Local rainbow colorings for various graphs

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Submitted: Feb 27, 2023; Accepted: May 16, 2024; Published: Jun 28, 2024

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Abstract

Motivated by a problem in theoretical computer science suggested by Wigderson, Alon and Ben-Eliezer studied the following extremal problem systematically one decade ago. Given a graph H, let $C(n,H)$ be the minimum number k such that the following holds. There are n colorings of $E(K_n)$ with k colors, each associated with one of the vertices of K_n, such that for every copy T of H in K_n, at least one of the colorings that are associated with $V(T)$ assigns distinct colors to all the edges of $E(T)$. In this paper, we obtain several new results in this problem including:

- For paths of short length, we show that $C(n,P_4)=\Omega(n^{1/5})$ and $C(n,P_t)=\Omega(n^{1/3})$ with $t\in\{5,6\}$, which significantly improve the previously known lower bounds $(\log n)^{\Omega(1)}$.
- We make progress on the problem of Alon and Ben-Eliezer about complete graphs, more precisely, we show that $C(n,K_r)=\Omega(n^{2/3})$ when $r\geqslant 8$, and $C(n,K_{s,t})=\Omega(n^{2/3})$ for all $t\geqslant s\geqslant 7$.
- When H is a star with at least 4 leaves, a matching of size at least 4, or a path of length at least 7, we give a new lower bound for $C(n,H)$. We also show that for any graph H with at least 6 edges, $C(n,H)$ is polynomial in n. All of these improve the corresponding results obtained by Alon and Ben-Eliezer.

Mathematics Subject Classifications: 05C35, 05C15

1 Introduction

One of the hardest problems of complexity theory is to prove nontrivial lower bounds on fundamental complexity measures for concrete computing problems. In 1993, Karchmer [12] gave a lower bound on non-deterministic circuit size and presented a new proof for the exponential monotone size lower bound for the clique function. Later, Wigderson [20]

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discussed the achievements, potential, and challenges of the elegant fusion method introduced by Karchmer [12], which unifies the previous approximation method of Razborov [16] and the topological method of Sipser [17]. In the same paper, Wigderson also provided several concrete open problems, one of which can be stated as follows.

Problem 1. Let $\chi_i : \{0,1\}^n \to [k]$, for $i \in [n]$ be a collection of k-colorings of the n-dimensional hypercube. For a triple of distinct vectors $X, Y, Z \in \{0,1\}^n$, say that coordinate $i \in [n]$ is interesting if not all three vectors agree in this coordinate. Say that χ_i is proper on this triple if the three colors $\chi_i(X)$, $\chi_i(Y)$ and $\chi_i(Z)$ are distinct. Define the collection of colorings good if for every triple of vectors, there is an interesting coordinate i for which χ_i is proper on this triple. The problem asks to, bound the smallest number k for which such a good collection exists.

Karchmer and Wigderson [13] later proved that, in the above problem, the smallest number k has to grow with n. Motivated by Problem 1, Alon and Ben-Eliezer [1] initiated the study of a new problem in extremal graph theory, which aims to find rainbow subgraphs under certain constraints. Formally, for a given graph H, let $C(n, H)$ be the minimum number k such that the following holds. There is a set of n colorings $\{f_v : E(K_n) \to [k] : v \in V(K_n)\}$, such that for every copy T of H in K_n, at least one of the colorings that are associated with $V(T)$ assigns distinct colors to all the edges of $E(T)$, that is, at least one vertex in T is associated with a coloring for which T is rainbow. Note that, we do not require each coloring to be proper edge coloring. Alon and Ben-Eliezer [1] remarked that a lower bound for $C(n, P_3)$ is also a lower bound for Problem 1. To see this, one can regard the n coordinates of n-dimensional hypercube as the n vertices of the complete graph K_n and regard $\binom{[n]}{2}$ as the set of edges in K_n. Then in Problem 1, $\chi_i(X)$, $\chi_i(Y)$ and $\chi_i(Z)$ are distinct means the edges of corresponding path of length 3 receive distinct colors. Based on this connection, Alon and Ben-Eliezer [1] improved the lower bound in [13] by showing $C(n, P_3) = \Omega((\frac{\log n}{\log\log n})^{1/4})$.

In recent years, there have been many other important problems in the field of extremal combinatorics involving finding rainbow structures in edge colorings of graphs. For example, the rainbow Turán problem [4, 5, 9, 11, 14, 18, 19] asks the maximum number of edges in a properly edge-colored graph that does not contain certain subgraph, all of whose edges have different colors. The anti-Ramsey problem [3, 7, 10, 21, 22] asks for the maximum number of colors in an edge coloring of a complete graph without a certain rainbow subgraph. Moreover, there was a famous conjecture of Ringel in 1963, one of whose statements involved finding a rainbow copy of any tree with n edges in a particular proper edge coloring of K_{2n+1}. This conjecture was recently confirmed by Montgomery, Pokrovskiy, and Sudakov [15] via many new techniques that are based on probabilistic methods. Return to the extremal problem of Alon and Ben-Eliezer, one can ask the following natural question.

Question 2. For a fixed graph H, determine the order of growth of $C(n, H)$ as $n \to \infty$.

Alon and Ben-Eliezer [1] characterized the set of all graphs H for which $C(n, H)$ is bounded by some absolute constant $c(H)$. Using the so-called Lovász local lemma [2, 6], they proved a general upper bound for any graph H. Moreover, they also obtained lower bounds for several graphs of special interests, including paths P_t, matchings I_t, and stars S_t, where t represents the number of edges. Here we list some known results in [1] as follows.

Theorem 3 ([1])**.**

- *$C(n,H) \leqslant c(H)$ if and only if H contains at most 3 edges and H is neither P_3 nor P_3 together with any number of isolated vertices. Moreover, in all these cases, we have $C(n,H) \leqslant 5$ for every n.*
- *Let H be a fixed graph with r vertices, then $C(n,H) = O(r^4 \cdot n^{\frac{r-2}{r}})$.*
- *$C(n,P_3) = \Omega((\frac{\log n}{\log\log n})^{1/4})$ and $C(n,P_t) = (\log n)^{\Omega(1)}$ for $t \in \{4,5,6\}$.*
- *$C(n,I_4) = \Omega(n^{1/6})$ and $C(n,I_t) = \Omega(n^{1/4})$ for $t \geqslant 5$.*
- *$C(n,S_4) = \Omega(n^{1/4})$ and $C(n,S_t) = \Omega(n^{1/3})$ for $t \geqslant 5$.*
- *$C(n,P_7)$, $C(n,P_8) = \Omega(n^{1/6})$ and $C(n,P_t) = \Omega(n^{1/4})$ for $t \geqslant 9$.*
- *For any graph H with at least 13 edges, there is a constant $b = b(H) > 0$ so that $C(n,H) = \Omega(n^b)$.*

In this paper, we show some new lower bounds on $C(n,H)$ for various graphs, including several sparse graphs such as paths, stars, and matchings, and dense graphs such as cliques and complete bipartite graphs.

For complete graphs, Alon and Ben-Eliezer [1] asked whether for every $\epsilon > 0$, there is an $r = r(\epsilon)$ such that $C(n,K_r) = \Omega(n^{1-\epsilon})$. However, they did not provide any good bound for $C(n,K_r)$. We make partial progress to their conjecture by showing the following result.

Theorem 4. *For any positive integer $r \geqslant 8$, we have*

$$C(n,K_r) = \Omega(n^{2/3}).$$

Furthermore, we can also prove a new bound for the complete bipartite graphs as follows.

Theorem 5. *For any positive integers $t \geqslant s \geqslant 7$, we have*

$$C(n,K_{s,t}) = \Omega(n^{2/3}).$$

Our improved lower bounds for sparse graphs can be listed as follows.

Theorem 6. *Let P_t be the path of length t, we have*

- *$C(n,P_4) = \Omega(n^{1/5})$.*

- $C(n, P_t) = \Omega(n^{1/3})$, *for* $t \in \{5, 6, 7\}$.
- $C(n, P_t) = \Omega(n^{1/2})$, *for* $t \geqslant 8$.

Theorem 6 resolves the problem of determining the function is polynomial or not for almost all paths, leaving only the case of P_3 open.

Theorem 7. *Let* S_t *be the star with* t *leaves, we have*

- $C(n, S_4) = \Omega(n^{1/3})$.
- $C(n, S_t) = \Omega(n^{1/2})$, *for* $t \geqslant 5$.

Theorem 8. *Let* I_t *be the matching of size* t, *we have*

- $C(n, I_4) = \Omega(n^{1/5})$.
- $C(n, I_t) = \Omega(n^{1/3})$, *for* $t \in \{5, 6\}$.
- $C(n, I_t) = \Omega(n^{1/2})$, *for* $t \geqslant 7$.

The next result shows that if H has at least 6 edges, then $C(n, H)$ must be polynomial in n.

Theorem 9. *For any graph* H *with at least* 6 *edges, there is a constant* $b = b(H) > 0$ *so that* $C(n, H) = \Omega(n^b)$.

This improves the result of Alon and Ben-Eliezer from 13 edges to 6 edges. Note that the first result in Theorem 3 tells that the constant cannot be improved to 3, thus our result is very close to being optimal.

Notation. Throughout this paper, we use f_v to denote the set of coloring functions associated with the vertices v. We will write $H_1 \sqcup H_2$ for the vertex-disjoint union of the graphs H_1 and H_2. In particular, we will write $H \sqcup \{p\}$ for the graph which consists of H plus an isolated vertex p. We always assume n is a sufficiently large number. The notations $O(\cdot)$, $\Omega(\cdot)$ and $o(\cdot)$ have their usual asymptotic meaning. We omit the floor and ceiling functions if are not essential.

Structure of the paper. The rest of this paper is organized as follows. We will present some auxiliary lemmas in Section 2. The proofs of new results for cliques and complete bipartite graphs are presented in Section 3. We prove the lower bounds for paths, stars and matchings separately in Section 4. We will show the polynomial lower bound for any graph with at least 6 edges in Section 5. Finally, we conclude this paper and pose some open problems in Section 6.

Note added: Recently, Janzer and Janzer [8] proved that $C(n, K_r) = \Omega(n^{1-\frac{4}{r+2}})$ for even $r \geqslant 4$ and $C(n, K_r) = \Omega(n^{1-\frac{10}{r-3}})$ for large odd $r \geqslant 3$, and they proved $C(n, P_3) = n^{o(1)}$.

2 Preliminaries

To show $C(n, H) > k$, our task is to show that for any set of n coloring functions with k colors, we can always find a copy of H such that none of its vertices induces a rainbow coloring on this copy of H. Moreover, one can see that if $H' \subseteq H$ is a subgraph of H on the same set of vertices, then every lower bound for $C(n, H')$ implies the same lower bound for $C(n, H)$. The property will help us show the improved bounds for paths of length at least 5, see Remarks 17 and 21.

Here we present the following simple lemmas, since the proofs of these lemmas are similar, for simplicity, we only give the proof of Lemma 10 in detail. By the first result in Theorem 3, we only consider the case where n is a sufficiently large number and $k \geqslant 6$ is an integer.

Lemma 10. *For any set of n k-colorings of K_n associated with n vertices, there exists a vertex $x \in V(K_n)$, a set S with $|S| = \frac{n-1}{k}$ and $f_x(xs_1) = f_x(xs_2)$ for all distinct $s_1, s_2 \in S$, and a set $P = V(K_n) \setminus (S \cup \{x\})$, such that the number of triples $(s, s', p) \in S \times S \times P$ with $f_p(xs) = f_p(xs')$ is at least $\frac{n^3}{24k^3}$.*

Proof of Lemma 10. Consider the complete graph K_n, for any set of n k-colorings of K_n, we can take an arbitrary vertex $x \in V(K_n)$, by pigeonhole principle, there exists a set S with $|S| = \frac{n-1}{k}$ such that f_x assigns the same color to all edges $xs \in E(K_n)$ with $s \in S$. We fix this set S, and then we set $P := V(K_n) \setminus (S \cup \{x\})$ and count the number of triples $(s, s', p) \in S \times S \times P$ with the property that $f_p(xs) = f_p(xs')$. Let $f_p^{-1}(i)$ be the set of vertices $s \in S$ such that $f_p(xs) = i$, as each vertex $p \in P$ contributes

$$\sum_{i=1}^{k} \binom{|f_p^{-1}(i)|}{2} \geqslant k \cdot \binom{|S|/k}{2} \geqslant \frac{n^2}{12k^3}$$

many such triples by the convexity of the function $\binom{x}{2}$. Moreover, since n is sufficiently large, we have $|P| \geqslant n - \frac{n-1}{k} - 1 \geqslant \frac{n}{2}$. Hence the total number of triples $(s, s', p) \in S \times S \times P$ with the desired property is at least $\frac{n^3}{24k^3}$. $\square$

Lemma 11. *For any set of $3n$ k-colorings of K_{3n} associated with $3n$ vertices, let $Y \in V(K_{3n})$ be a subset with $|Y| = n$ and $M = \{e_1, e_2, \ldots, e_n\}$ be a matching on vertex set $V(K_{3n}) \setminus Y$. Then the number of triples $(e, e', p) \in M \times M \times Y$ with $f_p(e) = f_p(e')$ is at least $\frac{n^3}{3k}$.*

Lemma 12. *For any set of $2n$ k-colorings of K_{2n} associated with $2n$ vertices, let $A, B \subseteq V(K_{2n})$ be two disjoint subsets of size n. Then the number of triples $(a, b_1, b_2) \in A \times B \times B$ with $f_a(ab_1) = f_a(ab_2)$ is at least $\frac{n^3}{3k}$.*

Lemma 13. *For any set of $3n$ k-colorings of K_{3n} associated with $3n$ vertices, let $X \subseteq V(K_{3n})$ be a subset of size n and K_{2n} be a complete subgraph on $V_{3n} \setminus X$. Then the number of triples $(b, e, e') \in X \times E(K_{2n}) \times E(K_{2n})$ with $f_b(e) = f_b(e')$ is at least $\frac{n^5}{3k}$.*

3 Dense graphs

In this section, we mainly focus on dense graphs such as complete graphs and complete bipartite graphs.

3.1 Cliques with at least 8 vertices

Here we first prove that $C(n, K_8) = \Omega(n^{2/3})$. Let $k := cn^{2/3}$, where the constant c is very small. We consider the complete graph with $2n$ vertices. For any set of $2n$ k-colorings of K_{2n}, our aim is to find a copy of K_8 such that none of its vertices induces a rainbow coloring. By Lemma 12, we can partition the vertex set of K_{2n} into two parts A and B with $|A| = |B| = n$ and then the number of triples $(a, b_1, b_2) \in A \times B \times B$ such that $f_a(ab_1) = f_a(ab_2)$ is at least $\frac{n^{7/3}}{3c}$. By pigeonhole principle, there exists a pair of vertices in B, called (b_1, b_2), such that there are at least $\frac{n^{1/3}}{3c}$ many distinct vertices $a \in A$ satisfying $f_a(ab_1) = f_a(ab_2)$. Then we choose a subset A' which consists of vertices satisfying the above property with size $|A'| = \frac{n^{1/3}}{3c}$.

Let $E(A')$ be the edge set of complete graph $K_{|A'|}$, then we use f_{b_1} to color the edges in $E(A')$. Note that there are $|E(A')| = \binom{n^{1/3}/3c}{2} > \frac{n^{2/3}}{19c} \gg 300k$ edges in the complete graph induced by A' as the constant c can be very small, hence by pigeonhole principle, we can recursively find a set of disjoint triples of the form $(h_{3i+1}, h_{3i+2}, h_{3i+3}) \in E(A') \times E(A') \times E(A')$ with the property that $f_{b_1}(h_{3i+1}) = f_{b_1}(h_{3i+2}) = f_{b_1}(h_{3i+3})$ till it covers 99% of the edges in $E(A')$. Eventually, we obtain a subset $F_1(A') = \{h_1, h_2, \ldots, h_{0.99|E(A')|}\} \subseteq E(A')$. Moreover, we do the same operation but change b_1 to b_2, then we obtain another subset $F_2(A')$ with $|F_2(A')| = 0.99|E(A')|$. Now we can choose three different edges e^1, e^2, e^3, where $e^1 \subseteq F_1(A') \cap F_2(A')$ satisfies $f_{b_1}(e^1) = f_{b_1}(e^2)$ and $f_{b_2}(e^1) = f_{b_2}(e^3)$. Next we select the vertices of $\{e^1, e^2, e^3\}$, if there are some common vertices of these edges, we arbitrarily add some vertices from A' to make sure there are 6 distinct vertices, denoted by $\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Since all of them belong to the set A', we have $f_{a_i}(a_i b_1) = f_{a_i}(a_i b_2)$ for $1 \leqslant i \leqslant 6$. As a consequence, we find a copy of K_8 induced by $\{b_1, b_2, a_1, a_2, a_3, a_4, a_5, a_6\}$, which does not admit a rainbow coloring. The proof of $C(n, K_8) = \Omega(n^{2/3})$ is finished.

Remark 14. To show $C(n, K_r) = \Omega(n^{2/3})$ for $r > 8$, we just need to add $r - 8$ vertices from A' to the selected vertex set of K_8.

3.2 Complete bipartite graph $K_{s,t}$ with $t \geqslant s \geqslant 7$

Here we first prove that $C(n, K_{7,7}) = \Omega(n^{2/3})$. Let $k := cn^{2/3}$, where the constant c is very small. We consider a complete graph with $3n$ vertices. For any set of $3n$ k-colorings of K_{3n}, we try to find a copy of $K_{7,7}$ such that none of its vertices induces a rainbow coloring. By Lemma 13, we can partition the vertex set of K_{3n} into two parts, say $V_L \cup V_R$, where $|V_L| = 2n$ and $|V_R| = n$, and writing E_L for the edge set of $K_{|V_L|}$, the number of the triples $(b, e, e') \in V_R \times E_L \times E_L$ with the property that $f_b(e) = f_b(e')$ is at least $\frac{n^{13/3}}{3c}$. As the number of edges in E_L is $\binom{2n}{2} \leqslant 2n^2$, by pigeonhole principle, there exists some pair

$(e_1, e_2) \in E_L \times E_L$ such that there are at least $\frac{n^{13/3}}{3c} \cdot \frac{1}{4n^4} \geqslant \frac{n^{1/3}}{12c}$ many vertices $b \in V_R$ satisfying $f_b(e_1) = f_b(e_2)$. Now let V'_R be a subset of V_R consisting of the vertices $b \in V_R$ such that $f_b(e_1) = f_b(e_2)$, and $|V'_R| \geqslant \frac{n^{1/3}}{12c}$. Moreover, we write the vertices of e_1 and e_2 as $e_1 = v_1v_2$ and $e_2 = u_1u_2$, respectively.

Next, we consider the coloring functions f_{v_1}, f_{v_2}, f_{u_1} and f_{u_2} on the edge set $E(V'_R)$. Here if e_1 and e_2 have a common vertex, then we can add an arbitrary vertex from V_L and regard it as the vertex u_2.

Note that $|E(V'_R)| = \binom{n^{1/3}/12c}{2} \geqslant \frac{n^{2/3}}{289c^2}$, by pigeonhole principle, we can recursively select a family $\mathcal{F}_1$ of edge-disjoint pairs of edges $(h_{2i+1}, h_{2i+2}) \in E(V'_R) \times E(V'_R)$ with the property that $f_{v_1}(h_{2i+1}) = f_{v_1}(h_{2i+2})$, till it covers 99% of the edges in $E(V'_R)$. This is possible since we can set c to be small enough so that $\frac{|E(V'_R)|}{100} \gg 29k$. Then we repeat the same operations but change v_1 to v_2 and change the pairs of edges to the sets of 7 edges, we can similarly obtain a family $\mathcal{F}_2$ of pairwise edge-disjoint sets of 7 edges, which covers 99% of the edges in $E(V'_R)$ and each set of 7 edges receives the same color from the function f_{v_2}. We continue to do the same operations twice, but replace the corresponding vertices with u_1 and with u_2, and also replace the sets of 7 edges to the sets of 16 edges and 29 edges respectively. Finally, we can obtain four families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ of internally disjoint sets of edges.

We can pick one edge such that for each family, there is some set containing this edge, moreover, we denote this edge as e^1. After we choose the edge e^1, then the edge e^2 with $(e^1, e^2) \in \mathcal{F}_1$ is determined. Next, consider the set of edges in $\mathcal{F}_2$ which contains e^1, we need to carefully pick some edge e^3 from this set, such that the vertices in $\{e^1, e^2, e^3\}$ do not form any odd cycle. Indeed, there are 7 choices but the number of potential edges which can lead to an odd cycle is at most $\binom{4}{2} = 6$. Similarly, we need to carefully choose some edge e^4 from the set in $\mathcal{F}_3$ which contains the edge e^1, such that the vertices $\{e^1, e^2, e^3, e^4\}$ does not form any odd cycle. This is also possible as there are 16 choices but the number of potential edges which may induce an odd cycle are at most $\binom{6}{2} = 15$. Also we pick an edge e^5 from the 29-element set in $\mathcal{F}_4$ which contains e^1 such that $\{e^1, e^2, e^3, e^4, e^5\}$ does not form any odd cycle for the similar reasons as $29 = \binom{8}{2} + 1$. We noted that the values 7, 16, 29 are sufficient for the above argument, we do not attempt to optimize these values.

Finally we can find 7 edges $e_1, e_2, e^1, e^2, e^3, e^4, e^5$ with the following properties.

- $f_{v_1}(e^1) = f_{v_1}(e^2)$; $f_{v_2}(e^1) = f_{v_2}(e^3)$; $f_{u_1}(e^1) = f_{u_1}(e^4)$; $f_{u_2}(e^1) = f_{u_2}(e^5)$.

The edges can be written as $e^1 = a_1a_2$, $e^2 = a_3a_4$, $e^3 = a_5a_6$, $e^4 = a_7a_8$ and $e^5 = a_9a_{10}$. If there is some common vertex between any pair of e^i and e^j for $1 \leqslant i \leqslant j \leqslant 5$, then we add some unused vertices from V'_R to guarantee there are 10 distinct vertices, namely, $\{a'_i\}_{i=1}^{10}$, where $\{a_i\}_{i=1}^{10} \subseteq \{a'_i\}_{i=1}^{10}$. We then find a copy of complete bipartite graph $K_{7,7}$, which contains edges $\{e_1, e_2, e^1, e^2, e^3, e^4, e^5\}$. Moreover the vertex set is $V(K_{7,7}) = \{v_1, v_2, u_1, u_2\} \cup \{a'_i\}_{i=1}^{10}$. Note that, for $1 \leqslant i \leqslant 10$, we have $f_{a'_i}(e_1) = f_{a'_i}(e_2)$ by the construction of V'_R. Hence, we find a copy of $K_{7,7}$ such that none of its vertices can induce a rainbow coloring. The proof of $C(n, K_{7,7}) = \Omega(n^{2/3})$ is finished.

Remark 15. To show $C(n, K_{s,t}) = \Omega(n^{2/3})$ for all $t \geqslant s \geqslant 7$, we just need to add $s + t - 14$

vertices from V_R' to the selected vertex set of $K_{7,7}$. Moreover, we actually proved that $C(n, P_3 \sqcup K_{5,5} \sqcup \overline{K_h}) = \Omega(n^{2/3})$ for every constant h.

4 Sparse Graphs

In this section, we prove the improved lower bounds on $C(n, H)$ when H are relatively sparse graphs.

4.1 Paths of short length

First we prove $C(n, P_4) = \Omega(n^{1/5})$. Let $k := cn^{1/5}$, where the constant $c > 0$ can be taken sufficiently small. For any set of $2n$ k-colorings of K_{2n}, it suffices to find a copy of P_4 such that there is no vertex that assigns distinct colors to all edges of this special copy of P_4.

We partition the vertex set of K_{2n} into two parts A and B with $|A| = |B| = n$. Consider the number of quadruples $(a, b_1, b_2, b_3) \in A \times B \times B \times B$ such that $f_a(ab_1) = f_a(ab_2) = f_a(ab_3)$. Let $f_a^{-1}(i)$ be the set of vertices $b \in B$ such that $f_a(ab) = i$. Observe that each element in A contributes

$$\sum_{i=1}^{cn^{\frac{1}{5}}} \binom{|f_a^{-1}(i)|}{3} \geqslant cn^{\frac{1}{5}} \cdot \binom{|B|/cn^{\frac{1}{5}}}{3} \geqslant \frac{n^{\frac{13}{5}}}{7c^2}$$

many such quadruples by the convexity of the function $\binom{x}{3}$. Moreover, since there are n choices for $a \in A$, totally there are at least $\frac{n^{18/5}}{7c^2}$ such quadruples in $A \times B \times B \times B$. By pigeonhole principle, there is a triple of vertices in B, called (b_1, b_2, b_3), such that there are at least $\frac{n^{3/5}}{7c^2}$ many vertices $a \in A$ satisfying $f_a(ab_1) = f_a(ab_2) = f_a(ab_3)$. Next let $A' \subseteq A$ be a subset with $|A'| \geqslant \frac{n^{3/5}}{7c^2}$, which consists of the vertices satisfying the above property. Then we consider the k^3-coloring $(f_{b_1}(b_2a), f_{b_2}(b_2a), f_{b_3}(b_2a))$ for all $a \in A'$. By pigeonhole principle, there are at least $\frac{1}{7c^5} \geqslant 2$ elements in A' receiving the same color, since c is sufficiently small. We choose two of them arbitrarily, and denote them as a_1 and a_2. Now through the above analysis, we can find a copy of P_4 on vertex set $\{b_1, a_1, b_2, a_2, b_3\}$ with the following properties:

- $f_{a_1}(a_1b_1) = f_{a_1}(a_1b_2)$; $f_{a_2}(a_2b_3) = f_{a_2}(a_2b_2)$;
- $f_{b_1}(b_2a_2) = f_{b_1}(b_2a_1)$; $f_{b_2}(b_2a_2) = f_{b_2}(b_2a_1)$; $f_{b_3}(b_2a_2) = f_{b_3}(b_2a_1)$.

That means, none of the vertices in this special copy of P_4 can induce a rainbow coloring. The proof of $C(n, P_4) = \Omega(n^{1/5})$ is finished.

4.2 Stars

For the star S_t with t leaves, we first consider the case of $t = 4$ and then obtain a better bound for any larger positive integer $t \geqslant 5$.

4.2.1 Star with 4 leaves

Here we give a proof of $C(n, S_4) = \Omega(n^{1/3})$. Let $k := cn^{1/3}$, and c be sufficiently small. Consider the complete graph K_n, for any set of n k-colorings of K_n, we need to find a copy of S_4 such that no vertex assigns distinct colors to all edges of this special copy of S_4. By Lemma 10, there exists a vertex $x \in V(K_n)$, a set S with $|S| = \frac{n-1}{k}$ and $f_x(xs_1) = f_x(xs_2)$ for all distinct $s_1, s_2 \in S$ and a set $P = V(K_n) \setminus (S \cup \{x\})$, such that the number of triples $(s, s', p) \in S \times S \times P$ with $f_p(xs) = f_p(xs')$ is at least $\frac{n^2}{24c^3}$.

Then by pigeonhole principle, there exists a pair of elements $(s, s') \in S \times S$ such that the number of vertices $p \in P$ with $f_p(xs) = f_p(xs')$ is at least $\frac{n^2}{24c^3} \cdot \frac{c^2}{n^{4/3}} = \frac{n^{2/3}}{24c}$. Let A be the subset consisting of the above vertices $p \in P$.

For the coloring function f_s, by pigeonhole principle, there is a subset $A' \subseteq A$ with $|A'| \geqslant \frac{n^{2/3}}{24c} \cdot \frac{1}{cn^{1/3}} = \frac{n^{1/3}}{24c^2}$, such that f_s assigns the same color to all edges xa with $a \in A'$. Similarly for $f_{s'}$, by pigeonhole principle again, there is a subset $A'' \subseteq A'$ with $|A''| \geqslant \frac{n^{1/3}}{24c^2} \cdot \frac{1}{cn^{1/3}} = 1/(24c^3)$, such that $f_{s'}$ assigns the same color to all edges xa with $a \in A''$. Since c is small enough, we have $1/(24c^3) \geqslant 2$. Then we can find a pair of distinct elements $(a, a') \in A'' \times A''$.

So, we have found a copy of S_4 on vertex set $\{x, a, a', s, s'\}$, where x is the center, which satisfies the following properties:

- $f_x(xs) = f_x(xs')$; $f_s(xa) = f_s(xa')$; $f_{s'}(xa) = f_{s'}(xa')$; $f_a(xs) = f_a(xs')$; $f_{a'}(xs) = f_{a'}(xs')$.

Hence, we find a copy of S_4, such that none of its vertices assigns distinct colors to all edges. The proof of $C(n, S_4) = \Omega(n^{1/3})$ is finished.

Remark 16. In the above proof, after we have found a set P with $|P| \geqslant \frac{n^{2/3}}{24c}$ and the pair $\{s, s'\} \in S \times S$ satisfying $f_x(xs) = f_x(xs')$, $f_p(xs) = f_p(xs')$ for all $p \in P$, we can find a matching M on the vertex set P and consider the k^2-coloring $(f_s, f_{s'})$. Note that $|P| \geqslant \frac{n^{2/3}}{24c} \gg k^2$, by pigeonhole principle, we can find a pair of edges e_1 and e_2 in the matching M such that $f_s(e_1) = f_{s'}(e_1)$ and $f_s(e_2) = f_{s'}(e_2)$. This gives a copy of $P_2 \sqcup K_2 \sqcup K_2$ with edge set $\{xs, xs', e_1, e_2\}$ such that no vertex induces a rainbow coloring. Thus we have $C(n, P_2 \sqcup K_2 \sqcup K_2) = \Omega(n^{1/3})$.

Remark 17. We can pick a vertex $p \in P$, and use the k^2-coloring $(f_s, f_{s'})$ to color all edges incident to vertex p in the graph induced by P. As $|P| \geqslant \frac{n^{2/3}}{24c} \gg k^2$, we can find a pair of vertices $a, b \in P$ such that the edges pa and pb receive the same color under the k^2-coloring $(f_s, f_{s'})$. This gives a copy of $P_2 \sqcup P_2$ with edges set $\{pa, pb, xs, xs'\}$ such that no vertex induces a rainbow coloring. Thus we have $C(n, P_2 \sqcup P_2) = \Omega(n^{1/3})$. Moreover, we can add arbitrary r isolated vertices from the set P to obtain same lower bound for $C(n, P_2 \sqcup K_2 \sqcup K_2 \sqcup rK_1)$ and $C(n, P_2 \sqcup P_2 \sqcup rK_1)$ for arbitrary non-negative integer r. Moreover, since any path of length at least 5 contains a copy of $P_2 \sqcup P_2$, the second result in Theorem 6 follows.

4.2.2 Stars with more than 4 leaves

We next consider the case of $t = 5$. Let $k := cn^{1/2}$ with sufficiently small constant $c > 0$. For any set of n k-colorings of K_n, by Lemma 10, we can find a vertex $x \in V(K_n)$, a set S of size $|S| \geqslant (n-1)/k$ with $f_x(xs_1) = f_x(xs_2)$ for all distinct $s_1, s_2 \in S$ and a subset $P = V(K_n) \setminus (S \cup \{x\})$, such that the number of triples $(s, s', p) \in S \times S \times P$ with $f_p(xs) = f_p(xs')$ is at least $\frac{n^{3/2}}{24c^3}$. Also using pigeonhole principle, we can find a pair of elements $(s, s') \in S \times S$ such that the number of vertices $p \in P$ with $f_p(xs) = f_p(xs')$ is at least $\frac{n^{3/2}}{24c^3} \cdot c^2/n = \frac{n^{1/2}}{24c}$. Let A consist of the above vertices $p \in P$, without loss of generality, we assume that $|A|$ is divided by 9.

Note that c is small enough, hence for $i = 1, 2, \ldots, \frac{2}{9} \cdot |A|$, by pigeonhole principle, we can recursively find disjoint triples $(p_{3i+1}, p_{3i+2}, p_{3i+3}) \in A \times A \times A$ with the property that $f_s(xp_{3i+1}) = f_s(xp_{3i+2}) = f_s(xp_{3i+3})$. Moreover, we do the same operations for the other vertex s', and then we will obtain two sets of internally edge-disjoint triples. Since the total number of vertices we obtain is larger than $|A|$, we can find one triple from each set respectively, such that they intersect, that is, there are distinct triples $\{p_t, p_j, p_k\} \subseteq A$ and $\{p_t, p_h, p_m\} \subseteq A$ such that $f_s(xp_t) = f_s(xp_j) = f_s(xp_k)$, $f_{s'}(xp_t) = f_{s'}(xp_h) = f_{s'}(xp_m)$, where $j = h$ or $k = m$ is also allowed. Finally, we can select a set of vertices $\{x, s, s', p_t, p_h, p_k\}$ to form a copy of star centered at vertex x with 5 leaves with the properties:

- $f_x(xs) = f_x(xs')$; $f_s(xp_t) = f_s(xp_k)$; $f_{s'}(xp_t) = f_{s'}(xp_h)$;
- $f_{p_t}(xs) = f_{p_t}(xs')$; $f_{p_h}(xs) = f_{p_h}(xs')$; $f_{p_k}(xs) = f_{p_k}(xs')$.

None of the vertices assigns distinct colors to all edges of this S_5. The proof of $C(n, S_5) = \Omega(n^{1/2})$ is finished.

Remark 18. For the remaining cases of $t > 5$, we can just take $t - 5$ vertices from A and add them to the selected set $\{x, s, s', p_t, p_h, p_k\}$ to obtain a copy of S_t such that none of its vertices assigns distinct colors to all its edges, we omit the details here.

4.3 Matchings

For the matching I_t, we first consider the cases of $t = 4$ and $t \in \{5, 6\}$, respectively. Furthermore, for any larger positive integer $t \geqslant 7$, we can even obtain a better lower bound.

4.3.1 Matching of size 4

First we prove that $C(n, I_4) = \Omega(n^{1/5})$. Let $k := cn^{1/5}$ with $\frac{1}{c^5} \gg 12$. We consider the complete graph with $3n$ vertices. For any set of $3n$ k-colorings of K_{3n}, it suffices to find a copy of I_4 such that there is no vertex that assigns distinct colors to all edges of this special copy of I_4. By Lemma 11, we can partition the vertex set of K_{3n} into two parts, say $X \cup Y$, where $|X| = 2n$ and $|Y| = n$ and there is a matching $M = \{e_1, e_2, \ldots, e_n\}$ in X, such that the number of triples $(e, e', p) \in M \times M \times Y$ with the property that $f_p(e) = f_p(e')$ is at least $\frac{n^{14/5}}{3c}$. By pigeonhole principle, there is a pair of edges in M, called (e_1, e_2), such that there are at least $\frac{n^{4/5}}{3c}$ many vertices $p \in Y$ satisfying $f_p(e_1) = f_p(e_2)$. Let $A \subseteq Y$ consist of all the vertices $p \in Y$ such that $f_p(e_1) = f_p(e_2)$, as we have mentioned above, $|A| \geqslant \frac{n^{4/5}}{3c}$. Write the vertices of e_1 and e_2 as $e_1 := v_1u_1$, $e_2 := v_2u_2$ respectively. As the size of A is at least $|A| \geqslant \frac{n^{4/5}}{3c}$, we can choose an arbitrary copy of matching $I_{|A|/2}$ in A, whose edge set is $H = \{h_1, h_2, \ldots, h_{|A|/2}\}$. Now we consider the coloring functions f_{v_1}, f_{u_1}, f_{v_2} and f_{u_2} of edges in H. Using pigeonhole principle for four times, we can find a subset $H' \subseteq H$ of edges with size $|H'| \geqslant \frac{n^{4/5}}{6c} \cdot \frac{1}{(cn^{1/5})^4} = \frac{1}{6c^5} \geqslant 2$ such that there is a pair of edges h_1 and h_2 in H' satisfying $f_{v_1}(h_1) = f_{v_1}(h_2)$, $f_{v_2}(h_1) = f_{v_2}(h_2)$, $f_{u_1}(h_1) = f_{u_1}(h_2)$ and $f_{u_2}(h_1) = f_{u_2}(h_2)$. Fix such a pair of edges h_1 and h_2 and write them as $h_1 := a_1b_1$ and $h_2 := a_2b_2$, respectively. By the above analysis, we can find a copy of I_4 in K_{3n} whose edge set is $\{e_1, e_2, h_1, h_2\}$ with the following properties:

- $f_{v_1}(h_1) = f_{v_1}(h_2)$; $f_{v_2}(h_1) = f_{v_2}(h_2)$; $f_{u_1}(h_1) = f_{u_1}(h_2)$; $f_{u_2}(h_1) = f_{u_2}(h_2)$;
- $f_{a_1}(e_1) = f_{a_1}(e_2)$; $f_{a_2}(e_1) = f_{a_2}(e_2)$; $f_{b_1}(e_1) = f_{b_1}(e_2)$; $f_{b_2}(e_1) = f_{b_2}(e_2)$.

The proof of $C(n, I_4) = \Omega(n^{1/5})$ is finished since none of the vertices in this I_4 we find can induce a rainbow coloring.

4.4 Matchings of sizes 5 and 6

In this part we show the better bounds for matchings of sizes 5 and 6. More precisely, we will show that $C(n, I_5) = \Omega(n^{1/3})$ in detail. The proof of $C(n, I_6) = \Omega(n^{1/3})$ is similar, so we just provide a simple remark.

Let $k := cn^{1/3}$, where the constant c is very small. We also focus on the complete graph with $3n$ vertices. For any set of $3n$ k-colorings of K_{3n}, we need to show that there is a copy of I_5 such that none of its vertices induces a rainbow coloring. By Lemma 11, there exists a partition of the vertex set of K_{3n} into $X \cup Y$ with $|X| = 2n$ and $|Y| = n$ and a matching $M = \{e_1, e_2, \ldots, e_n\}$ in X, such that the number of triples $(e, e', p) \in M \times M \times Y$ with the property that $f_p(e) = f_p(e')$ is at least $\frac{n^{8/3}}{3c}$. Using pigeonhole principle, we can find a pair of edges in $M := (e_1, e_2)$ with $e_1 = v_1u_1$ and $e_2 = v_2u_2$, such that there are at least $\frac{n^{2/3}}{3c}$ many vertices $p \in Y$ satisfying $f_p(e_1) = f_p(e_2)$. Let A be the subset of Y consisting of the vertices $p \in Y$ such that $f_p(e_1) = f_p(e_2)$ and $|A| = \frac{n^{2/3}}{3c}$. As $|A| = \frac{n^{2/3}}{3c}$, then we can choose an arbitrary copy of matching $I_{|A|/2}$ in A, whose edge set is $H = \{h_1, h_2, \ldots, h_{|A|/2}\}$.

Now we consider the k^2-coloring function (f_{v_1}, f_{u_1}) of edges in H. By pigeonhole principle, we can recursively find disjoint triples $(h_{3i+1}, h_{3i+2}, h_{3i+3}) \in H \times H \times H$ with the property $f_{v_1}(h_{3i+1}) = f_{v_1}(h_{3i+2}) = f_{v_1}(h_{3i+3})$ and $f_{u_1}(h_{3i+1}) = f_{u_1}(h_{3i+2}) = f_{u_1}(h_{3i+3})$ till they cover 99% of the elements in H (this is possible because c is small enough such that $\frac{1}{100}|H| \geqslant 3k^2$), then we do same operation but change v_1 to v_2 and u_1 to u_2 respectively, we will also obtain many disjoint triples cover 99% of the elements in H and every such chosen triple is colored same by the k^2-coloring (f_{v_2}, f_{u_2}). Then we can find two triples from the set of triples such that they intersect, namely, we can find two triples $\{h_t, h_j, h_k\} \subseteq H$ and $\{h_t, h_q, h_m\} \subseteq H$ such that $f_{v_1}(h_t) = f_{v_1}(h_j) = f_{v_1}(h_k)$,

$f_{u_1}(h_t) = f_{u_1}(h_j) = f_{u_1}(h_k)$, $f_{u_2}(h_t) = f_{u_2}(h_q) = f_{u_2}(h_m)$ and $f_{v_2}(h_t) = f_{v_2}(h_q) = f_{v_2}(h_m)$, where $j = q$ or $k = m$ is allowed. Then we can find five edges $\{e_1, e_2, h_t, h_j, h_m\}$ and denote the vertices of h_t, h_j, h_m as $\{a_t, b_t\}$, $\{a_j, b_j\}$, $\{a_m, b_m\}$, respectively. Through the above argument, we can see the following properties hold.

- $f_{v_1}(h_t) = f_{v_1}(h_j)$; $f_{v_2}(h_t) = f_{v_2}(h_m)$; $f_{u_1}(h_t) = f_{u_1}(h_j)$; $f_{u_2}(h_t) = f_{u_2}(h_m)$;
- $f_{a_t}(e_1) = f_{a_t}(e_2)$; $f_{a_j}(e_1) = f_{a_j}(e_2)$; $f_{a_m}(e_1) = f_{a_m}(e_2)$;
- $f_{b_t}(e_1) = f_{b_t}(e_2)$; $f_{b_j}(e_1) = f_{b_j}(e_2)$; $f_{b_m}(e_1) = f_{b_m}(e_2)$.

That means, none of the vertices in this I_5 we find above, can induce a rainbow coloring. Thus we have $C(n, I_5) = \Omega(n^{1/3})$. The proof is finished.

Remark 19. If we want to find a special copy of I_6 which does not admit the rainbow coloring, we just need to add one edge from H to the selected set $\{e_1, e_2, h_t, h_j, h_m\}$. We omit the details here.

4.4.1 Matchings with larger size

We mainly prove that $C(n, I_7) = \Omega(n^{1/2})$ here. Let $k := cn^{1/2}$, where the constant c is chosen to be very small. We still consider the complete graph with $3n$ vertices. For any set of $3n$ k-colorings of K_{3n}, using similar arguments as in Sections 4.3.1 and 4.4, there is a partition of the vertex set of K_{3n} into $X \cup Y$ with $|X| = 2n$ and $|Y| = n$ and a matching $M = \{e_1, e_2, \ldots, e_n\}$ in X, such that the number of triples $(e, e', p) \in M \times M \times Y$ with the property that $f_p(e) = f_p(e')$ is at least $\frac{n^{5/2}}{3c}$. We can also find a pair of edges in $M :=$ (e_1, e_2), such that there are at least $\frac{n^{1/2}}{3c}$ many vertices $p \in Y$ satisfying $f_p(e_1) = f_p(e_2)$, where $e_1 = v_1u_1$, $e_2 = v_2u_2$. Let A be a subset of Y consisting of the vertices $p \in Y$ such that $f_p(e_1) = f_p(e_2)$ and $|A| = \frac{n^{1/2}}{3c}$, we can form an arbitrary copy of matching $I_{|A|/2}$ in A, whose edge set is $H = \{h_1, h_2, \ldots, h_{|A|/2}\}$.

Now we consider the k-coloring function f_{v_1} of edges in H. By pigeonhole principle, we can recursively find a set of disjoint quintuples of the form $(h_{5i+1}, h_{5i+2}, h_{5i+3}, h_{5i+4}, h_{5i+5}) \in H \times H \times H \times H \times H$ with the property that $f_{v_1}(h_{5i+1}) = f_{v_1}(h_{5i+2}) = f_{v_1}(h_{5i+3}) = f_{v_1}(h_{5i+4}) = f_{v_1}(h_{5i+5})$ till it covers 99% of the elements in H (this is possible because we can set c to be small enough such that $1/100|H| \gg 5k$). Then we do same operations but change v_1 to u_1, we can also obtain another set of disjoint quintuples, which covers 99% of the elements in H. We continue the same operations twice, but replacing the corresponding vertices with u_2 and v_2. As a consequence, we can obtain four sets of internally disjoint quintuples, all of which have the desired properties. Next, we pick one quintuple from each set respectively, such that they contain some common element. That means there are $\{h_t, h_{k_1}, h_{k_2}, h_{k_3}, h_{k_4}\} \subseteq H$, $\{h_t, h_{k_5}, h_{k_6}, h_{k_7}, h_{k_8}\} \subseteq H$, $\{h_t, h_{k_9}, h_{k_{10}}, h_{k_{11}}, h_{k_{12}}\} \subseteq H$ and $\{h_t, h_{k_{13}}, h_{k_{14}}, h_{k_{15}}, h_{k_{16}}\} \subseteq H$ such that $f_{v_1}(h_t) = f_{v_1}(h_{k_1}) = f_{v_1}(h_{k_2}) = f_{v_1}(h_{k_3}) = f_{v_1}(h_{k_4})$, $f_{u_1}(h_t) = f_{u_1}(h_{k_5}) = f_{u_1}(h_{k_6}) = f_{u_1}(h_{k_7}) = f_{u_1}(h_{k_8})$, $f_{v_2}(h_t) = f_{v_2}(h_{k_9}) = f_{v_2}(h_{k_{10}}) = f_{v_2}(h_{k_{11}}) = f_{v_2}(h_{k_{12}})$ and $f_{u_2}(h_t) = f_{u_2}(h_{k_{13}}) = f_{u_2}(h_{k_{14}}) = f_{u_2}(h_{k_{15}}) = f_{u_2}(h_{k_{16}})$. Moreover, we can find one edge form each set of the above four quintuples such that they are pairwise disjoint, without loss of generality, let $\{h_{k_1}, h_{k_6}, h_{k_{11}}, h_{k_{16}}\}$ be the set of edges with the disjoint property. Then we pick seven edges $\{e_1, e_2, h_t, h_{k_1}, h_{k_6}, h_{k_{11}}, h_{k_{16}}\}$ and denote the vertices of $h_t, h_{k_1}, h_{k_6}, h_{k_{11}}, h_{k_{16}}$ as $\{a_t, b_t\}$, $\{a_{k_1}, b_{k_1}\}$, $\{a_{k_6}, b_{k_6}\}$, $\{a_{k_{11}}, b_{k_{11}}\}$ and $\{a_{k_{16}}, b_{k_{16}}\}$, respectively. Then we can see that the following properties hold.

- $f_{v_1}(h_t) = f_{v_1}(h_{k_1})$; $f_{v_2}(h_t) = f_{v_2}(h_{k_9})$; $f_{u_1}(h_t) = f_{u_1}(h_{k_5})$; $f_{u_2}(h_t) = f_{u_2}(h_{k_{16}})$;
- $f_{a_t}(e_1) = f_{a_t}(e_2)$; $f_{b_t}(e_1) = f_{b_t}(e_2)$;
- $f_{a_{k_1}}(e_1) = f_{a_{k_1}}(e_2)$; $f_{a_{k_5}}(e_1) = f_{a_{k_5}}(e_2)$; $f_{a_{k_9}}(e_1) = f_{a_{k_9}}(e_2)$; $f_{a_{k_{16}}}(e_1) = f_{a_{k_{16}}}(e_2)$;
- $f_{b_{k_1}}(e_1) = f_{b_{k_1}}(e_2)$; $f_{b_{k_5}}(e_1) = f_{b_{k_5}}(e_2)$; $f_{b_{k_9}}(e_1) = f_{b_{k_9}}(e_2)$; $f_{b_{k_{16}}}(e_1) = f_{b_{k_{16}}}(e_2)$.

That means, none of the vertices in this I_7 we find above, can give a rainbow coloring. Thus the proof of $C(n, I_7) = \Omega(n^{1/2})$ is finished.

Remark 20. when $t > 7$, if we want to find an I_t that does not admit a rainbow coloring, we just need to add $t-7$ edges from H to the selected set $\{e_1, e_2, h_t, h_{k_1}, h_{k_6}, h_{k_{11}}, h_{k_{16}}\}$.

Remark 21. The improved lower bounds for $C(n, P_{2t-1})$ and $C(n, P_{2t})$ can be obtained from the lower bounds for $C(n, I_t)$, this is because if $H' \subseteq H$ is a subgraph of H on the same set of vertices, then every lower bound for $C(n, H')$ implies the same lower bound for $C(n, H)$. This already shows that $C(n, P_t) = \Omega(n^{1/2})$ for $t \geqslant 13$. Actually we can slightly improve this result further via combining the ideas in the proof of $C(n, I_7) = \Omega(n^{1/2})$ and in Remark 16 to show that $C(n, P_2 \sqcup K_2 \sqcup K_2 \sqcup K_2) = \Omega(n^{1/2})$. As the proof is very similar, we omit the details here. Note that any path of length at least 8 contains $P_2 \sqcup K_2 \sqcup K_2 \sqcup K_2$, hence we have $C(n, P_t) = \Omega(n^{1/2})$ for $t \geqslant 8$. The third result in Theorem 6 follows.

5 Graphs with at least 6 edges have polynomial lower bounds

In this section, we will prove that for any graph H with at least 6 edges, there exists some constant $b = b(H) > 0$ such that $C(n, H) = \Omega(n^b)$. First, we need the following auxiliary lemma.

Lemma 22. *If a graph H has at least 6 edges, then H must contain at least one member of the family $\mathcal{H}_6$ as a subgraph, where $\mathcal{H}_6 = \{C_4,\ P_4,\ I_4,\ S_4,\ P_2 \sqcup P_2,\ P_2 \sqcup K_2 \sqcup K_2\}$.*

Proof. It suffices to prove the lemma when H has exactly 6 edges. Without loss of generality, we can assume that H has no isolated vertex. Our proof is based on the number of connected components of H.

1. If H has more than 4 connected components, then it must contain a copy of I_4.
2. If H has 3 connected components, then by pigeonhole principle, there exists some component with at least 2 edges, then H must contain a copy of $P_2 \sqcup K_2 \sqcup K_2$.

3. If H has 2 connected components, suppose both components have at least 2 edges, then H must contain a copy of $P_2 \sqcup P_2$. Next we assume that some component H_1 has exactly 5 edges. If H_1 contains a triangle, then H_1 contains either a copy of $P_2 \sqcup K_2$ or a copy of C_4, which means H contains either a copy of $P_2 \sqcup K_2 \sqcup K_2$ or C_4. If $H_1 \cong C_5$, then we know $C(n, C_5) \geqslant C(n, P_4) = \Omega(n^{1/5})$. Therefore it remains to consider the case that H_1 is a tree, if H_1 does not contain S_4 or P_4, then it is easy to check that H_1 contains a copy of $P_2 \sqcup K_2$, thus H contains a copy of $P_2 \sqcup K_2 \sqcup K_2$.

4. If H has only one connected component, suppose H does not contain a cycle, then either H contains a copy of S_4 or the longest path in H has at least 4 edges. Then we consider the case that H contains at least one cycle. Note that if H does not contain a copy of C_4 or P_4, then H contains a triangle. In this case, if the remaining three edges are incident to the same vertex on the triangle, then H contains a copy S_4, otherwise, H will contain a copy of P_4.

□

It remains to show that $C(n, C_4)$ also has the polynomial lower bound, we prove this result as follows.

Proposition 23.

$$C(n, C_4) = \Omega(n^{1/3}).$$

Proof of Proposition 23. Let $k := cn^{1/3}$, where the constant c is very small. We consider a complete graph with $2n$ vertices. For any set of $2n$ k-colorings of K_{2n}, we aim to find a copy of C_4 such that none of its vertices can induce a rainbow coloring.

By Lemma 12, we can partition the vertex set of K_{2n} into two parts A and B with $|A| = |B| = n$ and the number of triples $(a, b_1, b_2) \in A \times B \times B$ such that $f_a(ab_1) = f_a(ab_2)$ is at least $\frac{n^{8/3}}{3c}$. By pigeonhole principle, there is a pair of vertices in B, called (b_1, b_2), such that there are at least $\frac{n^{2/3}}{3c}$ many distinct vertices $a \in A$ satisfying $f_a(ab_1) = f_a(ab_2)$. Then we choose a subset A' which consists of the vertices satisfying the above property. It is obvious that $|A'| \geqslant \frac{n^{2/3}}{3c}$. Consider the k^2-coloring $(f_{b_1}(b_1a), f_{b_2}(b_2a))$ for all $a \in A'$, by pigeonhole principle, there are $\frac{1}{3c^3} \gg 2$ elements in A' which receive the same color. Then we pick two vertices and write them as a_1 and a_2. Now we find a copy of C_4 with vertex set $\{a_1, b_2, a_2, b_1\}$ with the following properties

- $f_{a_1}(a_1b_1) = f_{a_1}(a_1b_2)$; $f_{a_2}(a_2b_1) = f_{a_2}(a_2b_2)$; $f_{b_1}(b_1a_2) = f_{b_1}(b_1a_1)$; $f_{b_2}(b_2a_2) = f_{b_2}(b_2a_1)$.

That means, none of the vertices in this C_4 we find here, can induce a rainbow coloring. The proof of $C(n, C_4) = \Omega(n^{1/3})$ is finished.

□

The proofs of Theorems 6, 7, 8, Proposition 23, and Remarks 16 and 17 together give the proof of Theorem 9, since we can easily deal with the isolated vertices in these cases.

6 Concluding remarks and open problems

One of the most interesting problems in this topic proposed by Alon and Ben-Eliezer [1] was to decide whether the order of $C(n, P_t)$ is polynomial in n with $t \in \{3, 4, 5, 6\}$. In this paper, we show that $C(n, P_4) = \Omega(n^{1/5})$, and provide better bounds for P_5 and P_6. Now the case of P_3 remains open, and although we cannot answer this question on P_3, we give the following polynomial lower bound $C(n, C_4) = \Omega(n^{1/3})$, which perhaps gives some evidence that $C(n, P_3)$ is order of n^c for some constant $c > 0$.

We are also interested in some other small graphs which have few vertices and edges. The constant 6 in Theorem 9 cannot be directly improved by our results. However, we suspect that the constant 6 is not the best possible, it will be interesting to improve further. Moreover, motivated by the first result in Theorem 3, one can further consider the graphs with four edges, for instance, the graph consists of a triangle plus a pendant edge and the trees with four edges.

Acknowledgements

The authors would like to express our gratitude to the anonymous reviewer for the detailed and constructive comments which are helpful to the improvement of the technical presentation of this paper. The authors are extremely grateful to Prof Hong Liu for providing an important idea in the proof of Theorem 4 and offering several kind suggestions which are very helpful to the improvement of the presentation of this paper. Zixiang Xu was supported by IBS-R029-C4.

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