Bender–Knuth involutions for types B and C

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Abstract

We show that the combinatorial definitions of King and Sundaram of the symmetric polynomials of types B and C are indeed symmetric, in the sense that they are invariant by the action of the Weyl groups. Our proof is combinatorial and inspired by Bender and Knuth's classic involutions for type A.

Mathematics Subject Classifications: 05E05, 05E10, 05E18, 20C33

1 Introduction

Symmetric polynomials are a central object of study in two branches of mathematics. On the one side, for combinatorialists, they are generating functions. On the other side, for representation theorists, they are characters of representations. The interplay between these two points of view is best presented with a diagram (see Figure 1).
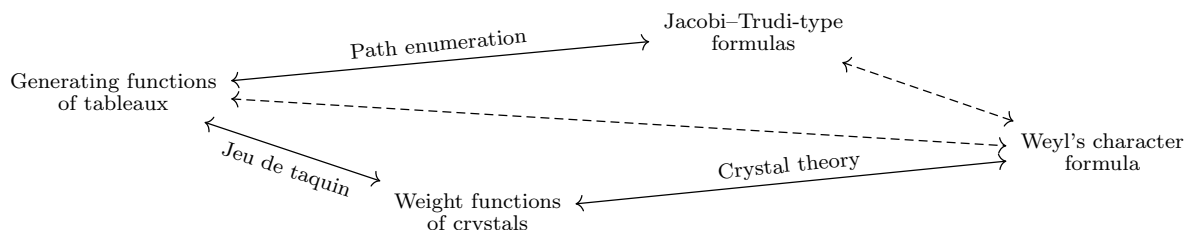

Figure 1: Different approaches to the topic in the literature.

In type A, corresponding to the representation theory of GL_n and its combinatorics, the objects and relations in Figure 1 are well understood. *Schur (symmetric) polynomials* arise as the generating functions of semistandard Young tableaux and as the (Weyl) characters of irreducible polynomial representations of GL_n. They are also the generating functions of Gelfand–Tsetlin patterns [Sta99]. For other Lie groups such as SO_{2n+1} (type B) and Sp_{2n} (type C), we have different candidates for tableaux and patterns whose generating

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functions are the *orthogonal (Schur) polynomials* and *symplectic (Schur) polynomials*. In this work, we focus on the tableau defined by King for type C [Kin76] and Sundaram for type B [Sun90], who show that their generating functions verify some recursive algebraic formulas and deduce that they recover the irreducible characters of the corresponding Lie groups [Sun86, Sun90].

Characters can be computed as the determinants of matrices whose entries are elementary symmetric polynomials. For type A this is the Jacobi–Trudi formula [Sta99]; for types B and C, see [FH91, Lec. 24 and App. A], [KT87]. These determinants enumerate tableaux after an argument of Gessel and Viennot for type A [GV85], [Sta99, Sec. 7.16]; see [FK97] for the type B and C analogues.

On the other hand, crystal bases allow Kashiwara and Nakashima to propose their own tableaux definitions [KN94]. For type A, these recover the above combinatorics; for type C, they are seen to be in bijection with King tableaux in [She99], via an analogue of the jeu de taquin algorithm. We are not aware of analogous bijections for type B in the literature.

Without direct proofs, it is not immediately obvious that generating functions of tableaux are the correct candidates for describing characters of representations. In particular, we ask whether these generating functions are invariant by the action of the Weyl group of corresponding type; $W(A_{n-1}) \cong S_n$ and $W(B_n) \cong S_2 \wr S_n \cong W(C_n)$. This is certainly true of Weyl characters, but we seek a direct and combinatorial proof.

For type A, a short argument by Bender and Knuth [BK72], [Sta99, Thm. 7.10.2] shows that the number of semistandard Young tableaux of a fixed shape λ with weight x^α coincides with the number of semistandard Young tableaux of shape λ with weight $(j \quad j+1).x^\alpha$ for any simple transposition $(j \quad j+1)$, $j = 1, ..., n-1$. This is done by constructing an involution BK_j^{A} which is known as a *(type A) Bender–Knuth involution*. Therefore, the Schur polynomial of shape λ is a *symmetric* polynomial; an element of $\mathbb{C}[x_1, ..., x_n]^{S_n}$. As a remark, these involutions do *not* induce an action of S_n on the set of tableaux of fixed shape, in general. Type A Bender–Knuth involutions are translated to Gelfand–Tsetlin patterns in [BK95]; we review these constructions later in this text.

We introduce *type C Bender–Knuth involutions* BK_j^{C} and show combinatorially and directly for the first time that there is an involutory action of $S_2 \wr S_n$ on the set of King patterns of top row λ. We conclude that symplectic polynomials lie in $\mathbb{C}[x_1^\pm, ..., x_n^\pm]^{S_2 \wr S_n}$. As a corollary, we get *type B Bender–Knuth involutions* and the analogous result for orthogonal polynomials. The maps are later translated to tableaux. As expected, our involutions do not define an action of $S_2 \wr S_n$ on the sets of patterns of a fixed shape (see [Gut23, Note 4.11]).

These results were first claimed by Sundaram in [Sun86]. However, as noted by Hopkins [Hop20], the original proof is incorrect and cannot be fixed in any simple way. (The claimed proof only shows that the functions are invariant by a set of elements of $S_2 \wr S_n$ which do not form a generating set for the group.) A corollary of our result, also noted by Sundaram [Sun86, Sun90], is that the symplectic and orthogonal polynomials define class functions on the set of diagonalizable elements in the algebraic groups of types B and C.

A first candidate for type C Bender–Knuth involutions is given as a composition of type A Bender–Knuth involutions. The resulting patterns may not be symplectic, so we post-compose with a *rectification* map. A 'locality' argument allows us to reduce our proof to computing that BK_2^C is an involution on a generic pattern when $n = 3$. This reduction is what enables us to conclude the result.

It is worth remarking that there are two ways of approaching this computation on generic patterns when $n = 3$. One is to argue directly and 'by hand', as we do. Alternatively, one could use a computer to check that BK_2^C (as a tropical rational map) is involutory. This appears to be beyond the reach of computer algebra systems at the time of writing. But we were able to check that $\text{Trop}^{-1}BK_2^C$ (as a rational map) is involutory. It remains to argue that the order of BK_2^C and the order of $\text{Trop}^{-1}BK_2^C$ coincide. This step is in general non-trivial (see e.g. [GR16]).

We recall some preliminary definitions in Section 2. We define tableaux and patterns for types A, B, and C in Section 3, and we define symmetric polynomials as their generating functions. We recall type A Bender–Knuth involutions and introduce the type B and C analogues in Section 4. Proving that these are involutions reduces to a computation, that we leave for Section 5.

2   Preliminary definitions

Fix a natural number $n \geqslant 1$ throughout this work. We work over $\mathbb{C}$. We will follow [Sta99, Ch. 7] for the standard concepts on symmetric polynomials, and [FH91] for Lie theory.

2.1   Symmetric polynomials and partitions

The *space of symmetric polynomials* in n variables is $\Lambda_n = \mathbb{C}[x_1, ..., x_n]^{S_n}$, where the symmetric group S_n acts by permuting the variables. A *partition* λ of length smaller or equal to n is an n-tuple of weakly decreasing non-negative integers. Let Par_n be the set of partitions of length smaller or equal to n. Bases of Λ_n are indexed by Par_n. We represent partitions through their *Young diagram*, which we draw following the English convention.

2.2   Lie groups and Weyl groups

The set $\text{Irr}(\text{GL}_n)$ of irreducible polynomial representations of GL_n is indexed by Par_n. The relationship between $\text{Irr}(\text{GL}_n)$ and Λ_n is explained through the following result: the (Weyl) characters of irreducible polynomial representations of GL_n form a basis of Λ_n. The character of the irreducible representation indexed by λ is the Schur polynomial $s_\lambda(x_1, ..., x_n)$, as defined combinatorially in Section 3.3.

The sets $\text{Irr}(\text{Sp}_{2n})$ and $\text{Irr}(\text{SO}_{2n+1})$ are also indexed by Par_n. (The set of irreducible representations of the Lie algebra $\mathfrak{so}(2n + 1)$ is richer, and indexed by the set of partitions and *half-partitions*. The representations indexed by half-partitions are called spin

representations, and will not be modelled by the combinatorics of this document.) The irreducible characters for SO_{2n+1} and Sp_{2n} are known as the *symmetric polynomials of types B and C*. These will be defined purely combinatorially in Section 3.3, and referred to as *orthogonal polynomials* and *symplectic polynomials*, respectively. Note that these are not in Λ_n. Rather, they lie in the ring $\mathbb{C}[x_1^{\pm}, ..., x_n^{\pm}]^W$ of Laurent polynomials invariant under the Weyl group of corresponding type (as a permutation group of the variables). In type A, these Laurent polynomials are polynomials, and the Weyl group of GL_n is $W(A_{n-1}) \cong S_n$; this is consistent with the above. The Weyl groups of type B and C coincide, and are isomorphic to the wreath product $S_2 \wr S_n$. We avoid wreath products in this work, and instead provide the following description of these Weyl groups as sub-groups of S_{2n}: if we interpret S_{2n} as the permutation group of the set $\{1, \bar{1}, 2, \bar{2}, ..., n, \bar{n}\}$, then $W(B_n)$ and $W(C_n)$ are isomorphic to the subgroup of S_{2n} generated by $(1\ \bar{1})$ and the permutations $(j\ \ j+1)(\bar{j}\ \ \overline{j+1})$ for $j \in [n-1]$.

3   Combinatorics and symmetric functions

3.1   Tableaux for types A, B, and C

Fix $\lambda \in \mathrm{Par}_n$. Let $[\lambda] := \{(i,j) \in [n] \times \mathbb{Z}_{>0} : j \leqslant \lambda_i\}$ be the set of its cells. Let $\mathcal{X}$ be a totally ordered set. A tableau of shape λ in the alphabet $\mathcal{X}$ is a function $T : [\lambda] \to \mathcal{X}$. We say a tableau is *semistandard* if $T(i,j) < T(i+1,j)$ and $T(i,j) \leqslant T(i,j+1)$ whenever this makes sense.

The (set-wise) co-restriction of a map $f : A \to B$ to a subset $C \subset B$ is defined to be the restriction of f to $f^{-1}(C)$.

Definition 1. Let $\mathcal{A} := \{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$, and $\mathcal{A}_\infty := \{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty\}$ be two ordered sets.

(A) A *semistandard Young tableaux* (on n letters) of shape λ is a semistandard tableau of shape λ in the alphabet $[n] = \{1 < 2 < \cdots < n\}$.

(B) A *(Sundaram) orthogonal tableau* T (on n letters) of shape λ is a tableau of shape λ in the alphabet $\mathcal{A}_\infty$ such that

- the co-restriction of T to $\mathcal{A}$ defines a symplectic tableau (see below), and
- at most one cell per row (the right-most cell) takes the value ∞.

(C) A *(King) symplectic tableau* T (on n letters) of shape λ is a semistandard tableau of shape λ in the alphabet $\mathcal{A}$ such that $T(i,j) \geqslant i$ for all $(i,j) \in [\lambda]$.

We let $\mathrm{SSYT}_n(\lambda)$, $\mathrm{SOT}_n(\lambda)$, and $\mathrm{KSpT}_n(\lambda)$ denote the sets of such tableaux.

The weight of a tableau $T : [\lambda] \to \mathcal{X}$ is the monomial $x^T = \prod_{a \in \mathcal{X}} x_{T^{-1}(a)}$ in the ring $\mathbb{C}[x_i : i \in \mathcal{X}]$. We take the conventions $x_{\bar{i}} = x_i^{-1}$ and $x_\infty = 1$. That is, weights of semi-standard Young tableaux are monomials in $\mathbb{C}[x_1, ..., x_n]$, whereas weights of orthogonal and symplectic tableaux lie in $\mathbb{C}[x_1^{\pm}, ..., x_n^{\pm}]$.

Example 2. We present a semistandard Young tableau, an orthogonal tableau, and a symplectic tableau of shape $(3, 3, 2)$ and their weights.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \quad x_1^2 x_2 x_3^3 x_4^2 \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & \infty \\ \hline 3 & 3 & \infty \\ \hline \bar{3} & \bar{3} \\ \cline{1-2} \end{array} \quad x_1 x_2 \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & \bar{2} \\ \hline 3 & 3 & \bar{3} \\ \hline \bar{3} & \bar{3} \\ \cline{1-2} \end{array} \quad x_1 x_3^{-1}$$

Note 3. Co-restriction to $\mathcal{A}$ defines a bijection $\mathrm{SOT}_n(\lambda) \to \bigcup_\mu \mathrm{KSpT}_n(\mu)$, where μ ranges over the partitions which may be formed from $[\lambda]$ by removing at most one cell per row. The order-preserving map $\mathcal{A} \to [2n]$ gives an inclusion $\mathrm{KSpT}_n(\lambda) \subseteq \mathrm{SSYT}_{2n}(\lambda)$.

3.2 Patterns for types A, B, and C

Definition 4. A *Gelfand–Tsetlin pattern* (or GT pattern) with n rows is a triangular array of non-negative integers $P = (P^{(n)}, ..., P^{(1)})$, with $P^{(k)} = (p_{1k}, ..., p_{kk})$ for $k \in [n]$, subject to the local inequalities of Figure 2. We say $P^{(k)}$ for $k \in [n]$ are the rows of P and we call $P^{(n)}$ the *top row*. Note that $P^{(k)} \in \mathrm{Par}_k$ for each k. Let $\mathrm{GT}_n(\lambda)$ be the set of GT patterns with n rows and top row λ.

$$\begin{array}{cccc} p_{14} & p_{24} & p_{34} & p_{44} \\ & p_{13} & p_{23} & p_{33} \\ & & p_{12} & p_{22} \\ & & & p_{11} \end{array} \qquad \begin{array}{ccc} p_{i,j+1} & & p_{i+1,j+1} \\ & \searrow \quad \nearrow \\ & p_{i,j} \\ & \nearrow \quad \searrow \\ p_{i-1,j-1} & & p_{i,j-1} \end{array}$$

Figure 2: Left: the arrangement of a GT pattern of size 4. Right: the local inequalities.

We have a bijection $\mathrm{SSYT}_n(\lambda) \to \mathrm{GT}_n(\lambda)$, by letting $P^{(k)}$ be the shape of $T^{-1}[k]$ (see [Sta99, Sec. 7.10]). In other words, $p_{i,j}$ counts the number of entries smaller or equal to j in the ith row of T. See Example 7.

Trough this bijection, the jth row sum $S_j := \sum_i p_{ij}$ of a pattern P counts the number of entries smaller or equal to j in the corresponding tableau. Therefore, if the weight of a pattern P is defined as the monomial $x^P := x_1^{S_1} x_2^{S_1 - S_2} \cdots x_n^{S_n - S_{n-1}}$, the bijection is weight-preserving.

Definition 5. A *(King) symplectic pattern* is a Gelfand–Tsetlin pattern P in which $p_{ij} = 0$ whenever $2i > j$ (see [Kin76]). We let $\mathrm{KSpP}_n(\lambda)$ be the set of symplectic patterns with $2n$ rows and top row λ.

We think of these as "half-triangular" arrays by omitting the entries p_{ij} with $2i > j$. See Example 7. We introduce the following definition.

Definition 6. A *(Sundaram) orthogonal pattern* is a symplectic pattern in which top row entries might be circled. Let P be an orthogonal pattern with N rows. The shape λ of P is defined by $\lambda_i := p_{iN} + 1$ if p_{iN} is circled and $\lambda_i := p_{iN}$ otherwise. For a partition λ, we let $\mathrm{SOP}_n(\lambda)$ be the set of orthogonal patterns with $2n$ rows and shape λ.

The maps from Note 3 together with the bijection $\mathrm{SSYT}_n(\lambda) \leftrightarrow \mathrm{GT}_n(\lambda)$ give maps

$$\mathrm{KSpT}_n(\lambda) \leftrightarrow \mathrm{KSpP}_n(\lambda) \subseteq \mathrm{GT}_{2n}(\lambda) \quad \text{and} \quad \mathrm{SOT}_n(\lambda) \leftrightarrow \mathrm{SOP}_n(\lambda) \leftrightarrow \bigcup_\mu \mathrm{KSpP}_n(\mu).$$

Through these bijections, given a pattern P of type B or C, the difference $S_{2j-1} - S_{2j-2}$ counts the number of entries equal to j in the corresponding tableau, whereas $S_{2j} - S_{2j-1}$ counts the number of entries equal to $\bar{j}$. Therefore, if the weight of a pattern P is defined as the monomial $x^P := x_1^{2S_1 - S_2} x_2^{2S_3 - S_4 - S_2} \cdots x_n^{2S_{2n-1} - S_{2n} - S_{2n-2}}$, the bijections are weight-preserving.

Example 7. We present patterns of top row (or shape) $\lambda = (3, 2)$ for types A, B, and C.

3.3 Symmetric polynomials as generating functions

Definition 8. Let $\lambda \in \mathrm{Par}_n$. The *Schur polynomial* s_λ, the *orthogonal polynomial* o_λ, and the *symplectic polynomial* sp_λ on n letters and of shape λ are defined as the generating functions of semistandard Young tableaux, orthogonal tableaux, and symplectic tableaux on n letters and of shape λ, respectively. Explicitly,

$$s_\lambda(x_1, ..., x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^T, \quad o_\lambda(x_1, ..., x_n) = \sum_{T \in \mathrm{SOT}_n(\lambda)} x^T, \quad \text{and} \quad sp_\lambda(x_1, ..., x_n) = \sum_{T \in \mathrm{KSpT}_n(\lambda)} x^T.$$

Equivalently, they are the generating functions of $\mathrm{GT}_n(\lambda)$, $\mathrm{SOP}_n(\lambda)$, and $\mathrm{KSpP}_n(\lambda)$.

4 Bender–Knuth involutions

We study type B and C analogues of the following elegant proof of Bender and Knuth.

Proposition 9. *Schur polynomials on n letters are $W(A_{n-1})$-symmetric.*

Sketch of proof. We follow [BK72], [Sta99, Thm. 7.10.2]. Let $(j \ \ j{+}1)$ be a simple transposition. Given a tableau T of shape λ and weight x^T, we produce a tableau $BK_j^{\mathrm{A}}(T)$ of shape λ and weight

$$x^{BK_j^{\mathrm{A}}(T)} = (j \ \ j{+}1).x^T,$$

where S_n acts on $\mathbb{C}[x_1, ..., x_n]$ by permuting the variables. To construct $BK_j^{\mathrm{A}}(T)$, begin by *freezing* each pair of entries $\boxed{j}, \boxed{j+1}$ that lie in the same column of T (these are usually referred to as $\{j, j{+}1\}$-vertical dominoes). The entries $\boxed{j}, \boxed{j+1}$ of T that are not yet frozen are called *mutable*. For each row, the mutable entries of T form a word $j^a(j+1)^b$, which is changed to a word $j^b(j+1)^a$ in $BK_j^{\mathrm{A}}(T)$. The map $BK_j^{\mathrm{A}} : \mathrm{SSYT}_n(\lambda) \to \mathrm{SSYT}_n(\lambda)$ is an involution. □

Figure 3: We illustrate BK_3^A. Highlighted, mutable entries of the tableau.

We refer to the map BK_j^A defined in this proof as the jth *type A Bender–Knuth involution*. See Figure 3.

The translation of type A Bender–Knuth involutions to GT patterns was studied in [BK95] and can be described as follows: it only affects the jth row $P^{(j)} = (p_{i,j})_{i \leqslant j}$, and sends each entry $p_{i,j}$ to

$$p'_{i,j} := \min\{p_{i,j+1}, p_{i-1,j-1}\} + \max\{p_{i+1,j+1}, p_{i,j-1}\} - p_{i,j},$$

where min and max ignore non-existing entries. That is,

$$BK_j^A\left(P^{(n)}, ..., P^{(j)}, ..., P^{(1)}\right) = \left(P^{(n)}, ..., (p'_{1,j}, ..., p'_{j,j}), ..., P^{(1)}\right).$$

Example 10. To motivate this translation of BK_j^A to GT patterns, we take a closer look at the example of Figure 3. Let T be the tableau on the left, and let $P = ((p_{i,n})_{i \leqslant n}, ..., (p_{i,1})_{i \leqslant 1})$ be its corresponding GT pattern. We consider the mutable $3^1 4^2$ word in the second row of T. It spans three columns, starting at (but not including) column 4 and ending at column 7.

We can compute its starting column (in this case 4) as the maximum of (a) the number of entries smaller than 3 in the second row, $p_{2,2} = 3$, and (b) the number of entries smaller or equal to 4 in the third row, $p_{3,4} = 4$. Similarly, its ending column, 7, is the minimum of (c) the number of entries smaller than 3 in the first row, $p_{1,2} = 7$, and (d) the number of entries smaller or equal to 4 in the second row, $p_{2,4} = 9$.

Correspondingly, $p_{2,3} = 5$ is sent by BK_3^A to $\min\{7, 9\} + \max\{3, 4\} - 5 = 6$.

We now show the analogue result for type C.

Proposition 11. *Symplectic polynomials in n letters are $W(C_n)$-symmetric.*

We begin by proposing a candidate for type C Bender–Knuth involutions. An involutory action of $(1\ \bar{1})$ on $\mathrm{KSpP}_n(\lambda) \subseteq \mathrm{GT}_{2n}(\lambda)$ is given by BK_1^A. For any other generator of $W(C_n)$, write this permutation as a product of simple transpositions with respect to the ordered set $\mathcal{A} = \{1 < \bar{1} < \cdots < n < \bar{n}\}$. We multiply permutations right-to-left, and thus we get

$$(j\ \ j{+}1)(\bar{j}\ \ \overline{j{+}1}) = (\bar{j}\ \ j{+}1)(j{+}1\ \ \overline{j{+}1})(j\ \ \bar{j})(\bar{j}\ \ j{+}1).$$

For each of these, perform a type A involution. Starting with any given pattern P_0, we get

$$P_0 \xmapsto[\;(\bar{j}\ j+1)\;]{\;BK^{\mathrm{A}}_{2j}\;} P_1 \xmapsto[\;(j\ \bar{j})\;]{\;BK^{\mathrm{A}}_{2j-1}\;} P_2 \xmapsto[\;(j+1\ \overline{j+1})\;]{\;BK^{\mathrm{A}}_{2j+1}\;} P_3 \xmapsto[\;(\bar{j}\ j+1)\;]{\;BK^{\mathrm{A}}_{2j}\;} P_4. \tag{1}$$

Thanks to the properties of type A Bender–Knuth involutions, the resulting pattern P_4 is of weight $(j\ j+1)(\bar{j}\ \overline{j+1}).x^{P_0}$, as desired. However, P_4 does not need to be symplectic: we might find an entry $p_{lk} \neq 0$ with $2l > k$.

Lemma 12. *Let P_4 be defined as above. Then $p_{lk} = 0$ for all (l,k) in*

$$\{(l,k)\ :\ 2l > k\} - \{(j+1, 2j)\}.$$

Proof. The type A Bender–Knuth involutions from Equation 1 only affect the rows $2j$ and $2j \pm 1$ of the pattern. The value of an entry of P_4 in position (l,k) with $2l > k$ and $k \notin \{2j-1, 2j, 2j+1\}$ is thus 0, since P_0 is symplectic.

Suppose $k = 2j - 1$. By the above, the value of p_{lk} in the pattern P_4 is computed as $\min\{p_{l,k+1}, 0\} + \max\{0, 0\} - 0$, which is 0. A similar argument applies to $k = 2j + 1$. Let $k = 2j$, let $l > j+1$. By the above, the value of p_{lk} is $\min\{0,0\} + \max\{0,0\} - 0 = 0$. $\square$

That is, the only possible obstruction to the symplectic property is the value of entry $p_{j+1,2j}$. We compose with the weight-preserving map rect (*rectification*) that subtracts $p_{j+1,2j}$ from the entries $p_{j+1,2j}$, $p_{j+1,2j+1}$, $p_{j,2j}$, and $p_{j,2j-1}$, and is the identity everywhere else. Indeed, the fact that it is weight-preserving follows from the definition of weight of a pattern, where the variable x_{j-1} is raised to the power $2S_{2j} - S_{2j+1} - S_{2j-1}$.

Define the jth type C Bender–Knuth involution as the composite

$$BK^{\mathrm{C}}_j := \mathrm{rect} \circ BK^{\mathrm{A}}_{2j} \circ BK^{\mathrm{A}}_{2j+1} \circ BK^{\mathrm{A}}_{2j-1} \circ BK^{\mathrm{A}}_{2j}.$$

Example 13. Let $j = 2$. We illustrate the 2nd type C Bender–Knuth involution on a symplectic pattern with 6 rows and top row $(3,3,2)$.

In the first step, we apply BK^{A}_4, corresponding to the permutation $(\bar{2}\ 3)$. This acts on the 4th row of the pattern, sending $(3,0,0,0)$ to $(2,2,0,0)$. For instance, 3 maps to $\min\{3\} + \max\{2,2\} - 3 = 2$. We keep doing this as indicated.

On the fourth step, the 4th row $(2,2,0,0)$ is sent to $(3,2,1,0)$. Indeed, the first 0 maps to $\min\{1,2\} + \max\{0,0\} - 0 = 1$. The array is no longer half-triangular. The rectification map corrects this by subtracting 1 from the four highlighted entries of the last pattern.

Proof of Proposition 11. It suffices to show that each BK_j^{C} is an involution. Consider the composites Φ and $(BK_j^{\mathrm{C}})^2$ given by

$$P_0 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_1 \xrightarrow{BK_{2j-1}^{\mathrm{A}}} P_2 \xrightarrow{BK_{2j+1}^{\mathrm{A}}} P_3 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_4 \;=\; P_5' \xrightarrow{BK_{2j}^{\mathrm{A}}} P_6' \xrightarrow{BK_{2j-1}^{\mathrm{A}}} P_7' \xrightarrow{BK_{2j+1}^{\mathrm{A}}} P_8' \xrightarrow{BK_{2j}^{\mathrm{A}}} P_9' \quad \text{and}$$

$$P_0 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_1 \xrightarrow{BK_{2j-1}^{\mathrm{A}}} P_2 \xrightarrow{BK_{2j+1}^{\mathrm{A}}} P_3 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_4 \xrightarrow{\mathrm{rect}} P_5 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_6 \xrightarrow{BK_{2j-1}^{\mathrm{A}}} P_7 \xrightarrow{BK_{2j+1}^{\mathrm{A}}} P_8 \xrightarrow{BK_{2j}^{\mathrm{A}}} P_9 \xrightarrow{\mathrm{rect}} P_{10}.$$

Note that Φ is the identity, and that both maps are identical on most entries. Indeed, in each step, the value of BK_i^{A} on an entry only depends on the value of its four neighbours. Starting with the four entries of P_5 that are perturbed by rect, the effect of this perturbation is only measured by the last two entries in rows $2j$ and $2j \pm 1$ of P_{10}.

Therefore, it is enough to show that BK_2^{C} is an involution on a generic pattern with 6 rows. Our strategy to tackle this final computation is to take the entry-wise differences $P_i - P_i'$ for $i = 5, ..., 9$. We have $P_9' = P_0$, and $P_{10} - P_9'$ is seen to vanish in Section 5. $\quad\square$

Corollary 14. *Orthogonal polynomials in n letters are $W(B_n)$-symmetric.*

Proof. We have $W(B_n) = W(C_n)$. From the weight-preserving bijection $\mathrm{SOP}_n(\lambda) \to \bigcup_\mu \mathrm{KSpP}_n(\mu)$, we get $o_\lambda = \sum_\mu sp_\mu$. The result now follows from Proposition 11. $\quad\square$

Combinatorially, type B Bender–Knuth involutions are defined on patterns by ignoring the circles and performing type C Bender–Knuth involutions, and later placing the circles back to where they were.

To describe type B and C Bender–Knuth involutions on tableaux, it remains to interpret rect. Consider a tableau T_0 and the composite

$$T_0 \xrightarrow[{(\bar{j}\ j+1)}]{BK_{2j}^{\mathrm{A}}} T_1 \xrightarrow[{(j\ \bar{j})}]{BK_{2j-1}^{\mathrm{A}}} T_2 \xrightarrow[{(j+1\ \overline{j+1})}]{BK_{2j+1}^{\mathrm{A}}} T_3 \xrightarrow[{(\bar{j}\ j+1)}]{BK_{2j}^{\mathrm{A}}} T_4.$$

Lemma 12 says T_4 is symplectic up to the existence of $\{j, \bar{j}\}$-vertical dominoes between rows j and $j + 1$. (For a proof of the lemma in the language of tableaux see [Gut23, Prop. 5.9].) The tableau $\mathrm{rect}(T_4)$ is constructed from T_4 by relabelling such dominoes into $\{j + 1, \overline{j+1}\}$-vertical dominoes, and sorting rows j and $j + 1$ into increasing order.

Example 15. Let $j = 2$. We translate Example 13 to tableaux.

Each of the four first maps are type A Bender–Knuth involutions, and the last map rectifies the tableau by getting rid of the highlighted $\{2, \bar{2}\}$-vertical domino.

Corollary 16. *Both symplectic polynomials and orthogonal polynomials (on n letters) form an integral basis of $\mathbb{Z}[x_1^\pm, ..., x_n^\pm]^{S_2 \wr S_n}$.*

Proof. Let $\leqslant$ be the lexicographic order on Par_n. Let $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{Z}[x_1^\pm, ..., x_n^\pm]^{S_2 \wr S_n}$. Define the leading term of f as the greatest λ such that $c_\lambda \neq 0$. Then $f - c_\lambda sp_\lambda$ has a lower leading term. Thus $\{sp_\lambda\}_{\lambda \in \mathrm{Par}_n}$ is spanning and linearly independent. Similarly for $\{o_\lambda\}_{\lambda \in \mathrm{Par}_n}$. $\qquad\square$

In particular, this shows that symplectic and orthogonal polynomials are integral linear combinations of Weyl characters of representations of Sp_{2n} and SO_{2n+1}, respectively.

5 A computation

To complete the proof of Proposition 11, we need to verify that the proposed map BK_2^{C} is an involution on the set of symplectic patterns with 6 rows and fixed shape.

To alleviate notation, we consider a pattern $a = (a^{(6)}, ..., a^{(1)})$, and denote with b, c, d, e, f the image of a under the following composite maps:

$$ a \xmapsto[(\bar{2}\,3)]{BK_4^{\mathrm{A}}} b \xmapsto[(2\,\bar{2})]{BK_3^{\mathrm{A}}} c \xmapsto[(3\,\bar{3})]{BK_5^{\mathrm{A}}} d \xmapsto[(\bar{2}\,3)]{BK_4^{\mathrm{A}}} e \xmapsto{\text{rect}} f. $$

Moreover, we set A as a copy of e, and A' as a copy of f, and define $B, B', ..., F'$ as follows:

$$ \Phi : a \xmapsto{(\bar{2}\,3)} b \xmapsto{(2\,\bar{2})} c \xmapsto{(3\,\bar{3})} d \xmapsto{(\bar{2}\,3)} e =: A \xmapsto{(\bar{2}\,3)} B \xmapsto{(2\,\bar{2})} C \xmapsto{(3\,\bar{3})} D \xmapsto{(\bar{2}\,3)} E = a, $$

$$ (BK_2^{\mathrm{C}})^2 : a \xmapsto{(\bar{2}\,3)} b \xmapsto{(2\,\bar{2})} c \xmapsto{(3\,\bar{3})} d \xmapsto{(\bar{2}\,3)} e \xmapsto{\text{rect}} A' \xmapsto{(\bar{2}\,3)} B' \xmapsto{(2\,\bar{2})} C' \xmapsto{(3\,\bar{3})} D' \xmapsto{(\bar{2}\,3)} E' \xmapsto{\text{rect}} F'. $$

We have $E = a$ as noted in Section 3.2, and we aim to show $F' = a$.

Let us start by comparing A and A'. We have $A'_{jk} = A_{jk}$ for all j, k except for

$$ A'_{35} = A_{35} - e_{34}, \quad A'_{24} = A_{24} - e_{34}, \quad A'_{23} = A_{23} - e_{34}, \quad \text{and} \quad A'_{34} = A_{34} - e_{34} = 0. $$

We may now turn to B and B', in which we thus find

$$ B'_{14} = B_{14}, \quad B'_{24} = B_{24}, \quad \text{and} \quad B'_{34} = B_{34}. $$

Indeed, we have

$$ \begin{aligned} B'_{24} &= \min\{A'_{25}, A'_{13}\} + \max\{A'_{35}, A'_{23}\} - A'_{24} \\ &= \min\{A_{25}, A_{13}\} + \max\{A_{35} - e_{34}, A_{23} - e_{34}\} - (A_{24} - e_{34}) = B_{24}, \text{ and} \\ B'_{34} &= \min\{A_{35} - e_{34}, A_{23} - e_{34}\} \\ &= \min\{A_{35}, A_{23}\} - e_{34} \\ &= \min\{A_{35}, A_{23}\} - A_{34} = B_{34}. \end{aligned} $$

In the next step, when comparing C and C', we therefore note

$$ \begin{aligned} C'_{13} &= C_{13}, \text{ and} \\ C'_{23} &= \min\{B_{24}, B_{12}\} + B_{34} - B'_{23} \\ &= \min\{B_{24}, B_{12}\} + B_{34} - A'_{23} \\ &= \min\{B_{24}, B_{12}\} + B_{34} - (A_{23} - e_{34}) = C_{23} + e_{34}. \end{aligned} $$

Similarly, in D, D',

$$D'_{15} = D_{15}, \quad D'_{25} = D_{25}, \quad \text{and} \quad D'_{35} = D_{35} + e_{34}.$$

Finally, comparing E and E' gives

$$E'_{14} = E_{14}, \quad E'_{24} = E_{24} + e_{34}, \quad \text{and} \quad E'_{34} = E_{34} + e_{34} = a_{34} + e_{34} = e_{34}.$$

And now, subtracting e_{34} from $E'_{34}, E'_{24}, D'_{35}$ and C'_{23} recovers the pattern E. This shows $F' = E = a$, as desired.

Note 17. For illustrative purposes, we give explicitly give the patterns $A', B', ..., F'$ in terms of $A, B, ..., C$ according to the computations above. To save space, we denote $x - e_{34}$ by x^- and $x + e_{34}$ by x^+.

$$A' = \begin{matrix} A_{16} & A_{26} & A_{36} & 0 & 0 & 0 \\ A_{15} & A_{25} & A^-_{35} & 0 & 0 \\ A_{14} & A_{24} & A^-_{34} & 0 \\ A_{13} & A_{23} & 0 \\ A_{12} & 0 \\ A_{11} \end{matrix} \quad \xmapsto{(\bar{2}\ 3)} \quad B' = \begin{matrix} B_{16} & B_{26} & B_{36} & 0 & 0 & 0 \\ B_{15} & B_{25} & B^-_{35} & 0 & 0 \\ B_{14} & B_{24} & B_{34} & 0 \\ B_{13} & B^-_{23} & 0 \\ B_{12} & 0 \\ B_{11} \end{matrix}$$

$$\xmapsto{(2\ \bar{2})} \quad C' = \begin{matrix} C_{16} & C_{26} & C_{36} & 0 & 0 & 0 \\ C_{15} & C_{25} & C^-_{35} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & 0 \\ C_{13} & C^+_{23} & 0 \\ C_{12} & 0 \\ C_{11} \end{matrix} \quad \xmapsto{(3\ \bar{3})} \quad D' = \begin{matrix} D_{16} & D_{26} & D_{36} & 0 & 0 & 0 \\ D_{15} & D_{25} & D^+_{35} & 0 & 0 \\ D_{14} & D_{24} & D_{34} & 0 \\ D_{13} & D^+_{23} & 0 \\ D_{12} & 0 \\ D_{11} \end{matrix}$$

$$\xmapsto{(3\ \bar{2})} \quad E' = \begin{matrix} E_{16} & E_{26} & E_{36} & 0 & 0 & 0 \\ E_{15} & E_{25} & E^+_{35} & 0 & 0 \\ E_{14} & E^+_{24} & E^+_{34} & 0 \\ E_{13} & E^+_{23} & 0 \\ E_{12} & 0 \\ E_{11} \end{matrix} \quad \xmapsto{\text{rect}} \quad F' = \begin{matrix} E_{16} & E_{26} & E_{36} & 0 & 0 & 0 \\ E_{15} & E_{25} & E_{35} & 0 & 0 \\ E_{14} & E_{24} & E_{34} & 0 \\ E_{13} & E_{23} & 0 \\ E_{12} & 0 \\ E_{11} \end{matrix}$$

Acknowledgements

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