Quasi-Random Boolean Functions

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Abstract

We examine a hierarchy of equivalence classes of local quasi-random properties of Boolean Functions. In particular, we prove an equivalence between a number of properties including balanced influences, spectral discrepancy, local strong regularity, subgraph counts in a Cayley graph associated to a Boolean function, and equidistribution of additive derivatives among many others. In addition, we construct families of quasi-random Boolean functions which exhibit the properties of our equivalence theorem and separate the levels of our hierarchy. Furthermore, we relate our properties to several extant notions of pseudo-randomness for Boolean functions.

Mathematics Subject Classifications: 94D10

1 Introduction

We consider *Boolean Functions* that map binary strings of length n to $\{True, False\}$. Boolean functions can encode a wide variety of mathematical and computational objects, such as decision problems, error-correcting codes, communication and cryptographic protocols, among others. These functions are extremely well-studied in coding theory, cryptography, and computational complexity among many other areas of computer science and data science. For each application, many researchers have developed tools and perspectives unique to each area to study these Boolean functions and have isolated key properties of Boolean functions, for instance the sensitivity of the function to changes in each coordinate, the size of its Fourier coefficients, or the distance of its support viewed as a binary code.

The goal of this paper is to organize a range of properties of Boolean functions into a hierarchy of equivalence classes in the same style as the quasi-random graphs and hypergraphs in [1, 2, 3] (for details, see section §3). Our properties are local in nature, forming a hierarchy depending on a local parameter d. For instance, one of our main properties, the Balanced Influences Property, concerns the influences of all vectors of

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Hamming weight at most d. Another property considers subgraph counts of 4-cycles in the associated Cayley graph with location restrictions depending on d. There is a second parameter in the descriptions of our properties, an error bound ϵ which controls our notion of equivalence between properties. For two properties $P_1(d, \epsilon)$ and $P_2(d, \epsilon)$, we say that P_1 implies P_2 if for every $\epsilon > 0$ there is a $\delta > 0$ such that $P_1(d, \delta)$ implies $P_2(d, \epsilon)$ where δ only depends on d and ϵ. If P_1 and P_2 imply each other, then we say that P_1 and P_2 are equivalent.

In our main theorem, we show how a number of known analytic properties of Boolean functions, such as the k-th order strict avalanche criterion, restrictions of the function having small Fourier coefficients, and discrepancy of the Fourier coefficients, can be either strengthened or weakened so as to become equivalent to one another. Motivated by the enumeration of "sub-patterns" within a larger object, we further show that several combinatorial properties of graphs built from our Boolean function are equivalent with these analytic properties. These combinatorial properties include local 4-cycle counts, a local sameness property, counts of rainbow embeddings of graphs and a co-degree condition on a Cayley graph defined from the Boolean function. Finally, we give an explicit construction of a family of Boolean functions which exhibits the properties in our main theorem. As it turns out, our construction depends crucially on the existence of good binary codes. As will be indicated throughout the paper, the properties that we discuss here are satisfied by a random Boolean function, and therefore are called *quasi-random* in the spirit of [1]

Our work continues the study of quasi-randomness of graphs and hypergraphs initiated in the work of Chung, Graham, and Wilson [1]. Quasi-randomness theorems exist for other combinatorial objects, including Griffiths' results on oriented graphs [4], Cooper's work on permutations [5], Chung and Graham's work on tournaments [6] and subsets of $\mathbb{Z}/N\mathbb{Z}$ [7], k-Uniform linear hypergraphs in works of Friedman and Widgerson [8] along with Rödl, Schacht, and Kohawakaya [9], Lenz and Mubayi [10], and k-Uniform general hypergraphs in papers of Chung and Graham [2, 3, 11, 12], Frankl, Rödl, Schacht, Kohawakaya, and Nagle in [13, 14, 15, 16, 17], surveyed in the papers of Gowers [18, 19]. There are also several extant theories of quasi-randomness for Boolean functions, implicitly in Chung and Graham's work on subsets of $\mathbb{Z}/N\mathbb{Z}$ [7] and explicitly in O'Donnell's textbook [20], Castro-Silva's monograph [21], and Chung and Tetali's work on communication complexity [22]. All of the theories mentioned above center on properties of a global nature, for instance the total number of copies of a fixed subgraph as considered in the first property of Chung, Graham, and Wilson's work [1]. By contrast, our properties here are local in nature. We shall later prove that our local theory of quasi-random Boolean functions is distinct from each of these global theories, stronger than several of the global theories, and incomparable with the others.

Our paper is organized as follows. In section §2, we give the preliminaries needed to state our quasi-random properties. In section §3, we state the main equivalence theorem of eleven quasi-random properties. Due to the large umber of properties and their rich connections, the proofs of the implications are divided into two sections. Section §4 considers influences of Boolean functions and several analytic properties. Section §5 considers a codegree property and 4-cycle counts amongst other combinatorial properties. We then give an explicit construction of quasi-random functions possessing the properties in our main theorem in section §6. These functions also separate the levels of the hierarchy of our equivalence classes. In section §7 we discuss several extant theories of quasi-random Boolean functions and compare these extant theories to our work in section §8. The main results are summarized in the flowcharts found in figures 3 and 6.

2 Preliminaries

We identify the set of binary strings $\{0,1\}^n$ with elements of $\mathbb{F}_2^n$ via a choice of basis for $\mathbb{F}_2^n$, and then define a *Boolean function* to be a map $f : \mathbb{F}_2^n \to \{1,-1\}$. We then equip the space of all maps $g : \mathbb{F}_2^n \to \mathbb{R}$ with the following inner product:

$$\langle f, g\rangle := \mathbb{E}_{x\in\mathbb{F}_2^n} f(x)g(x) = \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} f(x)g(x).$$

For each $\gamma \in \mathbb{F}_2^n$, the *Fourier character* $\chi_\gamma : \mathbb{F}_2^n \to \{1,-1\}$ is $\chi_\gamma(x) := (-1)^{\gamma\cdot x}$ where $\gamma \cdot x := \sum_{i=1}^n \gamma_i x_i$ is the usual dot product. The Fourier characters form an orthonormal basis for the space of all maps $g : \mathbb{F}_2^n \to \mathbb{R}$ with the inner product as defined above. Therefore, every function $g : \mathbb{F}_2^n \to \mathbb{R}$ has unique *Fourier coefficients* $\widehat{g}(\gamma)$ where $\widehat{g}(\gamma) = \langle g, \chi_\gamma\rangle$. Notice that $\widehat{f}(0) = \langle f, 1\rangle = \mathbb{E}_{x\in\mathbb{F}_2^n} f(x)$ is simply the average value of f. The *density* of a Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$, denoted by $\mathrm{dens}(f)$, is $\frac{\left|f^{-1}(\{-1\})\right|}{2^n}$, which we note is precisely $\frac{1-\widehat{f}(0)}{2}$. The *convolution* of two functions g and $h : \mathbb{F}_2^n \to \mathbb{R}$ is

$$(g * h)(x) := \mathbb{E}_{y\in\mathbb{F}_2^n} g(x+y)h(y).$$

Note that $\widehat{g*h}(\gamma) = \widehat{g}(\gamma)\,\widehat{h}(\gamma)$.

A Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$ can be equivalently defined as a multilinear polynomial from $\mathbb{F}_2^n \to \mathbb{F}_2$, [20] where $1 \in \mathbb{F}_2$ denotes True and $0 \in \mathbb{F}_2^n$ denotes False. As the multilinear expansion of a Boolean function is unique (see [20]) each Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$ has a well-defined $\mathbb{F}_2$-*degree*, given by the size of the largest monomial in its multilinear expansion over $\mathbb{F}_2$.

We will also need to track the size of individual vectors in $\mathbb{F}_2^n$. For a vector $x \in \mathbb{F}_2^n$, its *Hamming weight*, denoted $|x|$, is the number of nonzero entries in x. Similarly, the *Hamming distance* between two vectors x and $y \in \mathbb{F}_2^n$ is $|x-y|$. For a subset $S \subseteq \mathbb{F}_2^n$, its *diameter* is $\mathrm{diam}(S) := \max_{x,y\in S}|x-y|$. The *Hamming ball* of radius d in $\mathbb{F}_2^n$ and centered at the vector $x \in \mathbb{F}_2^n$, denoted by $B_d(n,x)$, is $\{y \in \mathbb{F}_2^n : |x-y| \leqslant d\}$.

For a proposition $P(x)$, let $[P(x)] := \begin{cases} 1 & P(x) \\ 0 & \neg P(x)\end{cases}$ denote the *indicator function* for $P(x)$. We will write 0 for the zero vector in $\mathbb{F}_2^n$ throughout, and write $\mathbf{1} \in \mathbb{F}_2^n$ for the all-ones vector. If μ is a distribution on a set Ω, and $P(x)$ is a proposition on the variable $x \in \Omega$, then $\mathbb{P}_{x\sim\mu}\left[P(x)\right]$ will denote the probability distribution that $P(x)$ holds when x

is drawn from the distribution μ. Whenever we write the expectation or probability over a set, such as $\mathbb{E}_{x\in\mathbb{F}_2^n}$, the expectation or probability is taken with respect to the uniform distribution. We refer the reader to O'Donnell's book [20] for any undefined terminology.

In the following subsections, §2.1 to §2.5, we state the definitions concerning various aspects of Boolean functions that will be used to define our various properties of Boolean functions.

2.1 The influences of Boolean functions

The notion of "influences" is prominent in both analysis of Boolean functions and cryptography.

Definition 1. For $\gamma \in \mathbb{F}_2^n$, the γ-*Influence* of f is

$$\mathrm{I}_\gamma[f] := \mathbb{P}_{x\in\mathbb{F}_2^n}\left[f(x) \neq f(x+\gamma)\right].$$

Note that $\mathrm{I}_0[f]$ is always 0. Furthermore, for $\gamma \in \mathbb{F}_2^n$ with $\gamma_i = 1$ and $\gamma_j = 0$ for $j \neq i$, $\mathrm{I}_\gamma[f]$ is precisely the influence of coordinate i as studied extensively in O'Donnell [20]. We note the work of Keevash et al [23] which considers a related generalization of influences in the context of hypercontractivity.

The following property of the γ-influences will be quite useful later.

Lemma 2. *For any fixed $\gamma \in \mathbb{F}_2^n$, a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies*

$$f * f(\gamma) = 1 - 2\,\mathrm{I}_\gamma[f].$$

Proof. By definition of γ-influence,

$$\begin{aligned} 1 - 2\,\mathrm{I}_\gamma[f] &= 1 - 2\,\mathbb{P}[f(x) \neq f(x+\gamma)] \\ &= \mathbb{E}_{x\in\mathbb{F}_2^n}\big(1 - 2[f(x) \neq f(x+\gamma)]\big) \\ &= \mathbb{E}_{x\in\mathbb{F}_2^n}\, f(x)f(x+\gamma) \qquad (1) \\ &= f * f(\gamma) \end{aligned}$$

where we use the fact that $f(x) \in \{1, -1\}$ in line (1). □

2.2 The spectral sampling of Boolean functions

Parseval's Theorem states that for $f : \mathbb{F}_2^n \to \{1, -1\}$,

$$\sum_{\gamma\in\mathbb{F}_2^n} \widehat{f}(\gamma)^2 = \mathbb{E}_{x\in\mathbb{F}_2^n}\left[f(x)^2\right] = 1.$$

Thus the Fourier coefficients of f define a probability distribution on $\mathbb{F}_2^n$ as follows:

Definition 3. For a fixed Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$, the *Spectral Sample* $\mathcal{S}_f$ is the distribution on $\mathbb{F}_2^n$ where

$$\mathbb{P}_{\gamma\sim\mathcal{S}_f}\left[\gamma = \delta\right] = \widehat{f}(\delta)^2$$

for each fixed $\delta \in \mathbb{F}_2^n$.

2.3 Subcubes and the counting of subcubes

Let $[n]$ denote the set $\{1, \ldots, n\}$, and for $S \subseteq [n]$, let $\overline{S}$ denote $[n] \setminus S$. Given a set $S \subseteq [n]$, and two vectors $x \in \mathbb{F}_2^S$, $y \in \mathbb{F}_2^{\overline{S}}$, let $x \underset{S}{\otimes} y$ denote the vector where

$$(x \underset{S}{\otimes} y)_i = \begin{cases} x_i & i \in S \\ y_i & i \in \overline{S} \end{cases}.$$

Definition 4. The *subcube* defined by a set $S \subseteq [n]$ and a vector $z \in \mathbb{F}_2^{\overline{S}}$ is the set

$$C(S, z) := \{x \underset{S}{\otimes} z : x \in \mathbb{F}_2^S\}.$$

We say that the *dimension* of the subcube $C(S, z)$ is $|S|$. Note that $C([n], \eta)$ where η is the empty string is precisely the hypercube Q_n. In Figure 1, we have two examples of subcubes.

We are also concerned about Boolean functions restricted to a subcube:

Definition 5. The *restriction* of $f : \mathbb{F}_2^n \to \{1, -1\}$ to the subcube $C(S, z)$ is the Boolean function $f|_{S,z} : \mathbb{F}_2^S \to \{1, -1\}$ defined by

$$f|_{S,z}(x) = f(x \underset{S}{\otimes} z)$$

If $S = \emptyset$, then $f|_{S,z}(x)$ is the constant function $f(z)$, and if $S = [n]$, then we recover f itself.

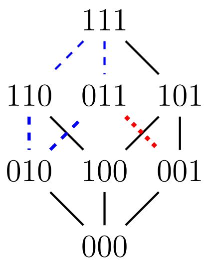

Figure 1: The blue dashed lines in the figure indicate the 2-dimensional subcube $C(\{1, 3\}, 1)$, i.e., the set of all length 3 binary strings with a 1 in the second coordinate. The red dotted line indicates the 1-dimensional subcube $C(\{2\}, 01)$.

We will need the following result from O'Donnell's book [20], translated into our notation.

Lemma 6. *[[20] Proposition 3.21] If $C(S, z)$ is a fixed subcube and $\gamma \in \mathbb{F}_2^S$, then*

$$\widehat{f|_{S,z}}(\gamma) = \sum_{\delta \in \mathbb{F}_2^{\overline{S}}} \widehat{f}\left(\gamma \underset{S}{\otimes} \delta\right) \chi_\delta(z).$$

2.4 Combinatorial aspects of Boolean functions

Here we give several useful combinatorial interpretations of Boolean functions that are of interest in their own right. For two sets A, B, let $A \hookrightarrow B$ denote the set of all injective functions from A to B. Let $A \sqcup B$ denote the disjoint union of the sets A and B.

2.4.1 Cayley Graphs

Given a group G and a set $S \subseteq G$, *Cayley graph* of G generated by S is the graph with vertex set G and $a, b \in G$ adjacent if $ab^{-1} \in S$. If $s \in S$ implies that $s^{-1} \in S$, then the Cayley graph with generating set S is undirected.

Of the many ways to define a graph from a Boolean function, the following first comes to mind.

Definition 7. For a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$, the *Cayley graph* of f, denoted $Cay(f)$, is the Cayley graph on $\mathbb{F}_2^n$ whose generating set is $f^{-1}(\{-1\})$.

As every element of $\mathbb{F}_2^n$ is its own additive inverse, it follows that $Cay(f)$ is an undirected graph.

2.4.2 Graph Homomorphisms

We will be interested in subgraph counts in $\mathrm{Cay}(f)$ which can be defined by graph homomorphisms.

Definition 8. A *graph homomorphism* from $H = (U, F)$ to $G = (V, E)$ is a map $\phi : V(H) \to V(G)$ such that

$$(u, v) \in F \implies (\phi(u), \phi(v)) \in E.$$

We will typically assume our graph homomorphisms are injective, and we denote the *normalized* number of injective graph homomorphisms via the following:

$$\mathrm{hom}(H, G) = \mathbb{E}_{\phi: V(H) \hookrightarrow V(G)} \prod_{(u,v) \in E(H)} [(\phi(u), \phi(v)) \in E(G)].$$

We will also make use of graph homomorphisms which may not be injective, and we denote the *normalized* number of such graph homomorphisms via the following:

$$\overline{\mathrm{hom}}(H, G) = \mathbb{E}_{\phi: V(H) \to V(G)} \prod_{(u,v) \in E(H)} [(\phi(u), \phi(v)) \in E(G)].$$

Note that the normalization factor in $\mathrm{hom}(H, G)$ is $\frac{1}{|V(G)|(|V(G)|-1)\ldots(|V(G)|-|V(H)|+1)}$ whereas in $\overline{\mathrm{hom}}(H, G)$ the normalization factor is $\dfrac{1}{|V(G)|^{|V(H)|}}$.

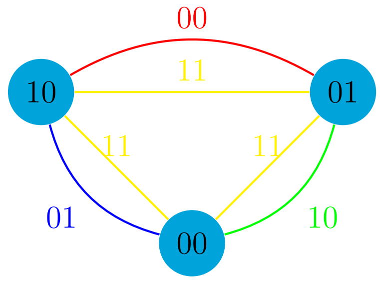

Figure 2: The rainbow Hamming graph $RHG(1,h)$ of the function $h(z) = (-1)^{1-z_1 z_2}$ where $z_1, z_2 \in \mathbb{F}_2$. Each edge is labeled by the string in $\mathbb{F}_2^2$ which defines its color.

2.4.3 Colored Multigraphs

The following definition is inspired by the work of Aharoni et al on rainbow extremal problems [24].

Definition 9. An *edge-colored multigraph* M with color set K is a multigraph with an edge-coloring using colors in K such that multiple edges between any two vertices u and v cannot have the same color.

We will typically think of the edges of an edge-colored multigraph as a subset of $V \times V \times K$.

Definition 10. For fixed $f : \mathbb{F}_2^n \to \{1, -1\}$ and $k \geqslant 1$, the *rainbow Hamming graph* $RHG(k, f)$ is the colored multigraph on the vertex set $B_k(n, 0)$ with color set $K = \mathbb{F}_2^n$ and edge set defined as

$$\{(u, v, x) \in V \times V \times K : f(u + x) = f(v + x)\}.$$

An explicit example of a rainbow Hamming graph is given in Figure 2.

2.4.4 Rainbow embeddings

We consider graph homomorphisms into a colored multigraph which agree with the coloring.

Definition 11. Let M be a colored multigraph with color set K and let G be a fixed (simple) graph. A *rainbow embedding* of G into M is an injective coloring $\chi : E(G) \hookrightarrow K$ and an injective map $\phi : V(G) \hookrightarrow V(M)$ such that

$$(u, v) \in E(G) \implies (\phi(u), \phi(v), \chi((u, v))) \in E(M).$$

These embeddings are also considered in the work of Alon and Marshall [25].

For a fixed graph G, a fixed colored multigraph M with color set K, let

$$\mathrm{chom}(G, M) := \mathbb{E}_{\phi:V(G)\hookrightarrow V(M)}\, \mathbb{E}_{\chi:E(G)\hookrightarrow K} \prod_{(u,v)\in E(G)} [(\phi(u), \phi(v), \chi((u, v))) \in E(M)]$$

be the normalized count of rainbow embeddings of G into M. If we additionally fix the injection $\phi : V(G) \hookrightarrow V(M)$, let

$$\mathrm{chom}_\phi(G, M) := \mathbb{E}_{\chi: E(G) \hookrightarrow K} \prod_{(u,v) \in E(G)} [(\phi(u), \phi(v), \chi((u, v))) \in E(M)]$$

be the normalized count of rainbow embeddings with a fixed map ϕ.

2.5 Bent Functions

We consider a specific class of Boolean functions originally defined by Rothaus [26].

Definition 12. [26] For n even, a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ is *bent* if for every $\gamma \in \mathbb{F}_2^n$ we have

$$\left|\widehat{f}(\gamma)\right| = 2^{-n/2}.$$

Note that bent functions only exist for n even.

Proposition 13. *[26] If $g : \mathbb{F}_2^n \to \{1, -1\}$ is bent, then $(g * g)(x) = [x = 0]$.*

Example 14. The **Inner Product** function $IP : \mathbb{F}_2^{2m} \to \{1, -1\}$ is defined by

$$IP(z) := (-1)^{\sum_{i=1}^m z_i z_{m+i}}.$$

For the sake of completeness, we show that IP is in fact a bent function. To calculate its Fourier coefficients, fix $\gamma \in \mathbb{F}_2^{2m}$, and let $\gamma_1, \gamma_2 \in \mathbb{F}_2^m$ denote the first m bits of γ and the last m bits respectively. For $x \in \mathbb{F}_2^{2m}$, define x_1, x_2 similarly. Then,

$$\begin{aligned} \widehat{IP}(\gamma) &= \mathbb{E}_{x \in \mathbb{F}_2^{2m}} IP(x) \chi_\gamma(x) \\ &= \mathbb{E}_{x_1 \in \mathbb{F}_2^m} \mathbb{E}_{x_2 \in \mathbb{F}_2^m} (-1)^{x_1 \cdot x_2 + \gamma_1 \cdot x_1 + \gamma_2 \cdot x_2} \\ &= \mathbb{E}_{x_1 \in \mathbb{F}_2^m} (-1)^{\gamma_1 \cdot x_1} \mathbb{E}_{x_2 \in \mathbb{F}_2^m} (-1)^{(x_1 + \gamma_2) \cdot x_2} \\ &= \mathbb{E}_{x_1 \in \mathbb{F}_2^m} (-1)^{\gamma_1 \cdot x_1} [x_1 = \gamma_2] \qquad (2) \\ &= (-1)^{\gamma_1 \cdot \gamma_2} 2^{-m} \end{aligned}$$

where we use the fact that Fourier characters are orthogonal in line (2). Thus IP is a bent function. We remark for later use that IP has $\mathbb{F}_2$-degree 2 as it is equal to the degree 2 polynomial $\sum_{i=1}^m z_i z_{m+i}$.

3 Quasi-random Properties and the Equivalence Theorem

In this section, we describe a number of quasi-random properties of Boolean functions. Each property involves two parameters, denoted by d and ϵ, where ϵ indicates the error bound and d is often related to the rank or dimension of patterns or objects in the property. We will typically think of ϵ and d as constants, but our results sometimes hold when ϵ and d depend on n. The proofs of the equivalence of these properties will be given in sections §4 and §5.

We begin with a basic property regarding the density of our Boolean functions. A random Boolean function will be -1 and 1 about equally often, i.e., it has density close to $\frac{1}{2}$. We say that a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ is **ϵ-balanced** if $\left|\text{dens}(f) - \frac{1}{2}\right| < \epsilon$. Since the density $\text{dens}(f)$ is equal to $\frac{1-\widehat{f}(0)}{2}$, any ϵ-balanced function f satisfies $\left|\widehat{f}(0)\right| < 2\epsilon$.

For the rest of the paper, we consider the following weaker density property:

Property P_0. A Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ is **weakly balanced** if the density of f is at least $\frac{3}{10}$ and at most $\frac{7}{10}$.

Equivalently, a weakly balanced function has $\left|\widehat{f}(0)\right| < \frac{2}{5}$. We remark that all of the quasi-random properties below will require a weakly balanced Boolean function. The assumption of weak balance is necessary, since there are Boolean functions which are not weakly balanced and satisfy some but not all of our quasi-random properties, as shown in Theorem 15. The specific value of $\frac{2}{5}$ is chosen for the sake of exposition and can be replaced by any constant strictly greater than $\frac{1}{2\sqrt{2}}$ and strictly less than $\frac{1}{2}$.

Our first property focuses on the directional influences defined in section §2.1. If $f : \mathbb{F}_2^n \to \{1, -1\}$ is chosen uniformly at random, we expect that the γ-influence (see Definition 1) should be close to $\frac{1}{2}$. Our first quasi-random property formalizes this notion for weakly balanced Boolean functions.

Property P_1. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Balanced Influences Property** $INF(d, \epsilon)$ if the γ-Influence of f is close to $\frac{1}{2}$ for every nonzero γ in the Hamming ball of radius d centered at 0, i.e.,

$$\left|\mathrm{I}_\gamma[f] - \frac{1}{2}\right| < \epsilon$$

for every γ such that $1 \leqslant |\gamma| \leqslant d$.

It is natural to assume that the Balanced Influences Property implies weak balance, but the implication does not hold for $d = 1$ and $d = 2$ as we shall prove in section §6.

Theorem 15. *For $d \in \mathbb{N}$ the following holds:*

- *For $d \geqslant 3$, if $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies the Balanced Influences Property $INF(d, \frac{2}{25} - 2^{-d-1})$, then f is weakly balanced.*
- *For $d \leqslant 2$, there exists a Boolean function such that* $\text{dens}(f) = \frac{1}{4}$ *but*

$$\mathrm{I}_\gamma[f] = \frac{1}{2}$$

for any $\gamma \in \mathbb{F}_2^n$ such that $0 < |\gamma| \leqslant d$.

For $f : \mathbb{F}_2^n \to \{1, -1\}$ drawn uniformly from all Boolean functions, the expected spectral sample (see Definition 3) is $\frac{1}{2^n}$ on each vector in $\mathbb{F}_2^n$. Rather than considering each vector in $\mathbb{F}_2^n$ individually, we will consider subcubes (see Definition 4). In particular, the total weight of the uniform distribution on a subcube of dimension k is exactly 2^{k-n}. Our next quasi-random property states that the spectral sample $\mathcal{S}_f$ assigns similar weight to each subcube as the uniform distribution does.

Property P_2. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Spectral Discrepancy Property** $SD(d, \epsilon)$ if the spectral sample of f has total weight close to 2^{l-n} on every subcube of dimension l where $l \geqslant n - d$, i.e.,

$$\left| \mathbb{P}_{z \sim \mathcal{S}_f}[z \in H] - 2^{\dim(H)-n} \right| < \epsilon$$

for every subcube H of dimension at least $n - d$.

Next, we have a counting property on subcubes via the notion of restricted functions (see Definition 5). As $f|_{S,z}$ is a map $\mathbb{F}^d \to \{1, -1\}$ for $|S| = d$, we can consider its Fourier coefficients. The next quasi-random property states that these Fourier coefficients are quite small on average.

Property P_3. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Restriction Fourier Property** $RF(d, \epsilon)$ if the average restriction of f is nearly a bent function on any subcube of dimension at most d, i.e.,

$$\left| \mathbb{E}_{z \in \mathbb{F}_2^{\overline{S}}} \left[\widehat{f|_{S,z}}(\gamma)^2 \right] - 2^{-\dim(C(S,z))} \right| < \epsilon$$

for every subcube $C(S, z)$ of dimension at most d and every $\gamma \in \mathbb{F}_2^S$.

The next property states that we can control certain patterns in the restrictions of f.

Property P_4. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Restriction Convolution Property** $RC(d, \epsilon)$ if the average self-convolution of restrictions of f to subcubes of dimension at most d is close to the indicator function of the 0 vector, i.e.,

$$\left| \mathbb{E}_{z \in \mathbb{F}_2^{\overline{S}}} (f|_{S,z} * f|_{S,z})(x) - [x = 0] \right| < \epsilon$$

for every set $S \subseteq [n]$ of size at most d and every $x \in \mathbb{F}_2^S$.

Convolutions are closely related to influences, so we have an additional influences property pertaining to an average restricted function:

Property P_5. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Restriction Influences Property** $RI(d, \epsilon)$ if the γ-Influences of the average restriction to subcubes of dimension at most d are close to $\frac{1}{2}$, i.e.,

$$\left| \mathbb{E}_{z \in \mathbb{F}_2^{\overline{S}}} \mathrm{I}_\gamma[f|_{S,z}] - \frac{1}{2} \right| < \epsilon$$

for every set $S \subseteq [n]$ of size at most d and every nonzero $\gamma \in \mathbb{F}_2^S$.

The *directional derivative* of a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ in the direction γ is $\Delta_\gamma f(x) = f(x+\gamma)f(x)$. Our next property states that pairs of multiplicative directional derivatives are equidistributed in the following sense:

Property P_6. A Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Equidistributed Derivatives Property** $EQD(d, \epsilon)$ if every pair of sufficiently close directional derivatives take each possible pair of values equally often, i.e., for every choice of $c_0, c_1 \in \{1, -1\}$ and for every $a, b \in \mathbb{F}_2^n$ such that $|a| \leqslant d$, $|b| \leqslant d$, and $0 < |a - b| \leqslant d$, we have

$$\left| \mathbb{E}_{x \in \mathbb{F}_2^n}[\Delta_a f(x) = c_0][\Delta_b f(x) = c_1] - \frac{1}{4} \right| < \epsilon.$$

Next we consider some combinatorial properties. Our first few combinatorial properties focus on the Cayley graph of a Boolean function $\mathrm{Cay}(f)$ (see Definition 7). For $v \in V(G)$, let $N_G(v)$ denote the neighborhood of v in G.

Property P_7. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Local Strong Regularity Property** $LSR(d, \epsilon)$ if any two vertices $u, v \in \mathbb{F}_2^n$ at Hamming distance most d have approximately the same number of common neighbors in the Cayley graph of f, i.e.,

$$\left| \frac{\left|N_{\mathrm{Cay}(f)}(x) \cap N_{\mathrm{Cay}(f)}(y)\right|}{2^n} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2} \right) \right| < \epsilon$$

for every pair of vertices x, y in $\mathrm{Cay}(f)$ such that $0 < |x - y| \leqslant d$.

We remark that the Local Strong Regularity Property is analogous to the co-degree property in Chung, Graham, and Wilson's work on quasi-random graphs [1]. Note that the term $\frac{\widehat{f}(0)}{2}$ allows for a range of edge densities in $\mathrm{Cay}(f)$, and in particular $\mathrm{Cay}(f)$ and $\mathrm{Cay}(-f)$ do not have the same edge density in general. Our next property states that nonetheless $\mathrm{Cay}(f)$ and $\mathrm{Cay}(-f)$ are somewhat interchangeable.

Property P_8. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Local Sameness Property** $SAME(d, \epsilon)$ if for any two vertices $u, v \in \mathbb{F}_2^n$ at Hamming distance most d, approximately half of all other vertices are a common neighbor of u and v either in the Cayley graph of f or the Cayley graph of $-f$, i.e.,

$$\left| \frac{\left|N_{\mathrm{Cay}(f)}(x) \cap N_{\mathrm{Cay}(f)}(y)\right| + \left|N_{\mathrm{Cay}(-f)}(x) \cap N_{\mathrm{Cay}(-f)}(y)\right|}{2^n} - \frac{1}{2} \right| < \epsilon$$

for every pair of vertices $x \neq y$ in $\mathrm{Cay}(f)$ such that $0 < |x - y| \leqslant d$.

Given the power of subgraph counts of 4-cycles in Chung, Graham, and Wilson's work on quasi-random graphs[1], we have an additional property regarding these 4-cycle counts. We say that a map $\psi : A \to \mathbb{F}_2^n$ has *diameter* at most k if $\left|\psi(u) - \psi(v)\right| \leqslant k$ for every $u, v \in A$.

Property P_9. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Local 4-Cycle Property** $L4C(d, \epsilon)$ if in the Cayley graph $\mathrm{Cay}(f)$, for any two vertices $u, v \in \mathbb{F}_2^n$ at Hamming distance at most d, the expected number of 4-cycles with u and v as antipodal points is close to the expected value, i.e.,

$$\left| \overline{\mathrm{hom}}_\phi(C_4, \mathrm{Cay}(f)) - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2} \right)^2 \right| < \epsilon$$

via the definition of $\overline{\mathrm{hom}}_\phi(H, G)$ for any injection $\phi : L(C_4) \hookrightarrow \mathbb{F}_2^n$ of diameter at most d.

Here we assume the function f is weakly balanced, an assumption which will be crucial in the proof of Theorem 24. We remark that Chung, Graham, and Wilson give a global count of C_4's, whereas we give a stronger condition which controls local appearances of C_4. This intuitive connection will be expanded upon in section §7 where we compare our properties to a number of previously known pseudo-random properties appearing in prior works.

Our final few combinatorial properties build on the Local 4-Cycle Property by giving strong control over local subgraph counts of an arbitrary graph. In particular, given a graph H, we fix the location of our desired subgraph in a larger graph derived from our Boolean function, and then ask for how many ways we can extend our choice of location to a homomorphism of H. To keep track of the extra information needed, these properties have a number of additional technical requirements and definitions.

We consider a count of rainbow embeddings in the rainbow Hamming graph (see sections §2.4.3 and §2.4.4).

Property P_{10}. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Rainbow Embeddings Property** $RAIN(d, \epsilon)$ if for every fixed simple graph G with at most $\max\{\sqrt{\epsilon} 2^{n/2-1}, 1\}$ edges and every choice of injection ϕ from G to the rainbow Hamming graph of f, there are close to a $2^{-|E(G)|}$-fraction of colorings of G which become rainbow embeddings of G under ϕ, i.e., the Rainbow Embeddings Property holds if

$$\left| \mathrm{chom}_\phi(G, RHG(d, f)) - 2^{-|E(G)|} \right| < \epsilon$$

for every fixed graph G such that $|E(G)| \leqslant \max\{\sqrt{\epsilon} 2^{n/2-1}, 1\}$, and every $\phi : V(G) \hookrightarrow V(RHG(d, f))$ of diameter at most d.

Property P_{11}. A weakly balanced Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ has the **Weak Rainbow Embeddings Property** $WRAIN(d, \epsilon)$ if for every choice of injection ϕ from K_2 to the rainbow Hamming graph of f, there are close to a $\frac{1}{2}$-fraction of colorings of K_2 which become rainbow embeddings of G under ϕ, i.e., the Rainbow Embeddings Property holds if

$$\left| \mathrm{chom}_\phi(K_2, RHG(d, f)) - \frac{1}{2} \right| < \epsilon$$

for every $\phi : V(K_2) \hookrightarrow V(RHG(d, f))$ of diameter at most d.

A map $\Delta : \mathbb{N} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a *loss function* if for each $d \in \mathbb{N}$, $\epsilon < \epsilon'$ implies that $\Delta(d,\epsilon) \leqslant \Delta(d,\epsilon')$. For properties $P(d,\epsilon)$ and $Q(d,\epsilon)$ and a loss function Δ, we say P Δ-*implies* Q, denoted $P(d,\epsilon) \overset{\Delta}{\Longrightarrow} Q(d,\epsilon)$, if for every $d \geqslant 1$, every $\epsilon > 0$, every $n > 0$ and every Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$

$$P(d,\Delta(d,\epsilon)) \implies Q(d,\epsilon).$$

Notice that $\Delta(d,\epsilon)$ does not depend on the function f or on the size of the domain, n. If

$$P(d,\epsilon) \overset{\Delta_1}{\Longrightarrow} Q(d,\epsilon) \text{ and } Q(d,\epsilon) \overset{\Delta_2}{\Longrightarrow} P(d,\epsilon)$$

for some loss functions Δ_1 and Δ_2, we say that P and Q are *equivalent*. Our main result is that P_1, P_2,...,P_{11} are all equivalent as stated below.

Theorem 16. *For any fixed d, the properties* $INF(d,\epsilon)$, $SD(d,\epsilon)$, $RF(d,\epsilon)$, $RC(d,\epsilon)$, $RI(d,\epsilon)$, $EQD(d,\epsilon)$, $LSR(d,\epsilon)$, $SAME(d,\epsilon)$, $L4C(d,\epsilon)$, $RAIN(d,\epsilon)$, *and* $WRAIN(d,\epsilon)$ *are all equivalent.*

If a Boolean function f satisfies the Balanced Influences Property $INF(d,\epsilon)$ for some d and ϵ, we say that f is *quasi-random* of rank d with error bound ϵ. Such a function f then satisfies all of the other properties in Theorem 16 with rank d and the appropriate value of ϵ.

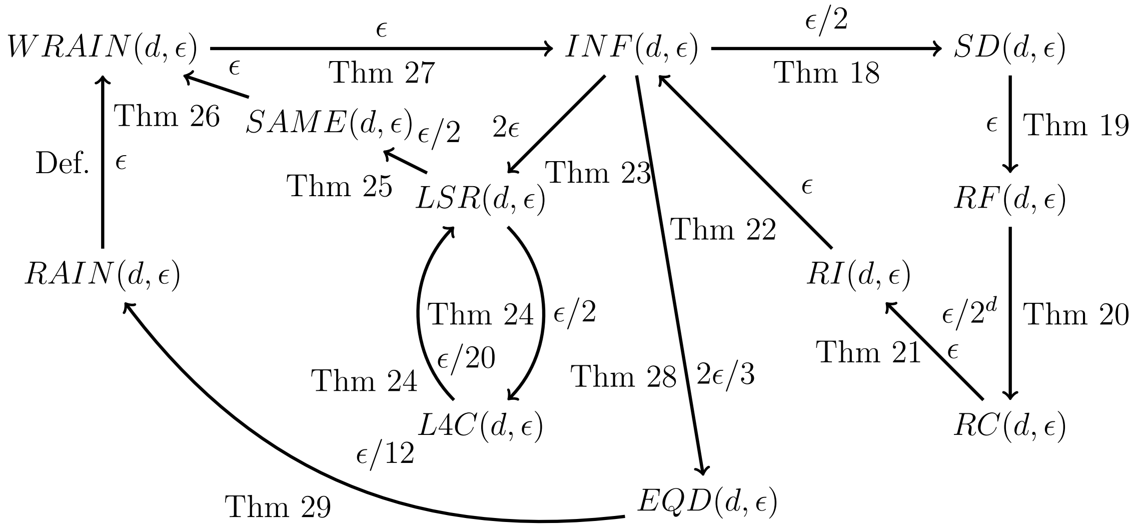

Figure 3: The implications in the Theorem 16. Each edge gives the loss in ϵ and the reference to the theorem in which the implication is shown.

We shall prove Theorem 16 via a series of theorems, each of which handles a specific implication between two properties. As we have a large number of properties and implications to prove, the proof of Theorem 16 is divided into two sections as follows:

- Section §4 considers the properties $INF(d, \epsilon)$, $SD(d, \epsilon)$, $RF(d, \epsilon)$, $RC(d, \epsilon)$, and $RI(d, \epsilon)$ which revolve around the Fourier expansion of a Boolean function.
- Section §5 considers the combinatorial properties $LSR(d, \epsilon)$, $L4C(d, \epsilon)$, $SAME(d, \epsilon)$, $EQD(d, \epsilon)$, $RAIN(d, \epsilon)$, and $WRAIN(d, \epsilon)$.

We can summarize the proof of our main theorem in figure 3, where each arrow is labeled with the relevant theorem and error bound.

One can easily observe that $P(d+1, \epsilon) \implies P(d, \epsilon)$ for each property P and every d and ϵ. Our second main result, proven in section §6, shows that these inclusions are strict, i.e., that there are functions which are quasi-random of rank d but not quasi-random of rank $d+1$.

Theorem 17. *For each $d \geqslant 1$ and any $0 < \epsilon < \frac{1}{8}$, there exists an explicit weakly balanced function $f_d : \mathbb{F}_2^n \to \{1, -1\}$ such that*

- *f_d satisfies the Balanced Influences Property $INF(d, \epsilon)$.*
- *f_d does not satisfy the Balanced Influences Property of rank $d+1$ for any $\epsilon < \frac{1}{2}$.*

4 Proof of Equivalence of Analytical Properties

In this section, we shall prove the equivalence of a number of analytic properties in Theorem 16. Figure 4 provides an outline of the implications proven in this section.

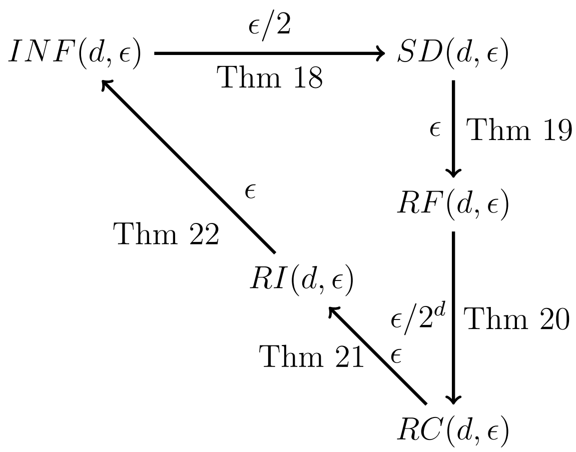

Figure 4: The implications in Theorem 16 proven in section §4. Each edge gives the loss in ϵ and the reference to the theorem in which the implication is shown.

First, we relate the Balanced Influences Property to the Spectral Discrepancy Property.

Theorem 18. *For any fixed integer $d \geqslant 1$ and any $\epsilon > 0$, the Balanced Influences Property $INF(d, \epsilon/2)$ implies the Spectral Discrepancy Property $SD(d, \epsilon)$.*

Proof. Fix a subcube $C(S, z_0)$ where $|S| = n - k$ for $k \leqslant d$. Let $M \in \mathbb{F}_2^{\overline{S} \times [n]}$ be the projection matrix which sends $z \in \mathbb{F}_2^n$ to $z|_{\overline{S}}$.

We observe that the indicator function $[\gamma \in C(S, z_0)]$ can be written as

$$[\gamma \in C(S, z_0)] = \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot (M\gamma - z_0)}. \tag{3}$$

Indeed, if $\gamma \in C(S, z_0)$, then $M\gamma = z_0$, and $\mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot (M\gamma - z_0)} = \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} 1 = 1$. If $\gamma \notin C(S, z_0)$, then $\gamma_j \neq (z_0)_j$ for some $j \in \overline{S}$. Therefore, $\mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot (M\gamma - z_0)} = \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot y}$ for some nonzero vector y. Hence, $\mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot (M\gamma - z_0)} = 0$. Let f be a function which satisfies the Balanced Influence Property $INF(d, \epsilon/2)$. We expand the definition of the spectral sample.

$$\begin{aligned}
\mathbb{P}_{\gamma \sim \mathcal{S}_f} \left[\gamma \in C(S, z_0) \right] &= \sum_{\gamma \in C(S, z_0)} \widehat{f}(\gamma)^2 \\
&= \sum_{\gamma \in \mathbb{F}_2^n} \widehat{f}(\gamma)^2 \left[\gamma \in C(S, z_0) \right] \\
&= \sum_{\gamma \in \mathbb{F}_2^n} \widehat{f}(\gamma)^2 \, \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot (M\gamma - z_0)} \qquad (4)
\end{aligned}$$

where we use equation (3) in line (4). Simplifying further, we have

$$\begin{aligned}
\mathbb{P}_{\gamma \sim \mathcal{S}_f} \left[\gamma \in C(S, z_0) \right] &= \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot z_0} \sum_{\gamma \in \mathbb{F}_2^n} \widehat{f}(\gamma)^2 (-1)^{v \cdot M\gamma} \\
&= \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot z_0} \sum_{\gamma \in \mathbb{F}_2^n} \widehat{f}(\gamma)^2 (-1)^{(M^\top v) \cdot \gamma} \\
&= \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot z_0} \sum_{\gamma \in \mathbb{F}_2^n} \widehat{f}(\gamma)^2 \chi_\gamma(M^\top v) \qquad (5) \\
&= \mathbb{E}_{v \in \mathbb{F}_2^{\overline{S}}} (-1)^{v \cdot z_0} f * f(M^\top v) \qquad (6)
\end{aligned}$$

where we use the definition of χ_γ in line (5) and Fourier expansion of $f * f$ in line (6). Notice that $f * f(M^\top 0) = (f * f)(0) = 1$, and that $x = 0$ is the only solution to $M^\top x = 0$.

Therefore, we can write

$$\begin{aligned}\left|\mathbb{P}_{\gamma\sim\mathcal{S}_f}\left[\gamma\in C(S,z_0)\right]-2^{-k}\right| &= \left|\sum_{v\in\mathbb{F}_2^k}(-1)^{v\cdot z_0}\frac{f*f(M^\top v)}{2^k}-2^{-k}\right| \\ &= \left|\sum_{v\in\mathbb{F}_2^k\setminus\{0\}}(-1)^{v\cdot z_0}\frac{f*f(M^\top v)}{2^k}\right| \\ &\leqslant \frac{1}{2^k}\sum_{v\in\mathbb{F}_2^k\setminus\{0\}}\left|f*f(M^\top v)\right| \\ &= \frac{1}{2^k}\sum_{v\in\mathbb{F}_2^k\setminus\{0\}}\left|1-2\,\mathrm{I}_{M^\top v}[f]\right|\end{aligned}$$

where we use Lemma 2 in the final line. As $k\leqslant d$, we have $|v|\leqslant d$. Since M is a projection matrix, $\left|M^\top v\right|=|v|\leqslant d$. Therefore, we may apply $INF(d,\epsilon/2)$ to find

$$\left|\mathbb{P}_{\gamma\sim\mathcal{S}_f}\left[\gamma\in C(S,z_0)\right]-2^{-k}\right|\leqslant\frac{1}{2^k}\sum_{v\in\mathbb{F}_2^k\setminus\{0\}}\epsilon\leqslant\epsilon$$

As $C(S,z_0)$ is arbitrarily chosen, f satisfies the Spectral Discrepancy Property $SD(d,\epsilon)$ as desired. ☐

Now we can relate the spectral sample to the Fourier coefficients of restricted functions.

Theorem 19. *For any fixed $d\geqslant 1$ and $\epsilon>0$ the Spectral Discrepancy Property $SD(d,\epsilon)$ implies the Restriction Fourier Property $RF(d,\epsilon)$.*

Proof. This proof is essentially the proof of Corollary 3.22 in [20], which we include here for completeness. Suppose $f:\mathbb{F}_2^n\to\{1,-1\}$ satisfies the Spectral Discrepancy Property $SD(d,\epsilon)$. Let $C(S,z)$ be an arbitrary subcube of dimension k where $k\leqslant d$. Then for a fixed $\gamma\in\mathbb{F}_2^S$, Lemma 6 gives us

$$\begin{aligned}\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\widehat{f|_{S,z}}(\gamma)^2 &= \mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\left(\sum_{\delta\in\mathbb{F}_2^{\overline{S}}}\widehat{f}\left(\gamma\underset{S}{\otimes}\delta\right)\chi_\delta(z)\right)^2 \\ &= \sum_{\delta_1,\delta_2\in\mathbb{F}_2^{\overline{S}}}\widehat{f}\left(\gamma\underset{S}{\otimes}\delta_1\right)\widehat{f}\left(\gamma\underset{S}{\otimes}\delta_2\right)\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\chi_{\delta_1}(z)\chi_{\delta_2}(z) \\ &= \sum_{\delta\in\mathbb{F}_2^{\overline{S}}}\widehat{f}\left(\gamma\underset{S}{\otimes}\delta\right)^2 \qquad (7) \\ &= \mathbb{P}_{\eta\sim\mathcal{S}_f}\left[\eta\in C(\overline{S},\gamma)\right] \qquad (8)\end{aligned}$$

where we use the orthogonality of the Fourier characters in line (7) and the definition of the spectral sample line (8). As $k \leqslant d$, $\left|\overline{S}\right| = n - k \geqslant n - d$. Thus we can apply Property $SD(d,\epsilon)$ to $C(\overline{S}, z)$ to find that

$$\left|\mathbb{P}_{\eta\sim\mathcal{S}_f}\left[\eta \in C(\overline{S},\gamma)\right] - 2^{-k}\right| < \epsilon$$

for every $\gamma \in \mathbb{F}_2^S$. Hence,

$$\left|\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\widehat{f|_{S,z}}(\gamma)^2 - 2^{-k}\right| < \epsilon$$

for every $\gamma \in \mathbb{F}_2^S$. As $C(S,z)$ is arbitrary, f satisfies the Restriction Fourier Property $RF(d,\epsilon)$. □

With a bound on the Fourier coefficients of restricted functions, we can bound the convolution of a restricted function with itself.

Theorem 20. *For any fixed d $\geqslant 1$ and $\epsilon > 0$ the Restriction Fourier Property $RF(d,\epsilon/2^d)$ implies the Restriction Convolution Property $RC(d,\epsilon)$*

Proof. Let $f : \mathbb{F}_2^n \to \{1,-1\}$ have the Restriction Fourier Property $RF(d,\epsilon/2^d)$, and note that f also satisfies $RF(k,\epsilon/2^k)$ for every $k \leqslant d$. Fix $k \in \mathbb{N}$ such that $k \leqslant d$ and a set $S \subseteq [n]$ where $|S| = k$.

We have

$$\begin{aligned}\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}(f|_{S,z} * f|_{S,z})(x) &= \mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\sum_{\delta\in\mathbb{F}_2^S}\widehat{f|_{S,z}}(\delta)^2\chi_\delta(x)\\ &= \sum_{\delta\in\mathbb{F}_2^S}\left(\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\widehat{f|_{S,z}}(\delta)^2\right)\chi_\delta(x)\end{aligned}$$

Using the Fourier expansion of the indicator function $[x=0]$, we then have

$$\begin{aligned}\left|\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}(f|_{S,z} * f|_{Sz})(x) - [x=0]\right| &= \left|\sum_{\delta\in\mathbb{F}_2^S}\left(\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\widehat{f|_{S,z}}(\delta)^2 - \frac{1}{2^k}\right)\chi_\delta(x)\right|\\ &\leqslant \sum_{\delta\in\mathbb{F}_2^S}\left|\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}}\widehat{f|_{S,z}}(\delta)^2 - \frac{1}{2^k}\right|\\ &\leqslant \sum_{\delta\in\mathbb{F}_2^S}\frac{\epsilon}{2^k}\\ &\leqslant \epsilon\end{aligned}$$

where we use $RF(k,\epsilon/2^k)$ in the penultimate line. Since k and S are arbitrary, we conclude that f satisfies the Restriction Convolution Property $RC(d,\epsilon)$. □

Theorem 21. *For any fixed $d \geqslant 1$ and $\epsilon > 0$ the Restriction Convolution Property $RC(d, 2\epsilon)$ implies the Restriction Influences Property $RI(d, \epsilon)$.*

Proof. Suppose f satisfies the Restriction Convolution Property $RC(d, 2\epsilon)$. Applying Lemma 2 to $f|_{S,z}$, we have

$$\mathrm{I}_\gamma[f|_{S,z}] = \frac{1 - f|_{S,z} * f|_{S,z}(\gamma)}{2}$$

for any fixed S and z. Now fix $k \in \mathbb{N}$ such that $k \leqslant d$ and $S \subseteq [n]$ where $|S| = k$. Then,

$$\left|\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \mathrm{I}_\gamma[f|_{S,z}] - \frac{1}{2}\right| = \left|\left(\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \frac{1 - f|_{S,z} * f|_{S,z}(\gamma)}{2}\right) - \frac{1}{2}\right| = \left|\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \frac{f|_{S,z} * f|_{S,z}(\gamma)}{2}\right|$$

If $\gamma \neq 0$, $RC(d, 2\epsilon)$ implies that the above is at most ϵ. Hence, f satisfies the Restriction Influences Property $RI(d, \epsilon)$. ☐

Theorem 22. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Restriction Influences Property $RI(d, \epsilon)$ implies the Balanced Influences Property $INF(d, \epsilon)$.*

Proof. Suppose $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies the Restriction Influences Property $RF(d, \epsilon)$. Fix $S \subseteq [n]$ with $|S| \leqslant d$ and a nonzero $\gamma \in \mathbb{F}_2^S$. Then,

$$\begin{aligned}\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \mathrm{I}_\gamma[f|_{S,z}] &= \mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \mathbb{E}_{x\in\mathbb{F}_2^S}[f|_{S,z}(x+\gamma) \neq f|_{S,z}(x)] \\ &= \mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \mathbb{E}_{x\in\mathbb{F}_2^S}[f(x \underset{S}{\otimes} z + \gamma \underset{S}{\otimes} 0)) \neq f(x \underset{S}{\otimes} z)]\end{aligned}$$

Let $y = x \underset{S}{\otimes} z$ and $\delta = \gamma \underset{S}{\otimes} 0$. Note that $|\delta| \leqslant d$ as $|S| \leqslant d$. Thus

$$\begin{aligned}\mathbb{E}_{z\in\mathbb{F}_2^{\overline{S}}} \mathrm{I}_\gamma[f|_{S,z}] &= \mathbb{E}_{y\in\mathbb{F}_2^n}[f(y+\delta) \neq f(y)] \\ &= \mathrm{I}_\delta[f]\end{aligned}$$

Since any vector of Hamming weight at most d can be represented as $\gamma \underset{S}{\otimes} 0$ for some set S with $|S| \leqslant d$ and $\gamma \in \mathbb{F}_2^S$, f satisfies the Balanced Influences Property $INF(d, \epsilon)$. ☐

5 Proof of Equivalence of Combinatorial Properties

In this section, we continue the proof of Theorem 16 and prove that several of our combinatorial properties are equivalent to the Balanced Influences Property. The diagram in figure 5 summarizes the proofs found in this section.

We begin by considering the relationship between γ-Influences and the Local Strong Regularity Property.

Theorem 23. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Balanced Influences Property $INF(d, 2\epsilon)$ implies the Local Strong Regularity Property $LSR(d, \epsilon)$.*

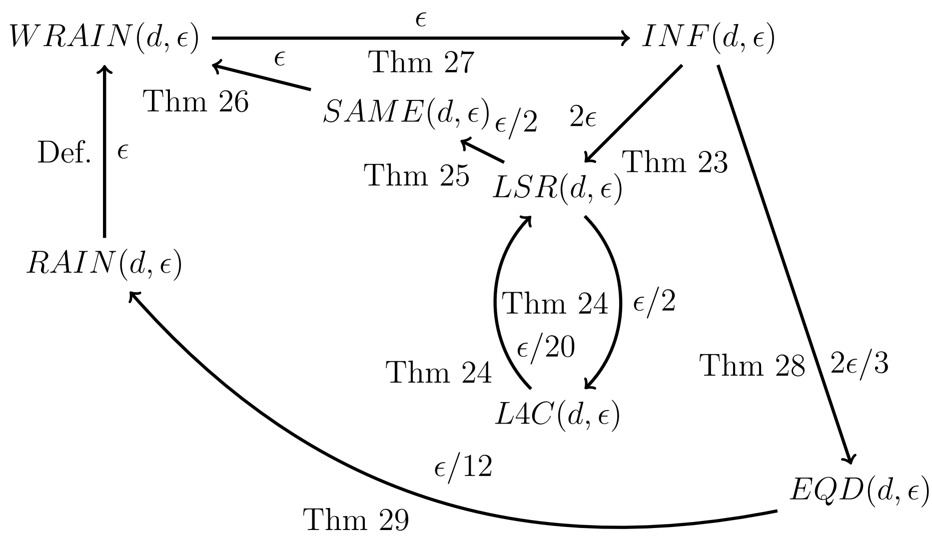

Figure 5: The implications in Theorem 16 proven in section §5. Each edge gives the loss in ϵ and the reference to the theorem in which the implication is shown.

Proof. Suppose $f : \mathbb{F}_2^n \to \{1,-1\}$ satisfies the Balanced Influences Property $INF(d, 2\epsilon)$. Fix u, v in the Cayley graph of f such that $0 < |u-v| \leqslant d$. Then,

$$\begin{aligned}
\left|\frac{\left|N(u)\cap N(v)\right|}{2^n} - \frac{1}{4} + \frac{\widehat{f}(0)}{2}\right| &= \left|\mathbb{E}_{z\in\mathbb{F}_2^n}\, \frac{(1-f(u+z))(1-f(v+z))}{4} - \frac{1}{4} + \frac{\widehat{f}(0)}{2}\right| \\
&= \left|\frac{\widehat{f}(0)}{2} - \frac{\widehat{f}(0)}{4} - \frac{\widehat{f}(0)}{4} + \frac{1}{4}\,\mathbb{E}_{z\in\mathbb{F}_2^n}\, f(u+z)f(v+z)\right| \\
&= \frac{1}{4}\left|f * f(u+v)\right| \\
&= \frac{1}{2}\,\mathrm{I}_{u+v}[f] \\
&\leqslant \epsilon
\end{aligned}$$

where we use Lemma 2 in the penultimate line and $INF(d, 2\epsilon)$ in the ultimate line. It follows that f satisfies the Local Strong Regularity Property $LSR(d, \epsilon)$. ☐

As Local Strong Regularity is a condition on common neighbors, we can use it to count 4-cycles.

Theorem 24. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Local 4-Cycle Property $L4C(d, \epsilon/20)$ implies the Local Strong Regularity Property $LSR(d, \epsilon)$ and the Local Strong Regularity Property $LSR(d, \epsilon/2)$ implies the Local 4-Cycle Property $L4C(d, \epsilon)$.*

Proof. Let u, v be the vertices in the left part of C_4, and fix an injective map $\phi : \{u, v\} \hookrightarrow$

$\mathbb{F}_2^n$. The key observation is the following:

$$\overline{\hom}_\phi(C_4, \mathrm{Cay}(f)) = \frac{\left|N(\phi(u)) \cap N(\phi(v))\right|^2}{2^{2n}}$$

Indeed, a (possibly non-injective) graph homomorphism of C_4 with a fixed left part is simply a choice of two vertices in the common neighborhood of $\phi(u)$ and $\phi(v)$ in $\mathrm{Cay}(f)$. Let $N(u,v) = \left|N(\phi(u)) \cap N(\phi(v))\right|$.

Hence,

$$\left|\overline{\hom}_\phi(C_4, \mathrm{Cay}(f)) - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right| = \left|\frac{N(u,v)^2}{2^{2n}} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right|$$

Factoring, we see that

$$\left|\frac{N(u,v)^2}{2^{2n}} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right| = \left|\frac{N(u,v)}{2^n} + \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right|\left|\frac{N(u,v)}{2^n} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right| \quad (9)$$

Now we prove both of the implications in the theorem. Assume first that f satisfies the Local 4-Cycle Property $L4C(d, \epsilon/20)$. By equation (9),

$$\frac{\epsilon}{20} \geqslant \left|\overline{\hom}_\phi(C_4, \mathrm{Cay}(f)) - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right| \geqslant \frac{1}{20}\left|\frac{N(u,v)}{2^n} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right| \quad (10)$$

where we use the fact that f is weakly balanced to show that

$$\left|\frac{N(u,v)}{2^n} + \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right| \geqslant \left|\frac{N(u,v)}{2^n} + \frac{1}{4}\right| - \left|\frac{\widehat{f}(0)}{2}\right| \geqslant \frac{1}{4} - \frac{1}{5} = \frac{1}{20}.$$

It follows that f satisfies the Local Strong Regularity Property $LSR(d, \epsilon)$.

Now assume that f satisfies the Local Strong Regularity Property $LSR(d, \epsilon/2)$, so that $\left|\frac{N(u,v)}{2^n} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right| < \epsilon/2$. Again using equation (9), we find that

$$\begin{aligned}\left|\overline{\hom}_\phi(C_4, \mathrm{Cay}(f)) - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right| &< \left|\frac{N(u,v)}{2^n} + \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right|\frac{\epsilon}{2} \\ &\leqslant \left(\left|\frac{N(u,v)}{2^n} - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)\right| + 2\left|\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right|\right)\frac{\epsilon}{2}\end{aligned}$$

By $LSR(d, \epsilon/2)$,

$$\begin{aligned}\left|\overline{\hom}_\phi(C_4, \mathrm{Cay}(f)) - \left(\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right)^2\right| &\leqslant \left(\frac{\epsilon}{2} + 2\left|\frac{1}{4} - \frac{\widehat{f}(0)}{2}\right|\right)\frac{\epsilon}{2} \\ &\leqslant \left(\frac{\epsilon}{2} + \frac{1}{2} + \left|\widehat{f}(0)\right|\right)\frac{\epsilon}{2} \\ &= \left(\frac{\epsilon}{2} + \frac{9}{10}\right)\frac{\epsilon}{2} \\ &\leqslant \epsilon\end{aligned}$$

where we use the facts that f is weakly balanced and $\epsilon \leqslant 1$. We conclude that f satisfies the Local 4-Cycle Property $L4C(d, \epsilon)$. □

Local Strong Regularity also allows us to consider the Cayley graph $\mathrm{Cay}(-f)$.

Theorem 25. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Local Strong Regularity Property $LSR(d, \epsilon/2)$ implies the Local Sameness Property $SAME(d, \epsilon)$.*

Proof. Let $f : \mathbb{F}_2^n \to \{1, -1\}$ be a Boolean function which satisfies the Local Strong Regularity Property $LSR(d, \epsilon/2)$. Fix $u, v \in \mathbb{F}_2^n$ such that $|u - v| \leqslant d$. Similarly to Theorem 24, let
$N^+(u, v) = \left|N_{\mathrm{Cay}(f)}(u) \cap N_{\mathrm{Cay}(f)}(v)\right|$ and let $N^-(u, v) = \left|N_{\mathrm{Cay}(-f)}(u) \cap N_{\mathrm{Cay}(-f)}(v)\right|$.
We observe that

$$\begin{aligned}\frac{N^-(u, v)}{2^n} &= \mathbb{E}_{x\in\mathbb{F}_2^n}\frac{1 + f(x + u)}{2}\frac{1 + f(x + v)}{2} \\ &= \frac{1}{4} + \frac{\widehat{f}(0)}{2} + \mathbb{E}_{x\in\mathbb{F}_2^n} f(x + u)f(x + v) \\ &= \frac{N^+(u, v)}{2^n} + \widehat{f}(0)\end{aligned}$$

Hence,

$$\left|\frac{N^+(u, v) + N^-(u, v)}{2^n} - \frac{1}{2}\right| = \left|2\frac{N^+(u, v)}{2^n} - \frac{1}{2} + \widehat{f}(0)\right| = 2\left|\frac{N^+(u, v)}{2^n} - \frac{1}{4} + \frac{\widehat{f}(0)}{2}\right| \leqslant \epsilon$$

where we use $LSR(d, \epsilon/2)$ in the final line. Hence, f has the Local Sameness Property $SAME(d, \epsilon)$. □

Since the rainbow Hamming graph has an edge uv with color x whenever $f(u + x) = f(v+x)$, the Local Sameness Property gives a natural way to control the rainbow Hamming graph.

Theorem 26. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Local Sameness Property $SAME(d, \epsilon)$ implies the Weak Rainbow Embeddings Property $WRAIN(d, \epsilon)$.*

Proof. Suppose $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies the Local Sameness Property $SAME(d, \epsilon)$. Fix $u, v \in B_d(n, 0)$, and let ϕ be an injection from $V(K_2)$ to $\{u, v\}$. By definition of rainbow embeddings, we have

$$\mathrm{chom}_\phi(K_2, RHG(d, f)) = \mathbb{E}_{\chi: E(K_2) \to \mathbb{F}_2^n}[(\phi(u), \phi(v), \chi(e)) \in E(RHG(d, f))]$$

Let $N^+(u, v) = \left|N_{\mathrm{Cay}(f)}(u) \cap N_{\mathrm{Cay}(f)}(v)\right|$ and let $N^-(u, v) = \left|N_{\mathrm{Cay}(-f)}(u) \cap N_{\mathrm{Cay}(-f)}(v)\right|$. Setting $x = \chi(e)$ and applying the definition of the edge set of $RHG(f)$, we have

$$\begin{aligned}\mathrm{chom}_\phi(K_2, RHG(d, f)) &= \mathbb{E}_{x \in \mathbb{F}_2^n}[f(\phi(u) + x) = f(\phi(v) + x)] \\ &= \mathbb{E}_{x \in \mathbb{F}_2^n}[f(\phi(u) + x) = f(\phi(v) + x) = 1] + \\ &\qquad \mathbb{E}_{x \in \mathbb{F}_2^n}[f(\phi(u) + x) = f(\phi(v) + x) = -1] \\ &= \frac{N^+(u, v) + N^-(u, v)}{2^n}\end{aligned}$$

Hence,
$$\left|\mathrm{chom}_\phi(K_2, RHG(d, f)) - \frac{1}{2}\right| = \left|\frac{N^+(u, v) + N^-(u, v)}{2^n} - \frac{1}{2}\right| < \epsilon$$
by the Local Sameness Property $SAME(d, \epsilon)$. Hence, f satisfies the Weak Rainbow Embeddings Property $WRAIN(d, \epsilon)$. □

Our next theorem is an immediate consequence of the Weak Rainbow Embeddings Property.

Theorem 27. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Weak Rainbow Embeddings Property $WRAIN(d, \epsilon)$ implies the Balanced Influences Property $INF(d, \epsilon)$.*

Proof. Suppose $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies the Weak Rainbow Embeddings Property $WRAIN(d, \epsilon)$. Fix $u \in B_d(n, 0)$, and let ϕ be an injection from $V(K_2)$ to $\{u, 0\}$. By definition of rainbow embeddings, we have

$$\mathrm{chom}_\phi(K_2, RHG(d, f)) = \mathbb{E}_{\chi: E(K_2) \to \mathbb{F}_2^n}[(u, 0, \chi(e)) \in E(RHG(d, f))]$$

Setting $x = \chi(e)$ and applying the definition of the edge set of $RHG(f)$

$$\begin{aligned}\mathrm{chom}_\phi(K_2, RHG(d, f)) &= \mathbb{E}_{x \in \mathbb{F}_2^n}[f(u + x) = f(x)] \\ &= \mathbb{P}_{x \in \mathbb{F}_2^n}\left[f(x + u) = f(x)\right] \\ &= 1 - \mathrm{I}_u[f]\end{aligned}$$

By Property $WRAIN(d, \epsilon)$, we have that $|\,\mathrm{chom}_\phi(K_2, RHG(d, f)) - \frac{1}{2}| < \epsilon$. Hence, it follows that $\left|\mathrm{I}_u[f] - \frac{1}{2}\right| < \epsilon$ and f satisfies the Balanced Influences Property $INF(d, \epsilon)$. □

Theorem 28. *For any fixed $d \geqslant 1$ and $\epsilon > 0$, the Balanced Influences Property $INF(d, 2\epsilon/3)$ implies the Equidistributed Derivatives Property $EQD(d, \epsilon)$.*

Proof. Fix $a, b \in \mathbb{F}_2^n$ such that $|a|, |b| \leqslant d$ and $0 < |a-b| \leqslant d$. Fix also $c_0, c_1 \in \{1, -1\}$. Let

$$X = \left| \mathbb{E}_{x \in \mathbb{F}_2^n} [\Delta_a f(x) = c_1][\Delta_b f(x) = c_0] - \frac{1}{4} \right|$$

We then have

$$\begin{aligned} X &= \left| \mathbb{E}_{x \in \mathbb{F}_2^n} \left(\frac{1 + c_1 \Delta_a f(x)}{2} \right) \left(\frac{1 + c_0 \Delta_b f(x)}{2} \right) - \frac{1}{4} \right| \\ &= \frac{1}{4} \left| c_1 \, \mathbb{E}_{x \in \mathbb{F}_2^n} \Delta_b f(x) + c_0 \, \mathbb{E}_{x \in \mathbb{F}_2^n} \Delta_a f(x) + c_0 c_1 \, \mathbb{E}_{x \in \mathbb{F}_2^n} \Delta_a f(x) \Delta_b f(x) \right| \\ &= \frac{1}{4} \left| \mathbb{E}_{x \in \mathbb{F}_2^n} \big(c_1 f(x) f(x+b) + c_0 f(x+a) f(x) \big) + c_0 c_1 \, \mathbb{E}_{x \in \mathbb{F}_2^n} \Delta_a f(x) \Delta_b f(x) \right| \end{aligned}$$

Note that $\Delta_a f(x) \Delta_b f(x) = f(x+a) f(x) f(x+b) f(x) = f(x+a) f(x+b)$ as $f(x) \in \{1, -1\}$. Therefore

$$\begin{aligned} X &= \frac{1}{4} \left| c_1 f * f(b) + c_0 f * f(a) + c_0 c_1 f * f(a+b) \right| \\ &\leqslant \frac{1}{4} \Big(\left| f * f(b) \right| + \left| f * f(a) \right| + \left| f * f(a+b) \right| \Big) \\ &= \frac{1}{2} \big(\mathrm{I}_b[f] + \mathrm{I}_a[f] + \mathrm{I}_{a+b}[f] \big) \\ &\leqslant \epsilon \end{aligned}$$

where we use Lemma 2 and $INF(d, 2\epsilon/3)$ thrice in the final line. Thus f satisfies the Equidistributed Derivatives property $EQD(d, \epsilon)$. ☐

Our final and most technical result connects equidistributed derivatives and rainbow embeddings.

Theorem 29. *For any fixed $d \geqslant 1$ and $1 \geqslant \epsilon > 0$, the Equidistributed Derivatives Property $EQD(d, \epsilon/12)$ implies the Rainbow Embeddings Property $RAIN(d, \epsilon)$.*

Proof. Let G be a fixed graph with at most $\max\{\sqrt{\epsilon} 2^{n/2-1}, 1\}$ edges. Let $\phi : V(G) \hookrightarrow B_d(n, 0)$ be an injection of diameter at most d.

We first consider the case where 1 maximizes the above. Let (u, v) be the single edge in G. By the definition of $RHG(f)$, we have

$$\begin{aligned} \mathrm{chom}_\phi(G, RHG(d, f)) &= \mathbb{E}_{\chi : E(G) \hookrightarrow \mathbb{F}_2^n} [(\phi(u), \phi(v), \chi((u, v))) \in E(RHG(d, f))] \\ &= \mathbb{E}_{x \in \mathbb{F}_2^n} [f(\phi(u) + x) = f(\phi(v) + x)] \\ &= \mathbb{E}_{x \in \mathbb{F}_2^n} [f(\phi(u) + x) f(x) = f(\phi(v) + x) f(x)] \\ &= \mathbb{E}_{x \in \mathbb{F}_2^n} [\Delta_{\phi(u)} f(x) = \Delta_{\phi(v)} f(x)] \\ &= \mathbb{E}_{x \in \mathbb{F}_2^n} [\Delta_{\phi(u)} f(x) = 1][\Delta_{\phi(v)} f(x) = 1] + \\ &\qquad \mathbb{E}_{x \in \mathbb{F}_2^n} [\Delta_{\phi(u)} f(x) = -1][\Delta_{\phi(v)} f(x) = -1] \end{aligned}$$

By $EQD(d, \epsilon/12)$ and the triangle inequality, we have

$$\begin{aligned}\left|\text{chom}_\phi(G, RHG(d, f)) - \frac{1}{2}\right| &\leqslant \left|\mathbb{E}_{x\in\mathbb{F}_2^n}[\Delta_{\phi(u)} f(x) = 1][\Delta_{\phi(v)} f(x) = 1] - \frac{1}{4}\right| + \\ &\qquad \left|\mathbb{E}_{x\in\mathbb{F}_2^n}[\Delta_{\phi(u)} f(x) = -1][\Delta_{\phi(v)} f(x) = -1] - \frac{1}{4}\right| \\ &\leqslant \frac{\epsilon}{6} \\ &\leqslant \epsilon\end{aligned}$$

so we turn to the case where G has more than one edge, but at most $\sqrt{\epsilon}2^{n/2-1}$ edges.

Recall that $\text{chom}_\phi(G, RHG(d, f))$ counts the normalized number of colorings χ such that ϕ becomes a rainbow embedding of G with the coloring χ in the rainbow Hamming graph $RHG(d, f)$. More formally, we have

$$\begin{aligned}\text{chom}_\phi(G, RHG(d, f)) &= \mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in E(G)} [(\phi(u), \phi(v), \chi((u, v))) \in E(RHG(d, f))] \\ &= \mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in E(G)} [f(\phi(u) + \chi((u, v))) = f(\phi(v) + \chi((u, v)))]\end{aligned}$$

We observe that the event $f(\phi(u) + \chi((u, v))) = f(\phi(v) + \chi((u, v)))$ is equivalent to the event that $\Delta_{\phi(u)} f(\chi((u, v))) = \Delta_{\phi(v)} f(\chi((u, v)))$. Let $P(u, v)$ denote the event that $\Delta_{\phi(u)} f(\chi((u, v))) = 1$ and $\Delta_{\phi(v)} f(\chi((u, v))) = 1$, and let $N(u, v)$ denote the event that $\Delta_{\phi(u)} f(\chi((u, v))) = -1$ and $\Delta_{\phi(v)} f(\chi((u, v))) = -1$. We then have

$$\text{chom}_\phi(G, RHG(d, f)) = \mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in E(G)} \Big([N(u, v)] + [P(u, v)]\Big)$$

Let

$$Z = \left|\text{chom}_\phi(G, RHG(d, f)) - 2^{-|E(G)|}\right|.$$

We then have

$$\begin{aligned}Z &= \left|\mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in E(G)} \Big([N(u, v)] + [P(u, v)]\Big) - 2^{-|E(G)|}\right| \\ &= \left|\mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in E(G)} \left([N(u, v)] + [P(u, v)] - \frac{1}{2} + \frac{1}{2}\right) - 2^{-|E(G)|}\right| \\ &= \left|\sum_{\emptyset\neq R\subseteq E(G)} 2^{-|E(G)\setminus R|}\, \mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in R} \left([N(u, v)] + [P(u, v)] - \frac{1}{2}\right)\right| \\ &\leqslant \sum_{\emptyset\neq R\subseteq E(G)} 2^{-|E(G)\setminus R|} \left|\mathbb{E}_{\chi: E(G)\hookrightarrow\mathbb{F}_2^n} \prod_{(u,v)\in R} \left([N(u, v)] + [P(u, v)] - \frac{1}{2}\right)\right|\end{aligned}$$

For $R \subseteq E(G)$, let $X_R = \sum_{\chi : R \hookrightarrow \mathbb{F}_2^n} \prod_{(u,v)\in R} \left([N(u,v)] + [P(u,v)] - \frac{1}{2} \right)$. Let Y_R be the analogous version of X_R which sums over all functions, not just injections, i.e., $Y_R = \sum_{\chi : R \to \mathbb{F}_2^n} \prod_{(u,v)\in R} \left([N(u,v)] + [P(u,v)] - \frac{1}{2} \right)$. We then have

$$\begin{aligned} Z &\leqslant \sum_{\emptyset \neq R \subseteq E(G)} 2^{-\left|E(G)\setminus R\right|} \frac{1}{(2^n)_{|R|}} \left|X_R\right| \\ &\leqslant \sum_{\emptyset \neq R \subseteq E(G)} 2^{-\left|E(G)\setminus R\right|} \frac{1}{(2^n)_{|R|}} \left(\left|X_R - Y_R\right| + \left|Y_R\right|\right) \end{aligned}$$

Fix $R \subseteq E(G)$.

$$\begin{aligned} \left|X_R - Y_R\right| &= \left| \sum_{\substack{\chi : R \to \mathbb{F}_2^n \\ \chi \text{not injective}}} \prod_{(u,v)\in R} \left([N(u,v)] + [P(u,v)] - \frac{1}{2} \right) \right| \\ &\leqslant \sum_{\substack{\chi : R \to \mathbb{F}_2^n \\ \chi \text{not injective}}} \prod_{(u,v)\in R} \left| \left([N(u,v)] + [P(u,v)] - \frac{1}{2} \right) \right| \end{aligned}$$

As $N(u,v)$ and $P(u,v)$ cannot occur simultaneously, we have

$$\begin{aligned} \left|X_R - Y_R\right| &\leqslant \sum_{\substack{\chi : R \to \mathbb{F}_2^n \\ \chi \text{not injective}}} \left(\frac{1}{2}\right)^{|R|} \\ &\leqslant \left(2^{n|R|} - (2^n)_{|R|}\right) \left(\frac{1}{2}\right)^{|R|} \\ &= (2^n)_{|R|} \left(\frac{2^{n|R|}}{(2^n)_{|R|}} - 1 \right) \left(\frac{1}{2}\right)^{|R|} \end{aligned}$$

Observe that $|R|^2 \leqslant \left|E(G)\right|^2 \leqslant \epsilon 2^{n-2}$. Thus $\frac{|R|^2}{2^n} \leqslant \frac{\epsilon}{4} \leqslant \frac{1}{4}$. We have

$$\begin{aligned} \frac{2^{n|R|}}{(2^n)_{|R|}} &\leqslant \left(\frac{2^n}{2^n - |R|} \right)^{|R|} \\ &= \left(1 - \frac{|R|}{2^n} \right)^{-|R|} \\ &\leqslant \exp\left(2 \frac{|R|^2}{n} \right) && (11) \\ &\leqslant 1 + 2\frac{|R|^2}{n} + \left(2\frac{|R|^2}{n} \right)^2 && (12) \end{aligned}$$

where we use the fact that $e^{-2x} \leqslant 1 - x$ for $x \in [0, 0.5]$ in line (11) and the fact that $e^x \leqslant 1 + x + x^2$ for $x \in [0, 1.79]$ in line (12). As $\frac{|R|^2}{2^n} \leqslant \frac{\epsilon}{4}$, it follows that

$$\frac{2^{n|R|}}{(2^n)_{|R|}} \leqslant 1 + \frac{\epsilon}{2}$$

and thus

$$|X_R - Y_R| \leqslant (2^n)_{|R|} \left(\frac{1}{2}\right)^{|R|} \frac{\epsilon}{2}$$

Now we turn to Y_R.

$$\begin{aligned}
|Y_R| &= \left| \sum_{\chi : E(G) \to \mathbb{F}_2^n} \prod_{(u,v) \in R} \left([N_\chi(u,v)] + [P_\chi(u,v)] - \frac{1}{2} \right) \right| \\
&= \left| \prod_{(u,v) \in R} \left(\left(\sum_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [N_\chi(u,v)] - \frac{1}{4} \right) + \left(\sum_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [P_\chi(u,v)] - \frac{1}{4} \right) \right) \right| \\
&= 2^{n|R|} \left| \prod_{(u,v) \in R} \left(\left(\mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [N_\chi(u,v)] - \frac{1}{4} \right) + \left(\mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [P_\chi(u,v)] - \frac{1}{4} \right) \right) \right|
\end{aligned}$$

By definition,

$$\begin{aligned}
\mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [P_\chi(u,v)] &= \mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [\Delta_{\phi(u)} f(\chi((u,v))) = 1][\Delta_{\phi(v)} f(\chi((u,v))) = 1] \\
\mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [N_\chi(u,v)] &= \mathbb{E}_{\chi : \{(u,v)\} \to \mathbb{F}_2^n} [\Delta_{\phi(u)} f(\chi((u,v))) = -1][\Delta_{\phi(v)} f(\chi((u,v))) = -1]
\end{aligned}$$

By assumption, ϕ is a map of diameter at most d from $V(G)$ to $B_d(n, 0)$. Thus, $\left|\phi(u)\right| \leqslant d$, $\left|\phi(v)\right| \leqslant d$, and $\left|\phi(u) - \phi(v)\right| \leqslant d$ for every $(u,v) \in E(G)$. Hence, we may apply $EQD(d, \epsilon/12)$ to find that

$$Y_R \leqslant 2^{n|R|} \left| \prod_{(u,v) \in R} \left(\frac{\epsilon}{12} + \frac{\epsilon}{12} \right) \right| \leqslant 2^{n|R|} \left(\frac{\epsilon}{6}\right)^{|R|} \leqslant (2^n)_{|R|} \left(1 + \frac{\epsilon}{2}\right) \left(\frac{\epsilon}{6}\right)^{|R|}$$

where we use the same bound on $2^{n|R|}$ as above. Now we can put everything back together as follows:

$$\begin{aligned}
Z &\leqslant \sum_{\emptyset \neq R \subseteq E(G)} 2^{-\left|E(G) \setminus R\right|} \left(\left(\frac{1}{2}\right)^{|R|} \frac{\epsilon}{2} + \left(1 + \frac{\epsilon}{2}\right) \left(\frac{\epsilon}{6}\right)^{|R|} \right) \\
&= \frac{\epsilon}{2} \left(1 - 2^{-\left|E(G)\right|}\right) + \left(1 + \frac{\epsilon}{2}\right) \left(\left(\frac{1}{2} + \frac{\epsilon}{6}\right)^{\left|E(G)\right|} - \frac{1}{2}^{\left|E(G)\right|} \right)
\end{aligned}$$

As $\frac{\epsilon}{3} \leqslant \frac{1}{2}$, we have the following:

$$\begin{aligned}
\left(\frac{1}{2}+\frac{\epsilon}{6}\right)^{|E(G)|} - \frac{1}{2}^{|E(G)|} &= \frac{1}{2}^{|E(G)|}\left(\left(1+\frac{\epsilon}{3}\right)^{|E(G)|} - 1\right) \\
&\leqslant \frac{1}{2}^{|E(G)|}\left(\frac{\epsilon}{3}\left|E(G)\right|\left(1+\frac{\epsilon}{3}\right)^{|E(G)|-1}\right) && (13) \\
&\leqslant \frac{\epsilon}{6}\left|E(G)\right|\left(\frac{3}{4}\right)^{|E(G)|-1} && (14) \\
&\leqslant \frac{\epsilon}{3} && (15)
\end{aligned}$$

where we use the fact that $(1+x)^m \leqslant 1 + mx(1+x)^{m-1}$ in line (13), the fact that $\frac{\epsilon}{3} \leqslant \frac{1}{2}$ in line (14), and the numerical fact that $m(3/4)^{m-1} \leqslant 2$ for every $m \geqslant 1$ in line (15). Therefore,

$$\left|\mathrm{chom}_\phi(G, RHG(d,f)) - 2^{-|E(G)|}\right| = Z \leqslant \frac{\epsilon}{2} + \left(1+\frac{\epsilon}{2}\right)\frac{\epsilon}{3} \leqslant \epsilon$$

as $\epsilon \leqslant 1$. Thus f also satisfies the Rainbow Embeddings Property $RAIN(d,\epsilon)$. □

6 Constructions of quasi-random Functions and Separation of the Hierarchy

In this section, we construct a large class of functions which separate the Balanced Influences Property $INF(d+1,\epsilon)$ from $INF(d,\epsilon')$.

An $[n,k,d]$-*binary linear code* is a subspace $\mathcal{C} \subseteq \mathbb{F}_2^n$ of dimension k such that $\min_{x\neq y}|x-y| = d$. An $[n,k,d]$-binary linear code may be specified by its *parity check matrix* $M \in \mathbb{F}_2^{(n-k)\times n}$ which has the property that $x \in \mathcal{C} \iff Mx = 0$. Note that the parity check matrix has rank $n-k$. We will need the following elementary fact regarding linear codes of distance d.

Lemma 30. *[[27], Proposition 2.3.5] If M is the parity check matrix of a code with distance strictly greater than d, then any nonzero $x \in \ker(M)$ must have $|x| > d$.*

Example 31. Let $\mathcal{C}$ be the $[8,4]$-Extended Hamming code with parity check matrix H

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

One can check that no vector of Hamming weight 3 or less can be an element of the kernel, as every set of 3 columns has at least one row with an odd number of 1's

The goal of this section is to demonstrate that a bent function composed with the parity check matrix of a distance d linear code is quasi-random of rank d with error ϵ for any $\epsilon > 0$.

Proof of Theorem 17. Let $\mathcal{C}$ be an $[n, k, d+1]$-binary linear code such that $n-k$ is even and $n \geqslant k+4$. Let $H \in \mathbb{F}_2^{(n-k)\times n}$ be a parity check matrix for $\mathcal{C}$. Let $g : \mathbb{F}_2^{n-k} \to \{1, -1\}$ be a bent function, and define$f : \mathbb{F}_2^n \to \{1, -1\}$ by

$$f(x) := g(Hx).$$

We claim that

$$\mathrm{I}_\gamma[f] = \begin{cases} \frac{1}{2} & \gamma \notin \ker(H) \\ 0 & \gamma \in \ker(H) \end{cases}$$

Indeed, by Lemma 2, we have

$$\begin{aligned} \mathrm{I}_\gamma[f] &= \frac{1}{2} - \frac{1}{2} f * f(\gamma) \\ &= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\delta \in \mathbb{F}_2^n} g(H\delta) g(H(\delta+\gamma)) \\ &= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\eta \in \mathrm{Range}(H)} g(\eta) g(\eta + H\gamma) \end{aligned} \tag{16}$$

where in line (16) we use the fact that $H\delta$ is uniformly distributed on $\mathrm{Range}(H)$ when δ is uniformly distributed on $\mathbb{F}_2^n$. As the parity check matrix is a surjective linear map from $\mathbb{F}_2^n \to \mathbb{F}_2^{n-k}$, we have

$$\begin{aligned} \mathrm{I}_\gamma[f] &= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\eta \in \mathbb{F}_2^{n-k}} g(\eta) g(\eta + H\gamma) \\ &= \begin{cases} \frac{1}{2} & \gamma \notin \ker(H) \\ 0 & \gamma \in \ker(H) \end{cases} \end{aligned} \tag{17}$$

where we use the fact that a $g * g(x) = 0$ for $x \neq 0$ (see Lemma 13) in line (17). Now we can apply Lemma 30 to conclude that if $|\gamma| \leqslant d$, $\gamma \notin \ker(H)$. It follows that $\mathrm{I}_\gamma[f] = \frac{1}{2}$ for every $\gamma \in \mathbb{F}_2^n$ with $0 < |\gamma| \leqslant d$.

Similarly, as $\mathcal{C}$ has distance $d+1$, there is some $\gamma' \in \mathbb{F}_2^n$ with Hamming weight $d+1$ such that $H\gamma' = 0$. Hence, $\mathrm{I}_{\gamma'}[f] = 0$ by equation (17) above. Thus $INF(d+1, \epsilon)$ cannot hold for f unless $\epsilon \geqslant \frac{1}{2}$.

It remains to show that $\left|\widehat{f}(0)\right| < \frac{2}{5}$, i.e., that f is weakly balanced. To that end we observe

$$\widehat{f}(0) = \mathbb{E}_{x \in \mathbb{F}_2^n} g(Hx) = \mathbb{E}_{y \in \mathbb{F}_2^{n-k}} g(y)$$

by the same reasoning as in line (16) above. Since g is bent, it follows that

$$\left|\widehat{f}(0)\right| = \left|\widehat{g}(0)\right| = 2^{-\frac{n-k}{2}}.$$

As $n \geqslant k+4$, we conclude that $\left|\widehat{f}(0)\right| \leqslant \frac{1}{4} < \frac{2}{5}$ and thus $INF(d, \epsilon)$ holds for f for any $\epsilon > 0$.

□

Finally, we show that the Balanced Influences Property implies weak balance. We will need the following lemma:

Lemma 32. *If $f : \mathbb{F}_2^n \to \{1, -1\}$ has the Balanced Influences property $INF(d, \epsilon/2)$, then $\widehat{f}(\gamma)^2 \leqslant 2^{-d} + \epsilon$ for every $\gamma \in \mathbb{F}_2^n$.*

Proof. By Theorem 18, if f has the Balanced Influences Property $INF(d, \epsilon/2)$, then f has the Spectral Discrepancy Property $SD(d, \epsilon)$. Fix $\gamma \in \mathbb{F}_2^n$ and let $C(S, z)$ be a subcube of dimension $n - d$ which contains γ. By $SD(d, \epsilon)$,

$$\widehat{f}(\gamma)^2 \leqslant \sum_{\delta \in C(S,z)} \widehat{f}(\delta)^2 \leqslant 2^{-d} + \epsilon.$$

Therefore, $\left|\widehat{f}(\gamma)\right| \leqslant \sqrt{2^{-d} + \epsilon}$ for every $\gamma \in \mathbb{F}_2^n$. □

Remark 33. The functions constructed in the proof of Theorem 17 show that the bound in Lemma 32 is tight. Indeed, these function have the property that every subcube of dimension $n - d$ contains exactly one nonzero Fourier coefficient of weight $2^{-d/2}$. Thus both of the above inequalities are tight for such functions.

Proof of Theorem 15. Fix $d \geqslant 3$. By Lemma 32, if $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfies $INF(d, \frac{2}{25} - 2^{-d-1})$ (note that $\frac{2}{25} > \frac{1}{16}$, so this expression is positive when $d \geqslant 3$), then

$$\left|\widehat{f}(\alpha)\right| < \sqrt{2^{-d} + \left(\frac{4}{25} - 2^{-d}\right)} = \frac{2}{5}$$

for every $\alpha \in \mathbb{F}_2^n$. Hence,

$$\left|\frac{|f^{-1}(\{-1\})|}{2^n} - \frac{1}{2}\right| = \frac{1}{2}\left|\widehat{f}(0)\right| < \frac{1}{5}$$

and f is weakly balanced.

For the second part of the Theorem, consider the function $f : \mathbb{F}_2^2 \to \{1, -1\}$ which is -1 if and only if its input is 11. One can easily verify that $\mathrm{I}_{10}[f] = \mathrm{I}_{01}[f] = \mathrm{I}_{11}[f] = \frac{1}{2}$, and $\frac{f^{-1}(\{-1\})}{2^2} = \frac{1}{4}$ by construction. □

7 Relating quasi-random Boolean Functions to Extant theories

There are various quasi-randomness theorems for Boolean functions implicitly or explicitly considered in several related works ranging from hypergraphs to analysis of Boolean functions. Typically, these theories capture global properties of a Boolean function while the quasi-random properties defined in section §3 are local. We will discuss an incomplete list of these extant theories and compare them with some of our local quasi-random properties.

We first consider Chung and Tetali's work on the relationship between k-uniform hypergraphs and Boolean functions in [22]. Their ideas also appear implicitly in the works of Gowers [18] on hypergraph regularity Lemmas and Szemeredi's Theorem, and in a survey paper by Castro-Silva [21]. These works convert a Boolean function to a k-uniform hypergraph via the following construction. Given a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$, its *Cayley hypergraph* H has the vertex set $\mathbb{F}_2^n$ and hyper-edges $\{x_1, \ldots, x_k\} \in E(H) \iff f(x_1 + \cdots + x_k) = -1$. Via the Cayley hypergraph, these authors transfer the theory of quasi-randomness for uniform hypergraphs to Boolean functions.

The main definition in these works is the following.

Definition 34. For $k \geqslant 1$, the *k-th Gowers uniformity norm* of a function $f : \mathbb{F}_2^n \to \{1, -1\}$, denoted $\|f\|_{U(k)}$, is defined as

$$\|f\|_{U(k)} := \left(\mathbb{E}_{x \in \mathbb{F}_2^n} \mathbb{E}_{v_1, \ldots, v_k \in \mathbb{F}_2^n} \prod_{\alpha_1, \ldots, \alpha_k \in \{0,1\}} f(x + \alpha_1 v_1 + \cdots + \alpha_k v_k) \right)^{2^{-k}}$$

We will typically use the following equivalent formula

$$\|f\|_{U(k)} = \left(\mathbb{E}_{x \in \mathbb{F}_2^n, M \in \mathbb{F}_2^{n \times k}} \prod_{v \in \mathbb{F}_2^k} f(x + Mv) \right)^{2^{-k}}$$

(see [28]). The Gowers uniformity norms are a direct translation of the properties in [11, 22] which count even and odd octahedra in k-uniform hypergraphs. For these theories, the key pseudo-random property is the following:

Property P_{12}. A Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ is **(ϵ, d)-Uniform** if

$$\|f\|_{U(d+1)} < \epsilon$$

As shown in Castro-Silva's monograph [21], $(\epsilon, k + 1)$-Uniformity ϵ-implies (ϵ, k)-Uniformity, and the implication is strict. Hence, just as we have a hierarchy of quasi-random properties in our Theorem 17, we can view (ϵ, k)-Uniformity as a similar hierarchy indexed by k. As shown in [28], the $k + 1$-st Gowers norm controls correlation of f with functions of $\mathbb{F}_2$-degree at most k (see section §2 for the definition of $\mathbb{F}_2$-degree).

We show the following theorem whose proof can be found in section §8:

Theorem 35. *For any $\epsilon > 0$, a Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ with (δ, d)-Balanced Influences is also $(\epsilon, 1)$-Uniform by setting $\delta = \frac{\epsilon^4}{4}$ and $d \geqslant 1 + \lceil \frac{4\ln(1/\epsilon)}{\ln(2)} \rceil$.*

Furthermore, (ϵ, d)-Balanced Influences and (ϵ, k)-Uniformity are incomparable for any $d \leqslant n$ and $k \geqslant 2$. More precisely,

(1) There is a function $f : \mathbb{F}_2^n \to \{1, -1\}$ with (δ, n)-Balanced Influences for any $\delta > 0$, yet f is not $(\epsilon, 2)$-Uniform for any $\epsilon < 1$.

(2) For any $k \geqslant 2$ and any $\delta > 0$, there is a function $g : \mathbb{F}_2^n \to \{1, -1\}$ which is (δ, k)-Uniform and does not have $INF(d, \epsilon)$ for any rank $d \geqslant 1$ or $\epsilon < \frac{1}{2}$.

Remark 36. Directional derivatives provide a third means of defining the Gowers uniformity norms [28], so one might then think that the Equidistributed Derivatives Property P_6 will have a close relationship with (δ, k)-Uniformity. However, the Equidistributed Derivatives Property only considers derivatives along vectors of Hamming weight at most k, whereas the Gowers uniformity norms consider all possible directional derivatives. As we shall see in the proof of Theorem 35 below, the Spectral Discrepancy Property P_2 is more applicable in comparing our work and the theory of (ϵ, k)-Uniformity.

O'Donnell presents several pseudo-random properties in [20] which center on the Fourier expansion defined in Section §2. The first pseudo-random property mentioned is the following:

Property P_{13}. A Boolean function $f : \mathbb{F}_2^n \to \{1, -1\}$ is **(ϵ, d)-Fourier Regular** if

$$\left| \widehat{f}(\gamma) \right| < \epsilon$$

for every $\gamma \in \mathbb{F}_2^n$ with $|\gamma| \leqslant d$.

By definition, $(\epsilon, d+1)$-Fourier Regularity ϵ-implies (ϵ, d)-Fourier Regularity, and a Fourier character χ_γ where $|\gamma| = d + 1$ shows that the implication is strict. Hence, just as we have a hierarchy of quasi-random properties in our Theorem 17, (ϵ, k)-Fourier Regularity can be viewed as forming an increasing hierarchy of pseudo-random properties indexed by k. Furthermore, (ϵ, n)-Fourier Regularity and $(\epsilon, 1)$-Uniformity are equivalent as is shown in Proposition 6.7 of [20].

As for the relationship between (ϵ, k)-Fourier regularity and our properties, we show the following theorem whose proof can be found in section §8:

Theorem 37. *For any $\epsilon > 0$, a Boolean Function $f : \mathbb{F}_2^n \to \{1, -1\}$ with (δ, d)-Balanced Influences is also (ϵ, k)-Fourier Regular for any $k \leqslant n$ by setting $\delta = \frac{\epsilon^2}{4}$ and $d = 1 + \lceil \frac{2\ln(1/\epsilon)}{\ln(2)} \rceil$.*

Conversely, for any $\delta > 0$ there is a function which is (δ, n)-Fourier Regular which does not have (ϵ, k) Balanced Influences for any rank $k \geqslant 1$ or error bound $\epsilon < \frac{1}{2}$.

Another collection of pseudo-random properties of Boolean functions appears implicitly in Chung and Graham's work on quasi-random subsets of $\mathbb{Z}/N\mathbb{Z}$ [7]. To apply their work to Boolean functions, we can identify the set of binary strings with elements of $\mathbb{Z}/2^n\mathbb{Z}$. Then a Boolean function can be identified with the set of elements of $\mathbb{Z}/2^n\mathbb{Z}$ on which it takes the value -1. Their key pseudo-random property is the following:

Property P_{14}. A Boolean function $f : \mathbb{Z}/2^n\mathbb{Z} \to \{1, -1\}$ is ϵ-**Cycle Regular** if f has correlation at most ϵ with all nonzero characters of $\mathbb{Z}/2^n\mathbb{Z}$, i.e., for every nonzero $j \in \mathbb{Z}/2^n\mathbb{Z}$,

$$\left|\mathbb{E}_{z\in\mathbb{Z}/2^n\mathbb{Z}}\, f(z)\exp(2\pi i j z/2^n)\right| < \epsilon.$$

As shown by Chung and Graham [7], ϵ-Cycle Regularity controls the correlations of a function f with a shifted copy of itself much like our Balanced Influences Property P_1. However, the arithmetic operations considered in ϵ-Cycle Regularity are carried out over $\mathbb{Z}/2^n\mathbb{Z}$ rather than $\mathbb{F}_2^n$ as in the Balanced Influences Property.

We prove the following theorem whose proof can be found in section §8:

Theorem 38. *For any $\delta > 0$ there is a δ-Cycle Regular function which is not $(\epsilon, k+1)$-Fourier Regular for any $\epsilon < 1$ where $k = C_0 \ln(1/\delta)$ for some absolute constant C_0.*

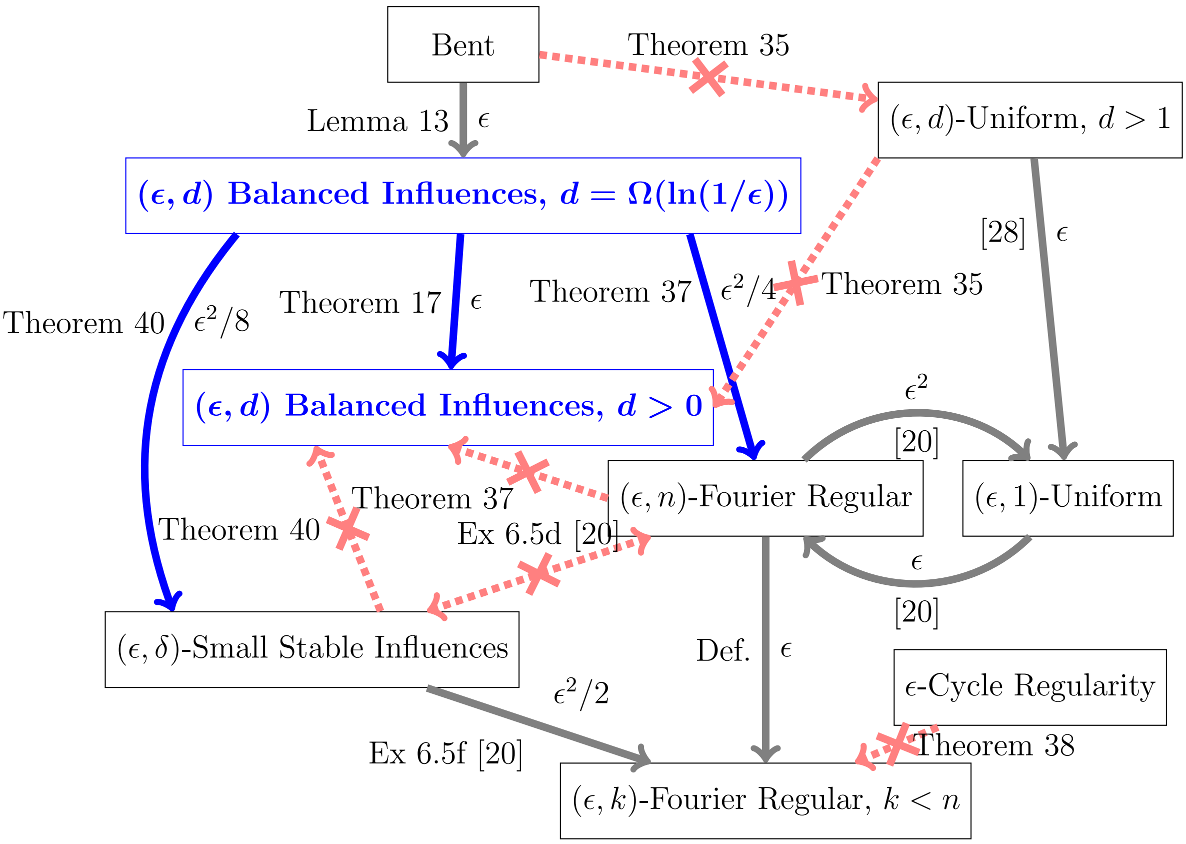

Figure 6: The relationships between different theories of quasi-randomness. Each box is a distinct theory of quasi-randomness. Each arrow is a strict implication. Beside each arrow we give a reference to the proof of the implication and the loss function. The results of this paper are in bold blue text and blue arrows. Non-implications are red dotted lines with an X in the middle, with a citation for each result.

O'Donnell [20] defines an additional pseudo-random property which uses a different generalization of influences as follows.

Definition 39. For a coordinate i and a parameter $\rho \in [0,1]$, the ***ρ*-stable influence** of a Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$ is

$$\mathrm{I}_i^{\rho}[f] := \sum_{\substack{\gamma \in \mathbb{F}_2^n \\ \gamma_i = 1}} \rho^{|\gamma|-1} \widehat{f}(\gamma)^2 .$$

The key pseudo-random property is:

Property P_{15}. A Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$ has **(ϵ, ρ)-Small Stable Influences** for some $\epsilon \in \mathbb{R}_{\geqslant 0}$ and $\rho \in [0,1]$ if

$$\mathrm{I}_i^{1-\rho}[f] < \epsilon$$

for every $i \in [n]$.

As shown by O'Donnell[20], ρ-Small Stable Influences measure the expected change in the function if the input bits are changed via a particular noise model. Thus, (ϵ, ρ) Small Stable Influences implies a form of noise stability.

We show the following theorem whose proof can be found in section §8:

Theorem 40. *For any $\epsilon > 0$ and $1 > \rho \geqslant 2 - \sqrt{2}$, a Boolean function $f : \mathbb{F}_2^n \to \{1,-1\}$ with (δ, d)-Balanced Influences also has (ϵ, ρ)-Small Stable Influences by setting $\delta = \frac{\epsilon^2}{8}$ and $d = \lceil \frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)} \rceil$.*

Conversely, there is a function which has $((1-\delta)^{n-1}, \delta)$-Small Stable Influences for any $\delta < 1$ but does not have (ϵ, k)-Balanced Influences for any k and any $\epsilon < \frac{1}{2}$.

Figure 6 illustrates the relationships between each theory of quasi-randomness and our results in section §3.

8 Proofs of relations among extant quasi-randomness theories for Boolean Functions

We will prove a series of lemmas which will be used to separate and relate the classes of quasi-random Boolean functions defined in section §7.

Lemma 41. *For even n, there is a function $f : \mathbb{F}_2^n \to \{1,-1\}$ which has $INF(d, \epsilon)$ for every $d \leqslant n$ and $\epsilon > 0$, but $\|f\|_{U(3)} = 1$.*

Proof. We consider the Inner Product function $IP(x) : \mathbb{F}_2^n \to \{1,-1\}$ defined in Example 14. As shown in example 14, IP is a bent function and therefore $\left|\widehat{IP}(\gamma)\right| = 2^{-n/2}$ for every $\gamma \in \mathbb{F}_2^n$. By Proposition 13 and Lemma 2. IP has the property $INF(d, \epsilon)$ for every $1 \leqslant d \leqslant n$ and $\epsilon > 0$. However, IP has $\mathbb{F}_2$-Degree 2. Since $\|g\|_{U(d+1)} = 1$ if g has $\mathbb{F}_2$-degree d (see[28]), we conclude that $\|IP\|_{U(3)} = 1$. ☐

Lemma 42. *Let $g : \mathbb{F}_2^n \to \{1,-1\}$ be a Boolean function. Let $M \in \mathbb{F}_2^{n\times(n+1)}$ be the projection matrix which sends $x \in \mathbb{F}_2^{n+1}$ to its first n coordinates, and let $w \in \mathbb{F}_2^{n+1}$ be the vector with a single 1 in the $n+1$st coordinate. Let $f : \mathbb{F}_2^{n+1} \to \{1,-1\}$ be defined by $f(x) = g(Mx)$. If g is (ϵ, k)-Uniform, then*

- f *is* (ϵ, k)*-Uniform*
- $\mathrm{I}_w[f] = 0$.

Proof. We first show that f is (ϵ, k)-Uniform. To that end, we have

$$\begin{aligned}
\|f\|_{U(k)} &= \left(\mathbb{E}_{x\in\mathbb{F}_2^{n+1}} \mathbb{E}_{N\in\mathbb{F}_2^{(n+1)\times k}} \prod_{v\in\mathbb{F}_2^k} f(x+Nv)\right)^{2^{-k}} \\
&= \left(\mathbb{E}_{x\in\mathbb{F}_2^{n+1}} \mathbb{E}_{N\in\mathbb{F}_2^{(n+1)\times k}} \prod_{v\in\mathbb{F}_2^k} g(M(x+Nv))\right)^{2^{-k}} \\
&= \left(\mathbb{E}_{x\in\mathbb{F}_2^{n+1}} \mathbb{E}_{N\in\mathbb{F}_2^{(n+1)\times k}} \prod_{v\in\mathbb{F}_2^k} g(Mx+MNv)\right)^{2^{-k}}
\end{aligned}$$

We write $y = Mx$ and $P = MN$. Since M is a projection matrix and x is uniformly distributed on $\mathbb{F}_2^{n+1}$, y is uniformly distributed on $\mathbb{F}^n$. Similarly, P is a uniformly distributed matrix in $\mathbb{F}_2^{n\times k}$. Hence,

$$\begin{aligned}
\|f\|_{U(k)} &= \left(\mathbb{E}_{y\in\mathbb{F}_2^n} \mathbb{E}_{P\in\mathbb{F}_2^{n\times k}} \prod_{v\in\mathbb{F}_2^k} g(y+Pv)\right)^{2^{-k}} \\
&= \|g\|_{U(k)} \\
&\leqslant \epsilon \qquad (18)
\end{aligned}$$

where we use our assumption on g in line (18). Thus f is (ϵ, k)-Uniform.

For the second claim, we observe that $f(x + w) = f(x)$ for every x. Therefore, $\mathrm{I}_w[f] = 0$. □

Now we will prove each of theorems relating our properties to extant theories.

Proof of Theorem 35. We have three claims to prove. First, we consider the relationship between (ϵ, n) Balanced Influences and $(\epsilon, 1)$-Uniformity. Let $f : \mathbb{F}_2^n \to \{1, -1\}$ satisfy $INF(d, \epsilon^4/4)$ where $d \geqslant 1 + \lceil \frac{4\ln(1/\epsilon)}{\ln(2)} \rceil$. By Lemma 32, f is $(\sqrt{2^{-d} + \epsilon^4/2}, n)$-Fourier Regular. Using the bound on d, we find that

$$2^{-d} + \epsilon^4/4 \leqslant \frac{1}{2} \exp\left(-4\ln(1/\epsilon)\right) + \epsilon^4/2 = \epsilon^4 \qquad (19)$$

Thus f is (ϵ^2, n)-Fourier Regular. By Proposition 6.7 in O'Donnell's book [20], $(\sqrt{\epsilon}, n)$-Fourier Regularity implies $(\epsilon, 1)$-Uniformity. Thus, f is $(\epsilon, 1)$-Uniform.

Next we show that (ϵ, k)-Balanced Influences is incomparable with (ϵ, d)-Uniformity for $d > 1$ and any k. Lemma 41 provides a function f which possesses $INF(k, \epsilon)$ for any $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ and any $\epsilon > 0$ yet has $\|f\|_{U(3)} = 1$.

Now we can show that (ϵ, d)-Uniformity cannot imply (ϵ, k)-Balanced Influences for any $k \geqslant 1$. Let $g : \mathbb{F}_2^n \to \{1, -1\}$ be a uniformly random Boolean function. For any $\epsilon > 0$, there is n sufficiently large such that g is (ϵ, k)-Uniform. By Lemma 42 if $f : \mathbb{F}_2^{n+1} \to \{1, -1\}$ is g composed with a projection matrix, f is (ϵ, k)-Uniform yet there is a vector $w \in \mathbb{F}_2^{n+1}$ such that $\mathrm{I}_w[f] = 0$ and $|w| = 1$. Thus, f cannot have the Balanced Influences Property $INF(k, \epsilon)$ for any $k \geqslant 1$ and $\epsilon < \frac{1}{2}$.

It follows that (ϵ, k)-Uniformity and quasi-randomness of rank d with error ϵ are incomparable for $k \geqslant 2$ and $d \geqslant 1$. $\square$

Proof of Theorem 37. Assume that f satisfies $INF(d, \epsilon^2/2)$ where $d \geqslant 1 + \lceil \frac{2\ln(1/\epsilon)}{\ln(2)} \rceil$. Lemma 32 implies that if $f : \mathbb{F}_2^n \to \{1, -1\}$ has $INF(d, \epsilon)$, then f is also $(2\sqrt{2^{-d} + \epsilon^2/2}, n)$-Fourier Regular. By the bound on d,

$$2^{-d} + \epsilon^2/2 \leqslant \frac{1}{2} \exp\big(-2\ln(1/\epsilon)\big) + \epsilon^2/2 = \epsilon^2$$

Thus, f is (ϵ, n)-Fourier Regular. If a function g is (δ, k)-Fourier Regular then g is also $(\delta, k-1)$-Fourier Regular by definition. Hence, if $f : \mathbb{F}_2^n \to \{1, -1\}$ has $(\epsilon^2/2, d)$-Balanced Influences then f is (ϵ, k)-Fourier Regular for any $k \leqslant n$.

For the second claim, we must show that (ϵ, k)-Fourier Regularity cannot imply (ϵ, d)-Balanced Influences for any $k \leqslant n$, $d \geqslant 1$ or $\epsilon < 1$. Consider the inner product function $IP : \mathbb{F}_2^{2n} \to \{1, -1\}$ defined in Example 2. By applying Lemma 42 to IP, we find a function $f : \mathbb{F}_2^{2n+1} \to \{1, -1\}$ which is $(2^{-n/2}, n)$-Fourier Regular and yet does not have $INF(d, \epsilon)$ for any $d \geqslant 1$ and $\epsilon < \frac{1}{2}$. As (ϵ, n)-Fourier Regularity implies (ϵ, k)-Fourier Regularity for $k < n$, IP is (ϵ, k)-Fourier Regular for any $k \leqslant n$. It follows that (ϵ, k)-Fourier Regularity does not imply $INF(d, \epsilon)$ for any choice of $k \leqslant n$, $d \geqslant 1$, and $\epsilon < \frac{1}{2}$. $\square$

Proof of Theorem 40. Assume f satisfies $INF(d, \epsilon^2/8)$ for $d = \lceil \frac{\ln(2/\epsilon)}{\ln(2-\rho)} \rceil$. By Theorem 18, f also satisfies $SD(d, \epsilon^2/4)$ for any $d \geqslant \lceil \frac{\ln(2/\epsilon)}{\ln(2-\rho)} \rceil$.

Recall that $1 > \rho \geqslant 2 - \sqrt{2} \approx 0.58$. We want to show that $\mathrm{I}_i^{1-\rho}[f] < \epsilon$ for each $i \in [n]$ via the Spectral Discrepancy Property. We observe that the set of $\gamma \in \mathbb{F}_2^n$ with $\gamma_i = 1$ is precisely the $n-1$-dimensional subcube $C(\overline{\{i\}}, 1)$, and the same subcube may be divided into 2^{d-1} subcubes of dimension $n - d$ as follows. Pick a set S of size d which contains i.

Then, $C(\overline{\{i\}},1) = \bigsqcup_{\substack{z\in\mathbb{F}_2^S\\ z_i=1}} C(\overline{S},z)$. Therefore,

$$\begin{aligned}
\mathrm{I}_i^{1-\rho}[f] &= \sum_{\substack{\gamma\in\mathbb{F}_2^n\\ \gamma_i=1}} (1-\rho)^{|\gamma|-1}\widehat{f}(\gamma)^2 \\
&= \sum_{\substack{z\in\mathbb{F}_2^S\\ z_i=1}} \left(\sum_{\gamma\in C(\overline{S},z)} (1-\rho)^{|\gamma|-1}\widehat{f}(\gamma)^2 \right) \\
&\leqslant \sum_{\substack{z\in\mathbb{F}_2^S\\ z_i=1}} \left(\max_{\gamma\in C(\overline{S},z)} (1-\rho)^{|\gamma|-1} \right) \left(\sum_{\gamma\in C(S,z)} \widehat{f}(\gamma)^2 \right) \\
&\leqslant \left(2^{-d}+\epsilon^2/4\right) \sum_{\substack{z\in\mathbb{F}_2^S\\ z_i=1}} \left(\max_{\gamma\in C(\overline{S},z)} (1-\rho)^{|\gamma|-1} \right)
\end{aligned}$$

where we use $SD(d,\epsilon^2/4)$ in the ultimate line. Now we can simplify further:

$$\begin{aligned}
\mathrm{I}_i^{1-\rho}[f] &\leqslant \left(2^{-d}+\epsilon^2/4\right) \sum_{\substack{z\in\mathbb{F}_2^S\\ z_i=1}} (1-\rho)^{|z|-1} \\
&= \left(2^{-d}+\epsilon^2/4\right) \left(\sum_{j=0}^{d-1} \binom{d-1}{j} (1-\rho)^j \right) \\
&= \left(2^{-d}+\epsilon^2/4\right) (2-\rho)^{d-1}
\end{aligned}$$

Since $d = \lceil \frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)} \rceil$, we have that $d \leqslant \frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)} + 1$. As $\rho < 1$, we have

$$(2-\rho)^{d-1} \leqslant (2-\rho)^{\frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)}} = \frac{2}{\epsilon}$$

Since $\rho \geqslant 2-\sqrt{2}$, $d = \lceil \frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)} \rceil \geqslant \frac{\ln\left(\frac{2}{\epsilon}\right)}{\ln(2-\rho)} \geqslant \frac{2\ln\left(\frac{2}{\epsilon}\right)}{\ln(2)}$. Therefore,

$$2^{-d} \leqslant 2^{-\frac{2\ln\left(\frac{2}{\epsilon}\right)}{\ln(2)}} = \frac{\epsilon^2}{4}$$

Thus,

$$\mathrm{I}_i^{1-\rho}[f] \leqslant \left(\frac{\epsilon^2}{4}+\frac{\epsilon^2}{4}\right)\frac{2}{\epsilon} = \epsilon$$

as desired.

Conversely, one can easily verify that $\chi_{\mathbf{1}}$ has $((1-\rho)^{n-1},\rho)$-Small Stable Influences, but $\mathrm{I}_\gamma[\chi_{\mathbf{1}}] = 1$ for every $\gamma \in \mathbb{F}_2^n$ with Hamming weight 1. Thus $\chi_{\mathbf{1}}$ does not have $INF(d,\epsilon)$ for any $d \geqslant 1$ unless $\epsilon = \frac{1}{2}$. □

The relationship between ϵ-Cycle Regularity and the other theories is more intricate than our other theories of quasi-randomness, largely due to the algebraic differences between $\mathbb{Z}/2^n\mathbb{Z}$ and $\mathbb{F}_2^n$. As Boolean functions in the sense of ϵ-Cycle Regularity are not functions on $\mathbb{F}_2^n$, we have the following definition to transfer results between these two theories:

Definition 43. Given $z \in \mathbb{Z}/2^n\mathbb{Z}$, let $z^* \in \mathbb{F}_2^n$ denote the binary expansion of z, i.e., the vector such that

$$z_i^* = a_i$$

where $z = \sum_{i=1}^n a_i 2^{i-1}$ is the binary expansion of z.

Chung and Graham [[7], Prop. 6.2] prove the following result.

Lemma 44. *[7] Let $g : \mathbb{Z}/2^n\mathbb{Z} \to \{1, -1\}$ be the function which is -1 if and only its input has an odd number of ones in its binary expansion. There is an absolute constant C such that g is ϵ-Cycle Regular where $\epsilon = C\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^n \approx 0.92^n$.*

In our notation, the function g considered in Lemma 44 can written as the composition of the binary expansion function defined in definition 43 with the Fourier character $\chi_{\mathbf{1}}$. As $\chi_{\mathbf{1}}$ is a Fourier character, $\chi_{\mathbf{1}}$ cannot be (ϵ, n)-Fourier Regular for any $\epsilon < 1$. Thus for any $\delta > 0$, δ-Cycle Regularity does not imply (ϵ, n)-Fourier Regularity for any $\epsilon < 1$. Here we generalize Lemma 44 to show that there is a Fourier character χ_γ where $|\gamma|$ is much smaller than n which is ϵ-Cycle Regular for any $\epsilon > 0$. As a consequence, we will show that for any $\delta > 0$, δ-Cycle Regularity cannot even imply (ϵ, k)-Fourier Regularity for a wide range of $k < n$ and $\epsilon < 1$.

Proof of Theorem 38. Set $k = \lceil C_0 \ln(1/\delta)\rceil$ for some absolute constant C_0 to be defined later. Define $S = \{1, \dots, k\}$. Define $\gamma \in \mathbb{F}_2^n$ by $\gamma := \mathbf{1} \underset{S}{\otimes} 0$ where $\mathbf{1} \in \mathbb{F}_2^S$ is the all-ones vector and $0 \in \mathbb{F}_2^{\overline{S}}$ is the zero vector. We show that χ_γ is δ-Cycle Regular.

Define $\omega_n := \exp\left(\frac{2\pi i}{2^n}\right)$. Now let $c \in \mathbb{Z}/2^n\mathbb{Z} \setminus \{0\}$ be arbitrary, and via the Euclidean algorithm, write $c = 2^{n-k}a + b$ where $0 \leqslant b < 2^{n-k}$. For $z \in \mathbb{Z}_{2^n}$, we write $z^* = y^* \otimes_S x^*$.

We then have $\chi_\gamma(z^*) = \chi_{\mathbf{1}}(y^*)\chi_0(x^*) = \chi_{\mathbf{1}}(y^*)$ by the definition of γ. Then,

$$\begin{aligned}
\mathbb{E}_{z\in\mathbb{Z}/2^n\mathbb{Z}}\,\chi_\gamma(z^*)\omega_n^{-cz} &= \mathbb{E}_{0\leqslant y<2^k}\,\mathbb{E}_{0\leqslant x<2^{n-k}}\,\chi_\gamma(y^* \bigotimes_S x^*)\omega_n^{-(2^{n-k}a+b)(2^kx+y)} \\
&= \mathbb{E}_{0\leqslant y<2^k}\,\mathbb{E}_{0\leqslant x<2^{n-k}}\,\chi_{\mathbf{1}}(y^*)\omega_n^{-2^nax-2^kxb-2^{n-k}ay-by} \\
&= \mathbb{E}_{0\leqslant y<2^k}\,\mathbb{E}_{0\leqslant x<2^{n-k}}\,\chi_{\mathbf{1}}(y^*)\omega_n^{-2^kxb-2^{n-k}ay-by} \\
&= \mathbb{E}_{0\leqslant y<2^k}\,\chi_{\mathbf{1}}(y^*)\omega_n^{-2^{n-k}ay-by}\,\mathbb{E}_{0\leqslant x<2^{n-k}}\,\omega_n^{-2^kxb} \\
&= \mathbb{E}_{0\leqslant y<2^k}\,\chi_{\mathbf{1}}(y^*)\omega_n^{-2^{n-k}ay-by}\,\mathbb{E}_{0\leqslant x<2^{n-k}}\,\omega_{n-k}^{-xb} \\
&= \begin{cases} 0 & b\neq 0 \\ \mathbb{E}_{0\leqslant y<2^k}\,\chi_{\mathbf{1}}(y^*)\omega_n^{-2^{n-k}ay} & b=0 \end{cases} \\
&= \begin{cases} 0 & b\neq 0 \\ \mathbb{E}_{0\leqslant y<2^k}\,\chi_{\mathbf{1}}(y^*)\omega_k^{ay} & b=0 \end{cases}
\end{aligned}$$

Observe that $\chi_{\mathbf{1}}(y^*)$ is precisely the function considered in Lemma 44 on the group $\mathbb{Z}_{2^k}$. Hence, we may apply Lemma 44 to conclude that

$$\begin{aligned}
\left|\mathbb{E}_{z\in\mathbb{Z}/2^n\mathbb{Z}}\,\chi_\gamma(z^*)\omega_n^{-cz}\right| &\leqslant C\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^k \\
&\leqslant C\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^{C_0\ln(1/\delta)} \\
&\leqslant \delta
\end{aligned}$$

where C and C_0 are sufficiently large absolute constants. Thus $z \to \chi_\gamma(z^*)$ is δ-Cycle Regular. However, $|\gamma| = k$, and so χ_γ cannot be $(\epsilon, k+1)$-Fourier Regular for any $\epsilon < 1$. Thus δ-Cycle Regularity does not imply $(\epsilon, k+1)$-Fourier Regularity for any $\epsilon < 1$. □

9 Problems and remarks

One overarching question which remains is that of unifying the various theories of quasi-randomness for Boolean functions. As seen in Theorems 35,37, 38, and 40, the various extant theories of quasi-random Boolean functions seem to express rather disparate properties. A natural question arises: is there is a theory of quasi-random Boolean functions which captures, for instance, both the Balanced Influences Property and (ϵ, k)-Uniformity?

One possible means of addressing this questions is suggested by the relationship between our work and (ϵ, d)-Uniformity as summarized in Figure 7. One can observe that our quasi-random Theorem takes a new direction off $(\epsilon, 1)$-Uniformity in the quasi-random hierarchy as defined in [21]. One possible direction towards finding a more general theory of quasi-random Boolean functions would be to find an analogue of the Balanced Influences Property for the ith level of the same hierarchy. As a Boolean function f is bent if and only if f has $INF(d, \epsilon)$ for every $\epsilon > 0$, any such generalization of Balanced

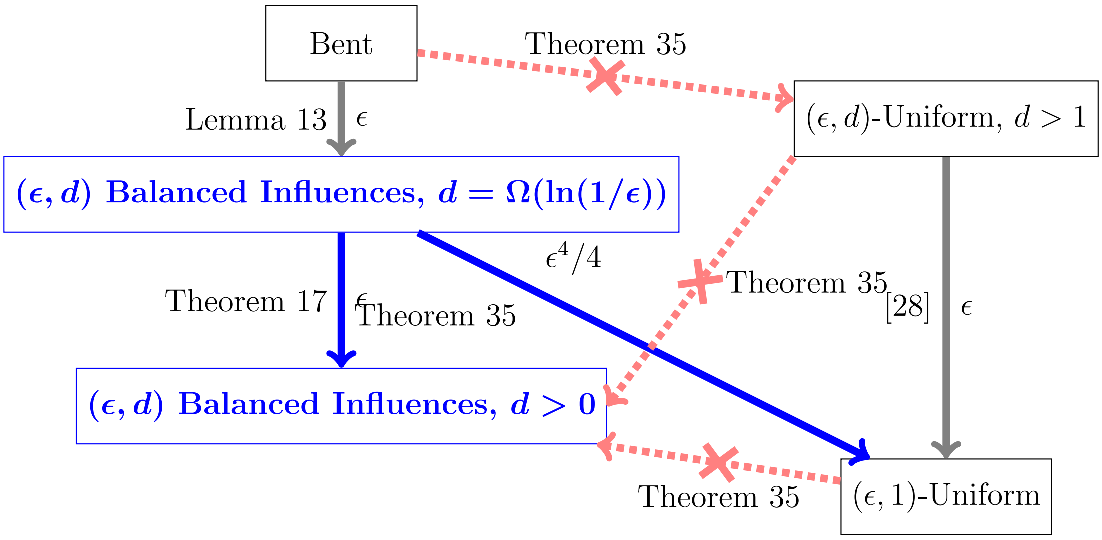

Figure 7: The relationships between the properties in Theorem 16 and (ϵ, d)-Uniformity. Each arrow is a strict implication. Each arrow has a label with a reference to the proof of the implication. The results of this paper are in bold blue text and dashed blue arrows. Incomparable properties are linked by red dotted lines.

Influences may provide a "higher order" analogue of bent functions. Such functions are likely to have very strong cryptographic properties and any construction of them would be interesting in its own right.

Another related direction is the relationship between (ϵ, k)-Fourier Regularity and ϵ-Cycle Regularity, which hinges upon the relation between Fourier analysis of $\mathbb{F}_2^n$ and Fourier analysis over $\mathbb{Z}/2^n\mathbb{Z}$. We have the following question, largely from numerical evidence:

Question 45. For any $\epsilon > 0$, is there a $k = k(\epsilon)$ and a $\delta = \delta(\epsilon) > 0$ such that any function f which is (δ, k)-Fourier Regular is also ϵ-Cycle Regular?

Another high-level question is the following: Is there a "local" theory of quasi-randomness for graphs, hypergraphs, tournaments, or other combinatorial objects which includes a local subgraph count property similar to the Rainbow Embeddings Property?

Many questions remain, some of which we include here. Our proof of Theorem 17 provides a large class of examples of functions which satisfy $INF(d, \delta)$ for any $\delta > 0$ but not $INF(d + 1, \epsilon)$ for $\epsilon < \frac{1}{2}$ and any $d \geqslant 1$. Furthermore, these functions have the property that $f * f(x)$ is the indicator function for some binary linear code.

Question 46. Let $f : \mathbb{F}_2^n \to \{1, -1\}$ be a Boolean function such that

$$f * f(x) = [x \in \mathcal{C}]$$

for some *nonlinear* binary code $\mathcal{C} \subseteq \mathbb{F}_2^n$ of distance d. Do such functions exist, and if so, is it true that f has $INF(d, \delta)$ for any $\delta > 0$ but not $INF(d + 1, \epsilon)$ for $\epsilon < \frac{1}{2}$?

Question 47. Is there a classification of functions $f : \mathbb{F}_2^n \to \{1, -1\}$ which satisfy

$$f * f(x) = [x \in \mathcal{C}]$$

for some $[n, k, d]$ binary linear code $\mathcal{C} \subseteq \mathbb{F}_2^n$?

We remark that any progress on this question will lead towards a solution of the problem of enumerating bent functions.

Our quasi-random properties all assume that a given function f is weakly balanced. In Theorem 15, we showed that for $d \geqslant 3$, the Balanced Influences Property $INF(d, \epsilon)$ implies weak balance for $d \geqslant 3$. For $d = 1$ and $d = 2$, we have a function $f : \mathbb{F}_2^n \to \{1, -1\}$ such dens$(f) = \frac{1}{4}$ i.e., f is not weakly ϵ-balanced, yet

$$\mathrm{I}_\gamma[f] = \frac{1}{2}$$

for every $\gamma \in \mathbb{F}_2^n$ such that $0 < |\gamma| \leqslant 2$. For $d = 1$, we can go much farther and construct an infinite family of functions with the same two properties as follows. Let α, β be two vectors such that $\alpha_i \neq 0$ or $\beta_i \neq 0$ for each $i \in [n]$. Then define $f(x) = \begin{cases} -1 & \alpha^\top x = 0 \text{ and } \beta^\top x = 0 \\ 1 & \text{otherwise} \end{cases}$. One can verify that dens$(f) = \frac{1}{4}$ and $\mathrm{I}_\gamma[f] = \frac{1}{2}$ if $|\gamma| = 1$. However, this construction cannot work for $d = 2$, as for any choice of $\alpha, \beta \in \mathbb{F}_2^n$, there are vectors of Hamming weight 2 which are orthogonal to both α and β. Any such vector will have influence 0. Hence, we ask

Question 48. Is there an infinite family of Boolean functions $f : \mathbb{F}_2^n \to \{1, -1\}$ such that dens$(f) = \frac{1}{4}$ and

$$\mathrm{I}_\gamma[f] = \frac{1}{2}$$

for every $\gamma \in \mathbb{F}_2^n$ such that $0 < |\gamma| \leqslant 2$?

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