4-Chromatic Graphs Have At Least Four Cycles of Length 0 mod 3

Sean Kim Michael E. Picollelli

Submitted: Dec 10, 2023; Accepted: May 25, 2024; Published: Jun 28, 2024

Abstract

A 2016 conjecture of Brewster, McGuinness, Moore, and Noel asserts that for $k \geqslant 3$, if a graph has chromatic number greater than k, then it contains at least as many cycles of length 0 mod k as the complete graph on $k+1$ vertices. Our main result confirms this in the $k = 3$ case by showing every 4-critical graph contains at least four cycles of length 0 mod 3, and that K_4 is the unique such graph achieving the minimum.

We make progress on the general conjecture as well, showing that $(k+1)$-critical graphs with minimum degree k have at least as many cycles of length 0 mod r as K_{k+1}, provided $k+1 \neq 0$ mod r. We also show that K_{k+1} uniquely minimizes the number of cycles of length 1 mod k among all $(k+1)$-critical graphs, strengthening a recent result of Moore and West and extending it to the $k = 3$ case.

Mathematics Subject Classifications: 05C15, 05C35, 05C38

1 Introduction

The study of cycles with a given length modulo an integer k began with work of Burr and Erdős [8], who conjectured that for odd values of k, graphs of sufficiently large average degree contain cycles of all lengths modulo k. This conjecture was proven by Bollobás [1], with the bound subsequently improved by Thomassen [16]. Further work, building in part on conjectures of Thomassen [16], has largely focused on the existence of such cycles under minimum degree or connectivity assumptions – see, for example, [5], [7], [9], or [15].

This paper is concerned with bounding the number of cycles of length 0 mod k and 1 mod k in non-k-colorable graphs. The first result in this direction is due to Tuza [17], who showed that graphs without cycles of length 1 mod k are k-colorable, generalizing König's classic characterization of bipartite graphs. For cycles of length 0 mod k, Chen and Saito [3] showed that graphs without cycles of length 0 mod 3 are 3-colorable; Dean, Lesniak and Saito [6] achieved the same conclusion for graphs without cycles of length

Department of Mathematics, California State University San Marcos, San Marcos, California, USA (`seankim1999@gmail.com`, `mpicollelli@csusm.edu`).

0 mod 4. Chen, Ma and Zang [4] established that any non-k-colorable graph is guaranteed cycles of every length $\ell \bmod k$ except possibly $\ell = 2$, settling the existence case in the $\ell = 0$ case. Very recent work has essentially settled the existence question entirely: Gao, Huo, and Ma [10] showed every $(k+1)$-critical non-complete graph has cycles of all lengths modulo k when $k \geqslant 6$. The same conclusion for the $k = 3$ case was implied by work in [3], [13], and [14], and the case where $k \in \{4, 5\}$ is the subject of recent work of Huo [11].

The question of how many such cycles must occur has received considerably less attention. Brewster, McGuinness, Moore and Noel [2] generalized a proof of Chen and Saito's result due to Wrochna to show that any graph which is not k-colorable has at least $(k-1)!/2$ cycles of length $0 \bmod k$. They further conjectured that the complete graph K_{k+1} achieves the minimum number of such cycles among all non-k-colorable graphs.

Conjecture 1.1 (Brewster, McGuinness, Moore, and Noel, [2])**.** If $\chi(G) > k$, then G contains at least $(k+1)(k-1)!/2$ cycles of length $0 \bmod k$.

Building on the arguments in [2], Moore and West [12] established bounds on the number of cycles of length $0 \bmod r$ and $1 \bmod r$ in non-k-colorable graphs, for $3 \leqslant r \leqslant k$, although those bounds fall short of yielding Conjecture 1.1.

The main contribution of this paper is to settle Conjecture 1.1 completely in the $k = 3$ case.

Theorem 1.2. *If G is 4-critical, then G has at least four cycles of length $0 \bmod 3$, with equality only for K_4.*

We also make progress in the general case by obtaining sharp lower bounds on the number of cycles of length $0 \bmod r$ in $(k+1)$-critical graphs which have minimum degree k, provided $k+1$ is not a multiple of r. This is summarized in the next result, which an immediate corollary of Theorem 2.4 in Section 2.

Theorem 1.3. *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph with $\delta(G) = k$. Then for each r, $2 \leqslant r \leqslant k$, with $k + 1 \neq 0 \bmod r$, G contains at least as many cycles of length $0 \bmod r$ as the complete graph K_{k+1}. In particular, G has at least $(k+1)(k-1)!/2$ cycles of length $0 \bmod k$.*

For cycles of length $1 \bmod k$ in non-k-colorable graphs, Moore and West [12] established a lower bound of $k!/2$, which is best possible as it is achieved by K_{k+1}. Their argument, a probabilistic application of Tuza's strengthening of Minty's Theorem, also yielded a structural condition for equality when $k \geqslant 4$.

Theorem 1.4 (Moore and West [12], Theorem 5)**.** *For $k \geqslant 3$, a non-k-colorable graph has at least $k!/2$ cycles with lengths congruent to 1 modulo k, with equality for $k \geqslant 4$ only when these cycles all have length $k+1$.*

By modifying their arguments, we achieve a slight strengthening of their result that extends to the $k = 3$ case, and relies on a simpler constructive approach.

Theorem 1.5. *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph. Then G has at least $k!/2$ cycles of length $1 \bmod k$, with equality only for the complete graph K_{k+1}.*

The remainder of this paper is organized as follows. In Section 2 we establish some results for $(k+1)$-critical graphs; Theorem 1.3 will follow immediately from Theorem 2.4. The proof of Theorem 1.5 follows in Section 3. Section 4 is then focused on the proof of Theorem 1.2.

2 Cycles in $(k+1)$-critical graphs

In this section we will establish our results on $(k+1)$-chromatic graphs. We remark that our notation largely follows [18], and we note that a graph G is **$(k+1)$-critical** if G has chromatic number $k+1$ but every proper subgraph has chromatic number at most k.

The results in this section are primarily established by modifying the approach to generalized Kempe chains used by Moore and West in [12]. To that end, we will use the the following definition from [12]:

Definition 2.1 (Moore and West [12])**.** Let G be a k-colorable graph and let φ be a proper k-coloring of G, where we assume the set of colors is $[k] = \{1, 2, \ldots, k\}$. For some r, $3 \leqslant r \leqslant k$, let σ be a cyclic permutation of r distinct elements from $[k]$. Then we define the ***σ*-subdigraph *D_σ*** to be the directed graph satisfying

1. $V(D_\sigma) = V(G)$.
2. For $u, v \in V(G)$, $uv \in E(D_\sigma)$ if and only if $uv \in E(G)$ and $\sigma(\varphi(u)) = \varphi(v)$.

For $v \in V(G)$, let A_v be the set of vertices accessible from v along directed paths in D_σ. A key observation in [12] is that we may "shift" the colors in A_v by recoloring each $w \in A_v$ with the color $\sigma(\varphi(w))$; the resulting k-coloring will still be a proper. Otherwise, a monochromatic edge xy under the recoloring would have one end $x \in A_v$ and the other end $y \notin A_v$, and must satisfy $\sigma(\varphi(x)) = \varphi(y)$, contradicting $y \notin A_v$.

As an illustration, we begin with a slight strengthening of Theorem 4 from [12] that bounds the number of cycles of a given modular length in a graph G such that G is $(k+1)$-chromatic but $G-e$ is not. We show further that the bound holds on the number of cycles through each vertex of e, although we do not claim that all such cycles found contain both vertices of e. We will use this stronger conclusion later in the proof of Theorem 1.2.

Theorem 2.2 (See Moore and West [12], Theorem 4)**.** *Fix $r, k \in \mathbb{N}$ with $3 \leqslant r \leqslant k$, and let e be an edge in a graph G. If $\chi(G) = k+1$ but $\chi(G-e) = k$, then letting x be an endpoint of e, $G-e$ contains at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles of length $0 \bmod r$ containing the vertex x.*

Proof. Letting $e = xy$, we fix a proper k-coloring φ of $G-e$ using colors $1, 2, \ldots, k$ and let σ be a cyclic r-permutation of $[k]$ that incl4udes $\varphi(x)$. We note that $\varphi(x) = \varphi(y)$ must hold, else φ is a proper k-coloring of G.

Consider the σ-subdigraph D_σ of G under φ and σ, and let A_x be the set of vertices accessible from x via directed paths. By construction of D_σ, the mapping φ' formed by setting $\varphi'(v) = \sigma(\varphi(v))$ for $v \in A_x$ and $\varphi'(v) = \varphi(v)$ otherwise is a proper coloring.

If $y \notin A_x$ then φ' yields a proper k-coloring of G, a contradiction. Therefore D_σ contains a directed path P from x to y. By symmetry, it also contains a directed path Q from y to x. The concatenation of P and Q yields a closed directed walk in D_σ. It's routine to show that the edges traversed by a closed directed walk can be decomposed into directed cycles, so every vertex on P and Q lies on a directed cycle in D_σ.

Thus, D_σ has a directed cycle containing x: let C_σ be its underlying cycle in G. Every directed cycle in D_σ visits every color from σ in order, implying that C_σ has length $0 \bmod r$, and thus the only other r-permutation which could produce the same cycle is the inverse σ^{-1}. Consequently, the number of cycles found is at least half the number of cyclic r-permutations, $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$. $\square$

As mentioned above, this bound generalizes a result from [2] in the case $r = k$ that establishes that $(k+1)$-chromatic graphs contain at least $(k-1)!/2$ cycles of length $0 \bmod k$. Our goal now is to show the stronger bound of Conjecture 1.1 holds for $(k+1)$-critical graphs of minimum degree k. We remark that while there exist $(k+1)$-critical graphs of arbitrarily large minimum degree for all $k \geqslant 3$, we are unaware of any results estimating what proportion of $(k+1)$-critical graphs have minimum degree greater than k. Since the minimum-degree-k case appears to be very 'typical' in the literature, and as such graphs are easily produced through the Hajós construction, we still think this result is of interest.

We begin with a technical lemma.

Lemma 2.3. *Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph with $\delta(G) = k$. Let v be a vertex of degree k in G with neighbors $v_1, \dots, v_k$, and let φ be a proper k-coloring of $G - v$ satisfying $\varphi(v_i) = i$. Finally, let σ be a cyclic r-permutation of elements in $[k]$. Then the σ-subdigraph of $G - v$ under φ contains directed $v_i, v_{\sigma^{-1}(i)}$-paths R^i for each color i in σ.*

Proof. Fixing i in σ, we extend φ to a proper k-coloring φ^i of $G - vv_i$ by setting $\varphi^i(v) = i$. Let D^i_σ denote the σ-subdigraph of $G - vv_i$ under φ^i, noting that it contains the σ-subdigraph of $G - v$ under φ. We claim that D^i_σ contains a directed path Q from v_i to v. Otherwise, we can shift the colors of the vertices accessible from v_i according to σ, yielding a proper k-coloring of G, a contradiction.

By definition of D^i_σ, the vertex preceding v on Q must be $v_{\sigma^{-1}(i)}$, and therefore $R^i = Q - v$ is a directed $v_i, v_{\sigma^{-1}(i)}$-path lying in the σ-subdigraph of $G - v$ as claimed. $\square$

Our next result yields Theorem 1.3 as an immediate corollary. To bound the number of cycles of length $0 \bmod r$ in a $(k+1)$-critical graph G with minimum degree k, we show that an injective mapping exists that sends cycles in K_{k+1} to cycles in G, such that the length of the image of a cycle is a multiple of that cycle's length.

Theorem 2.4. *For a graph H, let $\mathcal{C}(H)$ denote the set of all cycles in H, and $\mathcal{C}_{\leqslant i}(H)$ the set of cycles of length at most i. Let $k \geqslant 3$ and let G be a $(k+1)$-critical graph* *with $\delta(G) = k$. Then there exists an injective mapping $f : \mathcal{C}_{\leqslant k}(K_{k+1}) \to \mathcal{C}(G)$ such that $|V(f(C))| \equiv 0 \bmod |V(C)|$.*

Proof. We begin by taking K_{k+1} to have vertex set $[k+1]$. Let v be a vertex in G of degree k, and let $v_1, \ldots, v_k$ be its neighbors. Since G is $(k+1)$-critical, let φ be a proper k-coloring of $G - v$ using colors $1, 2, \ldots, k$. Since each neighbor v_i of v must receive a different color under φ, without loss of generality we assume $\varphi(v_i) = i$.

We turn now to defining our mapping f. Let C be a cycle in K_{k+1} of length $r \leqslant k$. Our construction of $f(C)$ differs depending on whether or not $k+1 \in V(C)$.

Suppose first that $k+1 \notin V(C)$: we orient C into a directed cycle, and let σ denote the corresponding cyclic r-permutation of $[k]$. Consider the σ-subdigraph of $G - v$ under φ: by Lemma 2.3, for each color i in σ it contains a directed $v_i, v_{\sigma^{-1}(i)}$-path R^i. The concatenation of these r paths, in the order $(R^i, R^{\sigma^{-1}(i)}, R^{\sigma^{-1}(\sigma^{-1}(i))}, \ldots, R^{\sigma(i)})$, produces a closed directed walk (see Figure 1), which must include the edges of at least one directed cycle $\widehat{C}$. We note that $\widehat{C}$ must follow the colors of σ in order, so letting $f(C)$ be its underlying cycle in $G - v$, $f(C)$ has length $0 \bmod r$, as required.

Suppose instead that $k+1 \in V(C)$: we construct $f(C)$ to include vertex v as follows. As above, we orient C into a directed cycle and let σ be the corresponding cyclic r-permutation, which includes color $k+1$. Since $r \leqslant k$, let $i \in [k+1]$ be a color *not* on σ and let σ' be the cyclic r-permutation of $[k]$ formed by replacing $k+1$ with i.

We next extend φ to a proper k-coloring φ^i of $G - vv_i$ by setting $\varphi^i(v) = i$, and let $D_{\sigma'}$ denote the σ'-subdigraph of $G - vv_i$ under φ^i. By Lemma 2.3, it contains directed paths $R^{\sigma'(i)}$ and R^i from $v_{\sigma'(i)}$ to v_i and v_i to $v_{\sigma'^{-1}(i)}$, respectively. Thus, the concatenation $(v, R^{\sigma'(i)}, R^i, v)$ is a closed directed walk in $D_{\sigma'}$, and therefore contains the edges of a directed cycle $\widehat{C}$ through vertex v. Let $f(C)$ be the underlying cycle of $\widehat{C}$, noting that $f(C)$ has length $0 \bmod r$.

Turning to injectivity, suppose that $f(C^1) = f(C^2)$ for some $C^1, C^2 \in \mathcal{C}^{\leqslant k}(K_{k+1})$, and let σ_1, σ_2 be the cyclic permutations of C^1, C^2 used in the construction above. We observe that $v \in V(f(C))$ if and only if $k+1 \in V(C)$, and we first suppose that $v \notin V(f(C^1))$. Then under φ, $f(C^1)$ can be oriented into a directed cycle so that it follows the colors of σ_1 in order, and it can be oriented so that follows the colors of σ_2 in order. But this implies that either $\sigma_1 = \sigma_2$, or σ_2 follows the colors of σ_1 in the reverse order ($\sigma_2 = \sigma_1^{-1}$), and in either case $C^1 = C^2$.

Suppose instead that $v \in f(C^1)$, and let σ'_1 and σ'_2 be the cyclic permutations in $[k]$ formed above from σ_1, σ_2 by replacing $k+1$ with some colors i_1, i_2, respectively. Then $f(C^1)$ can be oriented into directed cycle(s) $\widehat{C}^1, \widehat{C}^2$ so that by giving vertex v color i_j, $\widehat{C}^j$ follows the colors of σ'_j in order. If $f(C^1) - v$ has no repeated color under φ, the directed path $\widehat{C}^j - v$ follows the colors of the permutation $\sigma'_j - \{i_j\} = \sigma_j - \{k+1\}$ in order. Thus, either $\sigma_2 = \sigma_1$ or $\sigma_2 = \sigma_1^{-1}$, and $C^1 = C^2$ follows.

If, instead, $f(C^1) - v$ has repeated colors under φ, then letting r be the least length

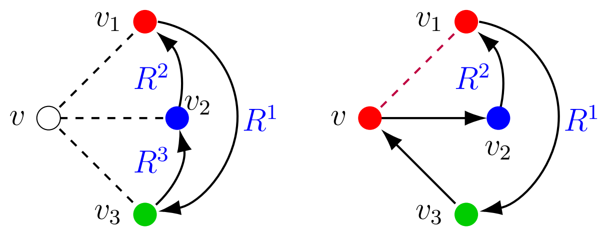

Figure 1: In the proof of Theorem 2.4 in the case $k = 3$, we give a view of the closed directed walks used to construct the cycles $f(C)$. On the left we have the case where C is the triangle on $\{1, 2, 3\}$ with $\sigma = (1\,2\,3)$, and on the right, the case where C is the triangle on $\{2, 3, 4\}$ where we take $\sigma = (1\,2\,4)$ and then let $\sigma' = (2\,3\,1) = (1\,2\,3)$. We remark that despite the depiction, the paths R^i may not be vertex-disjoint.

between repeated entries, it must hold that σ'_1, σ'_2 are cyclic r-permutations, the first $r-1$ entries of $\widehat{C}^j - v$ form the permutation $\sigma'_j - \{i_j\}$, and the rth vertex of $\widehat{C}^j - v$ is i_j. But this implies $i_1 = i_2$, and that either $\sigma'_2 = \sigma'_1$ or $\sigma'_2 = \sigma_1'^{-1}$, which then implies either $\sigma_2 = \sigma_1$ or $\sigma_2 = \sigma_1^{-1}$, and $C^1 = C^2$ follows, completing the proof that f is injective. $\square$

3 Proof of Theorem 1.5

Our aim now is to prove Theorem 1.5: let $k \geqslant 3$ and let G be a $(k+1)$-critical graph. Let v be an arbitrary vertex, and let φ be a k-coloring of $G - v$ using colors $1, 2, \dots, k$. We first show that G contains at least $k!/2$ cycles of length 1 mod k that include v.

For each $i \in [k]$, let N_i be the set of neighbors of v with color i under φ, noting $N_i \neq \varnothing$ since G is not k-colorable. Let G_i be the subgraph of G formed by deleting the edges between v and N_i, and let φ^i denote the extension of φ to a proper k-coloring of G_i by setting $\varphi^i(v) = i$.

Let σ be a cyclic permutation of $[k]$, and consider the σ-subdigraph D^i_σ of G_i under φ^i. We claim that D^i_σ must contain a directed path from v into N_i. Otherwise, letting W denote the set of vertices in D^i_σ accessible from v, we can recolor G_i by shifting the colors in W according to σ, producing a proper k-coloring φ' of G_i in which all vertices in N_i have color i and v has color $\sigma(i)$. But in that case φ' is a proper coloring of G, a contradiction.

With foresight we let $v_i \in N_i$ be a vertex on a shortest directed path P in D^i_σ from v to N_i, noting P has length 0 mod k since it begins and ends on a vertex of color i. Then P along with the edge $v_i v$ forms a directed cycle $\widehat{C}^i_\sigma$ in G of length 1 mod k; let C^i_σ be its underlying cycle in G. Since P visits all the colors of σ in order and contains the edge $v_i v$, $\widehat{C}^i_\sigma$, viewed now as an oriented subgraph of G, determines i and σ. Consequently, C^i_σ could also be the underlying cycle of $\widehat{C}^{\sigma(i)}_{\sigma^{-1}}$, but of no other such directed cycle constructed this way. So as i and σ vary, this construction produces at least $k \cdot (k-1)!/2 = k!/2$ distinct cycles of length 1 mod k containing v.

Suppose now that G has exactly $k!/2$ cycles of length 1 mod k in total. Our argument shows every vertex lies on at least $k!/2$ such cycles, so every such cycle is spanning. Fixing a cyclic permutation σ of $[k]$ and an $i \in [k]$, the constructed cycle C^i_σ is spanning, and therefore so is the shortest directed path from v to N_i in D^i_σ. Consequently, $|N_i| = 1$, implying that v has degree k and thus G is k-regular. Brooks' Theorem then implies $G = K_{k+1}$, completing the proof.

4 Proof of Theorem 1.2

This section is focused on providing the proof of Theorem 1.2. Our argument consists of the following three claims, noting that any 4-critical graph has minimum degree at least 3.

1. A 4-critical graph G with $\delta(G) = 3$ has at least four cycles of length 0 mod 3.
2. K_4 is the only 4-critical graph with $\delta(G) = 3$ and exactly four cycles of length 0 mod 3.
3. A 4-critical graph G with $\delta(G) \geqslant 4$ has at least five cycles of length 0 mod 3.

The first claim follows from Theorem 1.3. The second claim is the subject of Lemma 4.1 below. The third claim, which is the most technical argument of this paper, follows from Lemmas 4.3 and 4.7 in Section 4.1 below.

Lemma 4.1. *K_4 is the only 4-critical graph with minimum degree 3 that has exactly four cycles of length* 0 mod 3.

Proof. Suppose G is 4-critical, $\delta(G) = 3$, and G has exactly four cycles of length 0 mod 3. Let v be a vertex in G with degree 3 and label its neighbors v_1, v_2, v_3. Let φ be a proper 3-coloring of $G - v$ satisfying $\varphi(v_i) = i$, and let $\sigma = (1\,2\,3)$. Finally, we let R^1, R^2, R^3 denote the three directed paths in the σ-subdigraph of $G - v$ guaranteed to exist by Lemma 2.3.

For $i \in [3]$, let φ^i be the extension of φ to $G - vv_i$ formed by setting $\varphi^i(v) = i$, and let D^i_σ be the σ-subdigraph of $G - vv_i$ under φ^i. Then the closed walk in D^i_σ formed by the concatenation $(v, R^{\sigma(i)}, R^i, v)$ includes the (oriented) edges of a cycle C^i in G containing edges vv_i and $vv_{\sigma^{-1}(i)}$ in G of length 0 mod 3, and the closed walk formed by the concatenation (R_1, R_3, R_2) includes the edges of a cycle C^4 in $G - v$ of length 0 mod 3. (The closed walks are also illustrated in Figure 1.)

As the cycles C^1, C^2, C^3, C^4 are distinct, these are the only cycles of length 0 mod 3 in G. It follows then that the closed walk (R_1, R_3, R_2) includes only the edges of C^4, and, by construction, C^4 contains the paths $C^i - v$, $i \in [3]$. We further claim that C^4 spans $G - v$, as any vertex x in $G - v$ not lying on C^4 must lie on a (fifth) cycle of length 0 mod 3 by Theorem 2.2, taking e to be any edge incident to x, a contradiction.

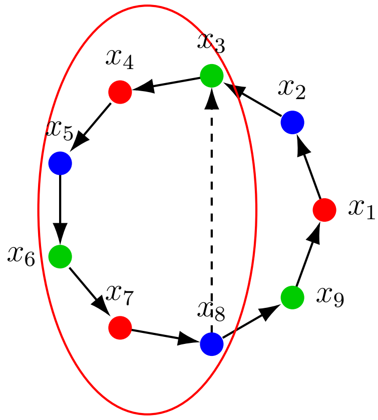

Figure 2: In the proof of Lemma 4.1, an illustration of the cycle C^4 containing a chord, which necessarily produces a fifth cycle of length 0 mod 3.

If C^4 is a triangle then it follows that $G = K_4$, so we suppose otherwise. Since $\sigma = (1\,2\,3)$, we can label C^4's vertices $x_1, x_2, x_3, \ldots, x_{3k}$ so that the edge set of C^4 is $E(C^4) = \{x_1x_2, x_2x_3, \ldots, x_{3k-1}x_{3k}, x_{3k}x_1\}$ and

$$\varphi(x_i) = \begin{cases} 1 & \text{if } i \equiv 1 \bmod 3, \\ 2 & \text{if } i \equiv 2 \bmod 3, \\ 3 & \text{if } i \equiv 0 \bmod 3. \end{cases}$$

Since G has minimum degree 3 and v has only three neighbors, C^4 contains chords. But chords can only connect vertices of different colors, and any such chord cuts C^4 into two cycles, one of which has length 0 mod 3, a contradiction that completes the proof. (See Figure 2.)

□

4.1 4-critical graphs with minimum degree at least 4

To finish the proof of Theorem 1.2, we must now show that every 4-critical graph G with minimum degree at least 4 contains at least five cycles of length 0 mod 3. In this case, the argument used in Theorem 2.4 does not directly adapt as it obtained cycles from closed directed walks. A key to constructing these walks was that every neighbor of a vertex v of degree 3 in a 4-critical graph receives a different color under a proper 3-coloring of $G - v$, which does not apply once $\delta(G) \geqslant 4$.

Instead, our arguments will rely on two other results. The first is a sufficient condition for cycles of length 0 mod 3 to exist due to Chen and Saito [3]:

Theorem 4.2 (Chen and Saito [3], Theorem 1)**.** *If G is a graph with $n \geqslant 2$ vertices and at most one vertex of degree 2 or less, then G contains a cycle of length* 0 mod 3.

We remark that in [3], the authors refer to cycles with length divisible by 3 as **good cycles**, and we will do so for the remainder of this section. Our first application of this will be to handle the case where a vertex lies on at least three good cycles.

Lemma 4.3. *If G is a 4-critical graph with $\delta(G) \geqslant 4$ and there exists a vertex $v \in V(G)$ such that v lies on at least three good cycles, then G contains at least five good cycles.*

Proof. Suppose $v \in V(G)$ lies on at least three good cycles, call them C^1, C^2, C^3. Since $\delta(G) \geqslant 4$, $\delta(G - v) \geqslant 3$, so $G - v$ contains at least one good cycle C^4 by Theorem 4.2.

If $V(C^4) \nsubseteq N(v)$ then C^4 contains an edge e with at most one endpoint in $N(v)$. Deleting e from $G - v$ produces a subgraph with at most one vertex of degree 2, and thus it contains a good cycle $C^5 \neq C^4$ by Theorem 4.2, yielding the result.

If $V(C^4) \subseteq N(v)$, then every edge of C^4 forms a triangle with v, and we're done if $|V(C^4)| \geqslant 5$. But if $|V(C^4)| < 5$, then $|V(C^4)| = 3$ since C^4 is good. This implies G contains a K_4 on $C^4 \cup \{v\}$, contradicting G's 4-criticality and completing the proof. □

We'll also appeal to a recent result of Gao, Huo, Liu and Ma [9], which was used to resolve a number of conjectures regarding the existence of cycles of prescribed lengths.

Definition 4.4 (Gao, Huo, Liu and Ma [9])**.** A collection of ℓ paths is **admissible** if the length of every path is at least two, and the lengths form an arithmetic progression with common difference one or two.

Theorem 4.5 (Gao, Huo, Liu, and Ma [9], Theorem 1.2)**.** *Let G be a 2-connected graph and let x, y be distinct vertices of G. If every vertex in G other than x and y has degree at least $\ell + 1$, then there exist ℓ admissible paths from x to y in G.*

We note that Theorem 2.2 implies that every vertex v in a 4-critical graph lies on at least two good cycles, by applying the Theorem first with an arbitrary edge incident with v, then with an edge incident with v on the found cycle.

Lemma 4.6. *If G is 4-critical with $\delta(G) \geqslant 4$ and every vertex is on at most two good cycles, then:*

1. *G is 4-regular, and*
2. *Every edge of G lies on exactly one good cycle.*

Proof. It suffices to argue that every edge lies on at least one good cycle. This will imply that every vertex lies on at least $d(v)/2$ good cycles, yielding a maximum degree of at most 4 under the assumptions given and therefore G is 4-regular. Furthermore, if any edge uv lies on at least two good cycles, then those two cycles only cover three edges incident with v, implying v lies on a third good cycle, a contradiction.

To that end, pick any edge $e = uv$. It is routine to show that a color-critical graph is 2-connected (e.g., see [18], Exercise 8.2.11), so by Theorem 4.5 there exist 3 admissible paths from u to v. Since the common difference is one or two, their lengths cover the congruence classes modulo 3, so one of these paths has length 2 mod 3 and forms a good cycle with e. □

The next lemma will complete the proof of Theorem 1.2.

Lemma 4.7. *If G is a 4-critical graph with minimum degree at least 4 and every vertex is on at most two good cycles, then G contains at least five good cycles.*

Proof. We first observe that under the given assumptions and by Theorem 2.2, every vertex lies on exactly two good cycles. Furthermore, deleting any vertex yields a subgraph with minimum degree at least 3, which contains a good cycle by Theorem 4.2. Thus, G has at least three good cycles, at least one of which is not spanning.

Let C^1 be a non-spanning good cycle, and let uv be an edge connecting C^1 to the rest of G, where $u \in V(C^1)$ and $v \in V(G) - V(C^1)$. Then, by Lemma 4.6, the edge uv lies on a good cycle C^2, and then v must lie on a second good cycle C^3. Our argument proceeds by considering two cases: whether or not C^2 is a triangle.

Case 1: C^2 is not a triangle. By Lemma 4.6, C^2 is the only good cycle containing edge uv, and G is 4-regular. Consequently, $N_G(u) \cap N_G(v) = \emptyset$, so let $U = N_G(u) - v$ and $V = N_G(v) - u$, and label their vertices u_1, u_2, u_3 and v_1, v_2, v_3, respectively. Observe that if $H = G - u - v$, then for any vertex $w \in U \cup V$, $d_H(w) = 3$, while for any vertex $x \in (V(H) - U \cup V)$, $d_H(x) = 4$. Thus, by Theorem 4.2, there exists a fourth good cycle that does not include u nor v; label it C. We have 2 subcases for C:

1. C has an edge e with an endpoint not in $U \cup V$
2. $V(C) \subseteq U \cup V$.

In Subcase 1, the edge e has at most one endpoint of degree 3. So, its removal will result in a graph with at most one vertex of degree 2. Therefore, by Theorem 4.2, we can see that there must exist another good cycle, call it C', that does not contain u, v, nor e. Thus, C' is our fifth good cycle in G.

For Subcase 2, note that no edge in C is contained in U or in V, else it lies on a triangle, contradicting the fact that every edge is on exactly one good cycle. Therefore, C is bipartite, and because C is good, it must be a 6-cycle that spans $U \cup V$.

Without loss of generality, we may assume $C = [u_1, v_1, u_2, v_2, u_3, v_3]$. Now, let $C' = [u, u_1, v_1, u_2, v_2, v]$. C and C' are two different good cycles that share common edges; a contradiction. (See Figure 3.) Thus, in Case 1, G has at least five good cycles.

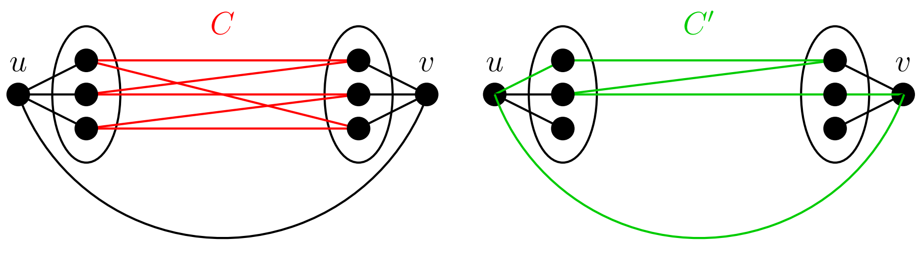

Figure 3: This image represents the fourth and fifth good cycles constructed in Subcase 2 of Case 1, noting this yields a contradiction as they share edges.

Case 2: C^2 is a triangle. Since C^2 is a triangle, there is a vertex w such that $V(C^2) = \{u, v, w\}$. Let $U = N_G(u) - v - w$, $V = N_G(v) - u - w$, and $W = N_G(w) - u - v$. We note that U, V, W are pairwise disjoint, else we contradict Lemma 4.6, and we may label their vertices $U = \{u_1, u_2\}, V = \{v_1, v_2\}$, and $W = \{w_1, w_2\}$. In the subgraph $H = G - u - v - w$, all vertices in U, V, and W will have degree 3, while all other vertices in H will have degree 4. So by Theorem 4.2, H has a good cycle C which does not use vertices u or v. Thus, C is a fourth good cycle in G, and we consider 2 subcases:

1. C has an edge e that has an endpoint not in $U \cup V \cup W$,

2. $V(C)$ is a subset of $U \cup V \cup W$.

In Subcase 1, similar to our argument in Case 1, removing e will result in a graph with at most one vertex with degree at most 2. Then Theorem 4.2 yields that G contains a fifth distinct good cycle.

So we will focus on Subcase 2. By similar reasoning as in Subcase 2 of Case 1, we know that C cannot have an edge between u_1 and u_2, between v_1 and v_2, or between w_1 and w_2. So, C is tripartite. Suppose first that C contains a vertex whose neighbors lie in different parts: without loss of generality, we may suppose that C contains u_1, v_1, w_1 and u_1 is adjacent to v_1, w_1. Then the cycle $C' = [u_1, v_1, v, u, w, w_1]$ is a different good cycle that shares edges with C; a contradiction. (See Figure 4.)

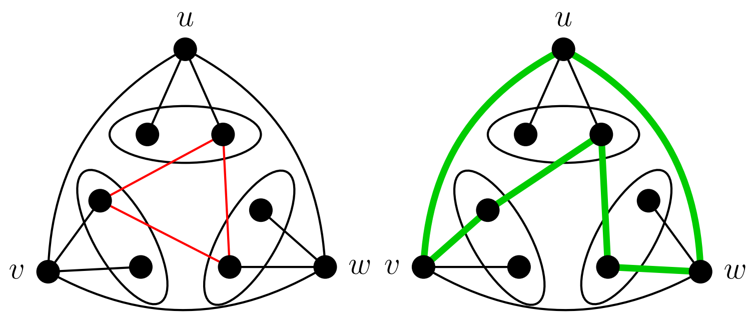

Figure 4: In Case 2, Subcase 2, the image on the left represents G with a good cycle C containing a vertex with neighbors in both other parts. The image on the right illustrates a second good cycle C' outlined in green, sharing edges with C.

If no vertex of C has neighbors in different parts, then letting $x \in V(C)$, x's neighbors lie in one part, and its neighbors' neighbors lie in the same part as x. That is, C must be contained in two parts and is therefore bipartite on at most 4 vertices, contradicting that C is good. Thus, in Case 2, G has at least five good cycles, completing the proof.

□

Acknowledgement

The authors wish to thank André Kündgen for a careful reading of an early draft and many helpful comments.

References

[1] B. Bollobás, Cycles modulo k, *Bull. London Math. Soc.* 9 (1977), no. 1, 97-98.

[2] R. C. Brewster, S. McGuinness, B. Moore, and J. A. Noel, A Dichotomy Theorem for Circular Colouring Reconfiguration. *Theoret. Comput. Sci.* 639 (2016), 1–13.

[3] G.T. Chen, A. Saito, Graphs with a Cycle of Length Divisible by Three. *J. Combinatorial Theory, Ser. B*, 60 (1994), 277–292.

[4] Z. Chen, J. Ma, and W. Zang, Colouring digraphs with forbidden cycles. *J. Combinatorial Theory, Ser. B* 115 (2015), 210–223.

[5] N. Dean, Which graphs are pancyclic modulo k?, Sixth Internat. Conf. on the Theory of Applications of Graphs, Kalamazoo, Michigan (1988) 315–326.

[6] N. Dean, L. Lesniak and A. Saito, Cycles of length 0 modulo 4 in graphs, *Discrete Math.* 121 (1993), 37–49.

[7] A. Diwan, Cycles of even lengths modulo k, *J. Graph Theory* 65 (2010), no. 3, 246–252.

[8] P. Erdős, Some recent problems and results in graph theory, combinatorics and number theory, *Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing* (Louisiana State Univ., Baton Rouge, La., 1976), Congress. Numer. XVII, pp. 3–14,

[9] J. Gao, Q. Huo, C. Liu, and J. Ma, A Unified Proof of Conjectures on Cycle Lengths in Graphs, *Intl. Math. Res. Notices* 10 (2021), 7615-7653.

[10] J. Gao, Q. Huo, and J. Ma, A strengthening on odd cycles in graphs of given chromatic number, *SIAM J. Discrete Math.* 35 (2021), 2317-2327.

[11] Q. Huo, A note on cycle lengths in graphs of chromatic number five and six, arXiv:2104.01382.

[12] B. Moore and D. B. West, Cycles in Color-Critical Graphs, *Electronic J. Combin.* 28(4) (2021), #P4.35.

[13] M. Lu and Z. Yu, Cycles of length 1 modulo 3 in graph, *Discrete Applied Math.* 113 (2001), 329-336.

[14] A. Saito, Cycles of length 2 modulo 3 in graphs, *Discrete Math.* 101 (1992), 285-289.

[15] B. Sudakov and J. Verstraëte, The extremal function for cycles of length ℓ mod k, *Electronic J. Combin.* 24(1) (2017), #P1.7.

[16] C. Thomassen, Graph decomposition with applications to subdivisions and path systems modulo k, *J. Graph Theory* 7 (1983), 261-271.

[17] Zs. Tuza, Graph coloring in linear time, *J. Combinatorial Theory, Ser. B* 55 (1992), no. 2, 236-243.

[18] D. West, *Combinatorial Mathematics*, Cambridge University Press, Cambridge, 2021.