Cubic graphs with colouring defect 3

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Abstract

The colouring defect of a cubic graph is the smallest number of edges left uncovered by any set of three perfect matchings. While 3-edge-colourable graphs have defect 0, those that cannot be 3-edge-coloured (that is, snarks) are known to have defect at least 3. In this paper we focus on the structure and properties of snarks with defect 3. For such snarks we develop a theory of reductions similar to standard reductions of short cycles and small cuts in general snarks. We prove that every snark with defect 3 can be reduced to a snark with defect 3 which is either nontrivial (cyclically 4-edge-connected and of girth at least 5) or to one that arises from a nontrivial snark of defect greater than 3 by inflating a vertex lying on a suitable 5-cycle to a triangle. The proofs rely on a detailed analysis of Fano flows associated with triples of perfect matchings leaving exactly three uncovered edges. In the final part of the paper we discuss application of our results to the conjectures of Berge and Fulkerson, which provide the main motivation for our research.

Mathematics Subject Classifications: 05C15, 05C21, 05C70, 05C75

1 Introduction

Every 3-edge-colourable cubic graph has a set of three perfect matchings that cover all of its edges. Conversely, three perfect matchings cover the edge set of a cubic graph only when they are pairwise disjoint and, therefore, the graph is 3-edge-colourable. It follows that if a cubic graph is not 3-edge-colourable, then any collection of three perfect matchings leaves some of its edges not covered. The minimum number of edges of a cubic graph G

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left uncovered by any set of three perfect matchings will be called the *colouring defect* of G and will be denoted by $\mathrm{df}(G)$. For brevity, we usually drop the adjective "colouring" and speak of the *defect* of a cubic graph. Clearly, a cubic graph has defect zero if and only if it is 3-edge-colourable, so defect can be regarded as a measure of uncolourability of cubic graphs.

The concept of colouring defect was introduced by Steffen [31] who used the notation $\mu_3(G)$ but did not coin any term for it. Among other things he proved that every 2-connected cubic graph which is not 3-edge-colourable – that is, a *snark* – has defect at least 3. He also proved that the defect of a snark is at least as large as one half of its girth. Since there exist snarks of arbitrarily large girth [23], there exist snarks of arbitrarily large defect.

The defect of a cubic graph was further examined by Jin and Steffen in [13] and was also discussed in the survey of uncolourability measures by Fiol et al. [9, pp.13-14]. In [13], Jin and Steffen studied the relationship of defect to other measures of uncolourability, in particular its relationship to oddness. The *oddness* of a cubic graph G, denoted by $\omega(G)$, is the minimum number of odd circuits in a 2-factor of G; it is correctly defined for any bridgeless cubic graph. Jin and Steffen proved [13, Corollary 2.4] that $\mathrm{df}(G) \geqslant 3\omega(G)/2$ and investigated the extremal case where $\mathrm{df}(G) = 3\omega(G)/2$ in detail.

In this paper we to continue the study of the colouring defect of snarks with emphasis on snarks with minimum possible defect, that is, defect 3. Snarks whose defect equals 3 have a remarkable property that they contain a 6-cycle [31, Corollary 2.5], which immediately implies that their cyclic connectivity does not exceed 6. This fact relates the study of colouring defect to a fascinating conjecture of Jaeger and Swart [12, Conjecture 2] which suggests that the cyclic connectivity of every snark is bounded above by 6. It is therefore a natural question to ask what structural properties of snarks ensure that their defect is 3, and conversely, what structural properties are implied by the fact that the defect is, or is not, this minimum possible value.

A natural approach to improving our understanding of the structure of snarks with defect 3 is through eliminating certain trivial features that they might posses. The main purpose of this paper is, therefore, to develop a theory of reductions for snarks with defect 3 analogous to standard reductions of short cycles and small cuts in general snarks. Snarks that have cycle-separating edge cuts of size smaller than 4 or circuits of length smaller than 5 are generally considered to be trivial. This is explained by the fact that if a snark contains a digon, a triangle, or a quadrilateral, one can easily remove it and subsequently restore 3-regularity to produce a smaller snark; similar reductions can be applied to small cuts [33]. Thus, a *nontrivial* snark must be cyclically 4-edge-connected and have girth at least 5.

The standard reductions to nontrivial snarks have been extremely useful in numerous investigations related to important conjectures in the area, such as the cycle double-cover conjecture [11], 5-flow conjecture [24], or Fulkerson's conjecture [25]. In this situation it is natural to attempt finding similar reductions within the class of snarks with defect 3. The expected aim would be to show that, given a snark with defect 3, one can eliminate cycles of length smaller than 5 and cycle-separating edge cuts of size smaller than 4 to produce

– in a certain natural manner – a snark whose defect is still 3 but lacks these "trivial" features. Our main results, Theorems 5.1 and 5.2, demonstrate that this expectation almost comes true.

In Theorem 5.1 we show that every snark G with defect 3 can be reduced to a snark G' with defect 3 such that either G' is nontrivial or G' contains a single triangle whose contraction produces a nontrivial snark with defect greater than 3. Such a triangle is called *essential*.

In Theorem 5.2 we further show that the reduced snark G' arises from a nontrivial snark by inflating a vertex lying on a 5-cycle that contains an edge uv such that $G' - \{u, v\}$ is 3-edge-colourable. Our main results thus indicate that in the study of colouring defect of cubic graphs one cannot completely avoid graphs with triangles.

In addition, in Theorem 5.3 we show that by contracting an essential triangle in a snark with defect 3 one can obtain a nontrivial snark with an arbitrarily high defect.

Proofs of these results require a detailed study of triples $\{M_1, M_2, M_3\}$ of perfect matchings in cubic graphs that leave the minimum number of uncovered edges, along with structures derived from them, especially hexagonal cores and Fano flows. A *hexagonal core* of a snark with defect 3 is a 6-cycle induced by the set of all edges that are not simply covered by the triple $\{M_1, M_2, M_3\}$; it alternates the uncovered edges with the doubly covered ones. A closely related concept is a that of a *Fano flow*. It is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow on G induced by the 2-factors complementary to M_1, M_2, and M_3. Its flow values can be identified with the points of the Fano plane and flow patterns around vertices with the lines of a configuration F_4 of four lines covering all seven points of the Fano plane (see Figure 1). Analysing Fano flows across small edge cuts or around short circuits makes up a substantial part of the proofs of Theorems 5.1 and 5.2.

Our paper is organised as follows. In the next section we collect the most important definitions and facts needed for understanding this paper. In Section 3 we introduce structures related to the colouring defect and investigate their properties. After establishing auxiliary results about reductions in Section 4, we prove our main results, Theorems 5.1 and 5.2, in Section 5. In Section 6 we discuss how the reductions established in the previous sections can be applied to verifying Berge's conjecture for snarks of defect 3. The conjecture states that five perfect matchings are sufficient to cover all edges of any bridgeless cubic graphs. We explain why every snark of defect 3 fulfils Berge's conjecture, moreover, we provide a structural characterisation of those snarks of defect 3 that require five perfect matchings to cover their edges (Theorem 6.5). This significantly strengthens a result of Steffen [31, Theorem 2.14], where Berge's conjecture was verified for cyclically 4-edge-connected cubic graphs of defect 3. The proof will appear in a separate article [19] (see also [18]). Finally, in Section 7 we summarise the outputs of computer-aided experiments directed towards defect and cores of nontrivial snarks of order up to 36. These results provide partial support for several conjectures that we propose at the end of this paper.

2 Preliminaries

All graphs in this paper are finite and for the most part cubic (3-valent). Multiple edges and loops are permitted. We use the term *circuit* to mean a connected 2-regular graph. An *m-cycle* is a circuit of length m. The length of a shortest circuit in a graph is its *girth*. If H is an induced subgraph of G, we let G/H denote the graph arising from G by contracting each component of H into a single vertex.

For a subgraph (or just a set of vertices) Y of a graph G we let $\delta(Y)$ denote the edge cut consisting of all edges joining Y to vertices not in Y. A connected graph G is said to be *cyclically k-edge-connected* if the removal of fewer than k edges from G cannot create a graph with at least two components containing circuits. An edge cut S in G that separates two circuits from each other is *cycle-separating*.

Large graphs are typically constructed from smaller building blocks called multipoles. Like a graph, each *multipole* M has its vertex set $V(M)$, its edge set $E(M)$, and an incidence relation between vertices and edges. Each edge of M has two ends, and each end may, but need not be, incident with a vertex of M. An end of an edge that is not incident with a vertex is called a *free end* or a *semiedge*. An edge with exactly one free end is called a *dangling edge*. An *isolated edge* is one whose both ends are free. All multipoles considered in this paper are *cubic*; this means that every vertex is incident with exactly three edge ends. An *n-pole* is a multipole with n free ends. Free ends of a multipole can be distributed into pairwise disjoint sets, called *connectors*. An *$(n_1, n_2, \ldots, n_k)$-pole* is an n-pole with $n = n_1 + n_2 + \cdots + n_k$ whose semiedges are distributed into k connectors $S_1, S_2, \ldots, S_k$, each S_i being of size n_i.

Consider an arbitrary n-pole M and choose two distinct free ends s_i and s_j belonging to edges e and e', respectively. We say that a multipole M' is formed by the *junction* of s_i and s_j if M' arises from M by identifying s_i and s_j while retaining the other ends of e and e'. The newly formed edge is denoted by $s_i * s_j$. If s_i and s_j are the free ends of the same isolated edge e, the junction amounts to the deletion of e. A *junction* of two n-poles M and N is a cubic graph, denoted by $M * N$, arising from M and N by performing the junction of their respective semiedge sets. If a bijection σ between the semiedges of M and those of N is specified, we write $M *_\sigma N$. Throughout the paper we will use the following convention: if a graph G can be expressed in the form $G = M * N$, then, depending on the context, $G - M$ will either mean the multipole N including its dangling edges or will denote the induced subgraph $G - M$. There is no danger of confusion.

An *edge colouring* of a multipole M is a mapping from the edge set of M to a set of colours. A colouring is *proper* if any two edge-ends incident with the same vertex carry distinct colours. A *k-edge-colouring* is a proper colouring where the set of colours has k elements. A cubic graph G is said to be *colourable* or *uncolourable* depending on whether it does or does not admit a 3-edge-colouring, respectively. A 2-connected uncolourable cubic graph is called a *snark*.

In the study of snarks it is useful to take the colours 1, 2, and 3 to be the nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. To be specific, one can identify a colour with its binary representation: $1 = (0,1)$, $2 = (1,0)$, and $3 = (1,1)$. With this choice, the condition that the three colours meeting at any vertex v are all distinct is equivalent to requiring that the sum of the colours at v is $0 = (0, 0)$. The latter condition coincides with the Kirchhoff law for a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a graph, or a multipole. Thus a proper 3-edge-colouring of a cubic multipole coincides the a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on it.

The following well-known statement is a direct consequence of Kirchhoff's law.

Lemma 2.1 (Parity Lemma). *Let $M = M(s_1, s_2, \ldots, s_k)$ be a k-pole endowed with a 3-edge-colouring ϕ. Then*

$$\sum_{i=1}^{k} \phi(s_i) = 0.$$

Equivalently, the number of free ends of M carrying any fixed colour has the same parity as k.

Our definition leaves the concept of a snark as wide as possible since more restrictive definitions may lead to overlooking certain important phenomena in cubic graphs. Our definition thus follows Cameron et al. [6], Nedela and Škoviera [29], Steffen [30], and others, rather than a more common approach where snarks are required to be cyclically 4-edge-connected and have girth at least 5, see for example [9]. In this paper, such snarks are called *nontrivial*.

The problem of nontriviality of snarks has been widely discussed in the literature, see for example [6, 29, 30]. Here we follow a systematic approach to nontriviality of snarks proposed by Nedela and Škoviera [29]. A set of vertices or an induced subgraph H of a snark G is called *non-removable* if the subgraph obtained by removing the vertices contained in H is colourable; otherwise, H is *removable*. A snark G is *critical* if every pair of distinct adjacent vertices of G is non-removable. A snark is *bicritical* if every pair of distinct vertices of G is non-removable. A snark G is *irreducible* if every induced subgraph H with at least two vertices is non-removable. It is known that a snark is irreducible if and only if it is bicritical, see [29, Theorem 4.4 and Corollary 4.6].

The following well-known lemma a is a straightforward consequence of Lemma 2.1.

Lemma 2.2. *In an arbitrary snark, every circuit of length at most* 4 *is removable.*

Parity Lemma has a remarkable consequence that colourable 4-poles extendable to a snark fall into two types – isochromatic and heterochromatic (see for example [7, Section 3]). Such a 4-pole is *isochromatic* if its semiedges can be partitioned into two pairs $\{x, x'\}$ and $\{y, y'\}$ such that for every 3-edge-colouring ϕ one has $\phi(x) = \phi(x')$ and $\phi(y) = \phi(y')$; otherwise it is *heterochromatic*. It can be shown that a colourable 4-pole is isochromatic M if and only if G arises from a snark G by removing two adjacent vertices u and v. Moreover, the two pairs $\{x, x'\}$ and $\{y, y'\}$ correspond to the edges formerly incident with the vertices u and v, respectively.

3 Arrays of perfect matchings and the defect of a snark

Various important structures in cubic graphs, such as Tait colourings, Berge or Fulkerson covers, Fan-Raspaud colourings, and several others, can be described in terms of sets or lists of perfect matchings obeying certain additional conditions. In this paper such lists will be called *arrays* of perfect matchings. To be more precise, a *k-array of perfect matchings* in a cubic graph G, briefly a *k-array* for G, is an arbitrary collection $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ of k not necessarily distinct perfect matchings of G. In particular, a *Berge cover* of G is a 5-array $\mathcal{B}$ such that each edge of G belongs to some member of $\mathcal{B}$; a *Fulkerson cover* is 6-array $\mathcal{F}$ such that each edge of G belongs to precisely two members of $\mathcal{F}$.

This paper focuses on the properties of snarks that can be expressed by means of k-arrays of perfect matchings with $k = 3$. Since every proper 3-edge-colouring gives rise to an array whose members are the three colour classes, 3-arrays can be regarded as approximations of 3-edge-colourings. An edge of G that belongs to at least one of the perfect matchings of the array $\mathcal{M} = \{M_1, M_2, M_3\}$ will be considered to be *covered*. An edge will be called *uncovered*, *simply covered*, *doubly covered*, or *triply covered* if it belongs, respectively, to zero, one, two, or three distinct members of $\mathcal{M}$.

Given a 3-array $\mathcal{M}$ for G, it is a natural task to maximise the number of covered edges, or equivalently, to minimise the number of uncovered ones. A 3-array that leaves the minimum number of uncovered edges will be called *optimal*. The minimal number of edges left uncovered by an optimal 3-array is the *colouring defect* of G, briefly, the *defect*, denoted by $\mathrm{df}(G)$.

In this section we establish several basic results concerning 3-arrays and the defect of a cubic graph with emphasis on optimal 3-arrays. We remark that some of the ideas and results have already appeared in the papers by Steffen and others [13, 14, 31, 32]. However, in order to make our exposition self-contained we need to accompany such results with proofs.

With each 3-array of perfect matchings one can associate several important structures which reside within the underlying graph. We discuss two of them: the characteristic flow and the core.

1. Characteristic flow. Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 3-array of a cubic graph G. One way to describe $\mathcal{M}$ is based on regarding the indices 1, 2, and 3 as colours. Since the same edge may belong to more than one member of $\mathcal{M}$, an edge of G may receive from $\mathcal{M}$ more than one element of the set $\{1, 2, 3\}$. To each edge e of G we can therefore assign the list $\phi(e)$ of elements of $\{1, 2, 3\}$ in lexicographic order it receives from $\mathcal{M}$. We let $w(e)$ denote the number of colours in the list $\phi(e)$ and call it the *weight*. In this way $\mathcal{M}$ gives rise to an edge-colouring

$$\phi\colon E(G) \to \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$$

where $\emptyset$ denotes the empty list. We call ϕ the *characteristic colouring* for $\mathcal{M}$. Obviously, such a colouring determines a 3-array if and only if, for each vertex v of G, all three indices from $\{1, 2, 3\}$ occur precisely once on the edges incident with v. In general, ϕ need not be a proper colouring. As we shall see below, ϕ is a proper edge-colouring if and only if G has no triply covered edge.

A different but equivalent way of representing a 3-array uses a mapping

$$\chi\colon E(G) \to \mathbb{Z}_2^3, \quad e \mapsto \chi(e) = (x_1, x_2, x_3)$$

defined by setting $x_i = 0$ if and only if $e \in M_i$ for $i \in \{1, 2, 3\}$. Since the complement of each M_i in G is a 2-factor, it is easy to see that χ is a $\mathbb{Z}_2^3$-flow. We call χ *the characteristic flow* for $\mathcal{M}$. Again, χ is a nowhere-zero $\mathbb{Z}_2^3$-flow if and only if G contains no triply covered edge. In the context of 3-arrays the characteristic flow was introduced in [14, p. 166], but the idea is older, see [27, Section 4]. Observe that the characteristic flow χ of a 3-array and the colouring ϕ uniquely determine each other. In particular, the condition on ϕ requiring all three indices from $\{1, 2, 3\}$ to occur precisely once in a colour around any vertex is equivalent to Kirchhoff's law.

The following result characterises 3-arrays with no triply covered edge.

Proposition 3.1. *Let $\mathcal{M}$ be a 3-array of perfect matchings of a cubic graph G. The following three statements are equivalent.*

(i) *G has no triply covered edge with respect to $\mathcal{M}$.*

(ii) *The associated colouring $\phi\colon E(G) \to \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$ is proper.*

(iii) *The characteristic flow χ for $\mathcal{M}$, with values in $\mathbb{Z}_2^3$, is nowhere-zero.*

Proof. (i) $\Leftrightarrow$ (ii): Consider an arbitrary vertex v of G and the three edges e_1, e_2, and e_3 incident with v. Since every perfect matching contains an edge incident with v, and since G has no triply covered edge, the distribution of weights on (e_1, e_2, e_3) is either $(1, 1, 1)$ or $(2, 1, 0)$ up to permutation of values. In both cases it is obvious that e_1, e_2, and e_3 receive distinct colours from ϕ. For the converse, if an edge $e = uv$ of G is triply covered, then $\phi(e) = 123$ and the other two edges incident with u receive colour $\emptyset$. Thus ϕ is not proper.

(ii) $\Leftrightarrow$ (iii): This is an immediate consequence of the fact that the number of vanishing coordinates in $\chi(e)$ coincides with $w(e)$. $\square$

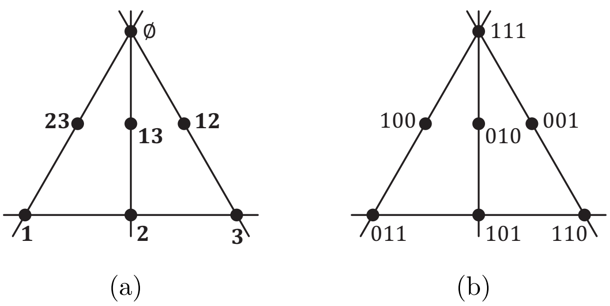


Figure 1: The configuration F_4 for 3-arrays with no triply covered edges

The fact that 3-arrays with no triply covered edge are associated to certain nowhere-zero flows and proper edge-colourings suggests that they deserve a special attention. If a

3-array $\mathcal{M}$ has no triply covered edge, the flow values of χ can be regarded as points of the Fano plane $PG(2,2)$ represented by the standard projective coordinates from $\mathbb{Z}_2^3 - \{0\}$. In this representation, the points of $PG(2,2)$ are the non-zero elements of $\mathbb{Z}_2^3$ and the lines of $PG(2,2)$ are the 3-element subsets $\{x,y,z\}$ of $\mathbb{Z}_2^3 - \{0\}$ such that $x+y+z=0$. The definition of the characteristic flow implies that for every vertex v of G the three flow values meeting at v form a line of the Fano plane. Since every vertex of G is incident with an edge of each member of $\mathcal{M}$, one can easily deduce that only four lines of the Fano plane can occur around a vertex, and these lines form a point-line configuration in $PG(2,2)$ which is shown in Figure 1; we denote this configuration by F_4. On the left-hand side of the figure the points are labelled with the corresponding colours – subsets of $\{1,2,3\}$. Based on this correspondence we will refer to the colouring ϕ and the flow χ as the *Fano colouring* and the *Fano flow* associated with a 3-array $\mathcal{M}$. The colours from $\{\emptyset,1,2,3,12,13,23\}$ and the corresponding elements of $\mathbb{Z}_2^3 - \{0\}$ will be used interchangeably.

2. Core. Another important structure associated with a 3-array is its core. The *core* of a 3-array $\mathcal{M} = \{M_1, M_2, M_3\}$ of G is the subgraph of G induced by all the edges of G that are not simply covered; we denote it by $\mathrm{core}(\mathcal{M})$. The core will be called *optimal* whenever $\mathcal{M}$ is optimal.

The edge set of $\mathrm{core}(\mathcal{M})$ thus coincides with $E_0(\mathcal{M}) \cup E_{23}(\mathcal{M})$, where $E_0(\mathcal{M})$ and $E_{23}(\mathcal{M})$ denote the set of all uncovered edges and the set of all doubly or triply covered edges with respect to $\mathcal{M}$, respectively. If $|E_0(\mathcal{M})| = k$, we say that $\mathrm{core}(\mathcal{M})$ is a *k-core*. It is worth mentioning that if G is 3-edge-colourable and $\mathcal{M}$ consists of three disjoint perfect matchings, then $\mathrm{core}(\mathcal{M})$ is empty. If G is not 3-edge-colourable, then every core must be nonempty. Figure 2 shows the Petersen graph endowed with a 3-array whose core is the "outer" 6-cycle. The hexagon is in fact an optimal core of the Petersen graph.

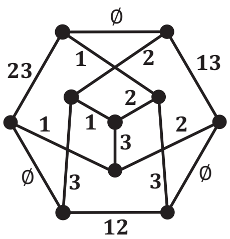


Figure 2: An optimal 3-array of the Petersen graph

The following proposition, much of which was proved by Steffen in [31, Lemma 2.2], lists the most fundamental properties of cores. We include the proof for the reader's convenience.

Proposition 3.2. *Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be an arbitrary 3-array of perfect matchings of a snark G. Then the following hold:*

(i) *Every component of* $\mathrm{core}(\mathcal{M})$ *is either an even circuit which alternates doubly covered and uncovered edges or a subdivision of a cubic graph. Moreover,* $\mathrm{core}(\mathcal{M})$ *is a collection of disjoint circuits if and only if G has no triply covered edge.*

(ii) *Every* 2*-valent vertex of* core($\mathcal{M}$) *is incident with one doubly covered edge and one uncovered edge, while every* 3*-valent vertex is incident with one triply covered edge and two uncovered edges.*

(iii) $E_{23}(\mathcal{M})$ *is a perfect matching of* core($\mathcal{M}$)*; consequently,* $|E_0| = |E_2(\mathcal{M})| + 2|E_3(\mathcal{M})|$.

(iv) $G - E_0(\mathcal{M})$ *is* 3*-edge-colourable.*

(v) *If* $\mathcal{M}$ *is optimal, then every component of* core($\mathcal{M}$) *is a simple graph which is either an even circuit of length at least* 6 *or a subdivision of a cubic graph.*

Proof. Set $H = \text{core}(\mathcal{M})$. We claim that every vertex of H has valency at least 2. Indeed, if a vertex v of G is incident with a triply covered edge, then the other two edges incident with v are uncovered. So, in this case, v has degree 3 in H. If v is incident with a doubly covered edge, then exactly one of the remaining edges is simply covered and the other one is uncovered. Thus v has valency 2 in H. If all three edges incident with v are simply covered, then v does not belong to H. It follows that every vertex in H has degree at least 2, and also that the set $E_{23}(\mathcal{M})$ of edges forms a perfect matching in H. Statements (i)–(iii) now follow immediately.

To prove (iv), assign colour i to every edge of G that is simply covered and belongs to M_i. If an edge of G is doubly covered, then both end-vertices are incident in G with one uncovered edge and one simply covered edge. It follows that we can colour such an edge with the colour $c \in \{1, 2, 3\}$ that does not occur on the simply covered edges adjacent to it. Finally, if an edge of G is triply covered, both end-vertices are incident with two uncovered edges. Thus we can colour such an edge with any colour from $\{1, 2, 3\}$. In this manner we have clearly produced a 3-edge-colouring of $G - E_0(\mathcal{M})$.

To prove (v) it is enough to argue that neither a digon nor a 4-cycle can occur as components of the core of an optimal 3-array. If core($\mathcal{M}$) contains a 4-cycle $Q = (e_0e_1e_2e_3)$, then two edges of Q are uncovered, say e_0 and e_2, and the other two are doubly covered. Clearly, one of the three perfect matchings covers both e_1 and e_3. Without loss of generality we may assume that it is M_1. The characteristic colouring ϕ then satisfies $\phi(e_1) = 1i$ and $\phi(e_3) = 1j$ for some $i, j \in \{2, 3\}$. We can modify ϕ to ϕ' by setting $\phi'(e_0) = 1$, $\phi'(e_2) = 1$, $\phi'(e_1) = i$, and $\phi'(e_0) = j$, and leaving all the remaining edges $\phi'(e) = \phi(e)$. However, ϕ' now clearly determines a 3-array with fewer uncovered edges, contradicting the minimality of $\mathcal{M}$. This proves that core($\mathcal{M}$) does not contain a quadrilateral. The argument that core($\mathcal{M}$) does not contain a digon is similar and therefore is omitted. □

We say that a 3-array $\mathcal{M}$ of G has a *regular core* if each component of core($\mathcal{M}$) is a circuit; otherwise the core is called *irregular*. By Proposition 3.2(ii), a core is regular if and only if G has no triply covered edge. (Steffen [31] calls such a core *cyclic*, but we believe that the letter term is somewhat misleading as it might suggest that the core is a single k-cycle for some k.) The well-known conjecture of Fan and Raspaud [8] states that every bridgeless cubic graph has three perfect matchings M_1, M_2, and M_3 with $M_1 \cap M_2 \cap M_3 = \emptyset$. Equivalently, the conjecture states that every bridgeless cubic graph has a 3-array with a regular core. The conjecture is trivially true for 3-edge-colourable graphs. Máčajová and Škoviera [28] proved this conjecture to be true for cubic graphs with oddness 2. We emphasise that neither the conjecture nor the proved facts suggest anything about optimal cores.

The following theorem characterises snarks with minimal colour defect. The lower bound 3 for the defect of a snark is due to Steffen [31, Corollary 2.5].

Theorem 3.3. *Every snark G has $\mathrm{df}(G) \geqslant 3$. Furthermore, the following three statements are equivalent for any cubic graph G.*

(i) $\mathrm{df}(G) = 3$.

(ii) *The core of any optimal* 3*-array of* G *is a* 6*-cycle.*

(iii) *G contains an induced 6-cycle C such that the subgraph $G - E(C)$ admits a proper 3-edge-colouring under which the six edges of $\delta(C)$ receive colours $1, 1, 2, 2, 3, 3$ or $1, 2, 2, 3, 3, 1$ with respect to a fixed cyclic ordering induced by an orientation of C.*

Proof. Let G be a snark. First observe that any two perfect matchings in G intersect. Indeed, if there were two disjoint perfect matchings in G, the set of remaining edges would be a third perfect matching, implying that G is 3-edge-colourable.

Now consider an optimal 3-array $\mathcal{M} = \{M_1, M_2, M_3\}$ of perfect matchings of G, and let H be the core of $\mathcal{M}$. Since G is a snark, we have $\mathrm{df}(G) > 0$, so G contains at least one uncovered edge and at least one multiply covered edge. To prove that $\mathrm{df}(G) \geqslant 3$ we consider two cases.

Case 1. *G contains a triply covered edge e.* Since G is bridgeless, each end-vertex of e is incident with two distinct uncovered edges. By Proposition 3.2(v), these four edges are all pairwise distinct, implying that $\mathrm{df}(G) \geqslant 4$.

Case 2. *G contains no triply covered edge.* Proposition 3.2(i) and (v) now implies that each component of H is a circuit of length at least 6, which means that there are at least three uncovered edges. Hence, $\mathrm{df}(G) \geqslant 3$ in this case.

So far we have shown that for every snark we have $\mathrm{df}(G) \geqslant 3$. We now finish the proof by proving that the statements (i)-(iii) are equivalent. Let G be an arbitrary cubic graph. Assume that $\mathrm{df}(G) = 3$. If we combine (ii) and (v) of Proposition 3.2, we can conclude that G contains no triply covered edge. Proposition 3.2(i) now tells us that each component of H is an even circuit that alternates uncovered and doubly covered edges. By Proposition 3.2(v), each such circuit has length at least 6, so H must be a single hexagon. This establishes the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Assume that the core of an optimal 3-array $\mathcal{M}$ of G is a 6-cycle C. Hence, G is a snark. By Proposition 3.2(i) and (v), G has no triply covered edge. It follows that on C the uncovered edges and the doubly covered edges alternate and that the edges leaving C are simply covered. Since G is a snark, any two perfect matchings intersect, which implies that the three doubly covered edges receive colours 12, 13, and 23 in some order. The colours of doubly covered edges determine the order of colours on the edges leaving C uniquely, and it is easy to see that they are as stated. Consequently, the resulting structure is as illustrated in Figure 3 up to permutation of the set $\{1,2,3\}$. In particular, $C=(e_0e_1\ldots e_5)$, and the edges leaving C are $f_0, f_1, \ldots, f_5$, where each f_i is adjacent to e_{i-1} and e_i with indices reduced modulo 6.

It remains to show that C is an induced 6-cycle. Suppose not. Then C has a chord, necessarily a simply covered edge. Without loss of generality we can assume that under the Fano colouring $\phi_{\mathcal{M}}$ corresponding to $\mathcal{M}$ the chord has colour 1. If we adopt the notation of Figure 3, the latter assumption means that $f_4 = f_5$. However, we can now extend the 3-edge-colouring of $G - E(C)$ determined by $\mathcal{M}$ to a 3-edge-colouring ψ of the entire G. Indeed, it is sufficient to set $\psi(e_1) = 1$, which further forces $\psi(e_0) = 2$, $\psi(e_2) = 3$, and $\psi(e_3) = \psi(e_5) = 1$. Thus we can define $\psi(e_4) = 2$ and $\psi(f_4) = 3$ and let $\psi(x) = \phi_{\mathcal{M}}(x)$ for all the remaining edges of G. Clearly, this is a proper 3-edge-colouring of G, which is a contradiction.

The implication (iii) $\Rightarrow$ (i) is trivial. □

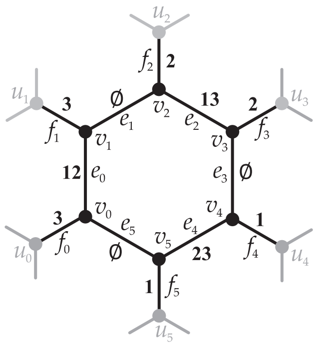


Figure 3: The core for df = 3 and its vicinity

If G is a snark with colouring defect 3, then by Theorem 3.3(iii) it contains an optimal array $\mathcal{M} = \{M_1, M_2, M_3\}$ whose core is an induced 6-cycle. Such a core will be referred to as a *hexagonal core* of G.

Consider an arbitrary induced 6-cycle $Q = (q_0q_1 \ldots q_5)$ in a snark G with $\mathrm{df}(G) = 3$, and let $r_1, \ldots, r_5$ be the edges of $\delta(Q)$, where each r_i is adjacent to q_{i-1} and q_i, with indices reduced modulo 6. In general, Q need not constitute the core of any optimal 3-array for G. If it does, we say that Q is a *core hexagon.* In such a case, there is an optimal array $\mathcal{M}$ for G such that $E_0(\mathcal{M})$ consists of three independent edges of Q. This can happen in two ways, either $E_0(\mathcal{M}) = \{q_0, q_2, q_4\}$ or $E_0(\mathcal{M}) = \{q_1, q_3, q_5\}$.

Assume that $E_0(\mathcal{M}) = \{q_1, q_3, q_5\}$. Let $\phi = \phi_{\mathcal{M}}$ be the Fano colouring of G induced by $\mathcal{M}$. Since permuting the indices of the perfect matchings M_1, M_2, and M_3 of $\mathcal{M}$ does not essentially change the 3-array, we can assume, without loss of generality, that $\phi(q_0) = 12$, $\phi(q_2) = 13$, and $\phi(q_4) = 23$. In other words, if $E_0(\mathcal{M}) = \{q_1, q_3, q_5\}$, we can assume that around Q the Fano colouring ϕ is as shown in Figure 3, with each q_i being identified with e_i. Similarly, if $E_0(\mathcal{M}) = \{q_0, q_2, q_4\}$, we can assume that the values of ϕ around Q

are rotated one step counter-clockwise with respect to Figure 3. The conclusion is that if Q is a core hexagon, then the cyclic sequence $(\phi(r_0), \phi(r_1), \ldots, \phi(r_5))$ of colours leaving Q is either $(3,3,2,2,1,1)$ or $(3,2,2,1,1,3)$, possibly after permuting the set $\{1,2,3\}$.

The previous considerations imply that in order to decide whether Q is, or is not, a core hexagon, it is sufficient to check whether at least one of the assignments $(3,3,2,2,1,1)$ or $(3,2,2,1,1,3)$ of colours to the edges of $\delta(Q)$ extends to a proper 3-edge-colouring of $G - E(Q)$. If both assignments extend, then Q is the core of two optimal 3-arrays with disjoint sets of uncovered edges. In this case we say that Q is a *double-core hexagon*. If only one of the assignments extends, then we say that Q is a *single-core hexagon*.

Observe that if a snark contains a double-core hexagon, then the two corresponding 3-arrays constitute a Fulkerson cover of G, that is, a collection of six perfect matchings that cover each edge precisely twice. The distribution of various types of hexagons in snarks will be discussed in Section 7.

4 Reduction to nontrivial snarks

Let G be a snark containing a k-edge-cut R with $k \geqslant 2$, which decomposes G into a junction $H * K$ of two k-poles H and K. If one of them, say H, is uncolourable, we can extend H to a snark G' of order not exceeding that of G by joining H with a suitable k-pole L of order $|L| \leqslant |K|$ (possibly $L = K$). Following [29], we call G' a *k-reduction* of G, and say that G can be *reduced* to G' along R. With a slight abuse of terminology, we also say that G' arises from G by a *reduction* with respect to R. More generally, we say that a snark G can be *reduced* to a snark G' if there exists a sequence $G = G_0, G_1, \ldots, G_t = G'$ of snarks such that for each $i \in \{0, \ldots, t-1\}$ the snark G_i can be reduced to G_{i+1} along some edge cut. A reduction G' of G is said to be *proper* if $|G'| < |G|$. A reduction $G' = H * L$ of $G = H * K$ where L has the minimum possible order is called a *completion* of H to a snark. Observe that if $G' = H * L = \bar{H}$ is a completion of H, then $|L| = 0$ or 1 depending on whether k is even or odd, respectively. Moreover, if $k \in \{2,3\}$, then H has a unique completion to a snark up to isomorphism.

The aim of the next two sections is to prove that every snark with defect 3 can be reduced to a nontrivial snark with defect 3, or else it has a very specific structure. In this section we gather auxiliary results needed for the proofs of our main results, which will be presented in Section 5.

A short reflection reveals that a reduction to a nontrivial snark of defect 3 is clearly not possible when the snark in question contains a triangle whose contraction produces a snark with defect greater than 3. A triangle with this property will be called *essential*. We show that the existence of an essential triangle is the only obstruction that prevents a snark with defect 3 from reduction, and that, in such a case, there is only one essential triangle in the graph. As we shall see later, an infinite family of snarks containing an essential triangle indeed exists (see Theorem 5.3). One such graph can be created from the graph in Figure 10 by inflating the central vertex z to a triangle.

Throughout this section we use the following notation: G is a snark with $\mathrm{df}(G) = 3$, $\mathcal{M} = \{M_1, M_3, M_3\}$ is an optimal 3-array of G whose core C is a 6-cycle, χ is the characteristic flow for $\mathcal{M}$, and ϕ is the associated Fano colouring. We further assume, without loss of generality, that the Fano colouring and the names of edges in the vicinity of C are those as in Figure 3.

Lemma 4.1. *Let G be a snark with defect 3, let C be a hexagonal core of G, and let T be an arbitrary triangle in G. The following statements hold.*

(i) *If $C \cap T = \emptyset$, then T is not essential.*

(ii) *If $C \cap T \neq \emptyset$, then $C \cap T$ consists of a single uncovered edge, and the edge cut $\delta(T)$ comprises three pairwise independent edges.*

(iii) *Every hexagonal core intersects at most one triangle.*

(iv) *G has at most one essential triangle.*

Proof. Let G be a snark with $\mathrm{df}(G) = 3$, let C be the core of an optimal 3-array $\mathcal{M}$ of G, and let T be an arbitrary triangle in G. Clearly, G/T is a snark. We claim that if $C \cap T = \emptyset$, then T is not essential. Indeed, if $C \cap T = \emptyset$, then $\delta(T)$ consists of simply covered edges, and by the Kirchhoff law for the characteristic flow these edges belong to three distinct members of $\mathcal{M}$. It follows that $\mathcal{M}$ induces a 3-array $\mathcal{M}'$ of G/T with $\mathrm{core}(\mathcal{M}') = C$. Hence $\mathrm{df}(G/T) = 3$, which means that the triangle T is not essential. This proves (i).

Now assume that $C \cap T \neq \emptyset$. Obviously, $C \cap T$ consists of a single edge e because C is an induced 6-cycle, by Theorem 3.3. If e was doubly covered, then $C \cap \delta(T)$ would consist of two uncovered edges, which in turn would violate the Kirchhoff law. Therefore e is uncovered. To finish the proof of (ii), suppose to the contrary that two edges of $\delta(T)$ are incident with the same vertex w outside T. Let f be the third edge of $\delta(T)$. Clearly, $\delta(T \cup \{w\})$ is a 2-edge-cut in G containing f, and both edges of $\delta(T \cup \{w\})$ are traversed by C. Kirchhoff's law implies that these two edges must carry the same value under the characteristic flow, so both of them are uncovered. In particular, f is uncovered. However, f is adjacent to e, so e is doubly covered, contrary to what we have already proved. This establishes (ii). (A typical triangle along with the associated Fano colouring is illustrated in Figure 4.)

Next we prove (iii). Suppose to the contrary that $C = (e_0e_1 \ldots e_5)$ intersects two distinct triangles T_1 and T_2. Adopting the notation of Figure 3, we can further assume that T_1 contains e_1 while T_2 contains e_5, both edges being uncovered. The remaining uncovered edge e_3 may, or may not, belong to a triangle. Clearly, $T_1 = (e_1f_1f_2)$ and $T_2 = (e_5f_5f_0)$. Let us contract each of T_1 and T_2 to a vertex thereby producing a cubic graph $G' = G/(T_1 \cup T_2)$. Note that G' is a snark and $C' = (e_0e_2e_3e_4)$ is a quadrilateral in G'. Moreover, $G' - V(C')$ is 3-edge-colourable which means that C' is non-removable. However, this fact contradicts Lemma 2.2 and establishes (iii).

Assume that G has an essential triangle, and let T be any of them. If C is an arbitrary hexagonal core, then C intersects all essential triangles, according to (i); in particular, C intersects T. However, (iii) implies that T is the only triangle intersected by C. Therefore, T is the only essential triangle of G, which proves (iv). $\square$

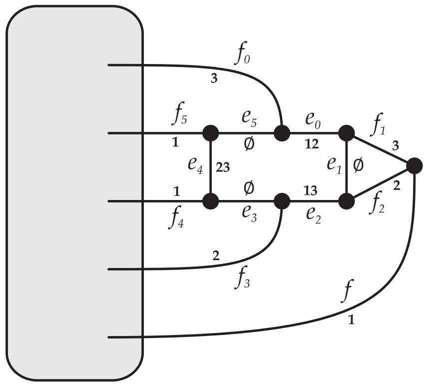


Figure 4: An essential triangle intersected by a hexagonal core

Lemma 4.2. *Let G be a snark with $\mathrm{df}(G) = 3$. If G contains a 2-edge cut R, then the hexagonal core C of any optimal array is disjoint from R. Moreover, C is inherited into the snark G' with $\mathrm{df}(G') = 3$ arising from the reduction of G with respect to R.*

Proof. Let R be a 2-edge-cut of G whose removal leaves components H and K. Consider a 3-array $\mathcal{M}$ for G whose core is a 6-cycle C of G. We show that C is wholly contained in one of the components of $G - R$. Suppose not. Then C intersects R, and hence $R \subseteq C$. Kirchhoff's law for the characteristic flow implies that the edges of R must have the same colour, which means that both edges are uncovered. Adopting the notation for C in accordance with Figure 3 we may assume that $R = \{e_3, e_5\}$. It follows that the edges e_0, e_1, and e_2 belong to one of the components, say H, and the edge e_4 belongs to the other component. The completion $\bar{H}$ of H contains a 4-cycle $(e_0e_1e_2f)$, where f is the edge resulting from the extension of H to $\bar{H}$. It is easy to see that if f is assigned colour 1, then the 3-edge-colouring of $H - E(C)$ induced by $\mathcal{M}$ uniquely extends to a 3-edge-colouring of $\bar{H}$. Similarly we can check that the completion $\bar{K}$ of K is 3-edge-colourable, too. With both $\bar{H}$ and $\bar{K}$ being 3-edge-colourable, we conclude that so is G, which is a contradiction. This proves that C is contained in one of the components of $G - R$, say H. All the edges of K are now simply covered, so K is 3-edge-colourable, and hence H is not. It is easy to see that the matchings $M_1 \cap H$, $M_2 \cap H$, and $M_3 \cap H$ of H extend to perfect matchings M_1', M_2', and M_3' of $\bar{H}$, which constitute a 3-array of $\bar{H}$ having C as its core. Therefore $\mathrm{df}(\bar{H}) = 3$, and $G' = \bar{H}$ is the sought reduction of G. □

Lemma 4.3. *Let G be a snark with $\mathrm{df}(G) = 3$. If G contains a cycle-separating 3-cut, then G admits a reduction to a smaller snark G' with $\mathrm{df}(G') = 3$, unless one of the components resulting from the removal of the cut is an essential triangle. Every hexagonal core of G that does not intersect a triangle is inherited into G'.*

Proof. Let $R = \{r_1, r_2, r_3\}$ be an arbitrary cycle-separating 3-edge-cut in G, and let H and K be the components of $G - R$. There are two cases to consider.

Case 1. *G admits an optimal 3-array $\mathcal{M}$ such that* $\mathrm{core}(\mathcal{M}) \cap R = \emptyset$. We show that contracting the component of $G - R$ not containing $\mathrm{core}(\mathcal{M})$ to a vertex produces a proper reduction G' of G with $\mathrm{df}(G') = 3$. Let C be the 6-cycle constituting the core of $\mathcal{M}$. The assumption guarantees that C is fully contained in a component of $G - R$, say H. Consider the completions $\bar{H}$ and $\bar{K}$ to cubic graphs. Since K is 3-edge-colourable, H is not. Therefore $\bar{K}$ is 3-edge-colourable, and $\bar{H}$ is not. By Theorem 3.3, $\mathrm{df}(\bar{H}) \geqslant 3$. As $C \subseteq H$, the edges of R are simply covered, and by the Kirchhoff law applied to the characteristic flow for $\mathcal{M}$ they belong to three distinct members of $\mathcal{M}$. It is now clear that $\mathcal{M}$ induces a 3-array $\mathcal{M}'$ of $\bar{H}$ with C as its core. Therefore $\mathrm{df}(\bar{H}) = 3$, and $G' = \bar{H}$ is the required reduction of G containing C. Note that G' is isomorphic to the graph G/K obtained from G by contracting K into a single vertex. This establishes Case 1.

Case 2. *The core of every optimal* 3*-array for G intersects the* 3*-edge-cut R.* We start with the following observation.

Claim 1. *$G - R$ has a unique component Q such G/Q is a snark. Moreover, $C \cap Q$ consists of a single uncovered edge.*

Proof of Claim 1. Clearly, $|C \cap R| = 2$, and we may assume that $C \cap R = \{r_1, r_2\}$. Kirchhoff's law yields that $\chi(r_1) + \chi(r_2) + \chi(r_3) = 0$. By Proposition 3.1, χ is a nowhere-zero $\mathbb{Z}_2^3$-flow, therefore the values $\chi(r_1)$, $\chi(r_2)$, and $\chi(r_3)$ constitute a line ℓ in the Fano plane. Since C intersects R, the line $\{(0,1,1),(1,0,1),(1,1,0)\} = \{1,2,3\}$ is excluded. There remain two possibilities for ℓ, which imply that either

- both r_1 and r_2 are doubly covered, or
- one of them is doubly covered and the other is uncovered.

Without loss of generality we may assume that in the former case $\phi(r_1) = 12$, $\phi(r_2) = 13$, and $\phi(r_3) = 1$, and in the latter case $\phi(r_1) = 12$, $\phi(r_2) = \emptyset$, and $\phi(r_3) = 3$, see Figure 1. We prove that the latter possibility does not occur.

Suppose to the contrary that $\phi(r_1) = 12$, $\phi(r_2) = \emptyset$, and $\phi(r_3) = 3$. Since the edges in R are independent, we conclude that $r_1 = e_0$ and $r_2 = e_3$, see Figure 3. Let H be the component of $G - R$ that contains e_1 and e_2. If we set $\phi'(e_0) = 2$, then the 3-edge-colouring of ϕ of $H - E(C)$ associated with $\mathcal{M}$ extends to a 3-edge-colouring ϕ' of $\bar{H}$. By symmetry, the 3-edge-colouring ϕ of $K - E(C)$ extends to a 3-edge-colouring of $\bar{K}$. It follows that G is 3-edge-colourable, which is a contradiction.

Therefore $\phi(r_1) = 12$, $\phi(r_2) = 13$, and $\phi(r_3) = 1$. Now $r_1 = e_0$ and $r_2 = e_2$, so one of the components of $G - R$, say H, contains the path $e_3e_4e_5$ and the other component K contains the uncovered edge e_1. Clearly, K is 3-edge-colourable. In other words, K is the required component Q of $G - R$ such that $Q \cap C$ consists of a single uncovered edge. This establishes Claim 1.

To finish the proof it remains to establish the following.

Claim 2. *If Q is not an essential triangle, then G has a proper reduction to a snark G' with* $\mathrm{df}(G') = 3$.

Proof of Claim 2. We keep the assumptions adopted in the course of the proof of Claim 1. In particular, $C \cap R = \{e_0, e_2\}$, $Q = K$, and the unique edge of $Q \cap C$ is e_1.

First assume that $Q = K$ is a triangle. Clearly, the graph G/K obtained by contracting K into a single vertex is a snark, so $\mathrm{df}(G/K) \geqslant 3$. As K is not essential, we conclude that $\mathrm{df}(G/K) = 3$, implying that G/K is the required reduction of G. Note that, in this case, every hexagonal core of G/K intersects the vertex of G/K resulting from the contraction of K, which means that it is not inherited from G.

Next assume that K is not a triangle. Consider the 3-edge-cut $R' = \{f_1, f_2, r_3\}$ in G. Let H' and K' be the components of $G - R'$, with K' being the one that does not contain e_1. Note that $C \subseteq H'$. Clearly, $\mathcal{M}$ induces a proper 3-edge-colouring of K', so H' is not 3-edge-colourable, and therefore the graph $G' = G/K'$ obtained from G by contracting K' into a single vertex is a snark. Clearly, G' is a proper reduction of G. Observe that the edges e_1, f_1, and f_2 form a triangle separated by R' from the rest of G'. To show that G' is the sought reduction it remains to check that $\mathrm{df}(G') = 3$. To this end it is sufficient to realise that $M_i' = M_i \cap G'$ is a perfect matching of G' for each $i \in \{1, 2, 3\}$. Thus $\mathcal{M}' = \{M_1', M_2', M_3'\}$ is a 3-array of G' with $\mathrm{core}(\mathcal{M}') = \mathrm{core}(\mathcal{M}) = C$, and we are done. This concludes the proof of Claim 2 as well as that of Lemma 4.3. $\square$

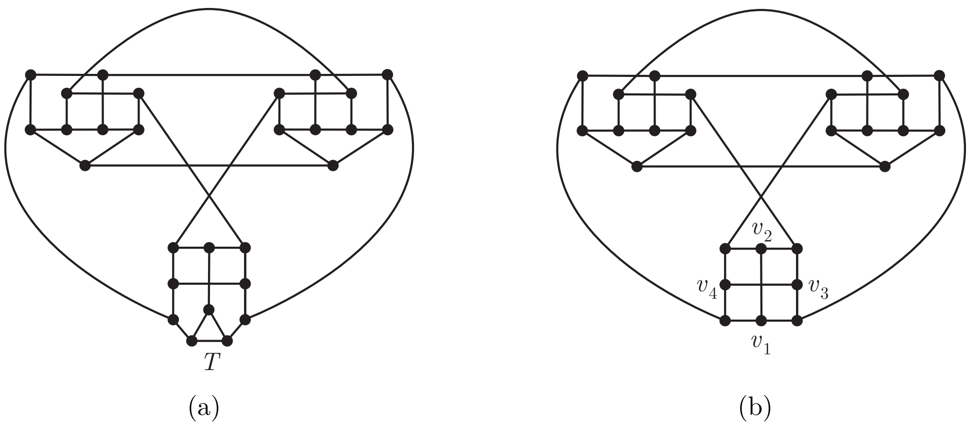


Figure 5: No hexagonal core of the snark G is inherited into G/T

Example 4.4. Lemma 4.1(i) informs us that if a snark with defect 3 contains an essential triangle, then the triangle must be intersected by every hexagonal core. Somewhat surprisingly, the converse is not true, which implies that the discussion following Claim 2 in the proof of Lemma 4.3 cannot be avoided. The graph G in Figure 5(a) has defect 3 and possesses three hexagonal cores, all of which intersect its only triangle T. The triangle T is *not* essential in G, because the graph G/T, shown in Figure 5(b), has defect 3 as well. The latter graph has two hexagonal cores, both containing the vertex resulting from contraction of T, which is denoted by v_1. Note that inflating G/T at any of the vertices v_i with $i \in \{1, 2, 3, 4\}$ produces a snark with a triangle having the same property as T has in G.

Lemma 4.5. *Let G be a snark with $\mathrm{df}(G) = 3$. If a hexagonal core of G intersects a quadrilateral, then the intersection consists of a single uncovered edge. Moreover, two edges leaving the quadrilateral are doubly covered and the other two are simply covered.*

Proof. Consider an arbitrary optimal 3-array $\mathcal{M}$ for G, and let C be its hexagonal core. We keep using the previous notation for the characteristic flow and the Fano colouring around C (cf. Figure 1 and Figure 3). Further, let D be a 4-cycle in G, and let $R = \delta(D)$.

Claim 1. $|C \cap R| = 2$.

Proof of Claim 1. Clearly, $|C \cap R|$ is even, so $|C \cap R| = 2$ or $|C \cap R| = 4$. The latter possibility cannot occur. Indeed, if we had $|C \cap R| = 4$, then $C \cap D$ would consist of two independent edges of D. However, the other two edges of D would constitute chords of C, contrary to Theorem 3.3.

By Claim 1, the edges of R naturally come in two pairs, those that belong to C and those that do not. Let us assume that $R = \{r_1, r_2, r_3, r_4\}$ where $C \cap R = \{r_1, r_2\}$. The other pair $\{r_3, r_4\}$ thus consists of simply covered edges.

Claim 2. $\phi(r_1) \neq \phi(r_2)$.

Proof of Claim 2. Suppose to the contrary that $\phi(r_1) = \phi(r_2)$. Clearly, this is only possible when both r_1 and r_2 are uncovered. Since $C \cap D$ is a path and C has no chords, we conclude that $C \cap D$ is not a path of length 3. Since we are assuming $\phi(r_1) = \phi(r_2)$, $C \cap D$ cannot be a path of length 2. We conclude that $C \cap D$ has one edge. Without loss of generality we may assume that $r_1 = e_5$ and $r_2 = e_1$, so the unique edge of $C \cap D$ is e_0. It follows that $D = (e_0 f_1 d_0 f_0)$, where d_0 is the edge joining the end-vertices u_0 and u_1 of f_0 and f_1, respectively, not lying on C. Since $\phi(r_1) = \phi(r_2)$, Kirchhoff's law implies that $\phi(r_3) = \phi(r_4)$. Recalling that $\phi(f_0) = \phi(f_1) = 3$, we conclude $\phi(r_3) \in \{1, 2\}$. Without loss of generality we may assume that $\phi(r_3) = \phi(r_4) = 1$. We are going to recolour the edges of $C \cup D$. If we set $\phi'(e_3) = 3$, we can uniquely extend the 3-edge-colouring of ϕ of $G - E(C \cup D)$ induced by $\mathcal{M}$ to a 3-edge-colouring ϕ' of the entire G with $\phi'(e_0) = 1$, $\phi'(e_1) = 3$, $\phi'(e_2) = 1$, $\phi'(e_4) = 2$, $\phi'(e_5) = 3$, $\phi'(f_0) = 2$, $\phi'(f_1) = 2$, and $\phi'(d_0) = 3$. This contradiction proves that $\phi(r_1) \neq \phi(r_2)$.

An important consequence of Claim 2 is that the set $\ell = \{\chi(r_1), \chi(r_2), \chi(r_3) + \chi(r_4)\}$ forms a line of the Fano plane. Replacing χ with ϕ, there are two possibilities for ℓ up to permutation of the index set $\{1, 2, 3\}$, just as in the proof of Lemma 4.3 (see Claim 1 therein): either $\ell = \{12, 13, 1\}$ or $\ell = \{\emptyset, 12, 3\}$. Next we show that the latter possibility does not occur.

Claim 3. $\{\phi(r_1), \phi(r_2), \phi(r_3) + \phi(r_4)\} = \{12, 13, 1\}$.

Proof of Claim 3. Suppose to the contrary that $\{\phi(r_1), \phi(r_2), \phi(r_3) + \phi(r_4)\} = \{\emptyset, 12, 3\}$. In view of symmetry, we can assume that $\phi(r_1) = 12$ and $\phi(r_2) = \emptyset$. It follows that $C \cap D$ coincides with the path $e_1 e_2$ or $e_5 e_4$. Without loos of generality we may assume the former, so $f_2 \in \{r_3, r_4\}$ and $D = (e_1 e_2 f_3 f_1)$. We may further assume that $f_2 = r_3$, whence $\phi(r_3) = 2$. Since r_4 shares a common vertex with f_1 and f_3, and one has $\phi(f_1) = 3$ and

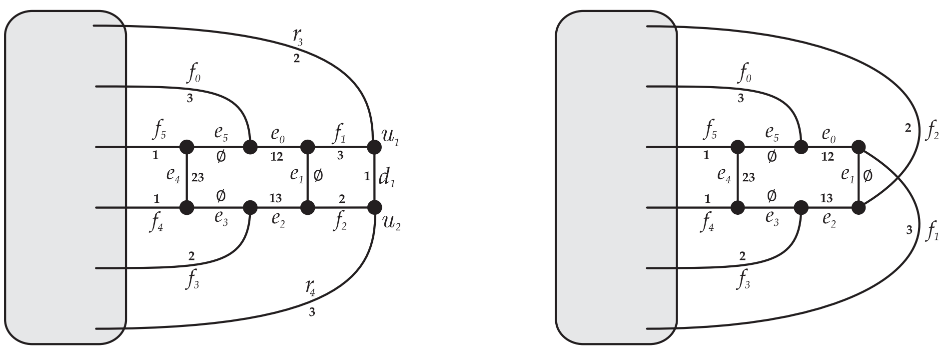
$\phi(f_3) = 2$, we conclude that $\phi(r_4) = 1$. This, in particular, confirms the assumption that $\phi(r_3)+\phi(r_4) = 3$. If we set $\phi'(e_5) = 2$, then the 3-edge-colouring of ϕ of $G - E(C)$ induced by $\mathcal{M}$ uniquely extends to a 3-edge-colouring ϕ' of the entire G with $\phi'(e_0) = 1$, $\phi'(e_1) = 3$, $\phi'(e_2) = 1$, $\phi'(e_3) = 2$, $\phi'(e_4) = 3$, $\phi'(f_1) = 2$, and $\phi'(f_3) = 3$. This contradiction establishes Claim 3.

We have just proved that $\{\phi(r_1), \phi(r_2), \phi(r_3)+\phi(r_4)\} = \{12, 13, 1\}$. In view of symmetry, we can assume that $\phi(r_1) = 12$, which implies that $r_1 = e_0$, $r_2 = e_2$, and that the unique edge of $C \cap D$ is the uncovered edge e_1. This proves the lemma. □

Proposition 4.6. *Let G be a snark with* $\mathrm{df}(G) = 3$*. If G contains a* 4*-cycle, then G can be reduced to a smaller snark G' with* $\mathrm{df}(G') = 3$*. Moreover, every hexagonal core of G is inherited into G'.*

Proof. Let D be a 4-cycle in G, and let $R = \delta(D) = \{r_1, r_2, r_3, r_4\}$. We may assume that the edges of R are independent, because otherwise G would have a cycle-separating 2-cut or 3-cut, and we could apply Lemmas 4.2 and 4.3 to conclude that G admits a proper reduction to a snark G' with $\mathrm{df}(G') = 3$. Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be an arbitrary optimal 3-array for G, and let C be its hexagonal core. There are essentially two possibilities for $C \cap D$: either $C \cap D = \emptyset$ or, by Lemma 4.5, $C \cap D$ consists of a single edge.

If $C \cap D = \emptyset$, then $C \cap R = \emptyset$. By Lemma 2.2, the graph $G - V(D)$ is not 3-edge-colourable. The four dangling edges of $G - V(D)$ are simply covered, and by the Kirchhoff law they occur in two equally coloured pairs. We join each pair into a single edge thereby producing a snark G' of order $|G'| = |G| - 4$. The Fano flow on G clearly induces one on G', which in turn determines the same core C. Hence, G' is the required proper reduction of G.



(a) (b)

Figure 6: A quadrilateral intersected by a hexagonal core and its reduction

Henceforth we may assume that $C \cap R \neq \emptyset$. From Lemma 4.5 we obtain that the intersection of the core and the quadrangle consists of a unique uncovered edge of C, say e_1.

Since $r_1 = e_0$ and $r_2 = e_2$, we conclude that $D = (e_1 f_1 d_1 f_2)$, where d_1 is the edge of D joining the end-vertices u_1 and u_2 of f_1 and f_2, respectively, not lying on C. Without loss of generality we may assume that r_3 and r_4 are incident with u_1 and u_2, respectively. Recall that $\phi(f_1) = 3$ and $\phi(f_2) = 2$, which implies that $\phi(d_1) = 1$, $\phi(r_3) = 2$, and $\phi(r_4) = 3$, see Figure 6(a). Now, we take the graph $G - \{u_1, u_2\}$, and keep the four dangling edges f_1, f_2, r_3, and r_4. We form G' from $G - \{u_1, u_2\}$ by performing the junctions $f_1 * r_4$ and $f_2 * r_3$. By Lemma 2.2, $G - V(D)$ is not 3-edge-colourable, so $\mathrm{df}(G') \geqslant 3$. If we define ϕ' by setting $\phi'(f_1 * r_4) = \phi(f_1) = \phi(r_4) = 3$, $\phi'(f_2 * r_3) = \phi(f_2) = \phi(r_3) = 2$, and $\phi'(x) = \phi(x)$ for all other edges x of G', we obtain a Fano colouring which determines a 3-array of G' whose core coincides with C, see Figure 6(b). The proof is complete. □

5 Main results

We are now ready to establish the main results of this paper.

Theorem 5.1. *Every snark G with $\mathrm{df}(G) = 3$ admits a reduction to a snark G' with $\mathrm{df}(G') = 3$ such that either G' is nontrivial or G' arises from a nontrivial snark K with $\mathrm{df}(K) \geqslant 4$ by inflating a vertex to a triangle; the triangle is essential in both G and G'.*

Proof. Consider an arbitrary snark G with $\mathrm{df}(G) = 3$. If G is nontrivial, then $G' = G$ is the required reduction. Assume that G is not nontrivial, but it cannot be reduced to a nontrivial snark with $\mathrm{df}(G) = 3$. We show that G has a reduction to a snark G' which arises from a nontrivial snark K with $\mathrm{df}(K) \geqslant 4$ by inflating a vertex to a triangle.

Let G' be a reduction of G with $\mathrm{df}(G') = 3$ such that G' is not nontrivial, but it has no reduction to a smaller snark with defect 3. Lemmas 4.2 and 4.6 imply that G' has no 2-edge-cuts and no quadrilaterals. Since G' is not nontrivial, it has a cycle-separating 3-edge-cut. By Lemma 4.3, one of the resulting components must be an essential triangle, which we denote by T. Set $K = G'/T$. Clearly, K has $\mathrm{df}(K) \geqslant 4$. Hence, to finish the proof it remains to prove that the graph K is cyclically 4-edge-connected and has no quadrilaterals.

Claim 1. *The graph K is cyclically 4-edge-connected.*

Proof of Claim 1. Suppose to the contrary that K is not cyclically 4-edge-connected. Clearly, K has no bridges and no 2-edge-cuts because these features would already be present in G'. Therefore K has a cycle-separating 3-edge-cut R. Observe that the same set R is a cycle-separating 3-edge-cut also in G'. Moreover, all of T is contained in the same component of $G' - R$; let us denote this component by L. Since L/T is a component of $K - R$ and L/T contains a circuit, L contains more than three vertices; in particular L is not an essential triangle. The other component of $G' - R$ cannot be an essential triangle either, because Lemma 4.1(iv) implies that G' has only one essential triangle, namely T. It follows that R is a cycle-separating 3-edge-cut in G' such that neither of the two components of $G' - R$ is an essential triangle. By Lemma 4.3, G' admits a reduction to a smaller snark G'' with $\mathrm{df}(G'') = 3$. This contradicts the choice of G' and proves that K is cyclically 4-edge-connected.

Claim 2. *The graph K has no quadrilateral.*

Proof of Claim 2. Suppose to the contrary that K contains a quadrilateral Q. The vertex v of K obtained by the contraction of T lies on Q, therefore G' has a 5-cycle $D = (d_0d_2d_1d_3d_4)$ sharing an edge with T, say d_0. Let g and h denote the other two edges of T, with g adjacent to d_1, and h adjacent to d_4. There is a 4-edge-cut S in G' that separates $D \cup T$ from the rest of G'. Let H be the other component of $G' - S$. Observe that S survives the contraction of T, so S separates Q from H in G'/T as well. Since G'/T is cyclically 4-edge-connected, S is an independent edge cut. Let $s_1 \in S$ be the edge of $\delta(T)$ not adjacent to d_0, and for $i \in \{2, 3, 4\}$ let s_i be the edge of S adjacent to both d_{i-1} and d_i, see Figure 7.

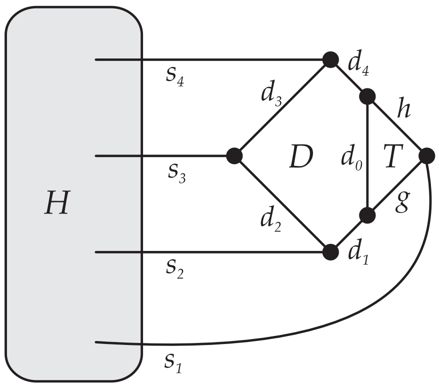


Figure 7: The structure of G' when the contraction of an essential triangle creates a 4-cycle

Because $\mathrm{df}(G') = 3$, there is a 3-array $\mathcal{M} = \{M_1, M_2, M_3\}$ in G' whose hexagonal core C intersects T. By Lemma 4.1(ii), $C \cap T$ consists of a single uncovered edge, so C cannot be the hexagon $(d_1d_2d_3d_4hg)$ fully contained in $D \cup T$. Therefore C intersects S. It is easy to see that $|C \cap S| = 2$. Without loss of generality we may assume that the common edge of C and T is the edge e_1 of the standard hexagonal core shown in Figure 3. There are two possibilities for the position of e_1: either e_1 is adjacent to s_1, or not. If e_1 is not adjacent to s_1, then $e_1 = d_0$. If e_1 is adjacent to s_1, then $e_1 \in \{g, h\}$, and in view of symmetry we may assume that $e_1 = g$. Accordingly, we have two cases to consider.

Case 1. $e_1 = d_0$. In this case both d_1 and d_4 belong to C; in view of symmetry we may clearly assume that $d_1 = e_2$ and $d_4 = e_0$. Moreover, s_1 does not belong to C, so $C \cap S$ consists of two edges from $\{s_2, s_3, s_4\}$. In view of symmetry we may assume that either $C \cap S = \{s_2, s_3\}$ or $C \cap S = \{s_2, s_4\}$. The former possibility does not occur because otherwise the edge d_2 would be a chord of C, contrary to Theorem 3.3. Hence $C \cap S = \{s_2, s_4\}$. Now $s_2 = e_3$ and $s_4 = e_5$, so the only edge of C in H is e_4. Recall that $\phi(e_4) = 23$. If we change this colour to 2 (or apply Proposition 3.2(iv)), we obtain a proper 3-edge-colouring of H. However, H is now 3-edge-colourable, and the other component of $G'/T - S$ is a quadrilateral. By Lemma 2.2, G'/T is 3-edge-colourable, and hence so is G'. This contradiction concludes Case 1.

Case 2. $e_1 = g$. In this case d_1 and s_1 belong C; in view of symmetry we may clearly assume that $d_1 = e_0$ and $s_1 = e_2$. It follows that $C \cap S = \{s_1, s_j\}$ for some $j \in \{2, 3, 4\}$. We show that $j = 2$. If $C \cap S = \{s_1, s_3\}$, then $e_5 = d_2$, $f_1 = d_0$, $f_2 = h$, and $f_5 = d_3$. Since $\phi(f_1) = 3$ and $\phi(f_2) = 2$, we have $\phi(d_4) = 1$. At the same time $\phi(f_5) = 1$, so d_3 and d_4 have the same colour, which is impossible, because they are adjacent. Therefore $j \neq 3$. Next, if $C \cap S = \{s_1, s_4\}$, then $d_4 = e_4$ and $s_4 = e_3$. In this situation, however, both e_2 and e_3 belong to S although they are adjacent in C. As S is independent, this is impossible. Therefore $j \neq 4$, and we conclude that $C \cap S = \{s_1, s_2\}$. Now $e_2 = s_1$ and $e_5 = s_2$, so e_3 and e_4 belong to H. Recall that $\phi(e_3) = \emptyset$ and $\phi(e_4) = 23$. If we change the colour of e_3 to 3 and the colour of e_4 to 2, we get a proper 3-edge-colouring of H. Again, by using Lemma 2.2 we can conclude that G' is 3-edge-colourable, which is a contradiction. This finishes Case 2 and proves that K has no quadrilaterals. The proof is complete. □

Our second theorem specifies conditions that must be satisfied by a snark K and a vertex v in order for the inflation of v to decrease the defect of K to 3. The formulation features an important concept of a cluster of 5-cycles in a snark that derives from relatively little known results of Kászonyi [20]-[22] and Bradley [1]-[3] concerning the structure of 3-edge-colourings of graphs. A *cluster of* 5*-cycles* in a cubic graph G, or simply a 5*-cluster* of G, is an inclusion-wise maximal connected subgraph of G formed by a union of 5-cycles. Kászonyi [21, 22] and later Bradley [1] proved that for each edge e of a snark G there exists a nonnegative integer $\psi_G(e)$ such that the number of 3-edge-colourings of $G \sim e$ equals $18 \cdot \psi_G(e)$. We refer to the function $\psi_G\colon E(G) \to \mathbb{N}$ as the *Kászonyi function* for G. In passing we mention that the Kászonyi function for the Petersen graph identically equals 1, see [3, Theorem 3.5].

One of the most remarkable properties of the Kászonyi function is that it is constant on each 5-cluster (see [1], [22], or the survey [3]). A 5-cluster H of a snark G will be called *heavy* if $\psi_G(e) > 0$ for each edge e of H; otherwise, H will be called *light*. Equivalently, a 5-cluster H is heavy if and only if $G \sim e$ is 3-edge-colourable for each edge e of H. In this context it is useful to recall that if $e = uv$, then $G \sim e$ is 3-edge-colourable if and only if $G - \{u, v\}$ is, see [29, Proposition 4.2].

Theorem 5.2. *Let K be a nontrivial snark with* $\mathrm{df}(K) \geqslant 4$*, let v be a vertex of K, and let G be the snark created from K by inflating v to a triangle. Then* $\mathrm{df}(G) = 3$ *if and only if v lies in a heavy cluster of* 5*-cycles of K.*

Proof. ($\Rightarrow$) Assume that $\mathrm{df}(G) = 3$. Let T denote the triangle of G obtained by the inflation of a vertex v of K. We show that v belongs to a heavy 5-cluster of K. Since $\mathrm{df}(K) \geqslant 4$ and $\mathrm{df}(G) = 3$, the triangle T is essential. Let C be a hexagonal core of G. Lemma 4.1(ii) implies that $C \cap T$ consists of a single uncovered edge, which we may assume to be the edge e_1 indicated in Figure 3. It follows that the Fano colouring ϕ around C is as illustrated in Figure 4. Let us contract T back to the vertex v and keep the colours of the edges of K. Clearly, $C/T = (e_0e_2e_3e_4e_5)$ is a 5-cycle containing v. To prove that v belongs to a heavy 5-cluster it is sufficient to show that $K \sim e_0$ admits a 3-edge-colouring. Recall that $e_0 = v_0v_1$ and consider the graph $K - \{v_0, v_1\}$. If we set

$\phi'(e_3) = 3$ and $\phi'(e_4) = 2$, we obtain a proper 3-edge-colouring of $K - \{v_0, v_1\}$. We infer that $K \sim e_0$ is 3-edge-colourable, too, and therefore v belongs to a heavy 5-cluster of K.

($\Leftarrow$) For the converse, assume that v belongs to a heavy 5-cluster H of K. Consider an (induced) 5-cycle $D = (d_0 d_1 d_2 d_3 d_4)$ of H such that v lies in D and is incident with the edges d_4 and d_0. Since H is heavy, removing any edge of D together with its end-vertices produces a 3-edge-colourable graph. Hence $K - E(D)$ is 3-edge-colourable, as well. Let g_i be the edge of $\delta(D)$ adjacent to d_{i-1} and d_i, with indices taken modulo 5. The edge of $\delta(D)$ incident with v is therefore g_0. From the Parity Lemma we deduce that every 3-edge-colouring σ of $K - E(D)$ colours three of the edges in $\delta(D)$ with the same colour, say 1, and the remaining two with colours 2 and 3, respectively. Moreover, if $\sigma(g_i) = 2$ and $\sigma(g_j) = 3$, then g_i and g_j are not adjacent to the same edge of D, otherwise σ would extend to a 3-edge-colouring of K. Even more, as can be deduced from Lemmas 6.1, 6.2, and 6.3 (iii) of [29], for any two edges g_i and g_j of $\delta(D)$ that are not incident with the same edge of D there exists a 3-edge-colouring τ of $K - E(D)$ such that $\tau(g_i) = 2$ and $\tau(g_j) = 3$. Set $\tau(g_1) = 2$ and $\tau(g_4) = 3$, so that all the remaining edges of $\delta(D)$ receive colour 1, see Figure 8 (left). Now, let us inflate v into a triangle T, thereby producing the graph G.

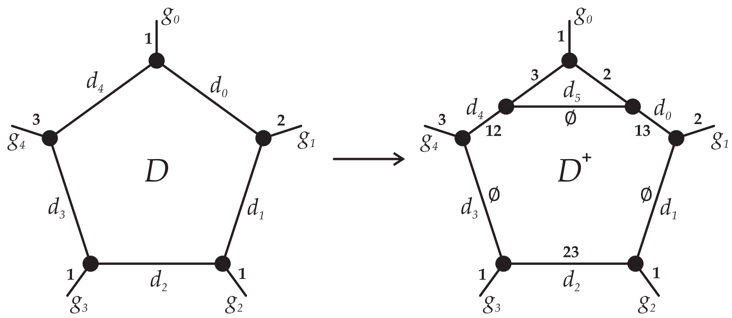


Figure 8: Creating an essential triangle by inflating a vertex on a non-removable 5-cycle

For each edge of K incident with v there is a unique corresponding edge of G leaving the triangle T; we let the latter edge keep the name of the former. Having made this agreement, let d_5 denote the edge of T adjacent to both d_0 and d_4. Clearly, $D^+ = (d_0 d_1 d_2 d_3 d_4 d_5)$ is an induced 6-cycle of G. Now we extend the colouring τ of $K - E(D)$ to a colouring of G. We start by setting $\tau(d_4) = 12$, $\tau(d_5) = \emptyset$, and $\tau(d_0) = 13$. This choice further enables setting $\tau(d_2) = 23$ and $\tau(d_1) = \tau(d_3) = \emptyset$, as well as assigning colours 2 and 3 to the remaining edges of T appropriately, see Figure 8 (right). It is easy to check that τ induces a 3-array $\mathcal{N}$ of G with $\mathrm{core}(\mathcal{N}) = D^+$. Therefore $\mathrm{df}(G) = 3$, as required. $\square$

Given a cubic graph G and a vertex v of G we let G^v denote the graph formed from G by inflating v to a triangle. We now show that a single vertex inflation can decrease defect from an arbitrarily large value to the minimal possible value of 3.

Theorem 5.3. *For every integer $n \geqslant 3$ there exists a nontrivial snark G with $\mathrm{df}(G) \geqslant n$ which contains a vertex whose inflation to a triangle produces a snark with defect* 3.

Proof. In [16, Theorems 5.1-5.3] it was proved that for every integer $n \geqslant 3$ there exists a cyclically 5-edge-connected snark $H = H_{2n}$ with girth $2n$ which contains a pair of adjacent vertices u and v such that $H - \{u, v\}$ is 3-edge-colourable. By [31, Corollary 2.5] (see also [16, Proposition 4.4]), the defect of H is at least n. Let I denote the isochromatic $(2, 2)$-pole arising from H_{2n} by removing the vertices u and v and forming connectors from the semiedges formerly incident with the same vertex. To construct $G = G_n$, we take three copies I_0, I_1, and I_2 of I and a 6-pole Z arising from the Petersen graph Pg by severing three independent edges p_0, p_1, and p_2 on a 6-cycle. We turn Z into a $(2, 2, 2)$-pole with connectors S_1, S_2, and S_3 by forming each S_i from the two half-edges of p_i, denoted by p_{i1} and p_{i2}. Next, for each $j \in \{0, 1, 2\}$, we join the input connector of I_j to S_j; we keep the notation p_{j1} and p_{j2} for the resulting edges which connect I_j to Z. Finally, we match the semiedges of the three output connectors in such a way that the two semiedges of each output connector lead to two other output connectors, see Figure 9 (left). The result is the required graph $G = G_n$.

We proceed to proving that G_n has the required properties. During our analysis we refer to Figure 9 (right) for the notation of vertices and edges of G_n. In particular, z denotes the central vertex of Z, the edges incident with z are e_0, e_1, and e_2, and $(f_0 f_1 \dots f_8)$ is the 9-cycle obtained by removing z from Z. The edges of $\delta(Z)$ leave Z in the order $p_{01}, p_{22}, p_{11}, p_{02}, p_{21}, p_{12}$ determined by a cyclic orientation of $Z - z$.

Claim 1. G_n is a nontrivial snark.

Proof of Claim 1. First we prove that G_n is a snark. Suppose that G_n admits a 3-edge-colouring ϕ. Since each I_j is an isochromatic $(2, 2)$-pole, the edges p_{j1} and p_{j2} receive the same colour from every 3-edge-colouring of G_n. Recall, that p_{j1} and p_{j2} arise by severing the edge p_j of Pg. It follows that the restriction of ϕ to Z extends to a 3-edge-colouring of Pg, which is a contradiction. Thus G_n is a snark.

It is clear from the construction that G_n has girth at least 5. We need to check that G_n is cyclically 4-edge-connected. To this end, it suffices to realise that the building blocks of G_n – the 6-pole Z and the 4-poles I_0, I_1, and I_2 – arise from nontrivial snarks and that the way in which the building blocks have been combined in G_n guarantees that no cycle-separating edge cut of size smaller than 4 can be created. Therefore every cycle-separating edge cut in G_n has size at least 4. In fact, all minimum size cycle-separating edge cuts in G_n separate one of the 4-poles I_j from the rest of G_n; hence, the cyclic connectivity of G_n equals 4.

Claim 2. G_n has no 3-array whose core is fully contained in Z.

Proof of Claim 2. Suppose to the contrary that G_n has a 3-array $\mathcal{M}$ with core $C \subseteq Z$. It follows that all the edges of $G_n - Z$ as well as those belonging to the edge cut $\delta(Z)$ connecting Z to $G_n - Z$ are simply covered. There are two cases to consider.

Case 1. *C contains a triply covered edge.* If x is a triply covered edge of C, then the four edges adjacent to x are uncovered, so x cannot be adjacent to an edge of $\delta(Z)$. Therefore x is one of the edges e_0, e_1, or e_2 incident with the central vertex z of Z, say e_0. Assume that the 9-cycle $(f_0 f_1 \dots f_8)$ has its edges enumerated cyclically in such a way that f_0 is

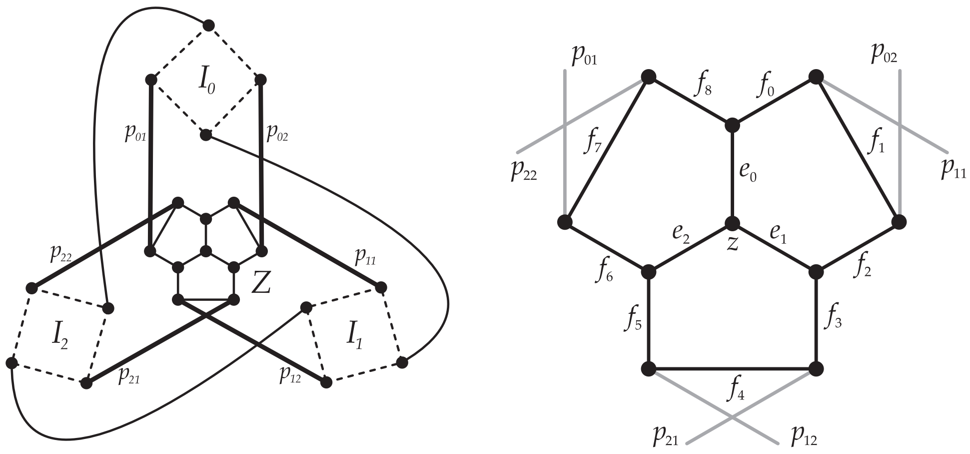


Figure 9: The graph G_n and its heavy cluster Z

adjacent to e_0 and f_2 is adjacent to e_1, see Figure 9 (right). All four edges adjacent to e_0 are uncovered, in particular, so are f_0 and e_1. Since the edges of $\delta(Z)$ are simply covered, it follows that f_1 is doubly covered and f_2 is uncovered. Now, e_1 and f_2 are adjacent uncovered edges, so f_3 must be triply covered. However, f_3 is adjacent to an edge of $\delta(Z)$, which is simply covered, and we have arrived at a contradiction.

Case 2. *C contains no triply covered edge.* In this case the core C is regular, and hence, by Proposition 3.2(i), it is a collection of disjoint even circuits. Observe that every circuit contained in Z is either a pentagon, an 8-gon, or a 9-gon. Therefore C must be a single 8-gon. Without loss of generality we may assume that $C = (e_0f_0f_1f_2f_3f_4f_5e_2)$. Now, one of e_0 and e_2 must be uncovered, say e_0. It follows that f_0 and f_4 are doubly covered and f_8 is simply covered. Consider the Fano colouring ϕ of G_n associated with the 3-array $\mathcal{M}$. Without loss of generality we may assume that $\phi(f_0) = 12$. This implies that $\phi(f_8) = \phi(p_{11}) = 3$, and since I_1 is isochromatic, we infer that $\phi(p_{12}) = \phi(p_{11}) = 3$. Note that f_4 is doubly covered and is adjacent to both $\phi(p_{12})$ and $\phi(p_{21})$, so $\phi(f_4) = 12$, and hence $\phi(p_{21}) = 3$. Using the isochromatic property of I_2 we conclude that $\phi(p_{22}) = 3$, which is impossible because p_{22} is adjacent to f_8 and $\phi(f_8) = 3 = \phi(p_{22})$. This contradiction completes the proof of Claim 2.

Claim 3. $\mathrm{df}(G_n) \geqslant n$.

Proof of Claim 3. Let D be a circuit of the core of any 3-array of G_n. By Claim 2, D must intersect at least one of I_0, I_1, and I_2. If D is contained in some I_j, then its length is clearly at least $2n$. Otherwise, D contains at least $2n-2$ vertices of I_j and at least two vertices of Z, and again its length is at least $2n$. Recall that the edges of D are of three kinds – uncovered, doubly covered and triply covered. Moreover, by Proposition 3.2(iii), the union of doubly and triply covered edges in D forms a matching of D. Therefore, there are at least n uncovered edges in D. In other words, the defect of G_n is at least n.

Claim 4. *The 5-cluster Z is heavy.*

Proof of Claim 4. It is sufficient to show that $G_n \sim x$ is 3-edge-colourable for some edge x of Z, say $x = e_0$. Let J denote the 6-pole obtained from G_n by removing the vertices of the 6-pole Z, so that $G_n = J * Z$. Recall that every isochromatic 4-pole admits a 3-edge-colouring where all four dangling edges receive the same colour, see for example [7, Section 3]). It follows that J admits a 3-edge-colouring where all six dangling edges receive the same colour. It is easy to check that such an assignment of colours to the dangling edges of Z extends to a 3-edge-colouring of $Z \sim e_0$; we leave the details to the reader. By combining these two 3-edge-colourings we obtain one for $G_n \sim e_0$. This proves that Z is a heavy 5-cluster.

Now we can finish the proof. Claim 4 states that $\mathrm{df}(G_n) \geqslant n$. On the other hand, Theorem 5.2 implies that the inflation of every vertex v of Z produces a graph G_n^v with $\mathrm{df}(G_n^v) = 3$. Both required properties of G_n are verified, and the proof is complete. □

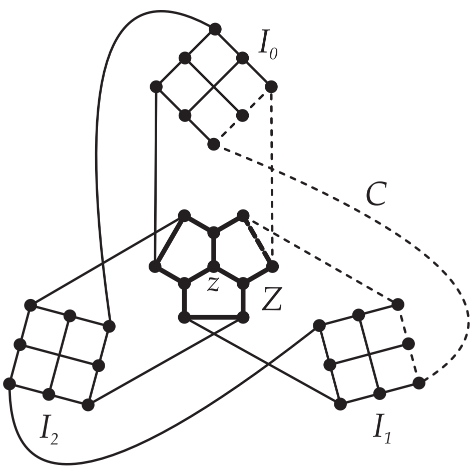


Figure 10: The smallest nontrivial snark with df $\geqslant 4$ that contains a heavy 5-cluster

Example 5.4. The smallest example of a nontrivial snark with defect greater than 3 containing a vertex whose inflation produces a snark with defect 3 has 34 vertices; it is depicted in Figure 10. Its structure is similar to the graphs constructed in Theorem 5.3. The isochromatic $(2,2)$-poles I_0, I_1, and I_2 arise from the Petersen graph by removing two adjacent vertices. The defect of this snark is 4, and the corresponding core C is an 8-cycle indicated in Figure 10 by dashed edges.

6 Berge covers of snarks with defect 3

To justify the importance of the results of the previous two sections, we briefly indicate how they apply to verifying Berge's conjecture for snarks with defect 3. We show that every bridgeless cubic graph with defect 3 can have its edges covered with four or five perfect matchings and we determine those that require five. In other words, we completely determine the perfect matching index of defect 3 graphs. A detailed proof will appear in [19]. Recall that the *perfect matching index* (also known as the *excessive index*) of a bridgeless cubic graph G, denoted by $\pi(G)$, is the smallest number of perfect matchings that cover all the edges of G. With this definition, Berge's conjecture states that $\pi(G) \leqslant 5$ for every bridgeless cubic graph G. Note that $\pi(G) \geqslant 3$, and the equality holds if and only if G is 3-edge-colourable.

The following theorem is proved in [17].

Theorem 6.1. *Let G be a cyclically 4-edge-connected cubic graph with defect 3. Then $\pi(G) = 4$, unless G is the Petersen graph.*

In order to be able to discuss the general situation in the class of defect 3 graphs we will make use of the following two well-known operations. Let G and H be cubic graphs with distinguished edges e and f, respectively. We define a 2-sum $G \oplus_2 H$ to be a cubic graph obtained by deleting e and f and connecting the 2-valent vertices of G to those of H. If instead of distinguished edges we have distinguished vertices u and v of G and H, respectively, we can similarly define a 3-sum $G \oplus_3 H$. We simply remove u and v and join the 2-valent vertices of $G - u$ to those of $H - v$ with three independent edges. Note that $G \oplus_3 H$ can be regarded as being obtained from G by inflating the vertex u to $H - v$.

A cubic graph G containing a cycle-separating 2-edge-cut or 3-edge-cut can be expressed as $G_1 \oplus_2 G_2$ or $G_1 \oplus_3 G_2$ uniquely, only depending on the chosen edge cut. It is easy to see that if two 2-edge-cuts or 3-edge-cuts intersect, the result of decomposition does not depend on the order in which the cuts are taken. As a consequence, we have the following result (see [10, Theorem 3.5]).

Theorem 6.2. *Every 2-connected cubic graph G admits a decomposition into a collection $\{G_1, \ldots, G_m\}$ of cyclically 4-edge-connected cubic graphs such that G can be reconstructed from them by repeated application of 2-sums and 3-sums. Moreover, this collection is unique up to ordering and isomorphism.*

Theorems 6.1 and 6.2, combined with results of the previous sections and with [17, Theorem 4.1] (see also [18, Theorem 2.1]) can now be used to prove that cubic graphs of defect 3 fulfil Berge's conjecture.

Theorem 6.3. *Every 2-connected cubic graph G with colouring defect 3 has $\pi(G) = 4$ or $\pi(G) = 5$. Moreover, if G has an essential triangle, then $\pi(G) = 4$.*

In order to characterise the cubic graphs of defect 3 that have perfect matching index equal to 5 we need to introduce a new concept. A bridgeless cubic graph Q is *quasi-bipartite* if it contains an independent set of vertices U such that the graph obtained by the contraction of each component of $Q - U$ to a vertex is a cubic bipartite graph where U is one of the partite sets. Roughly speaking, a quasi-bipartite cubic graph arises from a bipartite cubic graph by inflating certain vertices in one of the partite sets to larger subgraphs, while preserving the edges between the partite sets.

The next theorem describes conditions under which a 3-sum of two graphs has perfect matching index at least 5. A 3-sum with one of the summands being quasi-bipartite will be called *correct* if the resulting graph is again quasi-bipartite.

Theorem 6.4. *Let G and H be 2-connected cubic graphs with distinguished vertices u and v, respectively, where $\pi(G) \geqslant 5$ and H is 3-edge-colourable. Assume that $\pi(G-u) = 4$. Then $\pi(G \oplus_3 H) \geqslant 5$ if and only if H is quasi-bipartite and the 3-sum is correct.*

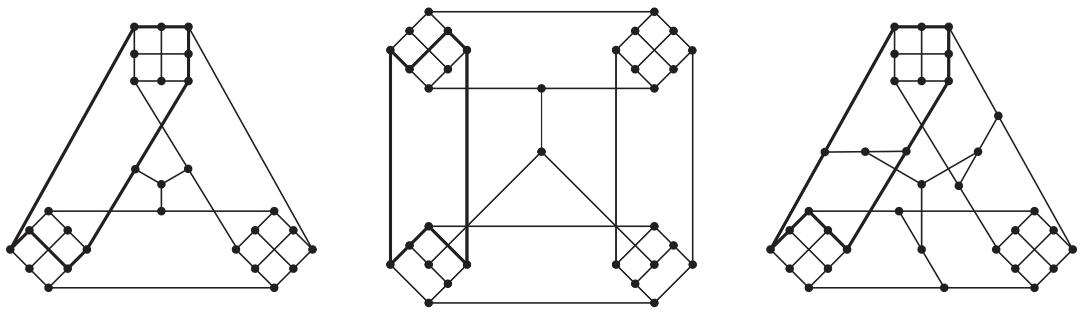
By Lemma 4.2, no hexagonal core can intersect a 2-edge-cut. Applying Theorem 6.2 we can now conclude that every 2-connected cubic graph with defect 3 arises from a 3-connected cubic graph H with defect 3 by performing repeated 2-sums of H with 3-edge-colourable graphs in such a way that a core of H is not affected by the 2-sums. It follows that we can restrict ourselves to 3-connected graphs.

The final result, for 3-connected graphs, reads as follows. Its proof involves the use of Theorem 6.1, the decomposition into cyclically 4-edge-connected graphs presented in Theorem 6.2, and a repeated application of Theorem 6.4.

Theorem 6.5. *Every 2-connected cubic graph G of defect 3 has perfect matching index at most 5. If G is 3-connected, then $\pi(G) = 5$ if and only if G arises from the Petersen graph by inflating any number of vertices of a fixed vertex-star by quasi-bipartite cubic graphs in a correct way.*

7 Concluding remarks

Here we analyse the defect and several related invariants of small snarks. Our analysis is computer-aided. We have computed the defect of all cyclically 4-edge-connected snarks of girth at least 5 and of order at most 36 from the database *House of Graphs: Snarks* [4]. We summarise the output in Table 1. As expected, the major part of the analysed snarks (in fact, around 99.999089%) have defect 3. The defect of all nontrivial snarks with at most 36 vertices takes values in the set $\{3, 4, 5, 6\}$.



(a) $\mathrm{df}(G_{28}) = 5$ (b) $\mathrm{df}(G_{32}) = 4$ (c) $\mathrm{df}(G_{34}) = 6$

Figure 11: Smallest nontrivial snarks with defect greater than 3

We shall briefly discuss the smallest nontrivial snarks of defect 4, 5, and 6 in more detail. The smallest nontrivial snark of defect greater than 3 has order 28. The graph is denoted by G_{28} and is displayed in Figure 11(a); it has defect 5. It is not difficult to understand the reason. If the defect of G_{28} was 3 or 4, then the core of an optimal

3-array would be a circuit of length 6 or 8, respectively. The graph G_{28} contains nine 6-cycles and three 8-cycles. However, it can be easily seen that each of the 6-cycles and 8-cycles is removable. The graph G_{28} appears also in another context in [26] as the smallest nontrivial snark different from the Petersen graph with circular flow number equal to 5. The smallest nontrivial snark with defect 4, denoted by G_{32}, has order 32, see Figure 11(b). The smallest order where cyclically 4-edge-connected snarks with defect 6 occur is 34; there are exactly seven such snarks. The most symmetrical of them, denoted by G_{34}, is depicted in Figure 11(c). By coincidence, G_{34} is the unique smallest nontrivial snark different from the Petersen graph whose edges cannot be covered with four perfect matchings, see [5]. In each of the graphs displayed in Figure 11 bold edges highlight the core of an optimal 3-array.

Order	Nontrivial	Critical	df = 3	df = 4	df = 5	df = 6
10	1	1	1	-	-	-
18	2	2	2	-	-	-
20	6	1	6	-	-	-
22	20	2	20	-	-	-
24	38	-	38	-	-	-
26	280	111	280	-	-	-
28	2900	33	2899	-	1	-
30	28399	115	28397	-	2	-
32	293059	29	293049	1	9	-
34	3833587	40330	3833538	24	18	7
36	60167732	14548	60167208	195	304	25
$\sum$	64326024	55172	64325438	220	334	32

Table 1: Defects of small nontrivial snarks

It transpires that among the nontrivial snarks of order up to 36 there are exactly three graphs of defect greater than 3 with a heavy cluster of 5-cycles. The smallest one is depicted in Figure 10; it has 34 vertices and defect 4. The remaining two have order 36 and defect 4 and 5, respectively. Recall that, by Theorem 5.2, the inflation of any vertex in a heavy cluster decreases the defect to 3.

We conclude this section with three remarks.

Remark 7.1. We have investigated properties of 6-cycles of all nontrivial snarks of order not exceeding 34. A 6-cycle C in a snark can be either removable or non-removable. If C is non-removable, then one of the following three possibilities occurs: C is a double-core hexagon, C is a single-core hexagon, or C does not constitute a hexagonal core. If the latter occurs, C is a *non-core hexagon.* Outputs of computations are summarised in Table 2. The four columns on the right-hand side of the table represent a partition of the set of all nontrivial snarks of defect 3 with at most 34 vertices according to the properties of their 6-cycles. The column with heading “Double-core” contains the numbers of nontrivial snarks in which every 6-cycle is a double-core hexagon for some 3-array. The column headed by "Double-core" and "Removable" enumerates nontrivial snarks in which every 6-cycle is either a double-core hexagon or it is removable, and both types of 6-cycles occur. The families of snarks enumerated in the remaining two columns are defined in a similar manner.

Order	Nontrivial	Critical	Double-core	Double-core Removable	Double-core Single-core	Double-core Single-core Removable
10	1	1	1	-	-	-
18	2	2	2	-	-	-
20	6	1	1	-	5	-
22	20	2	3	-	17	-
24	38	-	1	6	22	9
26	280	111	112	63	21	84
28	2900	33	126	706	1374	693
30	28399	115	907	9126	10798	7566
32	293059	29	3693	133046	53799	102511
34	3833587	40330	55144	2095876	192684	1489834
$\sum$	4158292	40624	59990	2238823	258720	1600697

Table 2: Hexagons in nontrivial snarks with defect 3

Every snark in the collection of tested snarks with defect 3 has been found to have at least one double-core hexagon, which is a remarkable phenomenon. This property, however, cannot be expected from trivial snarks. The graph G from Example 4.4, shown in Figure 5(a), contains a triangle that is intersected by all core hexagons. According to Lemma 4.1(ii), each hexagonal core of G is single-core. It is therefore natural to ask the following question.

Problem 7.2. Does there exist a nontrivial snark with defect 3 in which every core hexagon is single-core?

This problem is particularly interesting from the point of view of Fulkerson's conjecture. If such a snark did exist, then either its Fulkerson cover would not consist of two complementary optimal 3-arrays, or else the snark would provide a counterexample to Fulkerson's conjecture.

Remark 7.3. There exist many snarks where every 6-cycle is a double-core hexagon, see Table 2, and critical snarks of order at most 36 are among them. This observation motivates the following conjecture.

Conjecture 7.4. Every critical snark has defect 3.

Recall that the existence of a double-core hexagon in a snark implies that the union of the corresponding two 3-arrays constitutes a Fulkerson cover. Therefore, we propose the following.

Conjecture 7.5. In a critical snark, every hexagon is double-core. In particular, every optimal 3-array of perfect matchings extends to a Fulkerson cover.

In [29] it is proved that the irreducible snarks coincide with the bicritical ones. It means that the removal of any pair of distinct vertices yields a 3-edge-colourable graph. Irreducible snarks thus constitute a subfamily of critical snarks. Hence, we have the following weaker conjecture.

Conjecture 7.6. Every irreducible snark G has defect 3.

If Conjecture 7.6 is confirmed, then the following long-standing conjecture proposed in [29] is verified as well.

Conjecture 7.7. There exist no irreducible snarks of girth greater than 6.

We note that Conjecture 7.7 can be viewed as an "improved" version of once famous girth conjecture for snarks. Jaeger [12, Conjecture 1] conjectured that there exist no (nontrivial) snarks of girth greater than 6, which was later disproved by Kochol in [23].

Conjectures 7.4 to 7.7 are related as follows:

$$\text{Conjecture 7.5} \Rightarrow \text{Conjecture 7.4} \Rightarrow \text{Conjecture 7.6} \Rightarrow \text{Conjecture 7.7}$$

In [31, Conjecture 4.1] Steffen conjectured that every hypohamiltonian snark has defect 3. Since every hypohamiltonian snark is easily seen to be irreducible [30], the validity of Conjecture 7.6 would imply that of Steffen's conjecture.

Remark 7.8. In the collection of tested snarks, every non-removable hexagon in a nontrivial snark is either single-core or double-core. In other words, non-removable non-core 6-cycles do not occur among the tested snarks. The following question suggests itself.

Problem 7.9. Does there exist a snark G of defect 3 which contains a 6-cycle C such that $G - V(C)$ is 3-edge-colourable, but C does not constitute a hexagonal core? (In other words, does there exist a snark containing a non-removable non-core hexagon?)

This problem is closely related to a problem discussed in [15]. It asks whether a certain theoretically derived colouring set, denoted by $\mathcal{E}_{13}$, admits a realisation by a suitable 6-pole. Answering this problem would represent a significant step towards the so-called 6-decomposition theorem for snarks.

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