Proof of a Conjecture of Nath and Sellers on Simultaneous Core Partitions

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Abstract

In 2016, Nath and Sellers proposed a conjecture regarding the precise largest size of $(s, ms-1, ms+1)$-core partitions. In this paper, we prove their conjecture. One of the key techniques in our proof is to introduce and study the properties of generalized-β-sets, which extend the concept of β-sets for core partitions. Our results can be interpreted as a generalization of the well-known result of Yang, Zhong, and Zhou concerning the largest size of $(s, s+1, s+2)$-core partitions.

Mathematics Subject Classifications: 05A15,05A17

1 Introduction

Recall that an *integer partition*, or simply a *partition*, is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell > 0$ (see [22, 30]). Here ℓ is called the *length*, λ_i $(1 \leqslant i \leqslant \ell)$ are the *parts* and $|\lambda| := \sum_{1\leqslant i\leqslant \ell} \lambda_i$ is the *size* of λ. Each partition λ can be visualized by its *Young diagram*, which is an array of boxes arranged in left-justified rows with λ_i boxes in the i-th row. By flipping the Young diagram of the partition along its main diagonal, we obtain another partition corresponding to the new Young diagram. Such partitions are said to be *conjugate* to each other. A partition is called *self-conjugate* if it is equal to its conjugate partition. For each box $\square = (i, j)$ in the i-th row and the j-th column of the Young diagram of λ, its *hook length* $h_\square = h_{ij}$ is defined to be the number of boxes exactly below, exactly to the right, or the box itself. Let $s > 0$ be a positive integer. A partition λ is called an *s-core partition* if its hook length set doesn't contain any multiple of s. Furthermore, λ is called an $(s_1, s_2, \ldots, s_m)$-core partition if it is simultaneously an s_1-core, an s_2-core, ..., and an s_m-core partition (see [1, 19]). For instance, Figure 1 gives the Young diagram and hook lengths of the partition $(6, 3, 2, 1)$ and Figure 2 gives the Young diagram and hook

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lengths of its conjugation. The partition $(6,3,2,1)$ is a $(4,6,11)$-core partition since its hook length set doesn't contain multiples of 4, 6 or 11. One can note the first column hook lengths are $9,5,3,1$ and they uniquely determine the partition.

9	7	5	3	2	1
5	3	1			
3	1				
1					

Figure 1: The Young diagram and hook lengths of the partition $(6,3,2,1)$.

9	5	3	1
7	3	1	
5	1		
3			
2			
1			

Figure 2: The Young diagram and hook lengths of the conjugation $(4,3,2,1,1,1)$ of the partition $(6,3,2,1)$.

Core partitions arise naturally in the study of modular representation theory and combinatorics. For example, core partitions label the blocks of irreducible characters of symmetric groups (see [27]). Therefore, simultaneous core partitions play important roles in the study of relations between different blocks in the modular group representation theory. Simultaneous core partitions are connected with rational combinatorics (see [16]). Also, simultaneous core partitions are connected with Motzkin paths and Dyck paths (see [9, 11, 39, 40]). Some statistics of simultaneous core partitions, such as numbers of partitions, numbers of corners, largest sizes and average sizes, have attracted much attention in the past twenty years (see [2, 3, 4, 7, 8, 10, 13, 14, 15, 18, 20, 21, 23, 24, 25, 29, 31, 33, 35, 37, 42, 43, 44]). For example, Anderson [3] showed that the number of (s_1,s_2)-core partitions is equal to $\frac{1}{s_1+s_2}\binom{s_1+s_2}{s_1}$ when s_1 and s_2 are coprime to each other. Armstrong [4] conjectured that the average size of (s_1,s_2)-core partitions equals $(s_1-1)(s_2-1)(s_1+s_2+1)/24$ when s_1 and s_2 are coprime to each other, which was first proved by Johnson [18] and later by Wang [33]. However, there are still a lot of unsolved problems in this field.

In this paper, we focus on the largest size of simultaneous core partitions. For (s_1,s_2)-core partitions, Olsson and Stanton [27] showed that the largest size of such partitions is $\frac{({s_1}^2-1)({s_2}^2-1)}{24}$ when s_1 and s_2 are coprime to each other. Straub [32] studied the largest size of $(s,s+2)$-core partitions with distinct parts, and conjectured that the largest size should be equal to $\frac{1}{384}(s^2-1)(s+3)(5s+17)$ in 2016. Straub's conjecture was first proved by Yan, Qin, Jin and Zhou [38]. Later, the second author [36] obtained the largest sizes of $(t, mt+1)$ and $(t, mt-1)$-core partitions with distinct parts. Recently, Nam and Yu [23] derived formulas for the largest sizes of $(s, s+1)$-core partitions whose all parts are odd or all parts are even.

For $m \geqslant 3$, $(s_1, s_2, \ldots, s_m)$-core partitions have also been widely studied. The largest size of $(s, s+1, s+2)$-core partitions was conjectured to be

$$\begin{cases} t\binom{t+1}{3} & \text{if } s = 2t-1; \\ t\binom{t}{3} + \binom{t+1}{3} & \text{if } s = 2t-2 \end{cases}$$

by Amdeberhan [1, Conjecture 11.2], and later proved by Yang, Zhong and Zhou [41]. Furthermore, the second author [34] derived the formula for the largest size of general $(s, s+1, s+2, \ldots, s+p)$-core partitions. In 2019, Baek, Nam and Yu [5] studied self-conjugate $(s, s+1, s+2)$-core partitions and obtained their largest size.

In this paper, we prove the following result, which verifies Nath and Sellers' conjecture [26] on the largest size of $(s, ms-1, ms+1)$-core partitions.

Theorem 1 (see Conjecture **57** of [26])**.** *The largest size of an $(s, ms-1, ms+1)$-core partition is*

$$\begin{cases} \frac{m^2 t(t-1)(t^2-t+1)}{6} & \text{if } s = 2t-1; \\ \frac{m^2(t-1)^2(t^2-2t+3)}{6} - \frac{m(t-1)^2}{2} & \text{if } s = 2t-2. \end{cases}$$

There are two such maximal partitions; one's β-set is $\mathcal{L}_m(s)$ (please see Section 2 for the definition of β-sets and Section 3 for the definition of $\mathcal{L}_m(s)$), and the other one is the conjugate of the first one.

Remark 2. Theorem 1 can be seen as a generalization of the well-known result of Yang, Zhong and Zhou [41] on the largest size of $(s, s+1, s+2)$-core partitions. Thus in the following discussion, we could assume that $s, m \geqslant 2$.

Remark 3. Nath and Sellers [26] computed the size of the longest $(s, ms-1, ms+1)$-core partition, i.e. the partition with the largest β-set. Our result shows that the longest of $(s, ms-1, ms+1)$-core partitions is also the largest one.

Next, we provide two examples for Theorem 1.

Example 4. (1) When $s = 5$ and $m = 3$, Figures 3 and 4 show the two β-sets of maximal $(s, ms-1, ms+1)$-core partitions (here the β-set is just the collection of first column hook lengths, visually represented as beads on the s-abacus, see Definition 5 for the formal definition of the β-set). Figure 3 corresponds to the partition $(12, 9, 9, 6, 6, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1)$. Figure 4 corresponds to the partition $(16, 12, 8, 5, 5, 5, 3, 3, 3, 1, 1, 1)$. Both of them have the size 63 and are conjugate to each other. In each figure, the β-set is shown by the circled positions.

(2) When $s = 6$ and $m = 3$, Figures 5 and 6 show the two β-sets of maximal $(s, ms-1, ms+1)$-core partitions. Figure 5 corresponds to the partition $(22, 17, 12, 12, 9, 9, 9, 6, 6, 6, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1)$. Figure 6 corresponds to the partition $(24, 19, 14, 10, 10, 10, 7, 7, 7, 4, 4, 4, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1)$. Both of them have the size 135 and are conjugate to each other.

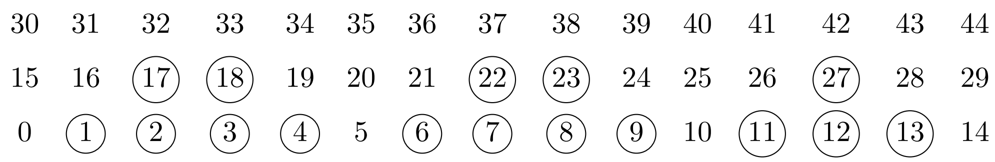

Figure 3: $\mathcal{L}_3(5)$: The β-set of a maximal $(5, 14, 16)$-core partition.

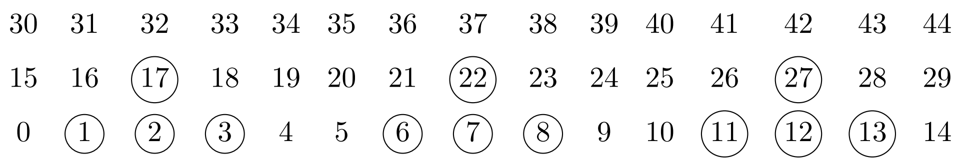

Figure 4: The β-set of the other maximal $(5, 14, 16)$-core partition.

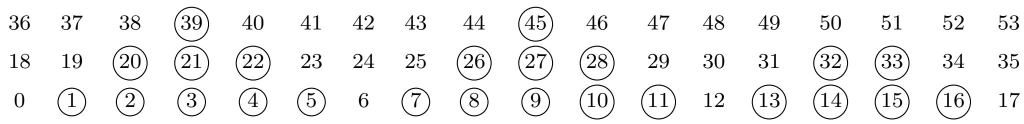

Figure 5: $\mathcal{L}_3(6)$: The β-set of a maximal $(6, 17, 19)$-core partition.

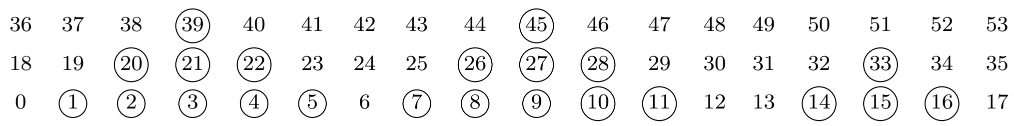

Figure 6: The β-set of the other maximal $(6, 17, 19)$-core partition.

The rest of the paper is structured as follows. In Section 2, we recall the definitions of β-sets and s-abacus diagrams, which serve as basic tools to study simultaneous core partitions. Following [26], we present the structure of ms-abacus diagrams of β-sets of $(s, ms-1, ms+1)$-core partitions in Section 3. Next, we define and study the properties of generalized-β-sets in Section 4. This will allow us to loosen the restriction of β-sets to make room for the adjustments we will need. In Section 5, we prove several lemmas that will be used repeatedly in the later sections. In the following sections, we will use adjustments step by step to find the necessary conditions of S for S being the generalized-β-set that maximizes $f(S)$ (the definition of $f(S)$ is given in Definition 8). In Section 6, we determine the possible shape of each row of S. In Section 7, we determine the possible shape of the entire S. Finally, we prove the main theorem in Section 8.

2 β-sets and s-abacus diagrams

In this section, we recall the definitions of β-sets and s-abacus diagrams for partitions.

Definition 5 ([6, 27, 28])**.** The β-set $\beta(\lambda)$ of a partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is denoted by

$$\beta(\lambda) := \{h_{i1} : 1 \leqslant i \leqslant \ell\},$$

which contains hook lengths of boxes in the first column of the corresponding Young diagram of λ.

Remark 6. It is easy to see that the β-set completely determines the partition.

Example 7. The β-set of the partition $(6, 3, 2, 1)$ is $\{9, 5, 3, 1\}$ (See Figure 1).

We introduce the notion of the size-counting function. It can be used to compute the size of a partition with the knowledge of its β-set.

Definition 8 (size-counting function, see [17])**.** Let S be a finite set of integers. We define

$$f(S) := \sum_{x \in S} x - \frac{|S|(|S|-1)}{2},$$

where $|S|$ is the cardinality of the set S.

Lemma 9 (see [28, 34])**.** *Let S be the β-set of a partition λ. Then the size of λ is equal to $f(S)$, i.e.,*

$$|\lambda| = f(S).$$

From now on we will focus on core partitions. The β-set of an s-core partition has the following property.

Lemma 10 (see [28, 34])**.** *Let s be a positive integer and S be the β-set of a partition λ. Then λ is an s-core partition, if and only if for any $x \in S$ and $x \geqslant s$, we have $x - s \in S$.*

Next, we recall the definition of s-abacus.

Definition 11 (s-abacus (see [17, 26]))**.** Let $X \subseteq \mathbb{N}$ be a set of positive integers and s be a positive integer. Then the s-abacus diagram S of X is defined to be the set

$$S := \{(i, j) : \quad i \geqslant 0, \quad 0 \leqslant j \leqslant s - 1, \quad si + j \in X\}.$$

Remark 12. Usually, we use a diagram like Figure 7 to represent an s-abacus. When $(i, j) \in S$, the number in the i-th row and j-th column is circled. We begin with Row 0 at the bottom and Column 0 at the far left.

Example 13. Figure 7 shows the 5-abacus diagram of $\{2, 4, 9\}$, which is equal to

$$S = \{(0, 2),\ (0, 4),\ (1, 4)\}.$$

Remark 14. The s-abacus is just a graphic representation of the β-set, thus we will use the terms β-set and s-abacus interchangeably. When we deal with the ms-abacus, we will write $is + j$ instead of (i, j) for $i \geqslant 0$ and $0 \leqslant j \leqslant s - 1$. When we mention a certain row of a given set, we mean the corresponding row in the ms-abacus of the set.

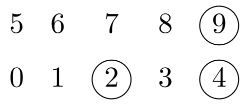

Figure 7: The 5-abacus diagram of {2,4,9}.

3 The β-sets of $(s, ms-1, ms+1)$-core partitions

In this section, we study the β-sets of $(s, ms-1, ms+1)$-core partitions. First, we recall the definition of s-pyramids.

Definition 15 (s-pyramid (see [26]))**.** Let S be an s-abacus. Assume that a and b are two integers satisfying $0 \leqslant a < b \leqslant s-1$. We call S an s-pyramid with base $[a, b]$, if

$$S = \{(i,j): \quad i \geqslant 0, \quad a+i \leqslant j \leqslant b-i\}.$$

For positive integers n, a and b, let $[n] := \{1, 2, \ldots, n\}$ and $[a, b] := \{a, a+1, a+2, \ldots, b-2, b-1, b\}$. The following theorem characterizes the shape of the β-set of an $(s, ms-1, ms+1)$-core partition.

Theorem 16 (see Lemmas **1.6** and **5.13** of [26])**.** *For given positive integers s and m, let P_k be an ms-pyramid with base $[(k-1)s+1, ks-1]$ for $1 \leqslant k \leqslant m-1$ and P_m be an ms-pyramid with base $[(m-1)s+1, ms-2]$. Furthermore, let $\mathcal{L}_m(s) = \cup_{k=1}^{m} P_k$ be the union of $P_1, P_2, \ldots, P_m$. Then for any set S, S is the ms-abacus of the β-set of an $(s, ms-1, ms+1)$-core partition, if and only if it satisfies:*

(1) $S \subseteq \mathcal{L}_m(s)$;

(2) If $(i,j) \in S$ and $i \geqslant 1$, then $(i-1, j-1) \in S$ and $(i-1, j+1) \in S$;

(3) If $(i,j) \in S$ and $j \geqslant s$, then $(i, j-s) \in S$;

(4) If $(i,j) \in S$, $i \geqslant 1$ and $j < s$, then $(i-1, j+(m-1)s) \in S$.

Example 17. (1) When $m = 3$ and $s = 5$, we have $\mathcal{L}_3(5) = P_1 \cup P_2 \cup P_3$ where $P_1 = \{1, 2, 3, 4, 17, 18\}$, $P_2 = \{6, 7, 8, 9, 22, 23\}$ and $P_3 = \{11, 12, 13, 27\}$. Figure 3 shows $\mathcal{L}_3(5)$.

(2) When $m = 3$ and $s = 6$, we have $\mathcal{L}_3(6) = P_1 \cup P_2 \cup P_3$ where the three 18-pyramids are $P_1 = \{1, 2, 3, 4, 5, 20, 21, 22, 29\}$, $P_2 = \{7, 8, 9, 10, 11, 26, 27, 28, 45\}$ and $P_3 = \{13, 14, 15, 16, 32, 33\}$. Figure 5 shows $\mathcal{L}_3(6)$.

(3) Consider the 15-abacus of the β-set S of a $(5, 14, 16)$-core partition. Assume that $23, 27 \in S$. By Theorem 16, we have $11, 13, 22 \in S$ since $27 \in S$; and $7, 9, 18 \in S$ since $23 \in S$. Repeating this process, we can deduce that $\mathcal{L}_3(5) \subseteq S$. Therefore, $S = \mathcal{L}_3(5)$.

Lemma 18. *For any positive integer s and m, the partition with the beta set $\mathcal{L}_m(s)$ is not self-conjugate.*

Proof. We prove it by contradiction. Otherwise, the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_u)$ that $\mathcal{L}_m(s)$ corresponds to is self-conjugate. Then $\lambda_1 = u$, here $u = |\mathcal{L}_m(s)|$. Thus we have $\max \mathcal{L}_m(s) = 2|\mathcal{L}_m(s)| - 1$.

When $s = 2t - 1$ is odd, it's obvious that

$$|\mathcal{L}_m(s)| = \sum_{i=1}^{m} |P_k| \sum_{i=1}^{m-1} (2 + 4 + \cdots + (2t - 2)) + (1 + 3 + \cdots + (2t - 3)) = (t - 1)(mt - 1).$$

On the other hand, $\max \mathcal{L}_m(s)$ equals

$$m(t - 1)(2t - 1) - t = 2|\mathcal{L}_m(s)| - 1 - (t - 1)(m - 1) < 2|\mathcal{L}_m(s)| - 1,$$

a contradiction.

When $s = 2t - 2$ is even, we can similarly prove that $\max \mathcal{L}_m(s) < 2|\mathcal{L}_m(s)| - 1$. A contradiction. □

4 Generalized-β-sets

Now we generalize the concept of the β-set of an $(s, ms - 1, ms + 1)$-core partition to the generalized-β-set. In the following sections, we will use this more general concept instead of the original β-sets studied in previous papers (see [17, 26, 28, 34]). To better formulate the concept of a generalized-β-set, we introduce the following notation. For nonempty finite set $S \subseteq [\lceil (s - 1)/2 \rceil ms - 1] = \{x \in \mathbb{N} : 1 \leqslant x \leqslant \lceil (s - 1)/2 \rceil ms - 1\}$, define

$$t(S) := \min\{i \geqslant 1 : S \subseteq [0, ims - 1]\}.$$

For $1 \leqslant i \leqslant \lceil (s - 1)/2 \rceil m$, define

$$\mathcal{B}_i(S) := S \cap [(i - 1)s, is - 1];$$

$$a_i(S) := |S \cap [(i - 1)s, is - 1]|;$$

$$n_i(S) := \max\ \{x \bmod s : x \in \mathcal{B}_i(S)\} \ \ \text{(we set } n_i(S) = 0 \text{ if } \mathcal{B}_i(S) = \emptyset).$$

For $1 \leqslant k \leqslant \lceil (s - 1)/2 \rceil$, define

$$A_k(S) := \sum_{j=1}^{m} a_{(k-1)m+j}(S).$$

When there is only one set S, we write $t, \mathcal{B}_i, a_i, n_i$ instead of $t(S), \mathcal{B}_i(S), a_i(S), n_i(S)$ for simplicity. When we deal with two sets S, S', the same is for S and we use $t', \mathcal{B}'_i, a'_i, n'_i$ instead of $t(S'), \mathcal{B}_i(S'), a_i(S'), n_i(S')$. The notation above is inspired by Example 4. Let S be the β-set of a maximal $(s, ms-1, ms+1)$-core partition. Observing Example 4, we find that S can be divided into several parts, namely $\mathcal{B}_i = S \cap [(i - 1)s, is - 1]$, each of which consists of consecutive integers. The shape of these consecutive integers is determined by the cardinality of $\mathcal{B}_i$, i.e. a_i and the maximal element, which can be represented by its modulo mod s, i.e. n_i. The notation t is just the number of non-zero rows that S has in $\mathcal{L}_m(s)$.

Example 19. Let $m = 3$, $s = 5$ and $S = \mathcal{L}_3(5)$. Then $t = 2$. We have (see Figure 3)

$$\mathcal{B}_1 = \{1, 2, 3, 4\}, a_1 = 4, n_1 = 4;$$

$$\mathcal{B}_2 = \{6, 7, 8, 9\}, a_2 = 4, n_2 = 4;$$

$$\mathcal{B}_3 = \{11, 12, 13\}, a_3 = 3, n_3 = 3;$$

$$\mathcal{B}_4 = \{17, 18\}, a_4 = 2, n_4 = 3;$$

$$\mathcal{B}_5 = \{22, 23\}, a_5 = 2, n_5 = 3;$$

$$\mathcal{B}_6 = \{27\}, a_6 = 1, n_6 = 2;$$

$$A_1 = a_1 + a_2 + a_3 = 11, A_2 = a_4 + a_5 + a_6 = 5.$$

Next, we give the definition of generalized-β-sets.

Definition 20 (generalized-$\boldsymbol{\beta}$-set)**.** Let $S \subseteq \mathcal{L}_m(s)$ be a nonempty set. Then, S is called a *generalized-β-set*, if it satisfies the following conditions:

(1) If $1 \leqslant i < \lceil (s-1)/2 \rceil m$ and $a_i = 0$, then $a_{i+1} = 0$;

(2) If $1 \leqslant i < tm$ and $m \nmid i$, then $a_i \geqslant a_{i+1}$;

(3) If $1 \leqslant i \leqslant (t-1)m$ and $a_{i+m} > 0$, then $a_{i+m} \leqslant a_i - 2$;

(4) If $1 \leqslant i < tm$, $m \nmid i$ and $a_{i+1} > 0$, then $n_i \geqslant n_{i+1}$;

(5) If $1 \leqslant i < tm$, $m \nmid i$ and $a_i = a_{i+1} > 0$, then $n_i = n_{i+1}$;

(6) $a_{(t-1)m} \geqslant a_{(t-1)m+1}$.

Example 21. When $m = 3$ and $s = 5$, $\mathcal{L}_3(5)$ is a generalized-β-set. In fact, it is obvious that (1) holds. Since $(a_1, a_2, a_3, a_4, a_5, a_6) = (4, 4, 3, 2, 2, 1)$ is non-increasing, (2) and (6) also hold. Since $a_4 = a_1 - 2$, $a_5 = a_2 - 2$ and $a_6 = a_3 - 2$, (3) is true. Since $(n_1, n_2, n_3, n_4, n_5, n_6) = (4, 4, 3, 3, 3, 2)$ is non-increasing, (4) is true. If $a_i = a_{i+1}$, then $i = 1$ or $i = 4$. Since $n_1 = n_2$ and $n_4 = n_5$, (5) is also true.

The following lemma is the reason why we use the term *generalized-β-set*.

Lemma 22. *Let λ be an $(s, ms-1, ms+1)$-core partition. Then the β-set $S := \beta(\lambda)$ of λ is a generalized-β-set.*

Proof. Recall that $\mathcal{B}_i = S \cap [(i-1)s, is-1]$ for $1 \leqslant i \leqslant tm$. For $i \geqslant 2$, assume that $\mathcal{B}_i \neq \emptyset$. Set $\mathcal{B}_i = \{x_1, x_2, \cdots, x_k\}$. By Theorem 16, $x_i - s \in \mathcal{B}_{i-1}$. Therefore,

$$a_{i-1} = |\mathcal{B}_{i-1}| \geqslant k = |\mathcal{B}_i| = a_i$$

and

$$n_{i-1} = \max \mathcal{B}_{i-1} (\text{mod } s) \geqslant \max(\mathcal{B}_i - s)(\text{mod } s) = \max \mathcal{B}_i (\text{mod } s) = n_i.$$

Therefore, Conditions (1), (2), (4) and (6) in Definition 20 are satisfied. If $a_{i-1} = a_i$, then $\mathcal{B}_{i-1} = \{x_1 - s, \cdots, x_k - s\}$, thus $n_{i-1} = n_i$. Therefore, Condition (5) in Definition 20 is satisfied. For $m+1 \leqslant i \leqslant tm$, notice that $\{x_1 - ms - 1, x_2 - ms - 1, \cdots, x_k - ms - 1, x_k - ms, x_k - ms + 1\} \subseteq \mathcal{B}_{i-m}$. Therefore,

$$|\mathcal{B}_{i-m}| \geqslant k + 2 = |\mathcal{B}_i| + 2.$$

Thus Condition (3) in Definition 20 is also satisfied. □

Remark 23. A generalized-β-set is not necessarily the β-set of an $(s, ms-1, ms+1)$-core partition. Let $s = 6$ and $m = 3$. See Figure 8 for example, $S = \{1, 2, 3, 4, 7, 8, 9, 10, 13, 14, 15, 16, 21, 22, 27, 33\}$ is not the β-set of a $(6, 17, 19)$-core partition, since $22 - 17 = 5 \notin S$, which violates Condition (2) in Theorem 16. Observe that $(a_1, a_2, a_3, a_4, a_5, a_6) = (4, 4, 4, 2, 1, 1)$ and $(n_1, n_2, n_3, n_4, n_5, n_6) = (4, 4, 4, 4, 3, 3)$. We verify that S is indeed a generalized-β-set. Obviously (1) holds. Both sequences are non-increasing, thus (2)(4)(6) hold. Since $a_1 - a_4 = 2$ and $a_2 - a_5 = a_3 - a_6 = 3$, (3) holds. Since $n_1 = n_2 = n_3$ and $n_5 = n_6$, (5) holds.

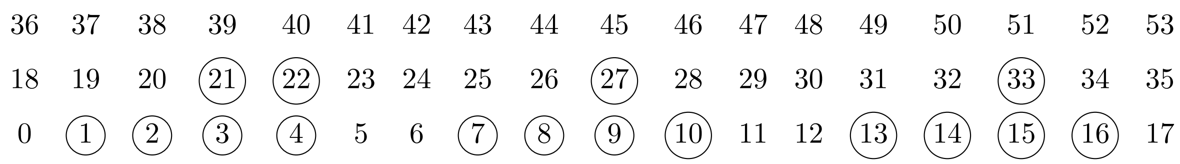

Figure 8: A generalized-β-set which is not a β-set.

The following proposition gives an upper bound of n_i.

Proposition 24. *Let $S \subseteq \mathcal{L}_m(s)$ be a generalized-β-set. For $0 \leqslant i \leqslant t-1$ and $1 \leqslant k \leqslant m$, we have*

$$n_{im+k} \leqslant \begin{cases} s - i - 2 & if\ a_{im+k} = a_{(i+1)m}; \\ s - i - 1 & if\ a_{im+k} > a_{(i+1)m}. \end{cases}$$

Proof. Since $S \subseteq \mathcal{L}_m(s)$, we have

$$n_{im+k} \leqslant \begin{cases} s - i - 2 & \text{if } k = m; \\ s - i - 1 & \text{if } 1 \leqslant k < m. \end{cases}$$

Then by Condition (5) in Definition 20, we obtain the result. □

The next proposition derives several inequalities for A_i.

Proposition 25. *Let $S \subseteq \mathcal{L}_m(s)$ be a generalized-β-set. Then,*

(1) $A_i - A_{i+1} \geqslant 2m$ for $1 \leqslant i \leqslant t-2$;

(2) If $A_{t-1} - A_t \leqslant 2m-1$, then there exists $1 \leqslant p < m$, such that $a_{(t-1)m} = a_{(t-1)m+1} = \cdots = a_{(t-1)m+p} = 1$ and $a_{(t-1)m+p+1} = \cdots = a_{tm} = 0$.

Proof. By the definition of t, we have $a_{(t-1)m+1} > 0$. By Condition (1) in Definition 20, we have $a_i > 0$ for $1 \leqslant i \leqslant (t-1)m+1$. Then Conclusion (1) can be directly derived by Condition (3) in Definition 20. Next, we prove Conclusion (2), assuming that $A_{t-1} - A_t \leqslant 2m-1$. By Condition (6) in Definition 20 we have $a_{(t-1)m} \geqslant a_{(t-1)m+1} > 0$. If $a_{(t-1)m} \geqslant 2$, then $a_{(t-2)m+k} \geqslant a_{(t-1)m} \geqslant 2$ for $1 \leqslant k \leqslant m$ by Condition (2) in Definition 20. Therefore, $a_{(t-2)m+k} - a_{(t-1)m+k} \geqslant 2$ for $1 \leqslant k \leqslant m$. This contradicts with $A_{t-1} - A_t < 2m$. Therefore, $a_{(t-1)m} = 1$. Note that $0 < a_{(t-1)m+1} \leqslant a_{(t-1)m}$ and $a_{tm} < a_{(t-1)m}$, we derive Conclusion (2) in this proposition. □

Example 26. (1) Let $m = 3$, $s = 5$, and $S = \mathcal{L}_3(5)$. Then we have $A_1 = 11$ and $A_2 = 5 = 11 - 2 \times 3 = A_1 - 2m$.

(2) Let $m = 3$, $s = 6$, and $S = \mathcal{L}_3(6)$. Then we have $A_1 = 14$, $A_2 = 8 = 14 - 2 \times 3 = A_1 - 2m$ and $A_3 = 2 = 8 - 2 \times 3 = A_2 - 2m$.

Let $\mathbb{A}_m(s)$ be the set of all β-sets of $(s, ms-1, ms+1)$-core partitions and $\mathbb{B}_m(s)$ be the set of all generalized-β-sets $S \subseteq \mathcal{L}_m(s)$. By Lemma 22, we know that

$$\max_{S' \in \mathbb{A}_m(s)} f(S') \leqslant \max_{S'' \in \mathbb{B}_m(s)} f(S'').$$

We aim to prove Nath-Sellers' conjecture by showing that

$$\max_{S' \in \mathbb{A}_m(s)} f(S') = \max_{S'' \in \mathbb{B}_m(s)} f(S'')$$

and

$$\arg \max_{S' \in \mathbb{A}_m(s)} f(S') = \arg \max_{S'' \in \mathbb{B}_m(s)} f(S''),$$

which will be proved in the following sections while we are searching for all generalized-β-sets S that maximize $f(S)$. Here and in the rest of the paper, for a function $g : D \to \mathbb{R}$,

$$\arg \max_{x \in D} g(x) := \{t \in D : g(x) = \max_{t \in D} g(x)\}.$$

We use generalized-β-sets instead of β-sets since the original definition of the β-set is too tight for the adjustments in Sections 5, 6 and 7. We can easily make a generalized-β-set remain a generalized-β-set after each adjustment in our following proofs, while a β-set may not remain a β-set after the adjustments.

5 Preliminaries for the proofs

In the next three sections, we will introduce several adjustments for a generalized-β-set S which doesn't maximize $f(S)$. After the adjustments, we will get a new set S', such that $f(S) < f(S')$. We wish to prove that S' is a generalized-β-set. The following lemma will be repeatedly used to verify Condition (3) in Definition 20 for S'.

Lemma 27. *Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be two non-increasing integer sequences such that $|x_n - x_1| \leqslant 1$ and $|y_n - y_1| \leqslant 1$. Assume that*

$$\sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} y_i - 2n,$$

then $x_i \leqslant y_i - 2$ for $1 \leqslant i \leqslant n$.

Proof. We prove it by contradiction. Assume that there exists $1 \leqslant i \leqslant n$, such that $x_i > y_i - 2$, then $x_i \geqslant y_i - 1$. For $j > i$, we have $x_j \geqslant x_i - 1 \geqslant y_i - 2 \geqslant y_j - 2$. For $j < i$, we have $x_j \geqslant x_i \geqslant y_i - 1 \geqslant y_j - 2$. Therefore, $\sum_{j=1}^{n} x_j > \sum_{j=1}^{n} y_j - 2n$. A contradiction. □

The following lemma shifts our focus from all generalized-β-sets to the ones with some specific structure.

Lemma 28. *Let S be a generalized-β-set that maximizes $f(S)$. Then*

$$\mathcal{B}_i = S \cap [(i-1)s, is-1] \tag{1}$$

consists of consecutive integers for $1 \leqslant i \leqslant tm$. Furthermore, for $0 \leqslant i \leqslant t-1$ and $1 \leqslant k \leqslant m$, we have

$$n_{im+k} = \begin{cases} s-i-2 & \text{if } a_{im+k} = a_{(i+1)m}; \\ s-i-1 & \text{if } a_{im+k} > a_{(i+1)m}. \end{cases} \tag{2}$$

Proof. First, we construct a generalized-β-set $S' \subseteq \mathcal{L}_m(s)$. For $0 \leqslant i \leqslant t-1$ and $1 \leqslant k \leqslant m$, if $a_{im+k} = a_{(i+1)m}$, let $\mathcal{B}'_{im+k} = \{(im+k)s-i-a_{im+k}-1, (im+k)s-i-a_{im+k}, \cdots, (im+k)s-i-2\}$. Otherwise, let $\mathcal{B}'_{im+k} = \{(im+k)s-i-a_{im+k}, (im+k)s-i-a_{im+k}+1, \cdots, (im+k)s-i-1\}$. Then $\mathcal{B}'_{im+k}$ consists of $a_{im+k} = |S \cap [(im+k-1)s, (im+k)s-1]|$ consecutive numbers, and $n'_i = \max\ \{x \bmod s : x \in \mathcal{B}_i(S')\}$ satisfies the equality in Proposition 24.

Therefore, Conditions (1)(2)(3)(6) in Definition 20 hold for S' since $a'_i = a_i$ for $1 \leqslant i \leqslant tm$. Conditions (4)(5) in Definition 20 are also true since the equality in Proposition 24 is achieved. Thus S' is also a generalized-β-set and $|S| = |S'|$.

For $1 \leqslant i \leqslant tm$, $\sum_{x \in \mathcal{B}_i} x \leqslant \sum_{x \in \mathcal{B}'_i} x$. Therefore,

$$f(S) = \sum_{i=1}^{tm} \sum_{x \in \mathcal{B}_i} x - \frac{|S|(|S|-1)}{2} \leqslant \sum_{i=1}^{tm} \sum_{x \in \mathcal{B}'_i} x - \frac{|S'|(|S'|-1)}{2} = f(S').$$

If $S \neq S'$, the equality above cannot be achieved, which means that $f(S) < f(S')$. However, at the beginning we assume that S is a generalized-β-set that maximizes $f(S)$, which means that $f(S) \geqslant f(S')$, a contradiction. Therefore, $S = S'$ and thus for $0 \leqslant i \leqslant t-1$ and $1 \leqslant k \leqslant m$, $\mathcal{B}_{im+k}$ consists of consecutive integers and

$$n_{im+k} = \begin{cases} s-i-2 & if\ a_{im+k} = a_{(i+1)m}; \\ s-i-1 & if\ a_{im+k} > a_{(i+1)m}. \end{cases}$$

□

Definition 29. Recall that $\mathbb{A}_m(s)$ is the set of all β-sets of $(s, ms-1, ms+1)$-core partitions and $\mathbb{B}_m(s)$ is the set of all generalized-β-sets $S \subseteq \mathcal{L}_m(s)$. Let $\mathbb{C}_m(s)$ be the set of all generalized-β-sets $S \subseteq \mathcal{L}_m(s)$ such that S satisfies Conditions (1) and (2) in Lemma 28. Let $\mathbb{D}_m(s)$ be the set of all nonempty sets $S \subseteq [\lceil (s-1)/2 \rceil ms - 1]$ such that the nonempty set $\mathcal{B}_i$ consists of a_i consecutive integers for $1 \leqslant i \leqslant \lceil (s-1)/2 \rceil m$.

The following lemma is obvious and the proof is omitted.

Lemma 30. *For the sets $\mathbb{C}_m(s)$ and $\mathbb{D}_m(s)$ defined in Definition 29, we have*

(1) $\mathbb{C}_m(s) \subseteq \mathbb{D}_m(s)$;

(2) $\mathbb{C}_m(s) \subseteq \mathbb{B}_m(s)$.

We use Figure 9 to visualize that $\mathbb{A}_m(s) \subseteq \mathbb{B}_m(s), \mathbb{C}_m(s) \subseteq \mathbb{B}_m(s), \mathbb{C}_m(s) \subseteq \mathbb{D}_m(s)$.

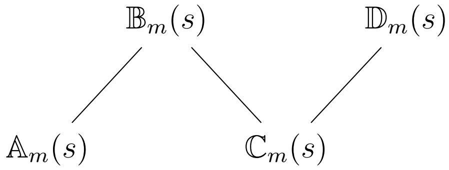

Figure 9: Relations between $\mathbb{A}_m(s), \mathbb{B}_m(s), \mathbb{C}_m(s)$, and $\mathbb{D}_m(s)$.

Remark 31. By Lemma 28 it is easy to see that

$$\arg \max_{S' \in \mathbb{B}_m(s)} f(S') = \arg \max_{S'' \in \mathbb{C}_m(s)} f(S'').$$

The following lemma gives a characterization of $\mathbb{D}_m(s)$.

Lemma 32. *There is a bijection $F : \mathbb{D}_m(s) \to V$, where $V \subseteq \mathbb{Z}^{2\lceil (s-1)/2 \rceil m}$, and for $Z = \left(x_1, \dots, x_{\lceil (s-1)/2 \rceil m}; y_1, \dots, y_{\lceil (s-1)/2 \rceil m}\right) \in \mathbb{Z}^{2\lceil (s-1)/2 \rceil m}$, $Z \in V$ if and only if Z satisfies the following conditions:*

(1) For $1 \leqslant i \leqslant \lceil (s-1)/2 \rceil m$, $0 \leqslant x_i \leqslant s-1$ and $0 \leqslant y_i \leqslant s$;

(2) For $1 \leqslant i \leqslant \lceil (s-1)/2 \rceil m$, $x_i \geqslant y_i - 1$.

Proof. Let $N = \lceil (s-1)/2 \rceil m$. For $S \in \mathbb{D}_m(s)$, let

$$F(S) = \left(n_1(S), n_2(S), \cdots, n_N(S); a_1(S), a_2(S), \cdots, a_N(S)\right).$$

Then for $1 \leqslant i \leqslant N$, by definition we have $0 \leqslant n_i(S) \leqslant s-1$, $0 \leqslant a_i(S) \leqslant s$ and $a_i(S) - 1 \leqslant n_i(S)$. Therefore, $f(S) \in V$ and F are well-defined.

We first prove that F is injective. Assume that $S, S' \in \mathbb{D}_m(s)$ such that $F(S) = F(S')$, then $(n_i, a_i) = (n'_i, a'_i)$ for $1 \leqslant i \leqslant N$. Then $\mathcal{B}_i = \{(i-1)s + n_i - a_i + 1, (i-1)s + n_i - a_i +$

$2, \cdots, (i-1)s+n_i\} = \{(i-1)s+n_i'-a_i'+1, (i-1)s+n_i'-a_i'+2, \cdots, (i-1)s+n_i'\} = \mathcal{B}_i'$. Therefore, we have $S = S'$.

Next, we prove that F is surjective. For any $Z = (x_1, \cdots, x_N; y_1, \cdots, y_N) \in V$, let $\mathcal{B}_i = \{(i-1)s+n_i-a_i+1, (i-1)s+n_i-a_i+2, \cdots, (i-1)s+n_i\}$. Then $\mathcal{B}_i \subseteq [(i-1)s, is-1]$ consists of consecutive integers. Therefore, we have $S = \cup_{i=1}^{N} \mathcal{B}_i \subseteq \mathbb{D}_m(s)$ and $F(S) = Z$.

From the discussion above, we obtain that F is bijective. □

Next, we show how to calculate $f(S)$ for $S \in \mathbb{D}_m(s)$.

Lemma 33. *Let $S \in \mathbb{D}_m(s)$. Then*

$$\sum_{x \in \mathcal{B}_i} x = a_i(i-1)s + n_i a_i - \frac{a_i(a_i-1)}{2}$$

and

$$f(S) = \sum_{i=1}^{tm} \left(a_i(i-1)s + n_i a_i - \frac{a_i(a_i-1)}{2} \right) - \frac{\sum_{i=1}^{tm} a_i \left(\sum_{i=1}^{tm} a_i - 1 \right)}{2}.$$

Proof. Since $f(S) = \sum_{i=1}^{tm} \sum_{x \in \mathcal{B}_i} x - |S|(|S|-1)/2$, we only need to prove the first equation. In fact, since $\mathcal{B}_i = \{(i-1)s+n_i-a_i+1, (i-1)s+n_i-a_i+2, \cdots, (i-1)s+n_i\}$, we have

$$\sum_{x \in \mathcal{B}_i} x = \sum_{j=0}^{a_i-1} ((i-1)s+n_i-j) = a_i((i-1)s+n_i) - \frac{a_i(a_i-1)}{2}.$$

□

Lemma 33 implies the following result.

Corollary 34. *Let $S, S' \subseteq \mathbb{D}_m(s)$ and $t = t'$. Assume that there exist positive integers $i < j$, such that the following three conditions hold:*

(1) $(n_i', a_i') = (n_i - 1, a_i - 1)$ or $(n_i, a_i - 1)$;

(2) $(n_j', a_j') = (n_j + 1, a_j + 1)$ or $(n_i, a_i + 1)$;

(3) For all $1 \leqslant k \leqslant tm$, $k \neq i, j$, we have $(n_k, a_k) = (n_k', a_k')$.

Then $f(S) < f(S')$.

Proof. If $(n_i', a_i') = (n_i-1, a_i-1)$ and $(n_j', a_j') = (n_j+1, a_j+1)$, then there exist some $x_0 \in \mathcal{B}_i$ and $y_0 \in (\mathcal{L}_m(s) \cap [(j-1)s, js-1]) \setminus \mathcal{B}_j$, such that $\mathcal{B}_i' = \mathcal{B}_i \setminus \{x_0\}$ and $\mathcal{B}_j' = \mathcal{B}_j \cup \{y_0\}$. Thus,

$$f(S') - f(S) = \left(\sum_{x \in \mathcal{B}_i'} x - \sum_{x \in \mathcal{B}_i} x \right) + \left(\sum_{y \in \mathcal{B}_j'} y - \sum_{y \in \mathcal{B}_j} y \right) = y_0 - x_0 > 0.$$

For other cases of (n_i', a_i') and (n_j', a_j'), the proof is similar. □

6 Adjustments in one row

The aim of this section is to show that if S maximizes $f(S)$, then $a_{(i-1)m+1} - a_{im} \leqslant 2$ for $1 \leqslant i \leqslant t$, just as Example 1.2 shows. The following lemma gives some properties of a generalized-β-set S that maximizes $f(S)$.

Lemma 35. *Assume that a generalized-β-set S maximizes $f(S)$. Then we have the following conclusions:*

(1) If $A_{t-1} - A_t \geqslant 2m$, then

$$a_{(i-1)m+1} - a_{im} \leqslant 1$$

for $1 \leqslant i \leqslant t$.

(2) If $A_{t-1} - A_t \leqslant 2m - 1$, by Proposition 25, there exists $1 \leqslant p < m$, such that $a_{(t-1)m} = a_{(t-1)m+1} = \cdots = a_{(t-1)m+p} = 1$ and $a_{(t-1)m+p+1} = \cdots = a_{tm} = 0$. Then

$$a_{(i-1)m+1} - a_{(i-1)m+p} \leqslant 1$$

and

$$a_{(i-1)m+p+1} - a_{im} \leqslant 1$$

for $1 \leqslant i \leqslant t$.

Remark 36. Note that the inequality $a_{i+m} \leqslant a_i - 2$ may be violated when $a_{i+m} = 0$. This needs to be discussed in the following proof.

Proof. By Lemma 28, $S \in \mathbb{C}_m(s)$. We prove Conclusion (2) here. The proof of Conclusion (1) is similar and omitted. Assume that Conclusion (2) is not true. Then either there exists some $0 \leqslant i \leqslant t-2$, such that $a_{im+p+1} - a_{(i+1)m} \geqslant 2$; or there exists some $0 \leqslant j \leqslant t-2$, such that $a_{jm+1} - a_{jm+p} \geqslant 2$. These two cases are similar. Without loss of generality, we consider the first case, i.e., there exists some $0 \leqslant i \leqslant t-2$, such that $a_{im+p+1} - a_{(i+1)m} \geqslant 2$.

Assume that $a_{im+p+1} = a_{im+p+2} = \cdots = a_{im+p+u} > a_{im+p+u+1}$ and $a_{(i+1)m} = a_{(i+1)m-1} = \cdots = a_{(i+1)m-v} < a_{(i+1)m-v-1}$ for some positive integers u and v. Then $u + v < m - p$. We will do adjustments to $S \cap [(im+p)s, (i+1)ms - 1]$, which can be divided into the following three cases.

(1) $v = 0$ and $a_{(i+1)m-1} \geqslant a_{(i+1)m} + 2$. Let $S' \in \mathbb{D}_m(s)$ satisfy the following conditions:

   (a) $t' = t$;

   (b) For $1 \leqslant k \leqslant tm$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq (i+1)m$ and $k \neq (i+1)m - 1$;

   (c) If $a_{(i+1)m-1} = a_{(i+1)m} + 2$, then

$$\left(n'_{(i+1)m-1}, a'_{(i+1)m-1}\right) = \left(n_{(i+1)m-1} - 1, a_{(i+1)m-1} - 1\right);$$

   If $a_{(i+1)m-1} \geqslant a_{(i+1)m} + 3$, then

$$\left(n'_{(i+1)m-1}, a'_{(i+1)m-1}\right) = \left(n_{(i+1)m-1}, a_{(i+1)m-1} - 1\right);$$

(d) $\left(n'_{(i+1)m}, a'_{(i+1)m}\right) = \left(n_{(i+1)m}, a_{(i+1)m} + 1\right)$.

It's easy to check that S' is well-defined and $|S| = |S'|$. Then by Corollary 34, we have $f(S') > f(S)$.

(2) $v = 0$ and $a_{(i+1)m-1} = a_{(i+1)m} + 1$. Assume that $a_{(i+1)m-1} = \cdots = a_{(i+1)m-w} < a_{(i+1)m-w-1}$. Then $w + u < m - p$ since $a_{im+p+1} - a_{(i+1)m} \geqslant 2$. Let $S' \in \mathbb{D}_m(s)$ satisfy the following conditions:

(a) $t' = t$;

(b) For $1 \leqslant k \leqslant tm$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \notin \{im+p+u, (i+1)m-w, (i+1)m-w+1, \cdots, (i+1)m\}$;

(c) If $a_{im+p+u} = a_{(i+1)m}+2$, then $\left(n'_{im+p+u}, a'_{im+p+u}\right) = (n_{im+p+u} - 1, a_{im+p+u} - 1)$; If $a_{im+p+u} \geqslant a_{(i+1)m} + 3$, then $\left(n'_{im+p+u}, a'_{im+p+u}\right) = (n_{im+p+u}, a_{im+p+u} - 1)$;

(d) For $1 \leqslant r \leqslant w$, $\left(n'_{(i+1)m-r}, a'_{(i+1)m-r}\right) = \left(n_{(i+1)m-r} - 1, a_{(i+1)m-r}\right)$;

(e) $\left(n'_{(i+1)m}, a'_{(i+1)m}\right) = \left(n_{(i+1)m}, a_{(i+1)m} + 1\right)$.

It's easy to check that S' is well-defined and $|S| = |S'|$. Let $D(i)$ denote $\sum_{x\in\mathcal{B}'_i} x - \sum_{x\in\mathcal{B}_i} x$, then

$$\begin{aligned} f(S') - f(S) &= D_{im+p+u} + D_{(i+1)m} + \sum_{r=1}^{w} D_{(i+1)m-r} \\ &\geqslant ws - w(a_{(i+1)m} + 1) \\ &= w(s - 1 - a_{(i+1)m}) \\ &> 0. \end{aligned}$$

Thus $f(S') > f(S)$. Notice that here S' may not be a generalized-β-set.

(3) $v > 0$. The construction is similar and omitted.

The adjustments when there exists some $0 \leqslant j \leqslant t-2$, such that $a_{jm+1} - a_{jm+p} \geqslant 2$ are similar. The process above can be repeated in S' if there exists some $0 \leqslant i \leqslant t-2$, such that $a'_{im+p+1} - a'_{(i+1)m} \geqslant 2$ or there exists some $0 \leqslant j \leqslant t-2$, such that $a'_{jm+1} - a'_{jm+p} \geqslant 2$. Notice that for any $T \in \mathbb{D}_m(s)$, $f(T) \leqslant \lceil (s-1)/2 \rceil^2 m^2 s^2$. Also

$$f(S') \geqslant f(S) + 1,$$

thus the adjustments will stop after finite steps, when we get a final set S'', such that $a''_{(i-1)m+1} - a''_{(i-1)m+p} \leqslant 1$ and $a''_{(i-1)m+p+1} - a''_{im} \leqslant 1$ for $1 \leqslant i < t$. For S, set

$$U_i = \sum_{j=1}^{p} a_{(i-1)m+j}$$

and

$$V_i = \sum_{j=p+1}^{m} a_{(i-1)m+j}$$

for $1 \leqslant i \leqslant t$. The analog notations U_i'' and V_i'' are for S''. By the discussion above, for $1 \leqslant i \leqslant t$, we have $U_i = U_i''$ and $V_i = V_i''$. Since S is a generalized-β-set, we obtain

$$U_i - U_{i+1} \geqslant 2p$$

for $1 \leqslant i \leqslant t-1$ and

$$V_i - V_{i+1} \geqslant 2(m-p)$$

for $1 \leqslant i \leqslant t-2$. From Lemma 27, S'' satisfies Condition (3) in Definition 20. It is easy to check that S'' satisfies Conditions (1)(2)(6) in Definition 20. Condition (2) in Lemma 28 hold for S' in every case of adjustment. So S'' satisfies Conditions (4)(5) in Definition 20. Thus S'' is also a generalized-β-set and $f(S) < f(S'')$, which contradict the assumption that S is a generalized-β-set that maximizes $f(S)$. □

With this lemma, we can control the shape of a certain row of a generalized-β-set S that maximizes $f(S)$.

Definition 37. Let $\mathbb{E}_m(s)$ be the set of all $S \in \mathbb{C}_m(s)$ such that S satisfies Conclusions (1) and (2) in Lemma 35.

By Lemma 35, we obtain the following corollary.

Corollary 38. *We have* $\mathbb{E}_m(s) \subseteq \mathbb{C}_m(s) \subseteq \mathbb{B}_m(s)$ *and*

$$\arg\max_{S' \in \mathbb{B}_m(s)} f(S') = \arg\max_{S'' \in \mathbb{C}_m(s)} f(S'') = \arg\max_{S''' \in \mathbb{E}_m(s)} f(S''').$$

Next, we give an example of the adjustments in Lemma 35.

Example 39. Let $m = 3$ and $s = 6$. Figure 10 shows the generalized-β-set S before the adjustment. Figure 11 shows the generalized-β-set S' after the adjustment.

Notice that $S' = S \setminus \{11\} \cup \{13\}$. We color $S \setminus S'$ blue and $S' \setminus S$ red. The adjustment is on the first row and follows case (1) in the proof, where $i = v = 0$ and $a_{(i+1)m-1} = a_{(i+1)m} + 2$. We can see that $(n_2, a_2) = (5, 5), (n_3, a_3) = (4, 3), (n_2', a_2') = (4, 4), (n_3', a_3') = (4, 4)$. Indeed, $(n_2', a_2') = (n_2 - 1, a_2 - 1)$ and $(n_3', a_3') = (n_3, a_3 + 1)$. We can verify that $f(S') - f(S) = 68 - 66 = 13 - 11 > 0$.

7 Adjustments in different rows

In this section, we aim to prove that if a generalized-β-set $S \subseteq \mathcal{L}_m(s)$ maximizes $f(S)$, then for nonzero a_i and a_{i+m}, we have $a_i = a_{i+m} + 2$, thus the equality of Condition (3) in Definition 20 holds.

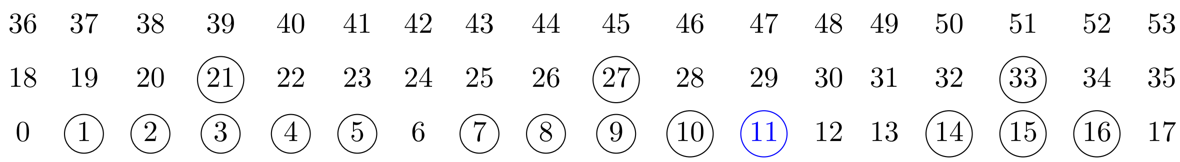

Figure 10: Before adjustments.

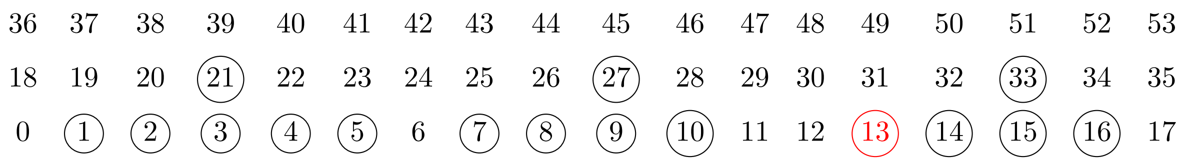

Figure 11: After adjustments.

Lemma 40. *Let S be a generalized-β-set that maximizes $f(S)$. Then we have the following conclusions:*

(1) If $A_{t-1} - A_t \geqslant 2m$, then

$$a_i = a_{i+m} + 2$$

for $1 \leqslant i \leqslant (t-1)m$.

(2) If $A_{t-1} - A_t \leqslant 2m-1$, by Proposition 25, assume that $a_{(t-1)m+1} = \cdots = a_{(t-1)m+p} = 1 > a_{(t-1)m+p+1}$. Then

$$a_i = a_{i+m} + 2$$

for $1 \leqslant i \leqslant (t-2)m + p$ and $a_{(t-2)m+p} = 3 > a_{(t-2)m+p+1}$.

Proof. By Lemma 35, we know that $S \in \mathbb{E}_m(s)$.

Step 1. First, we prove the following weaker version of Lemma 40:

(1)' If $A_{t-1} - A_t \geqslant 2m$, then $2m \leqslant A_i - A_{i+1} \leqslant 2m+1$ for $1 \leqslant i \leqslant t-1$. The equation $A_i - A_{i+1} = 2m+1$ only holds by at most one i.

(2)' If $A_{t-1} - A_t < 2m$, then $2m \leqslant A_i - A_{i+1} \leqslant 2m+1$ for $1 \leqslant i \leqslant t-2$. The equation $A_i - A_{i+1} = 2m+1$ only holds by at most one i.

First, we deal with the case $A_{t-1} - A_t \geqslant 2m$. We prove Conclusion (1)' by contradiction. Assume that Conclusion (1)' is not true. There are two possible cases.

(1) There exists $1 \leqslant i \leqslant (t-1)m$ such that $A_i - A_{i+1} \geqslant 2m+2$. By Lemma 35, $0 \leqslant a_{(i-1)m+1} - a_{im} \leqslant 1$ and $0 \leqslant a_{im+1} - a_{(i+1)m} \leqslant 1$. Let $S' \in \mathbb{D}_m(s)$ satisfy $(n'_k, a'_k) = (n_k, a_k)$ for $k < (i-1)m+1$ and $k > (i+1)m$, and furthermore satisfy the following conditions:

(a) If $a_{(i-1)m+1} = a_{im}$, then let $\left(n'_{(i-1)m+l}, a'_{(i-1)m+l}\right) = \left(n_{(i-1)m+l}+1, a_{(i-1)m+l}\right)$ for $1 \leqslant l \leqslant m-1$ and $(n'_{im}, a'_{im}) = (n_{im}, a_{im}-1)$;

(b) If $a_{(i-1)m+1} = a_{im}+1$, assume that $a_{im} = a_{im-1} = \cdots = a_{im-u+1} < a_{im-u}$. Let $\left(n'_{im-l}, a'_{im-l}\right) = (n_{im-l}, a_{im-l})$ for $0 \leqslant l \leqslant m-1$ and $l \neq u$ and $\left(n'_{im-u}, a'_{im-u}\right) = (n_{im-u}-1, a_{im-u}-1)$;

(c) If $a_{(i+1)m-1} = a_{(i+1)m}+1$, let $\left(n'_{im+l}, a'_{im+l}\right) = (n_{im+l}-1, a_{im+l})$ for $1 \leqslant l \leqslant m-1$ and $\left(n'_{(i+1)m}, a'_{(i+1)m}\right) = \left(n_{(i+1)m}, a_{(i+1)m}+1\right)$;

(d) If $a_{(i+1)m-1} = a_{(i+1)m}$ and $a_{im+1} = a_{(i+1)m}+1$, then we can assume that $a_{(i+1)m} = a_{(i+1)m-1} = \cdots = a_{(i+1)m-v} < a_{(i+1)m-v-1}$ for positive integer v. Let
$$\left(n'_{(i+1)m-v}, a'_{(i+1)m-v}\right) = \left(n_{(i+1)m-v}+1, a_{(i+1)m-v}+1\right)$$
and $\left(n'_{(i+1)m-l}, a'_{(i+1)m-l}\right) = \left(n_{(i+1)m-l}, a_{(i+1)m-l}\right)$ for $0 \leqslant l \leqslant m-1$ and $l \neq v$;

(e) If $a_{im+1} = a_{(i+1)m}$, let
$$\left(n'_{im+1}, a'_{im+1}\right) = (n_{im+1}+1, a_{im+1}+1)$$
and $\left(n'_{im+l}, a'_{im+l}\right) = (n_{im+l}, a_{im+l})$ for $2 \leqslant l \leqslant m$.

Adjustments in the i-th row involve Cases (a) and (b). Adjustments in the $i+1$-th row involve Cases (c), (d) and (e). Obviously $S' \in \mathbb{E}_m(s)$. If (c) doesn't hold, then $f(S) < f(S')$. If (c) holds, there are two possibilities: (b) and (c) hold simultaneously or (a) and (c) hold simultaneously. Recall that $D(i)$ denote $\sum_{x\in\mathcal{B}'_i} x - \sum_{x\in\mathcal{B}_i} x$.

Assume that (b) and (c) hold simultaneously. Notice that $a_{(i+1)m} \leqslant s-2-i$, thus

$$\begin{aligned}
& f(S') - f(S) \\
= \; & D\left((i+1)m\right) + D\left(im-u\right) + \sum_{j=1}^{m-1} D\left(im+j\right) \\
\geqslant \; & ms - (m-1)\left(a_{(i+1)m}+1\right) \\
\geqslant \; & ms - (m-1)(s-1-i) \\
> \; & ms - ms = 0.
\end{aligned}$$

Therefore, $f(S') > f(S)$. A contradiction.

Assume that (a) and (c) hold simultaneously. Similarly,

$$\begin{aligned}
& f(S') - f(S) \\
= \;& D(im) + D((i+1)m) + \sum_{k=1}^{m-1} D((i-1)m+k) + \sum_{k=1}^{m-1} D(im+k) \\
\geqslant \;& (m-1)s + (m-1)a_{im} - (m-1)\left(a_{(i+1)m}+1\right) \\
\geqslant \;& (m-1)s + (m-1)\left(a_{(i+1)m}+2\right) - (m-1)\left(a_{(i+1)m}+1\right) \\
> \;& 0.
\end{aligned}$$

Therefore, $f(S') > f(S)$. A contradiction.

(2) There exists $i < j$, such that $A_i - A_{i+1} \geqslant 2m+1$ and $A_j - A_{j+1} \geqslant 2m+1$. Then we construct $S' \in \mathbb{D}_m(s)$ similar to Case (1). We first do the adjustments to the i-th row of S similar to the adjustments to the i-th row in Case (1), and then do the adjustments to the $(j+1)$-th row of S similar to the adjustments to the $i+1$-th row in Case (1). Then we obtain S' after the adjustment. It is obvious that $S' \in \mathbb{E}_m(s)$. Similar to the discussion in Case (1), $f(S') > f(S)$. A contradiction.

Next, we deal with the case $A_{t-1} - A_t < 2m$. In this case, by Proposition 25, there exists $1 \leqslant p < m$, such that $a_{(t-1)m} = a_{(t-1)m+1} = \cdots = a_{(t-1)m+p} = 1$ and $a_{(t-1)m+p+1} = \cdots = a_{tm} = 0$. We will prove Conclusion (2)' by contradiction. Assume that Conclusion (2)' isn't true. Then either there exists $1 \leqslant i \leqslant t-2$, such that $A_i - A_{i+1} \geqslant 2m+2$; or there exist $1 \leqslant i < j \leqslant t-2$, such that $A_i - A_{i+1} = 2m+1$ and $A_j - A_{j+1} = 2m+1$. We will deduce a contradiction for each case.

Define

$$\mathcal{L}_i = \bigcup_{j=1}^{p} \mathcal{B}_{(i-1)m+j}$$

and

$$\mathcal{R}_i = \bigcup_{j=p+1}^{m} \mathcal{B}_{(i-1)m+j}.$$

for $1 \leqslant i \leqslant t$. Then their cardinal numbers are

$$L_i = |\mathcal{L}_i| = \sum_{j=1}^{p} a_{(i-1)m+j},$$

$$R_i = |\mathcal{R}_i| = \sum_{j=p+1}^{m} a_{(i-1)m+j}.$$

Firstly, we deal with the case $A_i - A_{i+1} \geqslant 2m+2$. We have the following three cases to consider.

(1) $L_i - L_{i+1} \geqslant 2p+1$ and $R_i - R_{i+1} \geqslant 2(m-p)+1$.

Consider $S' \in \mathbb{D}_m(s)$ satisfying the following conditions:

(a) $(n'_k, a'_k) = (n_k, a_k)$ unless $(i-1)m+1 \leqslant k \leqslant (i+1)m$;

(b) If $a_{(i-1)m+p+1} = a_{im}$, assume that $a_{(i-1)m+p+1} = a_{(i-1)m+p} = \cdots = a_{(i-1)m+u} < a_{(i-1)m+u-1}$. Then $(n'_k, a'_k) = (n_k+1, a_k)$ for $(i-1)m+u \leqslant k \leqslant im-1$ and $(n'_{im}, a'_{im}) = (n_{im}, a_{im}-1)$. Let $(n'_k, a'_k) = (n_k, a_k)$ for $(i-1)m+1 \leqslant k \leqslant (i-1)m+u-1$.

(c) If $a_{(i-1)m+p+1} = a_{im}+1$, assume that $a_{im} = a_{im-1} = \cdots = a_{im-v-1} < a_{im-v}$. For $(i-1)m+1 \leqslant k \leqslant im$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im-v$ and $\left(n'_{im-v}, a'_{im-v}\right) = (n_{im-v}-1, a_{im-v}-1)$;

(d) If $a_{im+1} = a_{(i+1)m}$, then $\left(n'_{im+1}, a'_{im+1}\right) = (n_k+1, a_k+1)$ and $(n'_k, a'_k) = (n_k, a_k)$ for $im+2 \leqslant k \leqslant (i+1)m$;

(e) If $a_{im+1} = a_{im+p} > a_{(i+1)m}$, then $\left(n'_{im+1}, a'_{im+1}\right) = (n_k, a_k+1)$ and $(n'_k, a'_k) = (n_k, a_k)$ for $im+2 \leqslant k \leqslant (i+1)m$;

(f) If $a_{im+1} > a_{im+p} = a_{(i+1)m}$, assume that $a_{im+1} = a_{im+2} = \cdots = a_{im+u-1} > a_{im+u}$. For $im+1 \leqslant k \leqslant (i+1)m$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im+u$ and $\left(n'_{im+u}, a'_{im+u}\right) = (n_{im+u}+1, a_{im+u}+1)$;

(g) If $a_{im+1} > a_{im+p} > a_{(i+1)m}$, assume that $a_{im+1} = a_{im+2} = \cdots = a_{im+u-1} > a_{im+u}$. For $im+1 \leqslant k \leqslant (i+1)m$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im+u$ and $\left(n'_{im+u}, a'_{im+u}\right) = (n_{im+u}, a_{im+u}+1)$.

Cases (b) and (c) are adjustments in $\mathcal{R}_i$. Cases (d),(e),(f) and (g) are adjustments in $\mathcal{L}_{i+1}$. Then $L'_i - L'_{i+1} \geqslant 2p$ and $R'_i - R'_{i+1} \geqslant 2(m-p)$. By Lemma 27, $a'_{im+k} \leqslant a'_{(i-1)m+k} - 2$ for $1 \leqslant k \leqslant m$. Condition (3) in Definition 20 won't be violated after the adjustments. We can verify that $S' \in \mathbb{E}_m(s)$. Similar to the discussion for the case $A_{t-1} - A_t \geqslant 2m$, we obtain $|S| = |S'|$ and $f(S) < f(S')$. Thus we reach a contradiction.

(2) $L_i - L_{i+1} = 2p$ and $R_i - R_{i+1} \geqslant 2(m-p)+2$.

Consider $S' \in \mathbb{D}_m(s)$ satisfying the following conditions:

(a) $(n'_k, a'_k) = (n_k, a_k)$ unless $(i-1)m+1 \leqslant k \leqslant (i+1)m$;

(b) If $a_{(i-1)m+p+1} = a_{im}$, assume that $a_{(i-1)m+u-1} > a_{(i-1)m+u} = \cdots = a_{im-1} = a_{im}$. then $(n'_k, a'_k) = (n_k, a_k)$ for $(i-1)m+1 \leqslant k \leqslant (i-1)m+u-1$, $(n'_k, a'_k) = (n_k+1, a_k)$ for $(i-1)m+u \leqslant k \leqslant im-1$ and $(n'_{im}, a'_{im}) = (n_{im}, a_{im}-1)$;

(c) If $a_{(i-1)m+p+1} = a_{im}+1$, assume that $a_{im} = a_{im-1} = \cdots = a_{im-v-1} < a_{im-v}$. For $(i-1)m+1 \leqslant k \leqslant im$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im-v$ and $\left(n'_{im-v}, a'_{im-v}\right) = (n_{im-v}-1, a_{im-v}-1)$;

(d) If $a_{im+p+1} = a_{(i+1)m}$, for $im+1 \leqslant k \leqslant (i+1)m$, let $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im+p+1$ and $\left(n'_{im+p+1}, a'_{im+p+1}\right) = (n_{im+p+1}+1, a_{im+p+1}+1)$;

(e) If $a_{(i+1)m-1} = a_{(i+1)m}$ and $a_{im+p+1} = a_{(i+1)m}+1$, assume that $a_{(i+1)m} = a_{(i+1)m-1} = \cdots = a_{(i+1)m-v} < a_{(i+1)m-v-1}$. For $im+1 \leqslant k \leqslant (i+1)m$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq im-v$ and

$$\left(n'_{(i+1)m-v}, a'_{(i+1)m-v}\right) = \left(n_{(i+1)m-v}+1, a_{(i+1)m-v}+1\right);$$

(f) If $a_{im+p+1} = a_{(i+1)m-1} = a_{(i+1)m}+1$, assume that $a_{(i+1)m-1} = a_{(i+1)m-2} = \cdots = a_{(i+1)m-w} < a_{(i+1)m-w-1}$. Let $\left(n'_{(i+1)m}, a'_{(i+1)m}\right) = \left(n_{(i+1)m}, a_{(i+1)m}+1\right)$ and $(n'_k, a'_k) = (n_k - 1, a_k)$ for $(i+1)m - w \leqslant k \leqslant (i+1)m-1$.

Case (b) and (c) are adjustments in $\mathcal{R}_i$. Case (d), (e) and (f) are adjustments in $\mathcal{R}_{i+1}$. We claim that Case (d) is valid. Otherwise, S' violates Condition (2) in Definition 20, then

$$a_{im+p} = a_{im+p+1} = \cdots = a_{(i+1)m}.$$

Notice that $L_i - L_{i+1} = 2p$ and $R_i - R_{i+1} \geqslant 2(m-p)+2$, we have $a_{(i-1)m+p} < a_{(i-1)m+p+1}$. A contradiction. Therefore, S' satisfies Condition (2) in Definition 20. We can easily prove that $S' \in \mathbb{E}_m(s)$. When (f) doesn't happen, obviously $f(S) < f(S')$. When (f) happens, similar to the discussions in Case (1) when $A_{t-1} - A_t \geqslant 2m$, we can prove that $f(S) < f(S')$. A contradiction.

(3) $L_i - L_{i+1} \geqslant 2p+2$ and $R_i - R_{i+1} = 2(m-p)$. It is similar to Case (2).

Therefore, we obtain $2m \leqslant A_i - A_{i+1} \leqslant 2m+1$ for $1 \leqslant i \leqslant t-2$. Now we deal with the case that there exists $1 \leqslant i < j \leqslant t-2$, such that $A_i - A_{i+1} = 2m+1$ and $A_j - A_{j+1} = 2m+1$. There are four cases to be considered.

(1) $R_i - R_{i+1} = 2(m-p)+1$ and $L_j - L_{j+1} = 2p+1$. This case is similar to Case (1) when $A_i - A_{i+1} \geqslant 2m+2$. We can still reach a contradiction.

(2) $L_i - L_{i+1} = L_j - L_{j+1} = 2p$ and $R_i - R_{i+1} = R_j - R_{j+1} = 2(m-p)+1$. This case is similar to Case (2) when $A_i - A_{i+1} \geqslant 2m+2$. We can still reach a contradiction.

(3) $L_i - L_{i+1} = L_j - L_{j+1} = 2p+1$ and $R_i - R_{i+1} = R_j - R_{j+1} = 2(m-p)$. This case is similar to Case (3) when $A_i - A_{i+1} \geqslant 2m+2$. We can still reach a contradiction.

(4) $L_i - L_{i+1} = 2p+1$, $R_i - R_{i+1} = 2(m-p)$, $L_j - L_{j+1} = 2p$, $R_j - R_{j+1} = 2(m-p)+1$. Assume that $a_{(i-1)m+1} = a_{(i-1)m+p+1}$. Since $L_i - L_{i+1} = 2p+1$ and $R_i - R_{i+1} = 2(m-p)$, $a_{im+p} = a_{(i-1)m+p} - 3 = a_{(i-1)m+p+1} - 3 = a_{im+p+1} - 1 < a_{im+p+1}$. A contradiction. Thus $a_{(i-1)m+1} > a_{(i-1)m+p+1}$. Similarly, $a_{jm+p} > a_{(j+1)m}$.

Consider $S' \in \mathbb{D}_m(s)$ satisfying the following conditions:

(a) $(n'_k, a'_k) = (n_k, a_k)$ unless $(i-1)m+1 \leqslant k \leqslant im$ or $jm+1 \leqslant k \leqslant (j+1)m$;

(b) If $a_{(i-1)m+1} = a_{(i-1)m+p} = a_{im}+1$, then $(n'_k, a'_k) = (n_k, a_k)$ for $(i-1)m+1 \leqslant k \leqslant (i-1)m+p-1$ or $(i-1)m+p+1 \leqslant k \leqslant im$ and $\left(n'_{(i-1)m+p}, a'_{(i-1)m+p}\right) = \left(n_{(i-1)m+p} - 1, a_{(i-1)m+p} - 1\right)$;

(c) If $a_{(i-1)m+1} = a_{(i-1)m+p} > a_{im}+1$, then $(n'_k, a'_k) = (n_k, a_k)$ for $(i-1)m+1 \leqslant k \leqslant (i-1)m+p-1$ or $(i-1)m+p+1 \leqslant k \leqslant im$ and $\left(n'_{(i-1)m+p}, a'_{(i-1)m+p}\right) = \left(n_{(i-1)m+p}, a_{(i-1)m+p} - 1\right)$;

(d) If $a_{(i-1)m+1} = a_{(i-1)m+p} + 1 = a_{im} + 1$, assume that $a_{(i-1)m+1} = a_{(i-1)m+2} = \cdots = a_{(i-1)m+u} > a_{(i-1)m+u+1}$. For $(i-1)m+1 \leqslant k \leqslant im$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq (i-1)m+u$ and $\left(n'_{(i-1)m+u}, a'_{(i-1)m+u}\right) = \left(n_{(i-1)m+u} - 1, a_{(i-1)m+u} - 1\right)$;

(e) If $a_{(i-1)m+1} = a_{(i-1)m+p} + 1 > a_{im} + 1$, assume that $a_{(i-1)m+1} = a_{(i-1)m+2} = \cdots = a_{(i-1)m+u} > a_{(i-1)m+u+1}$. For $(i-1)m+1 \leqslant k \leqslant im$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq (i-1)m+u$ and $\left(n'_{(i-1)m+u}, a'_{(i-1)m+u}\right) = \left(n_{(i-1)m+u}, a_{(i-1)m+u} - 1\right)$;

(f) If $a_{jm+p+1} = a_{(j+1)m}$, for $jm+1 \leqslant k \leqslant (j+1)m$, let $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq jm+p+1$ and $\left(n'_{jm+p+1}, a'_{jm+p+1}\right) = (n_{jm+p+1} + 1, a_{jm+p+1} + 1)$;

(g) If $a_{(j+1)m-1} = a_{(j+1)m}$ and $a_{jm+p+1} = a_{(j+1)m} + 1$, assume that $a_{(j+1)m} = a_{(j+1)m-1} = \cdots = a_{(j+1)m-v} < a_{(j+1)m-v-1}$. For $jm+1 \leqslant k \leqslant (j+1)m$, $(n'_k, a'_k) = (n_k, a_k)$ if $k \neq jm-v$ and

$$\left(n'_{(j+1)m-v}, a'_{(j+1)m-v}\right) = \left(n_{(j+1)m-v} + 1, a_{(j+1)m-v} + 1\right);$$

(h) If $a_{jm+p+1} = a_{(j+1)m-1} = a_{(j+1)m} + 1$, assume that $a_{(j+1)m-1} = a_{(j+1)m-2} = \cdots = a_{(j+1)m-w} < a_{(j+1)m-w-1}$. Let $\left(n'_{(j+1)m}, a'_{(j+1)m}\right) = \left(n_{(j+1)m}, a_{(j+1)m} + 1\right)$ and $(n'_k, a'_k) = (n_k - 1, a_k)$ for $(j+1)m - w \leqslant k \leqslant (j+1)m - 1$.

Cases (b), (c), (d) and (e) are adjustments in $\mathcal{L}_i$. Cases (f), (g), (h) are adjustments in $\mathcal{R}_{j+1}$. By the discussion before, Cases (b) and (f) are valid. Similarly we can prove that $S' \in \mathbb{E}_m(s)$ and $f(S) < f(S')$. A contradiction.

Step 2. Now we provide the proofs of Conclusions (1)(2). First, we prove Conclusion (1) by contradiction. Assume that $a_i = a_{i+m} + 2$ for $1 \leqslant i \leqslant (t-1)m$ doesn't hold. Then by Conclusion (1)', there exists a unique $1 \leqslant l \leqslant t-1$, such that $A_l - A_{l+1} = 2m+1$. Thus there exists a unique u, such that $a_u - a_{u+m} = 3$ and $a_i - a_{i+m} = 2$ for $i \neq u$. Assume that $q \leqslant m$ is the largest positive integer such that $a_1 = \cdots = a_q$. By Condition (2) in Definition 20 and Conclusion (2) in Lemma 35, we obtain

$$u \equiv q \pmod{m}.$$

Therefore, $u = (l-1)m + q$. By Lemma 35, $a_{(i-1)m+1} - a_{im} \leqslant 1$ for $1 \leqslant i \leqslant t$, so we have two cases: $a_1 = a_m + 1$ or $a_1 = a_m$.

Firstly, consider the case $a_1 = a_m + 1$. For $0 \leqslant w \leqslant t$, let $S_w \in \mathbb{D}_m(s)$ satisfy the following conditions. Here we use (n_i^w, a_i^w) to denote $(n_i(S_w), a_i(S_w))$.

(1) $(n_i^w, a_i^w) = (n_i, a_i)$ for $i \neq q, 2q, \cdots, (t-1)m+q$;

(2) If $0 \leqslant k \leqslant l-1$, then $(n_i^w, a_i^w) = (n_i - 1, a_i - 1)$ for $i = km+q, (k+1)m+q, \cdots, (l-1)m+q$;

(3) If $l \leqslant k \leqslant t$, then $(n_i^w, a_i^w) = (n_i + 1, a_i + 1)$ for $i = lm+q, (l+1)m+q, \cdots, (k-1)m+q$.

Then $S=S_l$. It's easy to check that $S_0,\cdots,S_t\in\mathbb{E}_m(s)$ and $|S_i|-|S_{i-1}|=1$ for $1\leqslant i\leqslant t$. Since

$$\begin{aligned} f(S_{n+1})-f(S_n) &= \sum_{x\in S_{n+1}} x-\frac{|S_{n+1}|\,(|S_{n+1}|-1)}{2}-\sum_{x\in S_n} x+\frac{|S_n|(|S_n|-1)}{2} \\ &= \sum_{x\in S_{n+1}} x-\sum_{x\in S_n} x-\frac{|S_n|(|S_n|+1)}{2}+\frac{|S_n|(|S_n|-1)}{2} \\ &= \sum_{x\in S_{n+1}} x-\sum_{x\in S_n} x-|S_n|, \end{aligned}$$

we have

$$\begin{aligned} & f(S_{n+2})-2f(S_{n+1})+f(S_n) \\ = & \sum_{x\in S_{n+2}-S_{n+1}} x-\sum_{x\in S_{n+1}-S_n} x-(|S_{n+1}|-|S_n|) \\ = & ms-1-1 \\ = & ms-2>0. \end{aligned}$$

Thus $f(S_{n+2})-f(S_{n+1})>f(S_{n+1})-f(S_n)$, and $f(S)=f(S_l)<\max\{f(S_{l-1}),f(S_{l+1})\}$. A contradiction.

Secondly, consider the case $a_1=a_m$. For $0\leqslant w\leqslant t$, let $S_w\in\mathbb{D}_m(s)$ satisfy the following conditions. Here we still use (n_i^w,a_i^w) to denote $(n_i(S_w),a_i(S_w))$.

(1) $(n_i^w,a_i^w)=(n_i,a_i)$ except when $i=q,2q,\cdots,(t-1)m+q$;

(2) Assume that $0\leqslant k\leqslant l-1$. For $km+1\leqslant i\leqslant lm$, $(n_i^w,a_i^w)=(n_i,a_i-1)$ if $i\equiv q \mod m$ and $(n_i^w,a_i^w)=(n_i+1,a_i)$ if $i\not\equiv q \mod m$. Otherwise, $(n_i^w,a_i^w)=(n_i,a_i)$.

(3) Assume that $l\leqslant k\leqslant t$. For $lm+1\leqslant i\leqslant km$, $(n_i^w,a_i^w)=(n_i,a_i+1)$ if $i\equiv q \mod m$ and $(n_i^w,a_i^w)=(n_i-1,a_i)$ if $i\not\equiv q \mod m$. Otherwise, $(n_i^w,a_i^w)=(n_i,a_i)$.

Then $S=S_l$. It's easy to check that $S_0,\cdots,S_t\in\mathbb{E}_m(s)$ and $|S_i|-|S_{i-1}|=1$ for $1\leqslant i\leqslant t$. Similarly,

$$\begin{aligned} & f(S_{n+2})-2f(S_{n+1})+f(S_n) \\ = & \sum_{x\in S_{n+2}} x-\sum_{x\in S_{n+1}} x-\left(\sum_{x\in S_{n+1}} x-\sum_{x\in S_n} x\right)-|S_{n+1}|+|S_n| \\ = & ms+1+2(m-1)-1=m(s+2)-2>0. \end{aligned}$$

This also contradicts the assumption of S.

Now we turn to the case where $A_{t-1}-A_t<2m$. We prove Conclusion (2) by contradiction. First, assume that the claim $a_i=a_{i+m}+2$ for $1\leqslant i\leqslant(t-2)m$ is not true. Similarly, there exist unique $1\leqslant l\leqslant t-1$ and $1\leqslant k\leqslant m$, such that $a_{(l-1)m+k}-a_{lm+k}=3$

and $a_i - a_{i+m} = 2$ for $i \neq (l-1)m+k$. We can make adjustments similar to the former case to the first p columns of S and the last $m-p$ columns of S separately. Still, we can reach a contradiction. Thus $a_i - a_{i+m} = 2$ for $1 \leqslant i \leqslant (t-2)m$.

Next, we prove that $a_{(t-2)m+1} = \cdots = a_{(t-2)m+p} = 3 > a_{(t-2)m+p+1}$ by contradiction. We know that $a_{(t-2)m+p} \geqslant a_{(t-1)m+p}+2 = 3$. Assume that $a_{(t-2)m+p+1} \geqslant 3$, then $p < m-1$ since $a_{(t-1)m} = 1$. Let $S' = S \cup \{((t-1)m+p+1)s-t\}$, then S' is still a generalized-β-set and $f(S) < f(S')$. A contradiction. Thus $a_{(t-2)m+p+1} \leqslant 2$. Assume that $a_{(t-2)m+1} \geqslant 4$. Set $a_{(t-2)m+1} = a_{(t-2)m+2} = \cdots = a_{(t-2)m+u} > a_{(t-2)m+u+1}$ for some $1 \leqslant u \leqslant p$ and $a_{(t-1)m} = a_{(t-1)m-1} = \cdots = a_{(t-1)m-v} < a_{(t-1)m-v-1}$ for some $0 \leqslant v \leqslant m-p-1$. Since $A_{t-1} - A_t < 2m$, we have $p < m-1$ and $v \geqslant 1$. Let $S' \in \mathbb{D}_m(s)$ satisfy the following conditions:

(1) $(n'_i, a'_i) = (n_i, a_i)$ if $i \equiv 1, 2, \ldots, u-1, u+1, \ldots, p \pmod{m}$;

(2) $\left(n'_{jm+u}, a'_{jm+u}\right) = (n_{jm+u}, a_{jm+u} - 1)$ for $0 \leqslant j \leqslant t-2$;

(3) Let
$$\left(n'_{jm-v}, a'_{jm-v}\right) = (n_{jm-v} + 1, a_{jm-v} + 1)$$
for $1 \leqslant j \leqslant t-1$ and $\left(n'_{(j-1)m+i}, a'_{(j-1)m+i}\right) = \left(n_{(j-1)m+i}, a_{(j-1)m+i}\right)$ if $1 \leqslant j \leqslant t-1$, $p+1 \leqslant i \leqslant m$ and $i \neq m-v$.

Since $a_i - a_{i+m} = 2$ for $1 \leqslant i \leqslant (t-2)m$, it's easy to verify that $S' \in \mathbb{E}_m(s)$. If (4) doesn't happen, obviously $f(S) < f(S')$, a contradiction. Otherwise, $v = 0$ and $a_{im} = a_{im-1} - 1$ for $1 \leqslant i \leqslant t-1$. Recall that $D(i)$ denote $\sum_{x \in \mathcal{B}'_i} x - \sum_{x \in \mathcal{B}_i} x$. Then

$$\begin{aligned} f(S') - f(S) &= \sum_{x \in S'} x - \sum_{x \in S} x \\ &= \sum_{i=0}^{t-2} \left(D(im+u) + D((i+1)m)\right) + \sum_{i=0}^{t-2} \sum_{j=p+1}^{m-1} D(im+j) \\ &\geqslant (t-1)(m-u)s - (t-1)(m-p-1)s > 0. \end{aligned}$$

So $f(S') > f(S)$. A contradiction. Thus for $(t-2)m+1 \leqslant i \leqslant (t-2)m+p$, we have $a_i - a_{i+m} = 2$. $\square$

Definition 41. Let $\mathbb{F}_m(s)$ be the set of all $S \in \mathbb{E}_m(s)$ such that S satisfies Conclusions (1) and (2) in Lemma 40.

By Lemma 40, we obtain the following corollary.

Corollary 42. *We have* $\mathbb{F}_m(s) \subseteq \mathbb{E}_m(s) \subseteq \mathbb{C}_m(s) \subseteq \mathbb{B}_m(s)$ *and*

$$\arg \max_{S' \in \mathbb{B}_m(s)} f(S') = \arg \max_{S'' \in \mathbb{C}_m(s)} f(S'') = \arg \max_{S''' \in \mathbb{E}_m(s)} f(S''') = \arg \max_{S'''' \in \mathbb{F}_m(s)} f(S'''').$$

Next, we give an example illustrating the adjustments in the proof of Lemma 40.

Example 43. Let $s = 6$ and $m = 3$. Figure 12 shows the generalized-β-set S before the adjustments. Figure 13 shows the generalized-β-set S' after the first adjustment. Figure 14 shows the same generalized-β-set S' with Figure 13. Figure 15 shows the generalized-β-set S'' after the second adjustment. The differences in the colors of some circles between Figure 13 and 14 are to highlight the changes of different adjustments. Here $t = 2, A_1 = 13, A_2 = 3$. Thus $A_{t-1} - A_t \geqslant 2m$.

Notice that $S' = S \cup \{22\} \setminus \{5\}$ is a generalized-β-set. We color $S \setminus S'$ blue and $S' \setminus S$ red. The first adjustment is in rows 1 and 2 and involves (1)(b) and (1)(e) in the case $A_{t-1} - A_t \geqslant 2m$ of Step 1 of the proof in Pages 15 and 16. We can find that $(n_1, a_1) = (5, 5), (n_4, a_4) = (1, 3), (n'_1, a'_1) = (4, 4), (n'_4, a'_4) = (2, 4)$. Indeed, we have $(n'_1, a'_1) = (n_1 - 1, a_1 - 1)$ and $(n'_4, a'_4) = (n_4 + 1, a_4 + 1)$. Obviously $f(S') - f(S) = 22 - 5 > 0$. Here $A'_1 = 12, A'_2 = 4$. So $A'_{t-1} - A'_t \geqslant 2m$.

Notice that $S'' = S' \cup \{5, 11, 28\} \setminus \{1, 7, 13\}$ is a generalized-β-set. We color $S' \setminus S''$ blue and $S'' \setminus S'$ red. The second adjustment is in rows 1 and 2 and involves (1)(a) and (1)(d) in the case $A'_{t-1} - A'_t \geqslant 2m$ of Step 1 of the proof in Lemma 40. We can find that $(n''_1, a''_1) = (n'_1 + 1, a'_1), (n''_2, a''_2) = (n'_2 + 1, a'_2), (n''_3, a''_3) = (n'_3, a'_3 - 1)$ and $(n''_5, a''_5) = (n'_5 + 1, a'_5 + 1)$. Obviously $f(S'') - f(S') = (5 - 1) + (11 - 7) + (28 - 13) > 0$.

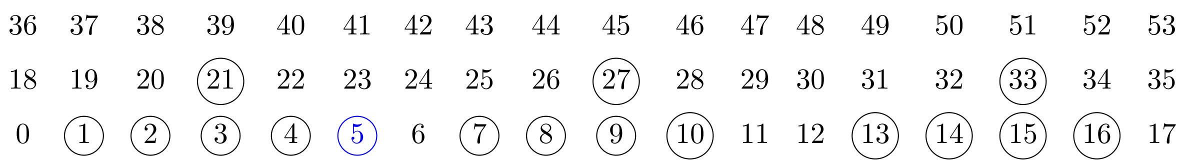

Figure 12: Before adjustments.

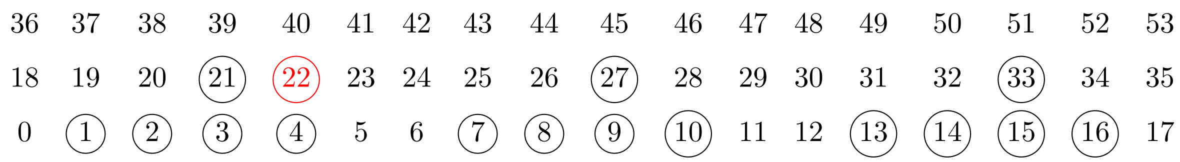

Figure 13: After the first adjustment.

8 Adjustments in columns

In this section, first we prove two useful results Lemma 44 and Lemma 46. These two lemmas deal with the cases $A_{t-1} - A_t \geqslant 2m$ and $A_{t-1} - A_t > 2m$ respectively.

Lemma 44. *Let S be a generalized-β-set that maximizes $f(S)$. Assume that $A_{t-1} - A_t \geqslant 2m$, then*

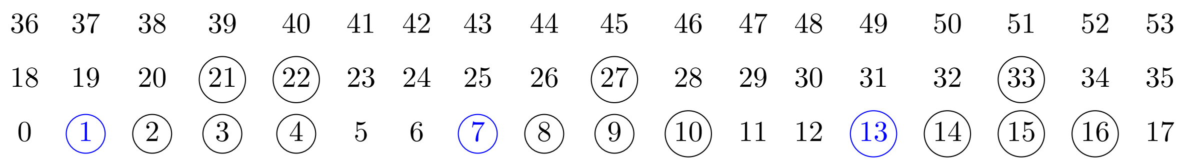

Figure 14: Before the second adjustment.

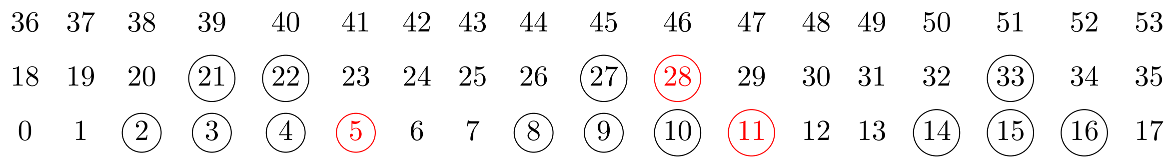

Figure 15: After the second adjustment.

(1) $a_1 = \cdots = a_m$ *or* $a_1 = \cdots = a_{m-1} = a_m + 1$*;*

(2) $a_{(t-1)m+1} \in \{1, 2\}$*.*

Proof. By Lemma 40, we obtain that $S \in \mathbb{F}_m(s)$ and $a_i - a_{i+m} = 2$ for $1 \leqslant i \leqslant (t-1)m$. We will prove Conclusion (1) by contradiction. Assume that Conclusion (1) is not true. Then there exists $1 \leqslant k \leqslant m-2$ and $a \geqslant 0$, such that $a_1 = \cdots = a_k = a+1$ and $a_{k+1} = \cdots = a_m = a$.

For $0 \leqslant n \leqslant m-1$, let $S_n \in \mathbb{C}_m(s)$ denote the t-row generalized-β-set such that $a_1 = \cdots = a_n = a+1$, $a_{n+1} = \cdots = a_m = a$ and $a_i - a_{i+m} = 2$ for all i. Then $S = S_k$ and we have

$$\begin{aligned} f(S_{n+1}) - f(S_n) &= \sum_{x \in S_{n+1}} x - \frac{|S_{n+1}|(|S_{n+1}|-1)}{2} - \sum_{x \in S_n} x + \frac{|S_n|(|S_n|-1)}{2} \\ &= \sum_{x \in S_{n+1}} x - \sum_{x \in S_n} x - \sum_{i=|S_n|}^{|S_{n+1}|-1} i. \end{aligned}$$

Therefore,

$$\begin{aligned} & f(S_{n+2}) - 2f(S_{n+1}) + f(S_n) \\ = & \left(\sum_{x \in S_{n+2}} x - \sum_{x \in S_{n+1}} x \right) - \left(\sum_{x \in S_{n+1}} x - \sum_{x \in S_n} x \right) - \left(\sum_{i=|S_{n+1}|}^{|S_{n+2}|-1} i - \sum_{i=|S_n|}^{|S_{n+1}|-1} i \right) \\ = & \; ts - t^2 = t(s-t) > 0. \end{aligned}$$

Thus we obtain $f(S) = f(S_k) < \max\{f(S_{k-1}), f(S_{k+1})\}$, a contradiction.

Next, we prove Conclusion (2) by contradiction. Assume that Conclusion (2) is not true. Then $a_{(t-1)m+1} \geqslant 3$, thus $a_{tm} \geqslant a_{(t-1)m+1} - 1 \geqslant 2$. Let $b_i = |\mathcal{L}_m(s) \cap [(i-1)s, is-1]|$ for $i \geqslant 1$. Then $b_{tm+1} = b_{(t-1)m+1} - 2 \geqslant a_{(t-1)m+1} - 2 \geqslant 1$. Let $R = S \cup \{(tm+1)s - t - 1\}$ and $T = S \cup \{(tm+1)s - t - 2\}$. Then R or T is a generalized-β-set while $f(S) < f(R)$ and $f(S) < f(T)$. A contradiction. □

Example 45. We give an example illustrating the proof of Lemma 44. Let $s = 5$ and $m = 3$, consider S_0 in Figure 16, S_1 in Figure 17 and S_2 in Figure 18. We color $S_0 \setminus S_1$ blue and $S_2 \setminus S_1$ red. Indeed $(a_1(S_0), a_2(S_0), a_3(S_0)) = (3, 3, 3)$, $(a_1(S_1), a_2(S_1), a_3(S_1)) = (4, 3, 3)$, $(a_1(S_2), a_2(S_2), a_3(S_2)) = (4, 4, 3)$. It's easy to verify that $S_0, S_1, S_2 \in \mathbb{F}_m(s)$ with $f(S_0) = 63$, $f(S_1) = 60$ and $f(S_2) = 63$. Therefore, $f(S_1) < \max\{f(S_0), f(S_2)\}$.

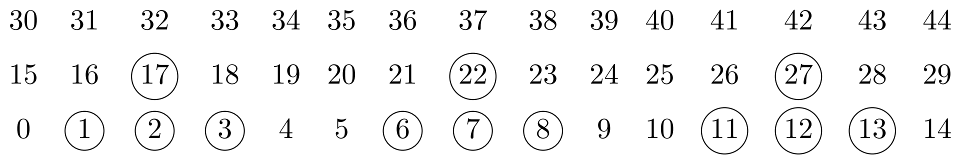

Figure 16: S_0 with $(a_1(S_0), a_2(S_0), a_3(S_0)) = (3, 3, 3)$.

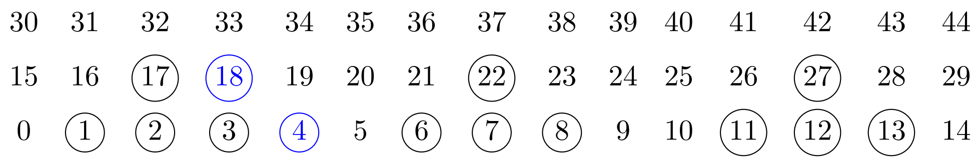

Figure 17: S_1 with $(a_1(S_1), a_2(S_1), a_3(S_1)) = (4, 3, 3)$.

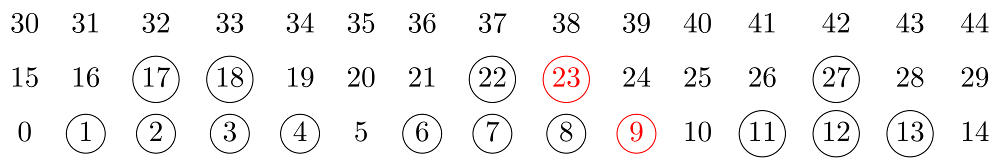

Figure 18: S_2 with $(a_1(S_2), a_2(S_2), a_3(S_2)) = (4, 4, 3)$.

Lemma 46. *Let S be a generalized-β-set that maximizes $f(S)$. Assume that $A_{t-1} - A_t < 2m$, then we have*

(1) s is even and $t = \frac{s}{2}$;

(2) $a_{(t-1)m} = \cdots = a_{tm-1} = 1$ *and* $a_{tm} = 0$.

Proof. By Corollary 42, we obtain that $S \in \mathbb{F}_m(s)$. Now we prove Conclusion (2). Since $A_{t-1} - A_t < 2m$, by Proposition 25 we set $a_{(t-1)m} = \cdots = a_{(t-1)m+p} = 1 > a_{(t-1)m+p+1}$, where $p \leqslant m-1$. First, we obtain that $a_{(t-2)m+1} = \cdots = a_{(t-2)m+p} = 3 > a_{(t-2)m+p+1}$ by Lemma 40. Now we claim that $a_{(t-2)m+p+1} = 1$.

Otherwise set $a_{(t-2)m+p+1} = \cdots = a_{(t-2)m+p+q} = 2$, $a_{(t-2)m+p+q+1} = \cdots = a_{(t-1)m} = 1$. Then $s \geqslant 2t$ and $q \leqslant m-p-1$. Let $S' \in \mathbb{E}_m(s)$ be a t-row set satisfying the following conditions:

(1) $(n'_i, a'_i) = (n_i, a_i)$ when $i \not\equiv p+1, p+2, \cdots, p+q \mod m$;

(2) $(n'_i, a'_i) = (n_i, a_i + 1)$ when $i \equiv p+1, p+2, \cdots, p+q \mod m$ and $i \leqslant (t-1)m$;

(3) $\mathcal{B}'_{(t-1)m+p+k} = \{((t-1)m+p+k)s - t\}$ for $1 \leqslant k \leqslant q$.

Then obviously $S' \in \mathbb{F}_m(s)$. Set the first $k+1$ elements of $\mathcal{B}'_{p+1}, \cdots, \mathcal{B}'_{p+q}$ as $l_1, \cdots, l_{q(k+1)}$. Therefore,

$$|S| = m(t-1)^2 + (2p+q)(t-1) + p$$

and

$$\begin{aligned} f(S') - f(S) &= qms\frac{t(t-1)}{2} + \sum_{i=1}^{qt} l_i - qt|S| - \sum_{i=1}^{qt}(i-1) \\ &= qt\left(\frac{(t-1)ms}{2} - m(t-1)^2 - (2p+q)(t-1) - p\right) + \sum_{i=1}^{qt}(l_i - i + 1) \\ &\geqslant qt(mt(t-1) - m(t-1)^2 - (m+p-1)(t-1) - p) + qt(ps+1) \\ &= q(p+1)t^2 > 0. \end{aligned}$$

A contradiction. Thus we can assume that $a_{(t-2)m+j} \in \{1, 3\}$ for $1 \leqslant j \leqslant m$.

Next, we prove $p = m-1$ by contradiction. Otherwise, we assume that $p < m-1$. Since $a_{(t-1)m+1} = \cdots = a_{(t-1)m+p} = 1$ for $1 \leqslant p < m-1$, $s \geqslant 2t$. Let $S' \in \mathbb{E}_m(s)$ be a t-row set satisfying the following conditions: for $1 \leqslant i \leqslant tm$,

(1) $(n'_i, a'_i) = (n_i, a_i)$ when $i \not\equiv p+1, p+2, \cdots, m-1 \mod m$;

(2) $(n'_i, a'_i) = (n_i + 1, a_i + 2)$ when $i \equiv p+1, p+2, \cdots, m-1 \mod m$ and $i \leqslant (t-1)m$;

(3) $\mathcal{B}'_{(t-1)m+k} = \{((t-1)m+k)s - t\}$ for $p+1 \leqslant k \leqslant m-1$.

Obviously $S' \in \mathbb{F}_m(s)$. Set the elements of $\mathcal{B}'_{p+1}, \mathcal{B}'_{p+2}, \cdots, \mathcal{B}'_{m-1}$ as $l_1, \cdots, l_{(m-p-1)(2t-1)}$.

Therefore, $|S| = m(t-1)^2 + 2pt - p$, and

$$\begin{aligned}
& f(S') - f(S) \\
= \quad & (m-p-1)(t-1)^2 ms + \sum_{i=1}^{(m-p-1)(2t-1)} l_i - (m-p-1)(2t-1)|S| \\
- \quad & \sum_{i=1}^{(m-p-1)(2t-1)} (i-1) \\
= \quad & (m-p-1)((t-1)^2 ms - (2t-1)(m(t-1)^2 + 2pt - p)) \\
+ \quad & \sum_{i=1}^{(m-p-1)(2t-1)} (l_i - i + 1) \\
\geqslant \quad & (m-p-1)(2mt(t-1)^2 - (2t-1)(m(t-1)^2 + 2pt - p)) \\
+ \quad & (m-p-1)(2t-1)(ps+1) \\
\geqslant \quad & (m-p-1)(m(t-1)^2 - p(2t-1)^2 + (2t-1)(2pt+1)) \\
= \quad & (m-p-1)(m(t-1)^2 + (p+1)(2t-1)) > 0.
\end{aligned}$$

A contradiction. Thus Conclusion (2) is proved.

Now we prove Conclusion (1). By Lemma 40, $a_{(t-2)m+1} = 3$. Let $S' \in \mathbb{E}_m(s)$ be a t-row set satisfying the following conditions: for $1 \leqslant i \leqslant tm$,

(1) $(n_i', a_i') = (n_i, a_i + 1)$ when $1 \leqslant i \leqslant (t-1)m$ and $m|i$;

(2) $(n_i', a_i') = (n_i, a_i)$ otherwise.

Obviously $S' \in \mathbb{F}_m(s)$. Since $|S| = m(t-1)^2 + 2(m-1)(t-1) + (m-1)$, we have

$$f(S') - f(S) = \frac{(t-1)(t-2)}{2} ms + \sum_{i=1}^{t-1} (ms - 2t + i) - (t-1)|S| - \sum_{i=1}^{t-1} (i-1)$$

$$= \quad (t-1)\left(\frac{t-2}{2} ms - m(t-1)^2 - 2(m-1)(t-1) - (m-1) + ms - 2t + 1\right)$$

$$= \quad \frac{mt(t-1)(s-2t)}{2} \geqslant 0.$$

The equality may only be achieved for $s = 2t$. Then s is even and $t = \frac{s}{2}$. Thus Conclusion (1) is proved. □

There are three kinds of adjustments in Lemma 46. We give an example to help visualize those adjustments.

Example 47. Let $s = 6$ and $m = 3$. It's easy to verify that $S_{11}, S_{12}, S_{21}, S_{22}, S_{31}, S_{32} \in \mathbb{F}_m(s)$ in Figure 19 to 24.

Figure 19 and Figure 20 display the first kind of adjustment. Notice that $S_{12} = S_{11} \cup \{7, 26, 45\}$. We color $S_{12} \setminus S_{11}$ red. Indeed $p = q = 1$, $(n_2(S_{12}), a_2(S_{12})) =$

$(n_2(S_{11}), a_2(S_{11})+1), (n_5(S_{12}), a_5(S_{12})) = (n_5(S_{11}), a_5(S_{11})+1)$ and $\mathcal{B}_8(S_{32}) = \{45\}$. We have $f(S_{11}) = 117 < 135 = f(S_{12})$.

Figure 21 and Figure 22 display the second kind of adjustment. Notice that $S_{22} = S_{21} \cup \{7, 11, 26, 28, 45\}$. We color $S_{22} \setminus S_{21}$ red. Indeed $p = 1$, $(n_2(S_{22}), a_2(S_{22})) = (n_2(S_{21})+1, a_2(S_{21})+2), (n_5(S_{22}), a_5(S_{22})) = (n_5(S_{21})+1, a_5(S_{21})+2)$ and $\mathcal{B}_8(S_{22}) = \{45\}$. We have $f(S_{21}) = 113 < 135 = f(S_{22})$.

Figure 23 and Figure 24 display the third kind of adjustment. Notice that $S_{32} = S_{31} \cup \{15\}$. We color $S_{32} \setminus S_{31}$ red. Indeed $t = 2$ and $(n_3(S_{32}), a_3(S_{32})) = (n_3(S_{31}), a_3(S_{31})+1)$. We have $f(S_{31}) = 72 < 78 = f(S_{32})$.

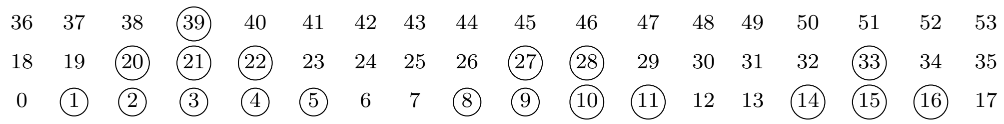

Figure 19: S_{11}: before the first kind of adjustment.

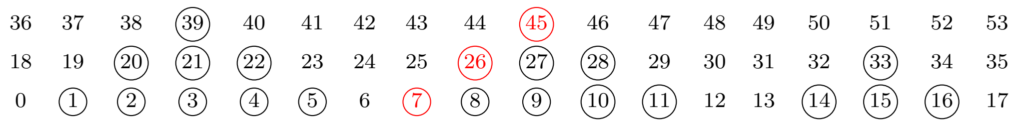

Figure 20: S_{12}: after the first kind of adjustment.

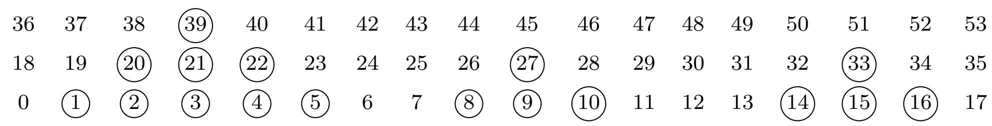

Figure 21: S_{21}: before the second kind of adjustment.

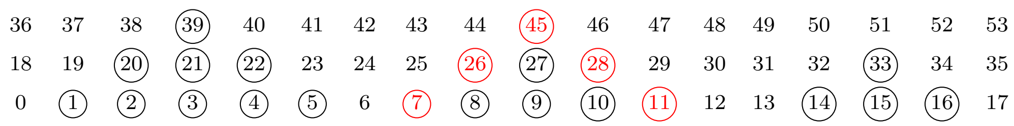

Figure 22: S_{22}: after the second kind of adjustment.

Lemmas 44 and 46 discuss about the necessary condition for the generalized-β-set S that maximizes $f(S)$. The following theorem gives a detailed description of S.

36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
18	19	20	21	$\textcircled{22}$	23	24	25	26	27	$\textcircled{28}$	29	30	31	32	33	34	35
0	1	2	$\textcircled{3}$	$\textcircled{4}$	$\textcircled{5}$	6	7	8	$\textcircled{9}$	$\textcircled{10}$	$\textcircled{11}$	12	13	14	15	$\textcircled{16}$	17

Figure 23: S_{31}: before the third kind of adjustment.

36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
18	19	20	21	$\textcircled{22}$	23	24	25	26	27	$\textcircled{28}$	29	30	31	32	33	34	35
0	1	2	$\textcircled{3}$	$\textcircled{4}$	$\textcircled{5}$	6	7	8	$\textcircled{9}$	$\textcircled{10}$	$\textcircled{11}$	12	13	14	$\textcircled{15}$	$\textcircled{16}$	17

Figure 24: S_{32}: after the third kind of adjustment.

Theorem 48. *Let S be a generalized-β-set that maximizes $f(S)$. If s is odd, then one of the following two cases holds:*

(1) $t = \frac{s-1}{2}$ and $a_{(t-1)m+1} = \cdots = a_{tm} = 1$;

(2) $t = \frac{s-1}{2}$, $a_{(t-1)m+1} = \cdots = a_{tm-1} = 2$ and $a_{tm} = 1$.

If s is even, then one of the following two cases holds:

(1) $t = \frac{s}{2}$ and $a_{(t-1)m} = \cdots = a_{tm-1} = 1$;

(2) $t = \frac{s}{2}$, $a_{(t-1)m+1} = \cdots = a_{tm-1} = 1$ and $a_{(t-1)m} = 2$.

Proof. First we discuss about the case where s is odd. By Corollary 42, we have $S \in \mathbb{F}_m(s)$. Define $S_k \in \mathbb{F}_m(s)$ as the corresponding generalized-β-set where $t = k$ and $a_{(t-1)m+1} = \cdots = a_{tm} = 1$, $1 \leqslant t \leqslant \frac{s-1}{2}$; $T_k \in \mathbb{F}_m(s)$ as the corresponding generalized-β-set where $t = k$ and $a_{(t-1)m+1} = \cdots = a_{tm} = 2$, $1 \leqslant t \leqslant \frac{s-3}{2}$; $P_k \in \mathbb{F}_m(s)$ as the corresponding generalized-β-set where $t = k$, $a_{(t-1)m+1} = \cdots = a_{tm-1} = 1$ and $a_{tm} = 0$, $1 \leqslant t \leqslant \frac{s-1}{2}$; and $Q_k \in \mathbb{F}_m(s)$ as the corresponding generalized-β-set where $t = k$, $a_{(t-1)m+1} = \cdots = a_{tm-1} = 2$ and $a_{tm} = 1$, $1 \leqslant t \leqslant \frac{s-1}{2}$. By the discussion in Lemma 44, we only need to compare $f(S_k), f(T_k), f(P_k)$ and $f(Q_k)$.

Given that $|S_k| = mk^2$, for $1 \leqslant k \leqslant \frac{s-3}{2}$, we have

$$\begin{aligned} f(S_{k+1}) - f(S_k) &= \sum_{x\in S_{k+1}} x - \sum_{x \in S_k} x - \sum_{i=|S_k|}^{|S_{k+1}|-1} i \\ &= (ms-1)mk^2 + \sum_{i=1}^{m}\sum_{j=2}^{2k+2}(is-j) - m(2k+1)mk^2 - \sum_{i=0}^{m(2k+1)-1} i \\ &\geqslant mk^2(m(s-2k-1)-1) + \sum_{i=0}^{m(2k+1)-1}(s-2k-2+i) - \sum_{i=0}^{m(2k+1)-1} i \\ &> 0. \end{aligned}$$

Therefore, $\left\{S_{\frac{s-1}{2}}\right\} = \arg\max_{1\leqslant k\leqslant (s-1)/2} f(S_k)$. With similar calculations, we have

$$\left\{T_{\frac{s-3}{2}}\right\} = \arg\max_{1\leqslant k\leqslant (s-3)/2} f(T_k);$$

$$\left\{P_{\frac{s-1}{2}}\right\} = \arg\max_{1\leqslant k\leqslant (s-1)/2} f(P_k);$$

$$\left\{Q_{\frac{s-1}{2}}\right\} = \arg\max_{1\leqslant k\leqslant (s-1)/2} f(Q_k).$$

Now we compare $f\left(S_{\frac{s-1}{2}}\right), f\left(T_{\frac{s-3}{2}}\right), f\left(P_{\frac{s-1}{2}}\right), f\left(Q_{\frac{s-1}{2}}\right)$.

For $1 \leqslant k \leqslant \frac{s-3}{2}$, set the first $k+1$ elements of each one of $\mathcal{B}_1(S_{k+1}),\cdots,\mathcal{B}_m(S_{k+1})$ as $l_1, \cdots, l_{m(k+1)}$. Then we have

$$\begin{aligned} f(S_{k+1}) - f(T_k) &= \sum_{x\in S_{k+1}} x - \sum_{x\in T_k} x - \sum_{i=|T_k|}^{|S_{k+1}|-1} i \\ &= ms\cdot\frac{1}{2}mk(k+1) + \sum_{i=1}^{m(k+1)} l_i - m(k+1)|T_k| - \sum_{i=1}^{m(k+1)}(i-1) \\ &= m^2k(k+1)\left(\left(\frac{s}{2}-k-1\right) + \sum_{i=1}^{m(k+1)}(l_i - i + 1)\right) \\ &> 0. \end{aligned}$$

Let $k = \frac{s-3}{2}$. Then $f\left(S_{\frac{s-1}{2}}\right) > f\left(T_{\frac{s-3}{2}}\right)$.

For $1 \leqslant k \leqslant \frac{s-1}{2}$, set the first k elements of each one of $\mathcal{B}_1(Q_k),\cdots,\mathcal{B}_m(Q_k)$ as

$l_1, \cdots, l_{mk}$. Given that $|P_k| = mk^2 - k$ and $|Q_k| = mk^2 + (m-1)k$, we have

$$\begin{aligned}
& f(Q_k) - f(P_k) \\
= \;& \sum_{x\in Q_k} x - \sum_{x\in P_k} x - \sum_{i=|P_k|}^{|Q_k|-1} i \\
= \;& ms\frac{m}{2}k(k-1) + \sum_{i=1}^{mk} l_i - mk(mk^2-k) - \sum_{i=0}^{mk-1} i \\
\geqslant \;& \frac{m^2 s}{2}k(k-1) + m\frac{k(k+1)}{2} + \frac{m(m-1)}{2}ks - mk(mk^2-k) - \frac{mk(mk-1)}{2} \\
= \;& \frac{mk}{2}((mk-1)s - 2mk^2 + (3-m)k + 2) \\
\geqslant \;& \frac{mk}{2}((mk-1)(2k+1) - 2mk^2 + (3-m)k + 2) \\
= \;& \frac{mk(k+1)}{2} > 0.
\end{aligned}$$

Let $k = \frac{s-1}{2}$. Then $f\left(Q_{\frac{s-1}{2}}\right) > f\left(P_{\frac{s-1}{2}}\right)$.

For $1 \leqslant k \leqslant \frac{s-1}{2}$, since $|S_k| = mk^2$ and $|Q_k| = mk^2 + (m-1)k$, we have

$$\begin{aligned}
& f(Q_k) - f(S_k) \\
= \;& \sum_{x\in Q_k} x - \sum_{x\in S_k} x - \sum_{i=|S_k|}^{|Q_k|-1} i \\
= \;& ms(m-1)\frac{k(k-1)}{2} + (m-1)\left(ks - \frac{k(k+1)}{2}\right) + \frac{(m-1)(m-2)}{2}ks \\
& -(m-1)kmk^2 - \frac{(m-1)k((m-1)k-1)}{2} \\
= \;& \frac{m(m-1)k^2}{2}(s-2k-1) \\
\geqslant \;& 0.
\end{aligned}$$

Let $k = \frac{s-1}{2}$. Then $f\left(Q_{\frac{s-1}{2}}\right) = f\left(S_{\frac{s-1}{2}}\right)$. From the discussion above, we know that $S_{\frac{s-1}{2}}$ and $Q_{\frac{s-1}{2}}$ are the only two generalized-β-sets that maximize $f(S)$.

Next, we discuss about the case where s is even. We first consider the case where $A_{t-1} - A_t \geqslant 2m$. Then by Lemma 40, $a_i - a_{i+m} = 2$ for $1 \leqslant i \leqslant (t-1)m$. Just like the former case, for $1 \leqslant k \leqslant \frac{s}{2} - 1$, we can define S_k, T_k, Q_k; for $1 \leqslant k \leqslant \frac{s}{2}$, we can define P_k. With similar discussions, we only need to compare $f\left(S_{\frac{s}{2}-1}\right), f\left(T_{\frac{s}{2}-1}\right), f\left(P_{\frac{s}{2}}\right), f\left(Q_{\frac{s}{2}-1}\right)$.

Since $f(Q_k) - f(S_k) = \frac{m(m-1)k^2}{2}(s-2k-1)$, we have $f\left(Q_{\frac{s}{2}-1}\right) > f\left(S_{\frac{s}{2}-1}\right)$. For

$1 \leqslant k \leqslant \frac{s}{2}-1$, since $|P_{k+1}| = mk^2+(2m-1)k+(m-1)$ and $|T_k| = mk(k+1)$, we have

$$\begin{aligned}
f(P_{k+1}) - f(T_k) &= \sum_{x\in P_{k+1}} x - \sum_{x\in T_k} x - \sum_{i=|T_k|}^{|P_{k+1}|-1} i \\
&= (m-1)\frac{k(k+1)}{2}ms + \sum_{i=1}^{m-1}\sum_{j=1}^{k+1}(is-j) - (m-1)(k+1)mk(k+1) \\
&\quad - \sum_{i=0}^{(m-1)(k+1)-1} i \\
&\geqslant \frac{mk}{2}(m-1)(k+1)(s-2k-2) + \sum_{i=0}^{(m-1)(k+1)-1}(s-k-1+i) \\
&\quad - \sum_{i=0}^{(m-1)(k+1)-1} i \\
&\geqslant (m-1)(k+1)\frac{s}{2} > 0.
\end{aligned}$$

Thus we have $f\left(P_{\frac{s}{2}}\right) > f\left(T_{\frac{s}{2}-1}\right)$. Similarly, we can prove that $f\left(P_{\frac{s}{2}}\right) > f\left(Q_{\frac{s}{2}-1}\right)$.

Therefore, $S = P_{\frac{s}{2}}$ is the only generalized-β-set that maximizes $f(S)$ when $A_{t-1}-A_t \geqslant 2m$. Let $P'_{\frac{s}{2}}$ be the corresponding generalized-β-set such that $t=\frac{s}{2}$, $a_{(t-1)m} = \cdots = a_{tm-1} = 1$ and $a_i - a_{i+m} = 2$ for $1 \leqslant i \leqslant (t-1)m-1$. By Lemma 46, $f\left(P_{\frac{s}{2}}\right) = f\left(P'_{\frac{s}{2}}\right)$, and we obtain that $P_{\frac{s}{2}}$ and $P'_{\frac{s}{2}}$ are the only two generalized-β-sets that maximize $f(S)$. □

Example 49. Figures 3 and 4 give an example where s is odd. Figures 5 and 6 give an example where s is even.

Finally, we can give the proof of the main result Theorem 1.

Proof of Theorem 1. By Lemma 22, we only need to consider the generalized-β-sets S that maximize $f(S)$.

When s is odd, $Q_{\frac{k-1}{2}} = \mathcal{L}_m(s)$. By Theorem 48, $S = Q_{\frac{k-1}{2}}$ or $S = S_{\frac{k-1}{2}}$. Both of them satisfy the conditions in Theorem 16. Thus they are both β-sets of certain $(s, ms-1, ms+1)$-core partitions. Let $M(s,m)$ be the corresponding partition of $\mathcal{L}_m(s)$. Since the conjugate of a partition has the same size as itself, by Lemma 18, $S_{\frac{k-1}{2}}$ is the β-set of the conjugate of $M(s,m)$.

Similarly, the conclusion is also true when s is even. □

9 Discussions

By extending the concept of β-sets to generalized-β-sets, we determine the possible structures of $(s, ms-1, ms+1)$-core partitions with the largest size, which proves the conjecture proposed by Nath and Sellers in [26]. Currently, most research results on simultaneous core partitions are about $(s_1, s_2, \ldots, s_m)$-core partitions with $m \leqslant 3$. When $m \geqslant 4$, it still lacks proper tools for studying such general simultaneous core partitions. We believe that the concepts and techniques related to generalized-β-sets introduced in this paper offer some insights for exploring statistics of general simultaneous core partitions, of which we know very little at this moment.

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