Hamilton Powers of Eulerian Digraphs

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Submitted: Jul 25, 2023; Accepted: May 2, 2024; Published: Jun 28, 2024

Abstract

In this note, we prove that the $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil^{th}$ power of a connected n-vertex Eulerian digraph is Hamiltonian, and provide an infinite family of digraphs for which the $\lfloor\sqrt{n}/2\rfloor^{th}$ power is not.

Mathematics Subject Classifications: 05C20, 05C45

1 Preliminaries

The k^{th} power of a (directed or undirected) graph G, denoted G^k, is the graph on the vertices of G in which there is an edge from a vertex u to a vertex v if there exists a uv-path in G of length at most k. It is well-known that the cube of any connected undirected graph is Hamiltonian (see [6, 11], also [3, Ex 10-14]). In 1974, Fleischner proved that the square of any two-connected undirected graph is Hamiltonian, solving the Plummer-Nash-Williams conjecture [4] (see [5] for a much simpler proof). Unfortunately, strongly-connected directed graphs (digraphs) may require the $\lceil n/2\rceil^{th}$ power to be Hamiltonian; even k-strong connectedness is only sufficient for guaranteeing that the $\lceil n/(2k)\rceil^{th}$ power is Hamiltonian [10]. For a general survey on Hamilton cycles in digraphs, we refer the reader to [7]. Interestingly, results for Eulerian digraphs are not nearly so bleak[1]. Through the study of minimally Eulerian digraphs (connected Eulerian digraphs with no proper connected Eulerian subgraph), we prove that

Theorem 1. *The $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil^{th}$ power of any n-vertex connected Eulerian digraph is Hamiltonian.*

In fact, we prove an even stronger result (in Theorem 4) that, given a minimally Eulerian digraph $G = (V, A)$, specifies an ordering $v_1, ..., v_n$ of V and an edge-disjoint

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[1] The first notable example of a class of digraphs requiring a "non-trivial" (say, $o(n)$) Hamiltonicity exponent are cacti, see [9] for details.

directed path (dipath) decomposition $P_1, ..., P_n$ of G, such that each P_i is a $v_i v_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil$. In addition, we provide an infinite family of minimally Eulerian digraphs for which the $\lfloor\sqrt{n}/2\rfloor^{th}$ power is not Hamiltonian (Example 5). For details regarding the importance of minimally Eulerian digraphs and their connection to the traveling salesman problem, we refer the reader to [2, 8].

1.1 Notation, Definitions, and Basic Results

Let $G = (V, A)$ be a simple digraph. If G contains a spanning directed cycle (dicycle), then G is Hamiltonian. If G contains an Euler circuit (a circuit containing every edge), then G is Eulerian. If G is connected, this is equivalent to the condition that, for every vertex $v \in V$, the indegree $d^-(v)$ equals the outdegree $d^+(v)$. If G is a connected Eulerian digraph and contains no proper connected Eulerian subgraph on the vertices of G, then G is minimally Eulerian; equivalently, a connected Eulerian digraph G is minimally Eulerian if, for any dicycle C of G, the graph $G-C := (V, A-A(C))$ is disconnected. If G contains no dicycle, then G is acyclic. For more details regarding graph theoretic definitions and notation, we refer the reader to [1]. Let

$$p_\#(G) := \frac{1}{2}\sum_{u\in V} |d^+(u) - d^-(u)|,$$

a measure of how “close” to Eulerian a digraph is, and a key ingredient in our proof. The quantity $p_\#(G)$ is exactly the minimal number of dipaths required in an edge-disjoint decomposition of G into dipaths and dicycles. That $p_\#(G)$ dipaths are required follows immediately from the definition of $p_\#(G)$ above. That $p_\#(G)$ dipaths are sufficient follows from a simple greedy algorithm (iteratively perform walks from vertices u with $d^+(u) > d^-(u)$, removing dicycles when they are formed, and only removing the dipath when a vertex v with $d^+(v) = 0$ is reached). The size of an acyclic digraph G is immediately bounded above by $p_\#(G)\,(|V|-1)$, and an even tighter estimate can be obtained relatively quickly:

Proposition 2. *Let $G = (V, A)$ be an acyclic digraph. Then $|A| \leqslant \sqrt{2p_\#(G)}\,|V|$.*

Proof. If $p_\#(G) = 0, 1, 2$, the result follows immediately, as $|A| \leqslant p_\#(G)(|V| - 1)$. Now, let $p_\#(G) > 2$, $V = \{v_1, ..., v_n\}$ be a topological sorting of G (i.e., $v_i v_j \in A$ implies that $i < j$), $k \in \mathbb{N}$ be the smallest number such that $p_\#(G) \leqslant \binom{k}{2}$, $\ell = \lceil n/k\rceil$, and $V_i = \{v_{(i-1)k+1}, ..., v_{ik}\}$, $i = 1, ..., \ell - 1$, $V_\ell = \{v_{(\ell-1)k+1}, ..., v_n\}$. There are at most $\binom{k}{2}$ edges within each of the subsets V_i, $i = 1, ..., \ell - 1$, and at most $\binom{n-k(\ell-1)}{2}$ within the subset V_ℓ. Our digraph G can be decomposed into $p_\#(G)$ edge-disjoint dipaths, and, by the topological sorting of V, each of the aforementioned $p_\#(G)$ dipaths has at most $\ell - 1$ edges between the subsets $V_1, ..., V_\ell$. Therefore, there are at most $(\ell-1)p_\#(G)$ total edges between the subsets $V_1, ..., V_\ell$. Combining these estimates gives

$$|A| \leqslant (\ell - 1)\left[\binom{k}{2} + p_\#(G)\right] + \binom{n-k(\ell-1)}{2}.$$

Dividing by $\sqrt{p_\#(G)}\,n$, we have

$$\frac{|A|}{\sqrt{p_\#(G)}\,n} \leqslant \frac{\ell-1}{n}\left(\frac{\binom{k}{2}}{\sqrt{p_\#(G)}} + \sqrt{p_\#(G)}\right) + \frac{\binom{n-k(\ell-1)}{2}}{\sqrt{p_\#(G)}n}.$$

The right hand side is convex w.r.t. $p_\#(G)$ and maximized when $p_\#(G)$ is as small as possible. We note that, by the definition of k, $p_\#(G) > \binom{k-1}{2}$. So the right hand side can be bounded above by replacing $p_\#(G)$ by $\binom{k-1}{2}$, giving

$$\frac{|A|}{\sqrt{p_\#(G)}\,n} < \frac{\ell-1}{n}\,\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n-k(\ell-1)\big)\big(n-k(\ell-1)-1\big)}{2\binom{k-1}{2}^{1/2}n}.$$

The right hand side is a convex quadratic function in the term ℓ (treating ℓ as a variable independent of n and k), and therefore achieves its maximum at one of the endpoints of the interval $[n/k, n/k+1]$. Setting $\ell = n/k$ gives

$$\frac{\ell-1}{n}\,\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n-k(\ell-1)\big)\big(n-k(\ell-1)-1\big)}{2\binom{k-1}{2}^{1/2}n} = \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}} - \frac{k^2-3k+2}{2n\binom{k-1}{2}^{1/2}},$$

and setting $\ell = n/k+1$ gives

$$\frac{\ell-1}{n}\,\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n-k(\ell-1)\big)\big(n-k(\ell-1)-1\big)}{2\binom{k-1}{2}^{1/2}n} = \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}}.$$

Noting that $k^2-3k+2 \geqslant 0$ for all $k \in \mathbb{N}$, we conclude that the maximum over the interval $[n/k, n/k+1]$ is obtained at $\ell = n/k+1$. Replacing ℓ by $n/k+1$, we have

$$|A| < \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}}\sqrt{p_\#(G)}\,n = \frac{(k-1)^{3/2}}{k(k-2)^{1/2}}\sqrt{2p_\#(G)}\,n \leqslant \sqrt{2p_\#(G)}\,n,$$

for $k \geqslant 3$ (recall, $p_\#(G) > 2$). □

From Proposition 2 we immediately obtain a bound (tight up to a multiplicative constant; see Example 5) on the maximum size of a minimally Eulerian digraph:

Proposition 3. *Let $G = (V, A)$ be a minimally Eulerian digraph. Then* $|A| \leqslant \sqrt{2(|V|-1)}\,|V| + |V| - 1$.

Proof. G is a connected Eulerian digraph, so it admits a rooted, directed subgraph T of G in which there is a unique path (in T) from the root to any other vertex of G. Every dicycle of G must intersect an edge of T, as the removal of any dicycle from a minimally Eulerian graph disconnects it. Therefore, $G - T$ is acyclic, and by Proposition 2, $|A| \leqslant |A(G-T)| + |A(T)| \leqslant \sqrt{2(|V|-1)}\,|V| + |V| - 1$. □

2 A Proof of Theorem 1 and a Lower Bound

To prove Theorem 1, we show an even stronger statement regarding minimally Eulerian digraphs.

Theorem 4. *Let $G = (V, A)$, $|V| = n > 1$, be a minimally Eulerian graph. Then there exists an ordering $v_1, ..., v_n$ of V and an n-dipath edge-disjoint decomposition $P_1, ..., P_n$ of G such that each P_i is a $v_i v_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $\lceil f(n)\sqrt{n}\,\rceil$, where*

$$f(n) = (\log_2 n)^{\log_{3/2} 2 + o(1)} \leqslant \frac{1}{2}\log_2^2 n.$$

Proof. We first show that there exists an ordering $v_1, ..., v_n$ of $V(G)$ such that there is an n-dipath edge-disjoint decomposition $P_1, ..., P_n$ of G such that each P_i is a $v_i v_{i+1}$-dipath. This ordering and decomposition can be constructed by picking a base vertex $v_1 \in V(G)$ and considering an Eulerian circuit W of G starting at v_1, ordering the remaining vertices based on the order of first appearance in this circuit, and taking each dipath P_i to be the walk in W between the first appearance of v_i and the first appearance of v_{i+1}. As G is minimally Eulerian, each such walk is a dipath. It suffices to consider $n \geqslant 6388$, as the length of a dipath is at most $n-1$ and $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\,\rceil \geqslant n-1$ for $n = 1, ..., 6387$.

Let $v_1, ..., v_n$ be an ordering of $V(G)$ and $P_1, ..., P_n$ a decomposition of G into edge-disjoint $v_i v_{i+1}$-dipaths P_i. We choose this ordering and decomposition so that the elements of the set $\{|A(P_1)|, ..., |A(P_n)|\}$ are lexicographically minimized (i.e., minimizes the length of the longest dipath, minimizes the length of the 2^{nd} longest dipath conditional on the minimality of the longest dipath, etc). Let $\widehat{P}$ be the longest dipath in the set $\{P_1, ..., P_n\}$, with length $|A(\widehat{P})| = \alpha\sqrt{n}$ for some $\alpha \geqslant \frac{1}{2}\big[\log_2 n\big]^{\log_{3/2} 2}$. We aim to build a sequence of subgraphs $H_0 (:= \widehat{P}) \subset H_1 \subset H_2 \subset ...$, bound the order of each subgraph from below using the lexicographic minimality of path lengths, and conclude that if α is too large then some H_i contains too many vertices, thus producing an upper bound on α.

Let $H_0 = \widehat{P}$. Let H_ℓ, $\ell > 0$, be the union of all P_i satisfying both $|A(P_i)| \geqslant \alpha\sqrt{n}/2^\ell$ and $\{v_i, v_{i+1}\} \cap V(H_{\ell-1}) \neq \varnothing$. Let n_ℓ, m_ℓ, and k_ℓ be the number of vertices, edges, and dipaths P_i in H_ℓ. We have $n_0 = \alpha\sqrt{n} + 1$, $m_0 = \alpha\sqrt{n}$, $k_0 = 1$ and, by construction, $m_\ell \geqslant k_\ell m_0 / 2^\ell$ for all $\ell \geqslant 0$.

We may produce a lower bound for the size of each H_ℓ by our lexicographic minimality condition. We claim that every vertex of H_ℓ is either the start- or end-vertex of a dipath P_i of length at least $m_0/2^{\ell+1}$. Suppose, to the contrary, that some $v_i \in V(H_\ell)$ satisfies $|A(P_{i-1})|, |A(P_i)| < m_0/2^{\ell+1}$. Let P_j be a dipath in H_ℓ containing v_i, and let us denote the $v_j v_i$ (resp. $v_i v_{j+1}$) portion of this path by P_j^1 (resp. P_j^2). By removing P_i, P_{i+1}, and P_j from our set $\{P_1, ..., P_n\}$ and replacing them with P_j^1, P_j^2, and $P_i \cup P_{i+1}$, we have replaced a path of length $|A(P_j)|$ ($|A(P_j)| \geqslant m_0/2^\ell$) with paths all of length strictly less than $|A(P_j)|$, a contradiction. Therefore, $k_{\ell+1} \geqslant n_\ell/2$ for all $\ell \geqslant 0$, as every vertex in $V(H_\ell)$ is the start- or end-vertex of a dipath P_i in $H_{\ell+1}$, and each dipath P_i has only one start- and one end-vertex.

The graph H_ℓ can be decomposed into the edge-disjoint union of two graphs $H_{\ell,a}$ and $H_{\ell,e}$, where $H_{\ell,a}$ is acyclic with $p_\#(H_{\ell,a}) \leqslant k_\ell$ (as H_ℓ is the edge-disjoint union of k_ℓ paths) and $H_{\ell,e}$ is the vertex-disjoint union of minimally Eulerian graphs $H_{\ell,e}^{(1)}, \ldots, H_{\ell,e}^{(p_\ell)}$ for some p_ℓ (if the Eulerian graph $H_{\ell,e}^{(j)}$ is not minimal, neither is G). By Proposition 2, $H_{\ell,a}$ has at most $\sqrt{2k_\ell}\, n_\ell$ edges. By Proposition 3, $H_{\ell,e}$ has at most

$$\sum_{j=1}^{p_\ell} \left(\sqrt{2(n_\ell^{(j)} - 1)}\, n_\ell^{(j)} + n_\ell^{(j)} - 1 \right) \leqslant \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1$$

edges, where $n_\ell^{(j)} := |V(H_{\ell,e}^{(j)})|$, $j = 1, \ldots, p_\ell$. Therefore,

$$m_\ell \leqslant \sqrt{2k_\ell}\, n_\ell + \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1.$$

Combining this inequality with the bound $m_\ell \geqslant k_\ell m_0/2^\ell$, we have

$$k_\ell m_0/2^\ell \leqslant \sqrt{2k_\ell}\, n_\ell + \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1. \tag{1}$$

Using Inequality (1), we produce a recursive lower bound on n_ℓ that gives an upper bound on α. In particular, we aim to show that

$$n_\ell \geqslant \left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^{2/3} \quad \text{for all } \ell \leqslant \log_2(5^2 \alpha). \tag{2}$$

If $n_\ell \geqslant \sqrt{2k_\ell}\, m_0/2^\ell$, then Inequality (2) immediately holds, as

$$n_\ell \geqslant \frac{\sqrt{2k_\ell}\, m_0}{2^\ell} = \left[\left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^2 \left(\frac{(2k_\ell)^{3/2} n^{1/2}}{n_{\ell-1}^2} \right) \left(\frac{5^2\, \alpha}{2^\ell} \right) \right]^{1/3} \geqslant \left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^{2/3}$$

for $\alpha \geqslant 2^\ell/5^2$. Now, suppose that $n_\ell < \sqrt{2k_\ell}\, m_0/2^\ell$. Then $k_\ell m_0/2^\ell - \sqrt{2k_\ell} n_\ell$ is monotonically increasing with respect to k_ℓ. Combining this fact with the bound $k_\ell \geqslant n_{\ell-1}/2$ and Inequality (1), we obtain

$$n_{\ell-1} m_0/2^{\ell+1} - \sqrt{n_{\ell-1}}\, n_\ell \leqslant k_\ell m_0/2^\ell - \sqrt{2k_\ell}\, n_\ell \leqslant \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1.$$

This implies that

$$n_{\ell-1} m_0/2^{\ell+1} \leqslant \sqrt{2(n_\ell - 1)}\, n_\ell + \sqrt{n_{\ell-1}}\, n_\ell + n_\ell - 1 < \frac{5}{2} n_\ell^{3/2},$$

for $n \geqslant 6388$, as $n_\ell \geqslant n_0 = \alpha\sqrt{n} + 1$, and so the claim holds in this case as well.

Using the initial bound $n_0 > m_0$ and Inequality (2), we obtain

$$\begin{aligned} n \geqslant n_\ell \geqslant n_0^{(2/3)^\ell} \prod_{i=1}^{\ell} \left(\frac{m_0}{5 \times 2^{\ell+1-i}} \right)^{(2/3)^i} &= \frac{n_0^{(2/3)^\ell}}{2^{2\ell}} \left(\frac{16 m_0^2}{25} \right)^{1-(2/3)^\ell} \\ &> \frac{16\, m_0^{2-(2/3)^\ell}}{25 \times 2^{2\ell}} \\ &= \frac{16\, \alpha^{2-(2/3)^\ell} n^{1-\frac{1}{2}(2/3)^\ell}}{25 \times 2^{2\ell}} \end{aligned}$$

for $\ell \leqslant \log_2(5^2\alpha)$. Taking the logarithm of both sides, we obtain the inequality

$$\log_2 \alpha < \frac{1}{2-(2/3)^\ell}\big(\log_2(25/16) + 2\ell + \tfrac{1}{2}(2/3)^\ell \log_2 n\big). \tag{3}$$

Setting $\ell = \lceil \log_{3/2}\big(\frac{3}{11}\log_2 n\big)\rceil$, we have $\ell < \log_2(5^2\alpha)$, as

$$\begin{aligned}\log_{3/2}\big(\tfrac{3}{11}\log_2 n\big) + 1 &= (\log_{3/2} 2)\log_2(\log_2 n) + \log_{3/2}(3/11) + 1 \\ &< (\log_{3/2} 2)\log_2(\log_2 n) + 2\log_2(5) - 1 \\ &= \log_2\big(\tfrac{5^2}{2}\log_2^{\log_{3/2} 2} n\big).\end{aligned}$$

For $\ell = \lceil \log_{3/2}\big(\frac{3}{11}\log_2 n\big)\rceil$, Inequality 3 implies that

$$\begin{aligned}\log_2 \alpha &< \frac{\log_2(25/16) + 2\big[\log_{3/2}\big(\frac{3}{11}\log_2 n\big) + 1\big] + \frac{1}{2}(2/3)^{\log_{3/2}\left(\frac{3}{11}\log_2 n\right)}\log_2 n}{2-(2/3)^{\log_{3/2}\left(\frac{3}{11}\log_2 n\right)}} \\ &= \frac{1}{1-\frac{11}{6\log_2 n}}\big[\log_2(5/2) + \log_{3/2}\big(\tfrac{3}{11}\log_2 n\big) + \tfrac{11}{12}\big].\end{aligned}$$

Taking the (base two) exponential of both sides, we obtain

$$\alpha < 2^{\frac{\log_2(5/2)-\log_{3/2}(11/3)+11/12}{1-11/(6\log_2 6388)}}\big[\log_2 n\big]^{\frac{\log_{3/2} 2}{1-11/6\log_2 6388}} \leqslant .46\big[\log_2 n\big]^{1.9995}.$$

This completes the proof. □

Finally, we give the following infinite class of digraphs to illustrate that Theorem 1 is tight up to a logarithmic factor.

Example 5. Let $G_k = (V_k, A_k)$, $k \in \mathbb{N}$, $k \geqslant 4$, where $V_k = \{u_1, ..., u_{\ell-1}, v_1, ..., v_\ell\}$, $\ell := k(k+1)/2$, and $u_iu_j \in A_k$ for $0 < j - i \leqslant k$, and $u_{\ell-\phi(i)}v_i, v_iu_{\phi(i)} \in A_k$ for all $i = 1, ..., \ell$, where $\phi(i)$ is the smallest number $p \in \mathbb{N}$ such that $\sum_{j=1}^{p}(k+1-j) \geqslant i$. This digraph is minimally Eulerian, as every dicycle contains some vertex v_i and $d^+(v_i) = d^-(v_i) = 1$ for all i. There are $n = k^2 + k - 1$ vertices and $k(k^2+2k-1)/2$ edges (i.e., about $n^{3/2}/2$). The distance between any pair v_i, v_j in the graph is at least $\lceil(\ell+1)/k\rceil = \lceil k/2\rceil + 1 \geqslant \lfloor\sqrt{n}/2\rfloor + 1$. In any Hamiltonian dicycle of a power of G_k, some pair v_i, v_j must be adjacent, and so at least the $[\lfloor\sqrt{n}/2\rfloor + 1]^{th}$ power is required. See Figure 1 for a visual example for $k = 4$.

Acknowledgements

This manuscript is the result of an undergraduate research project supported by Professor Michel Goemans through the MIT undergraduate research opportunities program (UROP), and we are thankful for his support. This project is also supported by the Lord Foundation through the MIT UROP program, and we are thankful for their support. This project is also supported by the MIT Office of Experiential Learning, and we are thankful for their support. This material is based upon work supported by the Institute for Advanced Study and the National Science Foundation under Grant No. DMS-1926686. The authors are grateful to Louisa Thomas for improving the style of presentation.

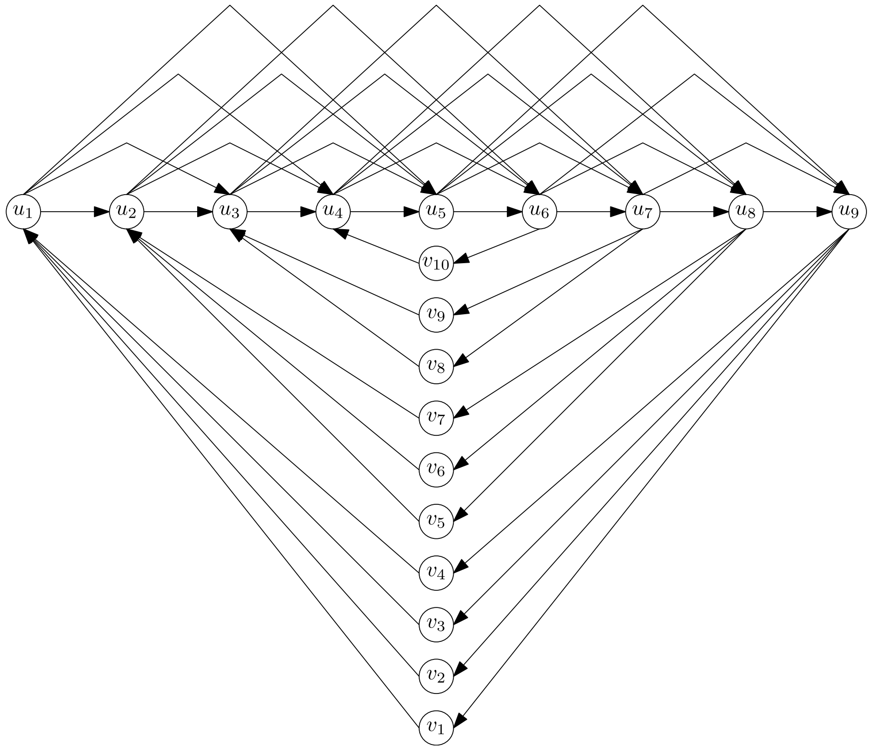

Figure 1: The minimally Eulerian graph G_k from Example 5 for $k = 4$.

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