(\log_2(ch) + 2)$. By Observation 5, there exists some $P_y \in \mathcal{P}$ with

$$\begin{aligned}
|B_{x_{t+1}} \cap P_y| &\geqslant |R \cap P_y| \\
&\geqslant \frac{|R|}{\operatorname{diam}_{G_h}(R) + 1} \\
&\geqslant \frac{2^t}{2t(\log_2(ch) + 2) + 1} \\
&= 2^{t - O(\log\log h)} \\
&= 2^{\Omega(\sqrt{\log h}) - O(\log\log h)} = 2^{\Omega(\sqrt{\log h})} \ .
\end{aligned}$$

□

Proof of Theorem 10. Suppose $G_h \subseteq T \boxtimes P \boxtimes K_c$ for some tree T and some path P. By Observation 2, G_h has a T-partition $\mathcal{T} := \{B_x : x \in V(T)\}$ and a path-partition $\mathcal{P} := \{P_y : y \in V(P)\}$ such that $|B_x \cap P_y| \leqslant c$ for each $(x, y) \in V(T) \times V(P)$. By Lemma 14, there exists $(x, y) \in V(T) \times V(P)$, such that $|B_x \cap P_y| \geqslant 2^{\Omega(\sqrt{\log h})}$. Combining these upper and lower bounds on $|B_x \cap P_y|$ implies that $c \geqslant 2^{\Omega(\sqrt{\log h})}$. □

Proof of Theorem 1. Let $n := |V(G_h)| = 2^{h+1} - 1$. Suppose that $G_h \subseteq H \boxtimes P \boxtimes K_c$ for some graph H of treewidth t and maximum degree Δ, some path P and some integer c. Then, by Theorem 3 and Observation 2, $G \subseteq T \boxtimes P \boxtimes K_{24c\Delta(t+1)}$ for some tree T. By Theorem 10 $24c\Delta(t + 1) \in \Omega(2^{\sqrt{\log h}})$, so $c\Delta t \geqslant 2^{\Omega(\sqrt{\log h})} = 2^{\Omega(\sqrt{\log\log n})}$. □

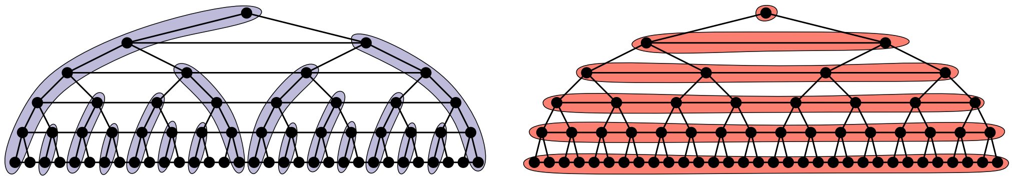

Figure 4: An outerplanar-partition $\mathcal{H}$ and a path-partition $\mathcal{P}$ of G_5 for which $|B \cap P| \leqslant 1$ for each $B \in \mathcal{H}$ and $P \in \mathcal{P}$.

3 Open Problems

We know that every planar graph G is contained in a product of the form $H \boxtimes P \boxtimes K_3$ where $\text{tw}(H) \leqslant 3$ [8]. Theorem 10 states that, for every c, there exists a planar graph of maximum degree 5 that is not contained in any product of the form $T \boxtimes P \boxtimes K_c$ where T is a tree and P is a path. This leaves the following open problem:

> Is every planar graph G of maximum degree Δ contained in a product of the form $H \boxtimes P \boxtimes K_c$ where the treewidth of H is 2, P is a path, and c is some function of Δ?

Figure 4 and Observation 2 show that G_h is a subgraph of $H \boxtimes P$ where H has treewidth 2 (and is even outerplanar) and P is a path. Our proof breaks down in this case because, unlike tree-partitions, outerplanar-partitions do not satisfy Observation 6. Indeed, the outerplanar-partition illustrated in Figure 4 contains a part B_x and a component X of $G_h - B_x$ with $|N_{G_h}(B_x) \cap V(X)| = h$. In a tree-partition this would imply that $|N_{G_h}(B_x) \cap V(X) \cap B_y| = h$ for some other part B_y of the partition. In contrast, for the outerplanar-partition shown in Figure 4, $|N_{G_h}(B_x) \cap V(X) \cap B_y| \leqslant 1$ for each $y \in V(H)$.

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