

Rotation systems and simple drawings in surfaces

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Abstract

Every simple drawing of a graph in the plane naturally induces a rotation system, but it is easy to exhibit a rotation system that does not arise from a simple drawing in the plane. We extend this to all surfaces: for every fixed surface Σ , there is a rotation system that does not arise from a simple drawing in Σ .

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

All graphs under consideration are simple, that is, they have no loops or parallel edges.

In a *drawing* of a graph, distinct vertices are represented by distinct points in the plane, and each edge $e = uv$ is represented by a Jordan arc whose endpoints are the points that represent u and v . It is also required that (D1) no edge contains a vertex other than its endvertices; (D2) every pair of edges intersect each other a finite number of times; (D3) every intersection of edges is either a common endvertex or a crossing (rather than a tangential intersection); and (D4) no three edges cross at a common point.

A drawing of a graph in an orientable surface determines a *rotation* $\pi(v)$ at each vertex v : this is the cyclic permutation that records the (clockwise) order in which the edges incident with v leave v . The set of the rotations of all the vertices is the *rotation system* of the drawing.

Rotation systems can be considered independently of their relationship with graph drawings. Let S be a set. A *rotation* of an element $s \in S$ is a cyclic permutation $\pi(s)$ of a subset of $S \setminus \{s\}$. A *rotation system* on S is a collection $\Pi = \{\pi(s)\}_{s \in S}$ of rotations of all the elements of S . If for each $s \in S$ we have that $\pi(s)$ is a cyclic permutation of the entire set $S \setminus \{s\}$, then Π is a *complete rotation system*. Thus a drawing of a complete graph in an orientable surface determines a complete rotation system.

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Figure 1: Two drawings with the same rotation system $\Pi = \{\pi(1) = (2, 3, 4), \pi(2) = (1, 3, 4), \pi(3) = (1, 4, 2), \pi(4) = (1, 2, 3)\}$. Note that the drawing on the left-hand side is not simple, since the adjacent edges 12 and 13 cross each other. It is straightforward to verify that this rotation system cannot be realized by a simple drawing in the plane. As illustrated on the right-hand side, Π can be realized by a simple drawing in the torus.

It is easy to see that every rotation system is the rotation system of a drawing of a graph in the plane; that is, every rotation system can be *realized* by a drawing in the plane. However, this does not hold if we slightly restrict the class of drawings under consideration.

One is often interested in drawing a graph so that the total number of edge crossings is as small as possible. It is easy to see that in this context we may focus on *simple* drawings, that is, drawings in which (D5) any two edges are either disjoint or have exactly one point in common (which, in view of (D3), is either a common endvertex or a crossing). For instance, the drawing of K_4 in the plane on the left-hand side in Figure 1 is not simple. It is worth noting that simplicity is a natural restriction beyond the context of crossing-minimal drawings, and it appears as a basic assumption in a variety of settings [2,4,6,8,11].

Simple drawings thus play a paramount role in topological graph theory, and so it is natural to ask: which rotation systems can be realized by a simple drawing in the plane? Equivalently: which rotation systems are *simply realizable* in the plane?

As far as we know, this question was first asked by Dan Archdeacon [3], for complete rotation systems. Archdeacon’s question was settled by Kynčl in [8] (see also [1] and [9]): the decision problem of whether a given complete rotation system is simply realizable in the plane is in **P**.

For arbitrary rotation systems little seems to be known, and serious difficulties seem to arise even if one imposes additional restrictions on the drawings under consideration [6]. One of the few positive results known is that $K_{2,n}$ has a simple realization for any rotation system (see [6, Figure 9] or [11, Lemma 4.8]). A good illustration of the intricacies of the notion of simplicity is a recent result by Arroyo et al. [5], who proved that testing whether a single edge can be added to a simple drawing while keeping simplicity is NP-hard. It is also worth mentioning that recently Schaefer proved that straight-line realizability (a stronger notion than simple realizability) of a graph with a given rotation system and bounded degree is $\exists\mathbb{R}$ -complete [12].

As we discuss in more detail in Section 8, Archdeacon also considered the simple realizability of rotation systems on compact orientable surfaces. For brevity, throughout this paper we shall refer to a compact orientable surface simply as a *surface*.

When we began to investigate the simple realizability of rotation systems in surfaces,

we quickly realized that we needed to start with a question that we had implicitly misjudged as a routine exercise: given a surface Σ , does there exist a rotation system that cannot be simply realized in Σ ? To our surprise, answering this question was far from trivial.

Theorem 1. *For each surface Σ , there is a rotation system that is not simply realizable in Σ .*

With the exception of the final discussion in Section 8, the rest of the paper is devoted to the proof of Theorem 1.

2 Reducing Theorem 1 to three lemmas

To prove Theorem 1 we define, for each positive integer n , a rotation system Π_n on the set $[n] \cup \{b, r\}$ and prove the following: if Σ is a fixed surface, and n is large enough, then Π_n cannot be realized in Σ .

As illustrated in Figure 2, the description of Π_n is quite simple:

$$\pi(b) = (n, \dots, 2, 1), \pi(r) = (1, 2, \dots, n), \text{ and } \pi(i) = (b, 1, 2, \dots, i-1, i+1, \dots, n, r) \text{ for } i \in [n],$$

with obvious adjustments for $i = 1$ and $i = n$. In a simple drawing that realizes Π_n , the stars with centers b, r , and each $i \in [n]$ are drawn as illustrated in Figure 2.

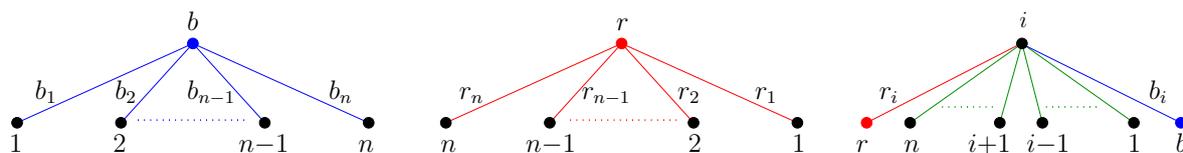


Figure 2: Illustration of the rotation system Π_n .

To help comprehension, in a drawing that realizes Π_n we say that b is the *blue* vertex, and the edge b_i that joins b to i is a *blue* edge, for $i = 1, \dots, n$. Similarly, vertex r is *red*, and the edge r_i that joins r to i is a *red* edge, for $i = 1, \dots, n$. Finally, for each distinct $i, j \in [n]$, the edge joining vertices i and j is a *green* edge. For each distinct $i, j \in [n]$, the edges b_i, b_j, r_j, r_i form a 4-cycle, the *canonical cycle* $C_{i,j}$.

Note that a drawing that realizes Π_n is a drawing of a graph L_n that is isomorphic to K_{n+2} minus one edge, since the only two elements in $[n] \cup \{b, r\}$ that are not adjacent to each other under Π_n are b and r .

It is easy to see that in a simple drawing that realizes Π_n , each canonical cycle is either crossing-free, or crosses itself exactly once, or crosses itself exactly twice. For $\ell = 0, 1, 2$, we say that an ℓ -drawing of L_n is a simple drawing that realizes Π_n in some surface, in such a way that every canonical cycle has exactly ℓ self-crossings. In Figure 3 we illustrate a 1-drawing of L_3 in the double torus. We use the standard polygonal representation of the double torus as an octagon whose sides are identified in pairs, according to the indicated labels and orientations.

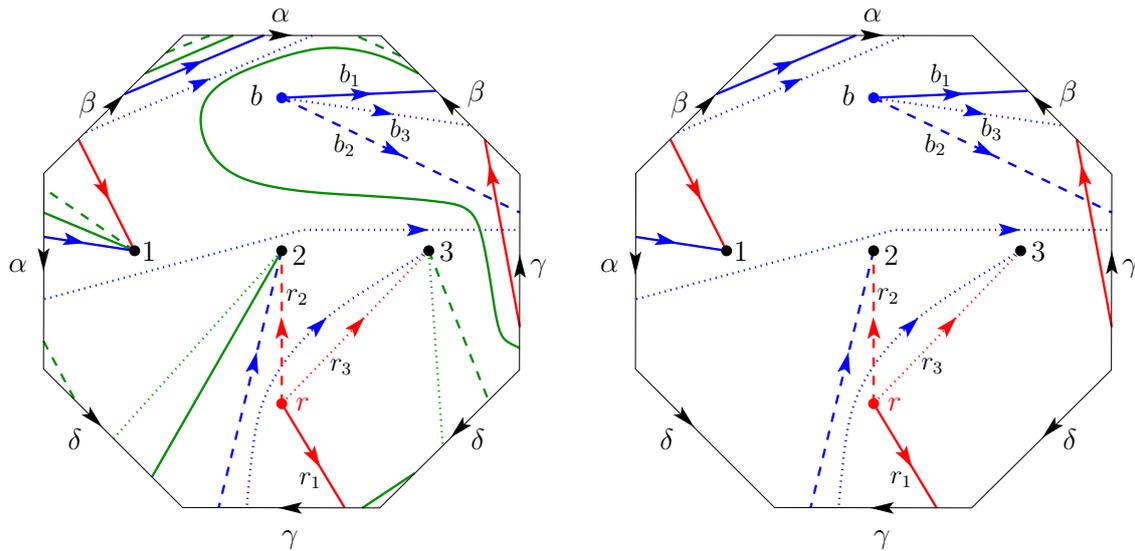


Figure 3: On the left-hand side we illustrate a simple drawing of L_3 that realizes Π_3 in the double torus, with three crossings. One crossing involves b_2 (dashed) and r_1 (solid), another crossing involves b_3 (dotted) and r_1 , and the other crossing involves b_3 and r_2 (dashed). Thus each canonical cycle has exactly one self-crossing, and so this is a 1-drawing of L_3 in the double torus. If we omit the green edges, we obtain the 1-drawing of $K_{2,3}$ in the double torus shown on the right-hand side.

If we discard the green edges from a drawing D_n that realizes Π_n , the result is a drawing E_n of $K_{2,n}$ in which the rotations of the vertices are $\pi(b) = (n, \dots, 2, 1)$, $\pi(r) = (1, 2, \dots, n)$, and $\pi(i) = (b, r)$, for $i \in [n]$. Note that the notion of a canonical cycle carries over naturally to E_n . If D_n is an ℓ -drawing for some $\ell \in \{0, 1, 2\}$, then clearly each canonical cycle also has ℓ self-crossings in E_n , and we also say that E_n is an ℓ -drawing of $K_{2,n}$. As it happens, most of the proof of Theorem 1 involves 1-drawings and 2-drawings of $K_{2,n}$.

The following proposition is crucial, as it will allow us to focus for the rest of the paper on drawings of L_n that are ℓ -drawings for some $\ell \in \{0, 1, 2\}$.

Proposition 2. *Let Σ be a surface. Suppose that for each positive integer m , there is a simple drawing of L_m that realizes Π_m in Σ . Then for each positive integer n , there is an ℓ -drawing of L_n in Σ , for some $\ell \in \{0, 1, 2\}$.*

Proof. Let n be a positive integer, and let m be the Ramsey number $R_2(n, n, n)$: if we colour each edge of the complete graph K_m with one of three available colours, then there is a complete subgraph of size n all of whose edges are of the same colour.

By hypothesis, there is a simple drawing D_m that realizes Π_m in Σ . We construct an auxiliary complete graph G with vertex set $[m]$, and colour an edge ij of G with colour $\ell \in \{0, 1, 2\}$ if the canonical cycle $C_{i,j}$ has exactly ℓ self-crossings in D_m . Since each canonical cycle has either 0, 1, or 2 self-crossings, this edge colouring of G is well-defined.

By Ramsey's theorem there is a complete subgraph H of G of size n , all of whose edges are of the same colour $\ell \in \{0, 1, 2\}$. Let i_1, i_2, \dots, i_n be the vertices of H , labelled so that $i_1 < i_2 < \dots < i_n$. Delete from D_m all the vertices in $[m] \setminus \{i_1, i_2, \dots, i_n\}$ and their incident edges, and for each $j = 1, \dots, n$ relabel the vertex i_j with j . Clearly, as a result we obtain an ℓ -drawing of L_n in Σ . \square

In view of Proposition 2, to prove Theorem 1 it suffices to show that, for each $\ell \in \{0, 1, 2\}$, no fixed surface can host ℓ -drawings of L_n for arbitrarily large values of n . This is achieved with the next three lemmas, which take care separately of the three possible values of ℓ . Actually, in Lemmas 4 and 5 we establish results that are strictly stronger than we need to prove Theorem 1, as they involve 1- and 2-drawings of $K_{2,n}$, a very sparse subgraph of L_n .

Lemma 3. *For each surface Σ , there is an n such that there is no 0-drawing of L_n in Σ .*

Lemma 4. *For each surface Σ , there is an n such that there is no 1-drawing of $K_{2,n}$ in Σ .*

Lemma 5. *For each surface Σ , there is an n such that there is no 2-drawing of $K_{2,n}$ in Σ .*

The rest of the paper is devoted to the proofs of these lemmas. Assuming the lemmas, Theorem 1 immediately follows. We include the proof for completeness.

Proof of Theorem 1, assuming Lemmas 3, 4, and 5. Let Σ be a surface. We note that Lemma 4 (respectively, Lemma 5) implies that there is an n such that there is no 1-drawing (respectively, 2-drawing) of L_n in Σ . Combining this with Lemma 3, we conclude that there is an N such that there is no ℓ -drawing of L_N in Σ , for any $\ell \in \{0, 1, 2\}$.

The contrapositive of Proposition 2 then implies that there is an integer m such that there is no simple drawing that realizes Π_m in Σ . In other words, Π_m is not simply realizable in Σ . \square

3 Proof of Lemma 3

To prove Lemma 3, we establish the next two propositions, where we use the following terminology. A *main cycle* of L_n is a canonical cycle of the form $C_{i,i+1}$, for some $i = 1, \dots, n$ (indices are read modulo n). In a 0-drawing of L_n in some surface, a main cycle is *good* if it bounds a region homeomorphic to a disk. Otherwise, it is *bad*. We note that the notions of a main cycle, of a good cycle, and of a bad cycle, carry over naturally to 0-drawings of $K_{2,n}$.

Proposition 6. *Let $n \geq 3$ be an integer. If D_n is a 0-drawing of L_n in a surface, then every main cycle is bad in D_n .*

In the next statement, and in the rest of the paper, we use $\text{gen}(\Sigma)$ to denote the genus of a surface Σ .

Proposition 7. *Let $n \geq 3$ be an integer. If E_n is a 0-drawing of $K_{2,n}$ in a surface Σ , then at least $n - 2 \operatorname{gen}(\Sigma)$ main cycles are good in E_n .*

Deferring the proofs of these statements for a moment, we note that Lemma 3 follows easily from them.

Proof of Lemma 3. If Σ has genus 0 we let $n = 3$. Otherwise, let $n = 2 \operatorname{gen}(\Sigma) + 1$. By way of contradiction, suppose there is a 0-drawing D_n of L_n in Σ . Note that D_n contains a 0-drawing E_n of $K_{2,n}$ in Σ . Thus by Proposition 7 it follows that there is at least one good main cycle in E_n , and so there is at least one good main cycle in D_n . But this contradicts Proposition 6. \square

We conclude the section with the proofs of Propositions 6 and 7.

Proof of Proposition 6. Let D_n be a 0-drawing of L_n in a surface. By way of contradiction, suppose that the main cycle $C_{i,i+1}$ is good in D_n , for some $i \in [n]$. Without loss of generality we may then assume that as we traverse in D_n the cycle $C_{i,i+1}$ visiting the vertices $b, i, r, i + 1$ in this order, we find that on the left-hand side of this traversal we bound a disk Δ . See Figure 4.

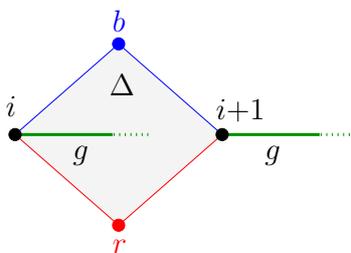


Figure 4: Illustration of the proof of Proposition 6.

The rotation at i in Π_n contains the cyclic subpermutation $(b, i + 1, r)$, and this means that at least part of the green edge g that joins i and $i + 1$ must lie inside Δ , as illustrated in Figure 4. On the other hand, the rotation at $i + 1$ in Π_n contains the cyclic subpermutation (b, i, r) , and so part of g must lie outside Δ , as we also illustrate in Figure 4. By the Jordan curve theorem, it follows that the edge g must cross the cycle $C_{i,i+1}$. But this contradicts that D_n is a simple drawing, since g joins i and $i + 1$ and each edge of $C_{i,i+1}$ is incident with either i or $i + 1$. \square

Proof of Proposition 7. We prove the proposition by induction on the genus of Σ . In the base case $\operatorname{gen}(\Sigma) = 0$, and so Σ is the sphere. In this case the proposition claims that all n main cycles in D_n are good. This is trivially true, since the host surface is the sphere.

The inductive hypothesis is that the proposition holds for surfaces of genus $k - 1$, for some integer $k \geq 1$. For the inductive step, let E_n be a 0-drawing of $K_{2,n}$ in a surface Σ of genus $\operatorname{gen}(\Sigma) = k$. Now E_n has n main cycles, and so we assume that $n > 2 \operatorname{gen}(\Sigma)$, as otherwise there is nothing to prove. Note that since E_n is a 0-drawing of $K_{2,n}$, we have in particular that no edge of $K_{2,n}$ is crossed in E_n .

If all main cycles in E_n are good we are done, and so we suppose that there is a main cycle $C_{i,i+1}$ in E_n that is bad. Thus $C_{i,i+1}$ is a noncontractible Jordan curve in Σ . As illustrated in Figure 5, we take two simple closed curves γ_1, γ_2 homotopic to $C_{i,i+1}$, drawn very close to $C_{i,i+1}$ so that E_n does not intersect the closed annulus bounded by γ_1 and γ_2 . Note that such a closed annulus (disjoint from E_n) must exist, as otherwise some edge of $C_{i,i+1}$ would be crossed in E_n .

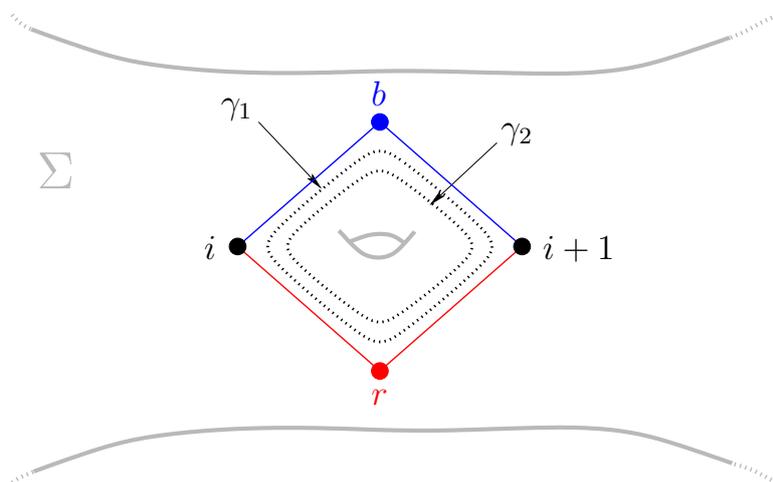


Figure 5: Illustration of the proof of Proposition 7.

We now remove the open annulus bounded by γ_1 and γ_2 . This turns Σ into a compact surface with two boundary components, namely γ_1 and γ_2 . We glue the boundary of a disk Δ_1 to γ_1 , and we glue the boundary of a disk Δ_2 to γ_2 . As a result, we obtain a (compact, without boundary) surface Σ' of genus $\text{gen}(\Sigma) - 1 = k - 1$.

The drawing E_n naturally induces a drawing E'_n in Σ' with the same properties as E_n . That is, E'_n is a 0-drawing of $K_{2,n}$ in Σ' . Thus by the induction hypothesis E'_n has at least $n - 2 \text{gen}(\Sigma') = n - 2(\text{gen}(\Sigma) - 1) = n - 2 \text{gen}(\Sigma) + 2$ main cycles that are good. To finish the proof, we show that at most two of these main cycles are bad in E_n .

Note that the main cycle $C_{i,i+1}$ is bad in E_n , but it became good in E'_n , as it now bounds a disk that contains Δ_1 . The key point is that there is at most one other main cycle that is bad in E_n but became good in E'_n : this would be a main cycle that in Σ was homotopic to $C_{i,i+1}$ (and hence it was bad in E_n), but became good in E'_n because in Σ' it bounds a disk that contains Δ_2 .

From this discussion, we conclude that the number of main cycles that are good in E_n is at least the number of main cycles that are good in E'_n minus two. Since this last number is at least $n - 2 \text{gen}(\Sigma) + 2$, it follows that the number of main cycles that are good in E_n is at least $n - 2 \text{gen}(\Sigma)$. \square

4 Narrowing down the class of 1-drawings we need to consider

At a high level, the strategy of the proof of Lemma 4 has two main steps. The first step, which is the goal of this section, is the following. Let Σ be a surface. Seeking a contradiction, we show that if for every n there is a 1-drawing of $K_{2,n}$ in Σ , then there is one such drawing that satisfies certain highly restrictive properties (to be specified shortly). The second step, achieved in the next section, will consist of showing that no fixed surface can host arbitrarily large 1-drawings that satisfy these properties, thus providing the desired contradiction.

Throughout this section and Section 5, whenever we refer to a 1-drawing we mean a 1-drawing of $K_{2,n}$ in a surface Σ , for some positive integer n . We use several times expressions such as “if there are arbitrarily large 1-drawings in Σ ”. This is to be interpreted as “if there exist 1-drawings of $K_{2,n}$ in Σ for arbitrarily large values of n ”.

Let D_n be a 1-drawing of $K_{2,n}$. Since D_n is simple, each crossing in D_n involves a blue edge b_i and a red edge r_j , for some distinct $i, j \in [n]$. We use $\times_{i,j}$ to denote this crossing. We assign a sign to $\times_{i,j}$ as follows. Orient each blue edge b_i from b to i , and orient each red edge r_j from r to j . As illustrated in Figure 6, a crossing is *positive* if the red edge crosses the blue edge from the left-hand side of the blue edge. Otherwise, the crossing is *negative*. If all the crossings in D_n are positive (respectively, negative), then we say that D_n itself is *positive* (respectively, *negative*). On the left-hand side of Figure 7 we illustrate a negative 1-drawing of $K_{2,3}$, and on the right-hand side we show a positive 1-drawing of $K_{2,3}$.



Figure 6: Every crossing in a simple drawing of $K_{2,n}$ is positive or negative, according to this convention.

We start to narrow down the 1-drawings of interest with the following statement.

Proposition 8. *Let Σ be a surface. If there exist arbitrarily large 1-drawings in Σ , then there exist arbitrarily large 1-drawings in Σ that are positive.*

Proof. Let n be any positive integer. Suppose that there exist arbitrarily large 1-drawings in Σ . Similarly as in the proof of Proposition 2, an application of Ramsey’s theorem yields that there exists a 1-drawing D_n of $K_{2,n}$ in Σ that is either positive or negative.

If D_n is positive, then we are done. If it is negative, we proceed as illustrated in Figure 7: we relabel the vertices $b \mapsto r, r \mapsto b$, and $i \mapsto n - i + 1$ for $i = 1, \dots, n$, and as a result we obtain a 1-drawing of $K_{2,n}$ that is positive. \square

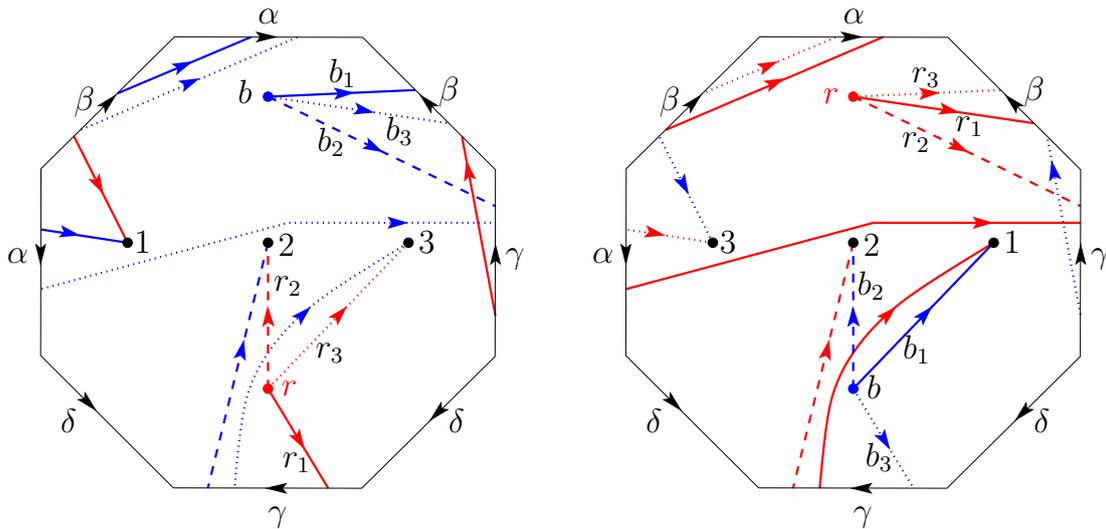


Figure 7: On the left-hand side we illustrate a 1-drawing of $K_{2,3}$ in the double torus. Each crossing in this drawing is negative, and so this 1-drawing is negative. The three crossings $\times_{2,1}, \times_{3,1}, \times_{3,2}$ in this drawing are backward crossings, and so this is a backward 1-drawing. If we perform the relabellings $b \mapsto r, r \mapsto b, 1 \mapsto 3$, and $3 \mapsto 1$, we obtain the 1-drawing on the right-hand side, which is also backward, but positive.

In order to continue the process of narrowing down the 1-drawings of interest, we introduce the notions of forward and backward crossings. Let $\times_{i,j}$ be a crossing in a 1-drawing D_n of $K_{2,n}$ in some surface. Recall that this is a crossing between the blue edge b_i and the red edge r_j , where $i \neq j$. If $i < j$, then $\times_{i,j}$ is a *forward* crossing, and if $i > j$ then it is a *backward* crossing. If all the crossings are forward (respectively, backward) then D_n itself is a *forward* (respectively, *backward*) drawing. For instance, both 1-drawings of $K_{2,3}$ shown in Figure 7 are backward drawings.

Proposition 9. *Let Σ be a surface. If there exist arbitrarily large 1-drawings in Σ that are positive, then there exist arbitrarily large 1-drawings in Σ that are positive and forward.*

Proof. Let n be a positive integer. Suppose that there exist arbitrarily large positive 1-drawings in Σ . Similarly as in the proof of Proposition 2, an application of Ramsey’s theorem yields that there exists a 1-drawing D_n of $K_{2,n}$ in Σ that is positive and either forward or backward.

If D_n is forward then we are done. If it is backward, we transform it as illustrated in Figure 8. First, we perform an orientation-reversing self-homeomorphism on Σ , and let D'_n denote the drawing that is the image of D_n under this mapping. Finally, we modify D'_n by exchanging the labels of b and r . The result is a 1-drawing of $K_{2,n}$ that is positive and forward. \square

To complete this process of narrowing down the 1-drawings that we need to consider, we introduce one final notion. Let D_n be a 1-drawing of $K_{2,n}$. As we traverse a blue edge

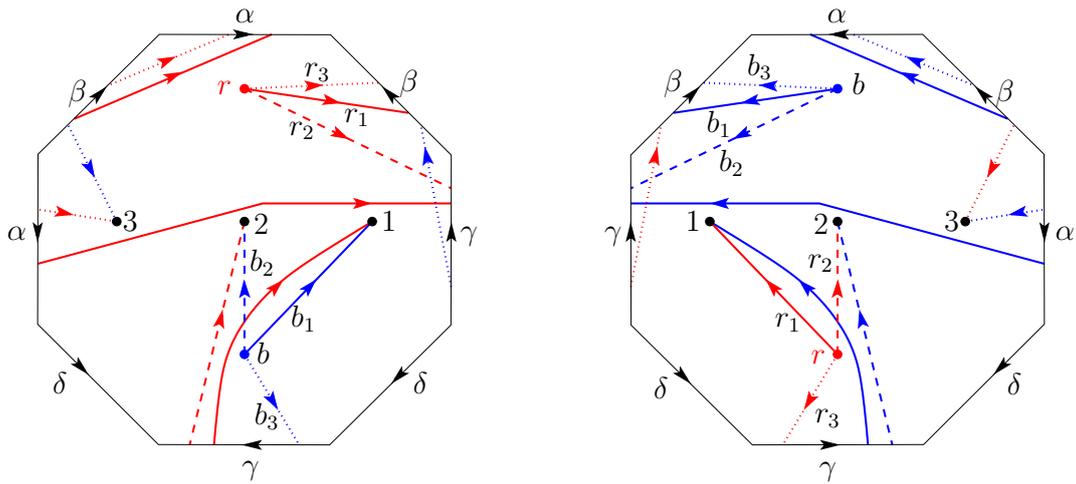


Figure 8: On the left-hand side we have a 1-drawing of $K_{2,3}$, in the double torus, that is positive and backward. If we perform a reflection on the polygon (that is, an orientation-reversing self-homeomorphism of the double torus) and exchange the labels of b and r , we obtain the drawing on the right-hand side. This is a 1-drawing of $K_{2,3}$ that is positive and forward.

b_i following its direction (we recall, from b to i), we encounter its crossings with red edges in some order $\times_{i,j_1}, \times_{i,j_2}, \dots, \times_{i,j_s}$. We say that j_1, j_2, \dots, j_s is the *crossing sequence* of b_i .

If the crossing sequence of b_i is increasing (respectively, decreasing), then we say that the edge b_i is *increasing* (respectively, *decreasing*). If every blue edge is increasing (respectively, decreasing), then D_n is *blue increasing* (respectively, *blue decreasing*).

Similarly, as we traverse a red edge r_j following its direction (from r to j) we encounter its crossings with blue edges in some order $\times_{i_1,j}, \times_{i_2,j}, \dots, \times_{i_t,j}$. Then i_1, i_2, \dots, i_t is the *crossing sequence* of r_j . If this sequence is increasing (respectively, decreasing), then r_j is *increasing* (respectively, *decreasing*). If every red edge is increasing (respectively, decreasing), then D_n is *red increasing* (respectively, *red decreasing*).

If D_n is blue increasing or blue decreasing, then it is *blue monotone*. If D_n is red increasing or red decreasing, then it is *red monotone*. Finally, if D_n is both blue monotone and red monotone, then it is *monotone*.

Proposition 10. *Let Σ be a surface. If there exist arbitrarily large 1-drawings in Σ that are positive and forward, then there exist arbitrarily large 1-drawings in Σ that are positive, forward, and monotone.*

Proof. We prove that if (*) there exist arbitrarily large 1-drawings in Σ that are positive and forward, then (**) there exist arbitrarily large 1-drawings in Σ that are positive, forward, and blue monotone. A totally analogous sequence of arguments shows that (**) implies that there exist arbitrarily large 1-drawings in Σ that are positive, forward, blue monotone, and red monotone (that is, positive, forward, and monotone, as claimed in the proposition).

Let n be any positive integer. Assuming (*), our aim is to show that there is a 1-drawing of $K_{2,n}$ in Σ that is positive, forward, and blue monotone.

Let m be the Ramsey number $R_3(n, n)$: if all the 3-edges in a complete 3-uniform hypergraph of size m are coloured with one of two available colours, then there is a complete subhypergraph of size n all of whose 3-edges are of the same colour.

By assumption, there is a 1-drawing D_m of $K_{2,m}$ in Σ that is positive and forward. Construct an auxiliary complete 3-uniform hypergraph G , whose vertex set is $[m]$. Let $e = \{p, q, s\}$ be a 3-edge of G , where $1 \leq p < q < s \leq m$. Note that since D_m is forward, the blue edge b_p is crossed by the red edges r_q and r_s . We colour e black if as we traverse the blue edge b_p in D_m from b to p , we encounter its crossing with r_q before its crossing with r_s , and we colour e yellow otherwise.

Clearly, every 3-edge of G is either black or yellow, and so by Ramsey's theorem there exist integers $1 \leq i_1 < i_2 < \dots < i_n \leq m$ such that in the complete subhypergraph H of G on $\{i_1, i_2, \dots, i_n\}$, all the 3-edges are of the same colour. Remove from D_m all the vertices in $[m] \setminus \{i_1, i_2, \dots, i_n\}$ and their incident edges, and relabel the remaining n vertices with the rule $i_j \mapsto j$ for $j = 1, \dots, n$. Let D_n be the resulting 1-drawing of $K_{2,n}$.

If all the 3-edges of H are black, then D_n is blue increasing, and if all the 3-edges of H are yellow then D_n is blue decreasing. That is, D_n is blue monotone. We finish the proof by noting that since D_m is positive and forward, then clearly D_n is also positive and forward. \square

Combining Propositions 8, 9, and 10, we obtain the following statement. This proposition will allow us, in the proof of Lemma 4, to focus our attention on 1-drawings that are positive, forward, and monotone.

Proposition 11. *Let Σ be a surface. If there exist arbitrarily large 1-drawings in Σ , then there exist arbitrarily large 1-drawings in Σ that are positive, forward, and monotone.*

5 Proof of Lemma 4

Lemma 4 claims that no fixed surface can host 1-drawings of $K_{2,m}$ for arbitrarily large values of m . In view of Proposition 11, in order to prove the lemma it suffices to show that *no fixed surface can host 1-drawings of $K_{2,n}$ that are positive, forward, and monotone, for arbitrarily large values of n .*

As we shall see shortly, Lemma 4 in this form follows easily from the next two claims.

Claim 12. *Let D_n be a 1-drawing of $K_{2,n}$ in a surface Σ , where n is even. Suppose that D_n is positive and forward. If D_n is blue increasing or red increasing, then $\text{gen}(\Sigma) \geq (n-2)/2$.*

Claim 13. *Let D_n be a 1-drawing of $K_{2,n}$ in a surface Σ . Suppose that D_n is positive and forward. If D_n is blue decreasing and red decreasing, then $\text{gen}(\Sigma) \geq n - 2$.*

Proof of Lemma 4, assuming Claims 12 and 13. Let Σ be a surface. By way of contradiction, suppose that for every m there is a 1-drawing of $K_{2,m}$ in Σ . By Proposition 11,

it follows that (\dagger) for every n there is a 1-drawing of $K_{2,n}$ in Σ that is forward, positive, and monotone.

Let $n > 2 + 2 \operatorname{gen}(\Sigma)$ be an even integer. By (\dagger) , there exists a 1-drawing D_n of $K_{2,n}$ in Σ that is forward, positive, and monotone. Thus D_n is forward, positive, and either (i) blue increasing or red increasing; or (ii) both blue and red decreasing. If (i) holds, we have a contradiction to Claim 12, and if (ii) holds, we have a contradiction to Claim 13. \square

To prove the claims, we make use of the common device of regarding a drawing as an embedding, by turning each crossing into a degree 4 vertex, a *crossing vertex* (coloured white in the figures). Under this perspective, we can use the topological graph theory machinery to investigate drawings.

Note that under this perspective, what used to be a blue edge b_i in a 1-drawing D_n of $K_{2,n}$ becomes a blue path, as the crossings of b_i are now regarded as vertices, which are the internal vertices of the path b_i . Similarly, what used to be a red edge r_i becomes a red path.

The proofs of Lemmas 4 and 5 rely heavily on facial walks, and so it seems worth to review this crucial notion from topological graph theory. Informally speaking, a facial walk is simply a walk that bounds a face in an embedding. To define this concept formally, we recall that all graphs under consideration have neither loops nor parallel edges, and so in this context a closed walk of a graph is simply a sequence $v_0 v_1 \cdots v_{k-1} v_0$ of vertices such that v_i is adjacent to v_{i+1} for $i = 0, \dots, k-1$ (indices are read modulo k).

Let G be a graph embedded on a surface Σ , and let $\Pi = \{\pi(v)\}_{v \in V}$ be the rotation system of the embedding. A *facial walk* of this embedding is a closed walk $v_0 v_1 v_2 \cdots v_{k-1} v_0$ where v_{i-1} and v_{i+1} appear consecutively in this order in $\pi(v_i)$, for $i = 0, \dots, k-1$. For an exhaustive discussion on facial walks we refer the reader to [10].

We make extensive use of the following fact (see for instance [10, Section 4.1]). If a graph with ν vertices and ε edges is embedded in Σ , and the embedding has ϕ facial walks, then

$$\operatorname{gen}(\Sigma) \geq \frac{1}{2} \left(2 - \nu + \varepsilon - \phi \right). \quad (1)$$

Proof of Claim 12. Suppose first that D_n is blue increasing, and recall that by assumption D_n is positive and forward. Let J_n be the restriction of D_n to the blue path b_1 and the red paths r_2, \dots, r_n (we suppress the crossing vertices of these red paths with other blue edges). As illustrated in Figure 9, the vertices of J_n are $r, b, 1, \dots, n$, and $\times_{1,2}, \dots, \times_{1,n}$, which are the crossings of b_1 with the red paths r_2, \dots, r_n , respectively.

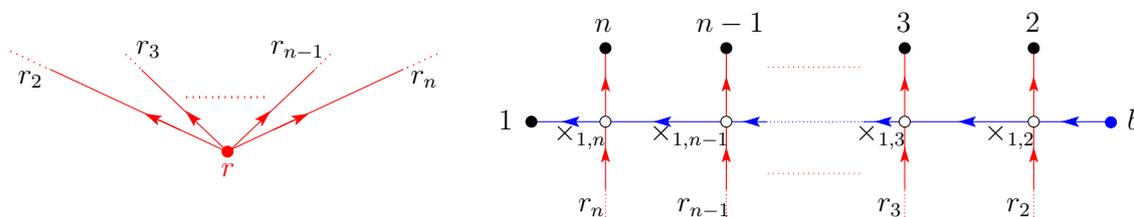


Figure 9: Illustration of the proof of Claim 12.

Thus J_n has $2n + 1$ vertices and $3n - 2$ edges, and it is straightforward to verify that it has only one facial walk. (Here is where we use that n is even; if n were odd, there would be two facial walks). Since Σ is the host surface of J_n , it follows from (1) that $\text{gen}(\Sigma) \geq (n - 2)/2$.

If D_n is red increasing, we consider instead the restriction J_n of D_n to the red path r_n and the blue paths b_1, \dots, b_{n-1} . As in the previous case, it is straightforward to verify that J_n has $2n + 1$ vertices, $3n - 2$ edges, and one facial walk, thus implying that $\text{gen}(\Sigma) \geq (n - 2)/2$. \square

Proof of Claim 13. As in the proof of Claim 12, the strategy is to find a restriction J_n of D_n that has exactly one face, and to apply (1) to obtain the required bound on $\text{gen}(\Sigma)$. As we shall see, the restriction J_n is as illustrated in Figure 10.

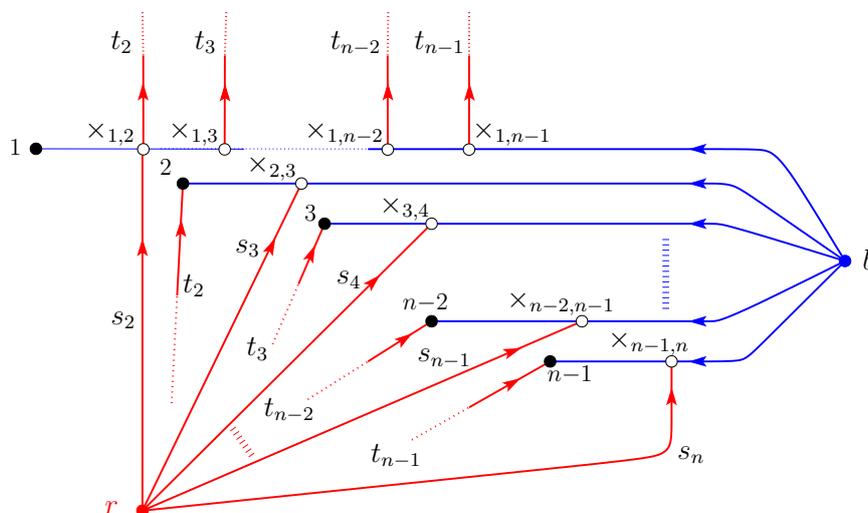


Figure 10: Illustration of the proof of Claim 13.

Note that since D_n is positive, forward, and red decreasing, for $i = 2, \dots, n$ the red path r_i crosses the blue paths $b_{i-1}, b_{i-2}, \dots, b_1$ in this order, from their left-hand sides. In particular, the first crossing of r_i as we traverse it from r to i is $\times_{i-1,i}$, and the last crossing is $\times_{1,i}$.

We also note that since D_n is blue decreasing, as we traverse b_1 from b to 1 we encounter the crossings $\times_{1,n-1}, \times_{1,n-2}, \dots, \times_{1,3}, \times_{1,2}$ in this order. See Figure 10.

In the restriction J_n we include the blue paths b_1, \dots, b_{n-1} , and for $i = 2, \dots, n - 1$, we include two portions of the red path r_i : the part s_i from r to its first crossing $\times_{i-1,i}$, and the part t_i from its last crossing $\times_{1,i}$ to vertex i . Finally, include from r_n its part s_n from r to its first crossing $\times_{n-1,n}$. See Figure 10. Thus J_n has $3n - 3$ vertices and $5n - 8$ edges, and it is straightforward to verify that it has only one facial walk. Thus it follows from (1) that $\text{gen}(\Sigma) \geq n - 2$. \square

6 Narrowing down the class of 2-drawings we need to consider

As we did for the proof of Lemma 4, we pave the way towards the proof of Lemma 5 by showing that it suffices to prove the lemma for a highly restricted class of 2-drawings of $K_{2,n}$.

Throughout this section and Section 7, whenever we refer to a 2-drawing we mean a 2-drawing of $K_{2,n}$ in a surface Σ , for some positive integer n . We use several times expressions such as “if there are arbitrarily large 2-drawings in Σ ”. This is to be interpreted as “if there exist 2-drawings of $K_{2,n}$ in Σ for arbitrarily large values of n ”.

We have an additional important remark. In the previous section, for the proof of Lemma 4, it was convenient to turn crossings into degree 4 vertices: by regarding drawings as embeddings, we were able to use (1). This device will also be used in the next section for the proof of Lemma 5. However, in the current section (similarly to Section 4) we are interested in properties of 2-drawings of $K_{2,n}$, and so we do *not* regard crossings as degree 4 vertices. Thus for $i = 1, \dots, n$, throughout this section b_i and r_i maintain their identities as edges.

We start by noting that in a 2-drawing D_n of $K_{2,n}$, for each $i \in [n]$ the red edge r_i crosses all the blue edges except for b_i . Thus the crossing sequence of r_i is a permutation of $[n] \setminus \{i\}$. We say that r_i is *ahead first* if in this sequence $i+1, i+2, \dots, n$ appear first in some order, followed by $1, 2, \dots, i-1$ in some order. On the other hand, if in this sequence $1, \dots, i-1$ appear first in some order, followed by $i+1, i+2, \dots, n$ in some order, then we say that r_i is *behind first*. If every red edge in D_n is ahead first (respectively, behind first), then D_n is *ahead first* (respectively, *behind first*).

If in the crossing sequence of r_i we have that $i+1, i+2, \dots, n$ appear in this order, then r_i is *ahead increasing*, and if they appear in descending order then it is *ahead decreasing*. If all the red edges are ahead increasing (respectively, decreasing), then the drawing D_n itself is *ahead increasing* (respectively, *ahead decreasing*). If D_n is ahead increasing or ahead decreasing, then it is *ahead monotone*.

Similarly, if in the crossing sequence of r_i we have that $1, 2, \dots, i-1$ appear in this order, then r_i is *behind increasing*, and if they appear in descending order then it is *behind decreasing*. If all the red edges are behind increasing (respectively, decreasing), then the drawing D_n itself is *behind increasing* (respectively, decreasing). If D_n is behind increasing or behind decreasing, then it is *behind monotone*.

Thus, for instance, if D_n is ahead first, ahead decreasing, and behind increasing, then for each $i \in [n]$ the crossing sequence of the red edge r_i is $n, n-1, \dots, i+2, i+1, 1, 2, \dots, i-1$ (with obvious adjustments for $i = 1$ and $i = n$).

If all the crossings of r_i with $b_{i+1}, b_{i+2}, \dots, b_n$ are positive (respectively, negative) then r_i is *ahead positive* (respectively, *ahead negative*). If all the red edges are ahead positive (respectively, if they are all ahead negative) then D_n itself is *ahead positive* (respectively, *ahead negative*). If D_n is ahead positive or ahead negative, then it is *ahead consistent*.

Similarly, if all the crossings of r_i with b_1, b_2, \dots, b_{i-1} are positive (respectively, negative) then r_i is *behind positive* (respectively, *behind negative*). If all the red edges are behind positive (respectively, if they are all behind negative) then D_n itself is *behind pos-*

itive (respectively, *behind negative*). If D_n is behind positive or behind negative, then it is *behind consistent*.

Regarding blue edges, we need only pay attention to b_1 and b_n . If in D_n the crossing sequence of b_1 is $2, \dots, n-1, n$ (respectively, $n, n-1, \dots, 2$) then we say that D_n is b_1 -*increasing* (respectively, b_1 -*decreasing*). In either case we say that D_n is b_1 -*monotone*. Similarly, if in D_n the crossing sequence of b_n is $1, 2, \dots, n-1$ (respectively, $n-1, \dots, 2, 1$) then D_n is b_n -*increasing* (respectively, b_n -*decreasing*). In either case we say that D_n is b_n -*monotone*.

Let Σ be a surface. An application of Ramsey's theorem for 3-uniform hypergraphs shows that if there exist arbitrarily large 2-drawings in Σ , then there exist arbitrarily large 2-drawings in Σ that are either ahead first or behind first. We note that a behind first drawing gets transformed into an ahead first drawing by performing the relabelling $i \mapsto n-i+1$ to each vertex $i \in [n]$. In view of this, the previous observation implies that if there exist arbitrarily large 2-drawings in Σ , then there exist arbitrarily large 2-drawings in Σ that are ahead first.

With successive applications of Ramsey's theorem we get a series of additional properties, ending up with the following.

Proposition 14. *Let Σ be a surface. If there exist arbitrarily large 2-drawings in Σ , then there exist arbitrarily large 2-drawings in Σ that are ahead first, ahead monotone, behind monotone, ahead consistent, behind consistent, b_1 -monotone, and b_n -monotone.*

7 Proof of Lemma 5

Lemma 5 claims that no fixed surface Σ can host 2-drawings of $K_{2,m}$ for arbitrarily large values of m . In view of Proposition 14, in order to prove the lemma it suffices to show that *no fixed surface can host 2-drawings of $K_{2,n}$ that are ahead first, ahead monotone, behind monotone, ahead consistent, behind consistent, b_1 -monotone, and b_n -monotone.*

Even though evidently the focus has been reduced to a finite number of possibilities, at first glance we seem to have a grueling task ahead. Indeed, since there are 2 ways in which a 2-drawing can be ahead monotone, 2 ways in which it can be behind monotone, 2 ways in which it can be ahead consistent, 2 ways in which it can be behind consistent, 2 ways in which it can be b_1 -monotone, and 2 ways in which it can be b_n -monotone, in principle we need to investigate $2^6 = 64$ different possibilities. Fortunately, as we shall see shortly, all cases are disposed of with a handful of simple arguments.

Each of the 64 cases is dealt with in one of the next three claims. For instance, the next statement swiftly takes care of 60 cases with the same easy observation used to prove Claim 12.

Claim 15. *Let D_n be a 2-drawing of $K_{2,n}$ in a surface Σ , where n is even. Suppose that D_n is either:*

- (i) *behind positive and b_1 -increasing; or*
- (ii) *behind negative and b_1 -decreasing; or*

- (iii) ahead positive and b_n -increasing; or
- (iv) ahead negative and b_n -decreasing; or
- (v) ahead increasing, and ahead positive; or
- (vi) ahead decreasing, and ahead negative; or
- (vii) behind increasing, and behind positive; or
- (viii) behind decreasing, and behind negative.

Then $\text{gen}(\Sigma) \geq (n - 2)/2$.

With the following claim, we will deal with 3 of the 4 remaining cases.

Claim 16. *Let D_n be a 2-drawing of $K_{2,n}$ in a surface Σ . Suppose that D_n is ahead first, and in addition it is either:*

- (i) ahead increasing, ahead negative, and b_n -increasing; or
- (ii) behind increasing, behind negative, and b_1 -increasing.

Then there exists a 1-drawing of $K_{2,n-1}$ in Σ .

Finally, with the next statement, we take care of the single remaining case.

Claim 17. *Let D_n be a 2-drawing of $K_{2,n}$ in a surface Σ , where $n \equiv 1 \pmod{3}$. Suppose that D_n is ahead first, ahead decreasing, behind decreasing, ahead positive, behind positive, and b_1 -decreasing. Then $\text{gen}(\Sigma) \geq (n - 1)/3$.*

We defer the proofs of the claims for the moment, and show that they imply Lemma 5.

Proof of Lemma 5, assuming Claims 15, 16, and 17. Let Σ be a surface. To prove the lemma we exhibit an integer n and show that there cannot exist a 2-drawing of $K_{2,n}$ in Σ .

To define n , first let N be the maximum integer such that there is a 1-drawing of $K_{2,N}$ in Σ . The existence of N is guaranteed from Lemma 4. We let n be any integer such that $n \geq \max\{3 \text{gen}(\Sigma) + 3, N + 2\}$ and $n \equiv 1 \pmod{3}$.

By Proposition 14 there is a 2-drawing D_n of $K_{2,n}$ that is ahead first, ahead monotone, behind monotone, ahead consistent, behind consistent, b_1 -monotone, and b_n -monotone.

A trivial analysis shows that D_n either (a) satisfies one of (i)–(viii) in Claim 15; or (b) it satisfies one of (i)–(ii) in Claim 16; or (c) it satisfies the hypotheses in Claim 17.

Suppose that (a) holds. Note that n is even. By Claim 15, we have that $\text{gen}(\Sigma) \geq (n - 2)/2$, that is, $n \leq 2 \text{gen}(\Sigma) + 2$. But this contradicts the choice of n , which in particular implies that $n \geq 2 \text{gen}(\Sigma) + 3$.

If (b) holds, Claim 16 guarantees the existence of a 1-drawing of $K_{2,n-1}$ in Σ , and from this it follows that $n - 1 \leq N$. But this contradicts the choice of n , which in particular implies that $n - 1 \geq N + 1$.

Finally, suppose that (c) holds. By Claim 17, we have that $\text{gen}(\Sigma) \geq (n - 1)/3$, that is, $n \leq 3 \text{gen}(\Sigma) + 1$. But this contradicts the choice of n , which in particular implies that $n \geq 3 \text{gen}(\Sigma) + 2$. \square

We reiterate that in the proofs of the claims we adopt once again the perspective used in the proof of Claims 12 and 13, where crossings are regarded as degree 4 vertices. Thus once again b_i and r_i are seen as paths, for $i = 1, \dots, n$.

Proof of Claim 15. This claim follows using an argument totally analogous to the one used in the proof of Claim 12, to find a restriction J_n of D_n with $2n + 1$ vertices, $3n - 2$ edges, and one facial walk.

For $i \in \{1, n\}$, we use $D_n|_{b_i}$ to denote the restriction of D_n to the blue path b_i and all the red paths with the exception of r_i . Similarly, $D_n|_{r_i}$ is the restriction of D_n to the red path r_i and all the blue paths with the exception of b_i . For instance, Figure 9 in Section 5 illustrates $D_n|_{b_1}$ if D_n is behind positive and b_1 -increasing, that is, if D_n is as in (i) in Claim 15.

As an additional example, in Figure 11 we illustrate $D_n|_{r_n}$ if D_n is behind decreasing and behind negative, as in (viii) in Claim 15.

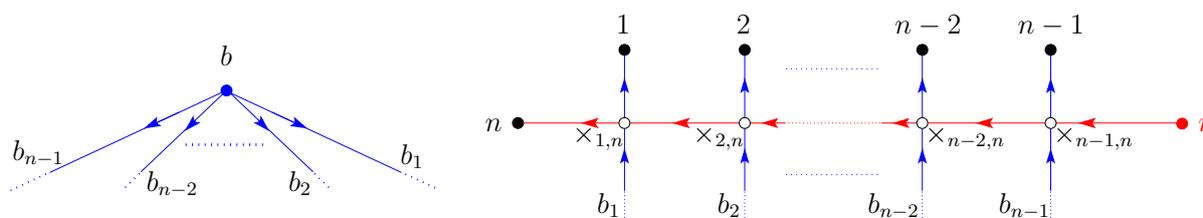


Figure 11: Illustration of the proof of Claim 15.

If (i) or (ii) holds, let $J_n := D_n|_{b_1}$. If (iii) or (iv) holds, let $J_n := D_n|_{b_n}$. If (v) or (vi) holds, let $J_n := D_n|_{r_1}$. Finally, if (vii) or (viii) holds, let $J_n := D_n|_{r_n}$. In all these cases, as in the proof of Claim 12, it is easily verified that J_n has $2n + 1$ vertices, $3n - 2$ edges, and one facial walk. Using (1), we conclude that the genus of the host surface of J_n (which is the host surface of D_n) is at least $(n - 2)/2$. \square

Proof of Claim 16. Suppose that (i) holds. The idea of the proof is quite simple. First, we remove certain parts of D_n , thus obtaining a drawing D'_n . We then contract a path in D'_n to a vertex, and show that as a result we obtain a 1-drawing of $K_{2,n-1}$.

Recall that in a 2-drawing of $K_{2,n}$, for each $i = 1, \dots, n$ the red path r_i crosses the blue path b_j for every $j \neq i$. In particular, for $i = 1, 2, \dots, n - 1$, the red path r_i crosses the blue path b_n . As illustrated in Figure 12(a), we let s_i be the part of the red path r_i from r to its crossing $\times_{n,i}$ with the blue path b_n , and we let t_i be the rest of r_i , that is, the part from $\times_{n,i}$ to vertex i . In this figure, the parts t_i are drawn thick.

As we also illustrate in Figure 12(a), the assumption that D_n is b_n -increasing implies that as we traverse b_n from b to n we find $\times_{n,1}, \times_{n,2}, \dots, \times_{n,n-1}$ in this order. Also note

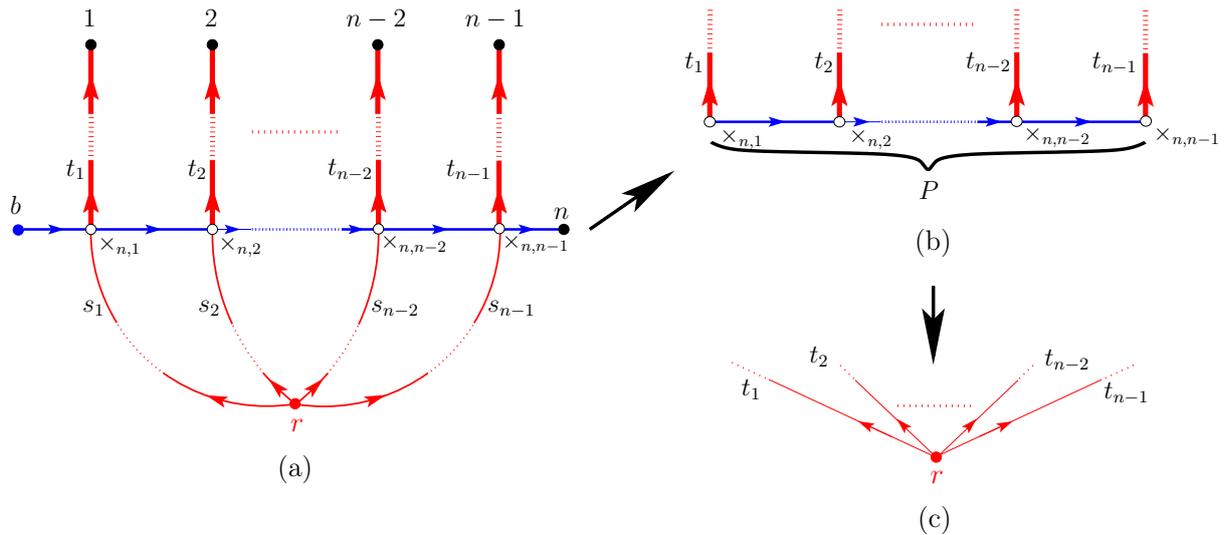


Figure 12: Illustration of the proof of Claim 16.

that the illustration in Figure 12(a) reflects that D_n is ahead negative, and so in particular $\times_{n,1}, \times_{n,2}, \dots, \times_{n,n-1}$ are negative crossings.

We now discard from D_n the red path r_n , the red vertex r , and for $i=1, 2, \dots, n-1$ we discard the part s_i of the red path r_i . Finally, we also discard the blue edge from b to $\times_{n,1}$ and also the blue edge from $\times_{n,n-1}$ to n . We let D'_n be the drawing obtained after removing these vertices and edges from D_n . In Figure 12(b) we illustrate the part of D'_n that consists of what remains of the blue path b_n (which is the blue path P from $\times_{n,1}$ to $\times_{n,n-1}$) and its incident red edges.

Finally, as illustrated in Figure 12(c), we contract in D'_n the whole blue path P to a vertex that we label r . Note that there is no ambiguity in the use of this label, since the original red vertex r was discarded at the beginning of the procedure. Let F_{n-1} denote the drawing thus obtained. We conclude the proof of Case (i) by showing that F_{n-1} is a 1-drawing of $K_{2,n-1}$, where the blue paths are b_1, b_2, \dots, b_{n-1} , and the red paths are t_1, t_2, \dots, t_{n-1} .

To prove that F_{n-1} is a 1-drawing of $K_{2,n-1}$, we need to show that (I) the paths t_1, t_2, \dots, t_{n-1} leave r in this clockwise cyclic order; and (II) for each pair of distinct $i, j \in \{1, 2, \dots, n-1\}$, there is exactly one crossing in the canonical cycle that consists of b_i, b_j, t_i , and t_j .

As illustrated in Figure 12(c), (I) follows simply because the original drawing D_n is b_n -increasing. Finally, to see that (II) holds it suffices to note that for $i = 1, 2, \dots, n-1$, the red path t_i crosses the blue paths b_1, b_2, \dots, b_{i-1} , and it does not cross any of the blue paths b_{i+1}, \dots, b_n : this follows since the original drawing D_n is ahead first and ahead increasing. Thus (I) and (II) hold, and so F_{n-1} is a 1-drawing of $K_{2,n-1}$ in Σ , as claimed.

If (ii) holds, we proceed in a totally analogous manner. In this case we discard the red vertex r , the red path r_1 , and for $i = 2, \dots, n$ the part of r_i from r to its crossing $\times_{1,i}$ with the blue path b_1 . Next, we discard the portions of b_1 from b to $\times_{1,2}$ and from $\times_{1,n}$

to 1. Finally, we contract the part of b_1 from $\times_{1,2}$ to $\times_{1,n}$ to a vertex we label r , and as a result we obtain a 1-drawing of $K_{2,n-1}$ in Σ . \square

Proof of Claim 17. As we illustrate in Figure 13, the assumption that D_n is ahead decreasing, ahead positive, behind decreasing, and behind positive, implies that for $i = 2, \dots, n-1$, as we traverse the red path r_i from r to i , we cross the blue paths $b_n, b_{n-1}, \dots, b_{i+1}, b_{i-1}, \dots, b_2, b_1$ in this order, and we cross all these paths from their left-hand sides.

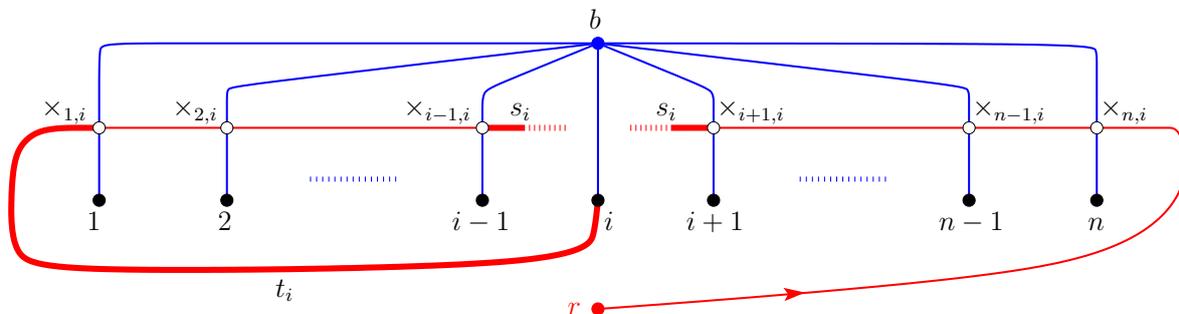


Figure 13: Illustration of the proof of Claim 17.

As we illustrate in this figure, after crossing b_{i+1} at $\times_{i+1,i}$, the path r_i reaches b_{i-1} at $\times_{i-1,i}$. Thus there is a red edge with endpoints $\times_{i+1,i}$ and $\times_{i-1,i}$; we use s_i to denote this red edge. We also note that as we traverse r_i , the last crossing we encounter is its crossing $\times_{1,i}$ with b_1 , and after this crossing we arrive in vertex i . Thus there is a red edge with endpoints $\times_{1,i}$ and i ; we use t_i to denote this red edge. Both s_i and t_i are thick in Figure 13.

We let J_n be the restriction of D_n to the part that consists of the blue paths b_1, \dots, b_n plus the segments s_i and t_i for $i = 3, 6, 9, \dots, n-1$. See Figure 14. As we illustrate in that figure, since D_n is b_1 -decreasing it follows that the crossing-vertices $\times_{1,n-1}, \dots, \times_{1,6}, \times_{1,3}$ appear in this order as we traverse b_1 from b to 1.

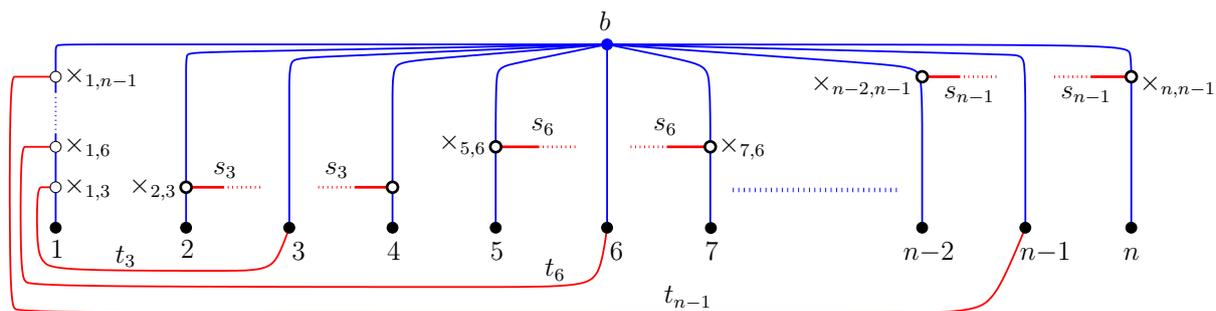


Figure 14: Illustration of the proof of Claim 17.

We note that J_n has $2n$ vertices and $8n/3 - 5/3$ edges, and it is straightforward to verify that it has exactly one facial walk. Using (1) we obtain that the genus of the host surface of J_n (which is the host surface of D_n) is at least $(n-1)/3$. \square

8 Concluding remarks and open questions

It is well known (see for instance [7, 10]) that every rotation system is realized by an embedding on a surface. Since an embedding is an (particularly nice) instance of a simple drawing, it immediately follows that every rotation system, and in particular every complete rotation system, is simply realizable in every surface whose genus is sufficiently large.

Archdeacon defined the *crossing genus* $\text{crg}(\Pi)$ of a rotation system Π as the smallest genus of a surface in which Π can be simply realized. He put forward the following.

Problem (Archdeacon [3]). *Given a complete rotation system, calculate its crossing genus.*

For each nonnegative integer g , let $f(g)$ be the least integer such that there exists a complete rotation system of size $f(g)$ that is not simply realizable in the surface of genus g . For instance, $f(0) = 4$, since there exist complete rotation systems of size 4 that are not simply realizable in the sphere (equivalently, the plane), but trivially every complete rotation system of size 3 is simply realizable in the sphere.

Theorem 1 implies that $f(g)$ is well-defined. On the other hand, in our arguments we use Ramsey's theorem repeatedly, and so the upper bound we can prove for $f(g)$ is multiple exponential in g . Can one prove a remarkably better upper bound for $f(g)$? For instance, is it true that $f(g) = O(2^g)$?

How about lower bounds for $f(g)$? Any rotation system of size n can be realized as a 2-cell embedding (and hence as a simple drawing) of K_n in some surface. If the rotation system defines ϕ facial walks, then the genus of this surface is $(1/2)(2 - |V(K_n)| + |E(K_n)| - \phi) \leq (1/2)(1 - n + \binom{n}{2}) = (1/4)(n - 1)(n - 2) < n^2/4$. Therefore every rotation system of size n can be simply realized in the surface of genus $n^2/4$, and this implies that $f(g) = \Omega(\sqrt{g})$. Can one prove a significantly better lower bound for $f(g)$? For instance, is it true that $f(g) = \Omega(g)$?

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References

- [1] B. Ábrego, O. Aichholzer, S. Fernández-Merchant, T. Hackl, J. Pammer, A. Pilz, P. Ramos, G. Salazar, and B. Vogtenhuber. All Good Drawings of Small Complete Graphs. In *Proc. 31st European Workshop on Computational Geometry EuroCG '15*, pages 57–60, Ljubljana, Slovenia, 2015.

- [2] P. Angelini, M. A. Bekos, F. J. Brandenburg, G. Da Lozzo, G. Di Battista, W. Didimo, M. Hoffmann, G. Liotta, F. Montecchiani, I. Rutter, and C. D. Tóth. Simple k -planar graphs are simple $(k + 1)$ -quasiplanar. *J. Comb. Theory, Ser. B*, 142:1–35, 2020.
- [3] D. Archdeacon. Problems in topological graph theory —questions I can’t answer—. *Yokohama Mathematical Journal*, 47:89–92, 1999.
- [4] A. Arroyo, J. Bensmail, and R. B. Richter. Extending drawings of graphs to arrangements of pseudolines. *J. Comput. Geom.*, 12(2):3–24, 2021.
- [5] A. Arroyo, F. Klute, I. Parada, B. Vogtenhuber, R. Seidel, and T. Wiedera. Inserting one edge into a simple drawing is hard. *Discrete Comput. Geom.*, 69(3):745–770, 2023.
- [6] J. Cardinal and S. Felsner. Topological drawings of complete bipartite graphs. *J. Comput. Geom.*, 9(1):213–246, 2018.
- [7] J. L. Gross and T. W. Tucker. *Topological graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1987.
- [8] J. Kynčl. Simple realizability of complete abstract topological graphs in \mathbf{P} . *Discrete and Computational Geometry*, 45:383–399, 2011.
- [9] J. Kynčl. Simple realizability of complete abstract topological graphs simplified. *Discrete and Computational Geometry*, 64:1–27, 2020.
- [10] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins series in the mathematical sciences. Johns Hopkins University Press, 2001.
- [11] M. Schaefer. *Crossing numbers of graphs*. Discrete Math. Appl. (Boca Raton). Boca Raton, FL: CRC Press, 2018.
- [12] M. Schaefer. On the complexity of some geometric problems with fixed parameters. *J. Graph Algorithms Appl.*, 25(1):195–218, 2021.