An Extension of Stanley's Symmetric Acyclicity Theorem to Signed Graphs

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Abstract

In 1995, Richard P. Stanley introduced the chromatic symmetric function X_G of a graph G and proved that, when written in terms of the elementary symmetric functions, it reveals the number of acyclic orientations of G with a given number of sinks. In this paper, we generalize this result to signed graphs, that is, to graphs whose edges are labeled with $+$ or $-$ and whose colorings and orientations can interact with their signs.

Additionally, we introduce a non-homogeneous basis which detects the number of sinks and which not only gives a Stanley-type result for signed graphs, but gives an analogous result of this form for unsigned graphs as well.

Mathematics Subject Classifications: 05C15, 05C22, 05C31

1 Introduction

A fundamental notion in the study of graphs is that of a *proper coloring* of a graph. This is a function which assigns a natural number to each vertex of a graph in such a way that no two vertices which are connected by an edge are assigned the same color. The *chromatic polynomial* $\chi_G(\lambda)$ of a graph G, is a polynomial in λ whose value is the number of ways to properly color a graph G in λ colors. In 1973, Richard P. Stanley [7, Corollary 1.3] proved that $|\chi_G(-1)|$ is the number of acyclic orientations of G, a surprising fact which extracts non-coloring information from the chromatic polynomial.

Stanley generalized his result in 1995 [8] by defining the *chromatic symmetric* function X_G of a graph G. If $\mathcal{P}(G)$ is the set of proper colorings of G in colors from $\mathbb{N}$, then

$$X_G(x_1, x_2, \dots) = \sum_{\kappa \in \mathcal{P}(G)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

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where $v_1, v_2, \ldots, v_n$ are the of vertices of G. Note that this makes X_G a formal power series over countably many variables $\{x_k : k \in \mathbb{N}\}$. The series X_G is called symmetric because for any permutation $\pi: \mathbb{N} \to \mathbb{N}$, we may observe that $X_G(x_{\pi(1)}, x_{\pi(2)}, \ldots) = X_G(x_1, x_2, \ldots)$. This allows us to state Stanley's result [8, Theorem 3.3]:

Theorem 1. *If the chromatic symmetric function of a graph G is written in terms of the elementary symmetric basis, then the number of acyclic orientations of G with k sinks is the sum of the coefficients of terms having k elementary symmetric function factors.*

Generalizing Stanley's work to *signed graphs*, Thomas Zaslavsky introduced the *chromatic polynomial* of a signed graph Σ and analogously proved that, when evaluated at -1, its absolute value is the number of acyclic orientations of Σ [9, Corollary 4.1]. In this paper, we will generalize both Zaslavsky's theorem on the chromatic polynomial and Stanley's theorem on the chromatic symmetric function to a result about the *chromatic B-symmetric function*, defined as follows.

Following Eric S. Egge [4], we define the chromatic B-symmetric function X_Σ of a signed graph Σ in direct analogy to the chromatic symmetric function, to be

$$X_\Sigma(\ldots, x_{-1}, x_0, x_1, \ldots) = \sum_{\kappa \in \mathcal{P}(\Sigma)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

where once again $v_1, v_2, \ldots, v_n$ are the vertices of Σ, and $\mathcal{P}(\Sigma)$ is the set of proper colorings of Σ, which we fully define in the next section. Any permutation $\pi: \mathbb{Z} \to \mathbb{Z}$ satisfying $\pi(k) = -\pi(-k)$ for all $k \in \mathbb{Z}$ will fix X_Σ, that is,

$$X_\Sigma(\ldots, x_{\pi(-1)}, x_{\pi(0)}, x_{\pi(1)}, \ldots) = X_\Sigma(\ldots, x_{-1}, x_0, x_1, \ldots).$$

Note that, differing from [4], we allow the vertex color 0, which is in line with Zaslavsky's work in [9]. In particular, for any permutation π which satisfies $\pi(k) = -\pi(-k)$, we must have $\pi(0) = 0$, which distinguishes 0 from the other colors. This group of permutations is isomorphic to a Coxeter group of type B; hence we call it the chromatic "B"-symmetric function, and generally define a function in $\mathbb{Z}$-indexed variables to be B-symmetric when it is fixed by all such permutations.

We will define a basis for B-symmetric functions which satisfies a Stanley-type result and we will call this the *augmented elementary B-symmetric basis*. Specifically, we have following main theorem.

Theorem 2. *If the chromatic B-symmetric function of a signed graph Σ is written in terms of the augmented elementary B-symmetric basis, then the number of acyclic orientations of Σ with k sinks is the sum of the coefficients of terms having k elementary symmetric function factors.*

Outline of the paper

In §2, we define various notions related to signed graphs which are used throughout. In §3, we state the main theorems of this paper and some corollaries. In §4, we introduce a deletion–contraction relation for the chromatic B-symmetric function, which we utilize in the proof of the main theorem.

The following two sections provide some technical results which constitute the heart of the proofs of the main theorems. In §5, we commence a detailed study of acyclic orientations of signed graphs and how they interact with proper colorings. In §6, we construct a very useful function φ which recovers information about the acyclic orientations of a signed graph from its chromatic B-symmetric function.

Finally, the proofs of the main theorems are given in §7, and proofs of some corollaries are described in §8.

2 Signed graphs

As defined by Zaslavsky [9], a *signed graph* is a graph whose edges have been labeled with either a plus sign or a minus sign (such edges are called positive or negative respectively). More specifically, we may think of a signed graph as a graph along with a function, sgn, which sends the edges of the graph to an element of the set $\{+,-\}$.

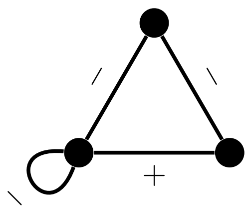


Figure 1: A signed graph with three vertices, three negative edges (one of which is a loop) and one positive edge.

As noted in [9], many definitions for unsigned graphs have analogous definitions which can be applied to signed graphs. Of particular importance here is the notion of what it means for a signed graph to be cyclic or acyclic. An *orientation* of a signed graph is a way of assigning an arrow to each half-edge (or equivalently, an arrow to each incidence between a vertex and an edge). Each arrow can point either toward or away from its vertex, subject to the condition that on a positive edge both arrows must point in the same direction and on a negative edge both arrows must point in opposite directions.

We will be primarily concerned with *acyclic* orientations. A cycle is a circuit such that every vertex in the circuit has at least one arrow pointing into it and one arrow pointing out of it, when only the edges in the circuit are considered. See Figure 4. An orientation is *acyclic* if it contains no cycles.

When a vertex has only arrows pointing toward it, it is called a *sink*. If it has only arrows pointing away from it, it is called a *source*.

We will also be considering *colorings* of signed graphs. A coloring of a signed graph Σ is a function from the vertex set of Σ to $\mathbb{Z}$. Using the notation $e\colon uv$ to denote that the endpoints of an edge e are the vertices u and v, we say that a coloring κ is *proper*

Figure 2: The four possible orientations of an edge.

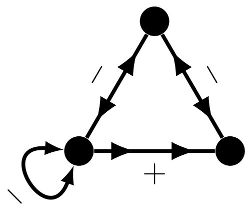


Figure 3: An oriented signed graph.

if for any $e\colon uv$, we have $\kappa(u) \neq \mathrm{sgn}(e) \cdot \kappa(v)$. Thus if two vertices are connected by a positive edge, then a proper coloring cannot assign them the same integer, and if they are connected by a negative edge, then a proper coloring cannot assign them integers which are negatives of each other.

Some more quick remarks can be made on the nature of signed graph colorings. If a signed graph Σ has a vertex with a positive loop, then it has no proper coloring. On the other hand, vertices with negative loops can be properly colored by any integer excluding zero. Finally, we note that a signed graph with all positive edges is virtually indistinguishable from an unsigned graph, and its chromatic B-symmetric function is, by re-indexing, identical to its chromatic symmetric function.

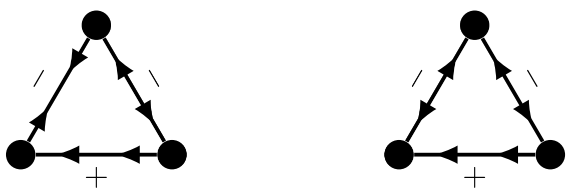


Figure 4: The orientation on the left is cyclic whereas the orientation on the right is acyclic. The acyclic orientation on the right has two sinks: the top vertex and the bottom left vertex.

3 Statements of the main theorems

Recall that as defined above, the chromatic B-symmetric function of a signed graph Σ is

$$X_\Sigma(\ldots, x_{-1}, x_0, x_1, \ldots) \coloneqq \sum_{\kappa \in \mathcal{P}(\Sigma)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)},$$

where $v_1, v_2, \ldots, v_n$ are the vertices of Σ and $\mathcal{P}(\Sigma)$ is the set of proper colorings of Σ. For notational convenience, we will put $x^\kappa \coloneqq x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$ whenever applicable.

We will use the notation $p_{a,b} \coloneqq \sum_{i \in \mathbb{Z}} x_i^a x_{-i}^b$ and refer to the set $\{p_{a,b} \colon a \geqslant 1, b \geqslant 0\} \cup \{x_0\}$ as the *p-basis* for the set of B-symmetric functions. It will be important for later to note that the p-basis is algebraically independent over $\mathbb{Q}$.

It turns out that X_Σ can always be written uniquely in terms of sums and products of elements of the p-basis.

Defining the *elementary symmetric functions* in the usual way,

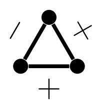
$$e_n \coloneqq \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

we can rewrite Newton's identities in the form $p_{n,0} = (-1)^{n+1} n e_n + \sum_{i=1}^{n-1} (-1)^{n+i-1} e_{n-i} p_{i,0}$. It is necessary to note that sums and products of the elementary symmetric functions do *not* span the space of B-symmetric functions, but only the subspace of symmetric functions in the variables $\{x_k \colon k \in \mathbb{Z}\}$.

Definition 3. For convenience we will put $q_{a,b} \coloneqq (-1)^{a+b+1} p_{a,b}$ and $z \coloneqq -x_0$. Now we have that any chromatic B-symmetric function can be written uniquely in terms of sums and products of elements of the set $\{e_n \colon n \geqslant 1\} \cup \{q_{a,b} \colon a, b \geqslant 1\} \cup \{z\}$. We will call this the *augmented elementary B-symmetric basis*.

Now we return to the main theorem stated in §1, whose proof is given in §7 (see Theorem 35).

Theorem 4. *If the chromatic B-symmetric function of a signed graph Σ is written in terms of the augmented elementary B-symmetric basis, then the number of acyclic orientations of Σ with k sinks is the sum of the coefficients of terms having k elementary symmetric function factors e_n.*

Example 5. Let Σ be the signed graph  .

As we will calculate later in Example 16, $X_\Sigma = p_{1,0}^3 - p_{1,0} p_{1,1} - 2p_{1,0} p_{2,0} + 2p_{2,1} + p_3 - x_0^3$. In the augmented elementary B-symmetric basis, this is $X_\Sigma = (e_1 e_2) + (3e_3 + q_{1,1} e_1) + (2q_{2,1} + z^3)$, and so Σ has 1 acyclic orientation with 2 sinks, $3 + 1 = 4$ acyclic orientations with, 1 sink and $2 + 1 = 3$ acyclic orientations with 0 sinks. Indeed, Σ has 8 orientations total.

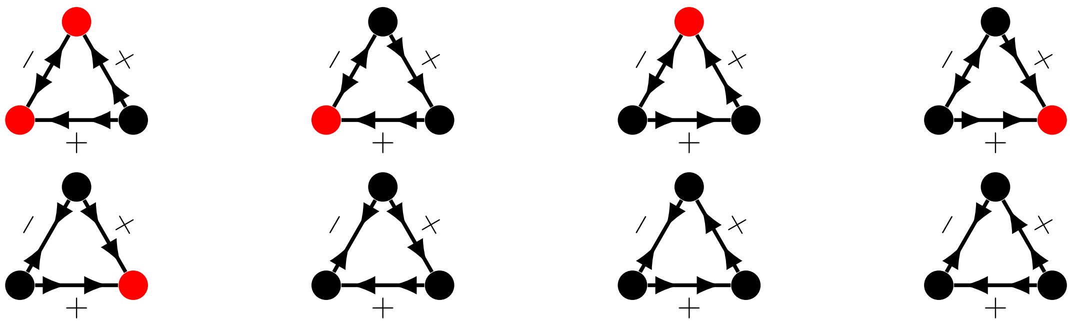
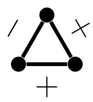


Figure 5: The 8 orientations of Σ, with sinks marked in red.

We may also consider a similar basis to arrive at a similar theorem, whose proof is also given in §7 (see Theorem 38).

Definition 6. Let $\xi_n := \sum_{a=1}^{n} \binom{n}{a} p_{a,0}$ for $n \geqslant 1$. Alternatively, we may write this as $p_{n,0} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{n-i} \xi_i$.

Theorem 7. *If the chromatic B-symmetric function of a signed graph Σ is written in terms of the basis $\{\xi_n : n \geqslant 1\} \cup \{q_{a,b} : a, b \geqslant 1\} \cup \{z\}$, then the number of acyclic orientations of Σ with k sinks is the sum of the coefficients of terms such that the sum of the indices of each ξ_n factor is equal to k.*

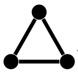
Example 8. Again, let Σ be the signed graph  .

We know that $X_\Sigma = p_{1,0}^3 - p_{1,0}p_{1,1} - 2p_{1,0}p_{2,0} + 2p_{2,1} + p_3 - x_0^3$. In the basis of Theorem 7, this is $X_\Sigma = (\xi_1\xi_1\xi_1 + \xi_3 - 2\xi_1\xi_2) + (4\xi_1\xi_2 - 3\xi_2) + (q_{1,1}\xi_1 + 3\xi_1) + (2q_{2,1} + z^3)$, and so Σ has $1 + 1 - 2 = 0$ acyclic orientations with 3 sinks, $4 - 3 = 1$ acyclic orientation with 2 sinks, $1 + 3 = 4$ acyclic orientations with 1 sink, and $2 + 1 = 3$ acyclic orientations with 0 sinks.

These theorems specialize to results about unsigned graphs. Specifically, we will show in §8 that Stanley's result about the elementary symmetric basis [8, Theorem 3.3] follows immediately from Theorem 4. Also, from Theorem 7 we obtain the following result, whose proof is given in §8 (see Corollary 39):

Definition 9. For $a > 0$, let $p_a := \sum_{i \geqslant 0} x_i^a$ and $\zeta_n := \sum_{a=1}^{n} \binom{n}{a} p_a$. Equivalently, we can say $p_a = \sum_{i=1}^{a} \binom{a}{i} (-1)^{a-i} \zeta_i$.

Corollary 10. *If the chromatic symmetric function of an unsigned graph G is written in terms of the basis $\{\zeta_n : n \geqslant 1\}$, then the number of acyclic orientations of G with k sinks is the sum of the coefficients of terms such that the sum of the indices of each ζ_n factor is equal to k.*

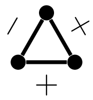
Example 11. Let G be the unsigned graph .

Then $X_G = p_1^3 - 3p_1p_2 + 2p_3$. This can also be written as

$$\begin{aligned} X_G &= \zeta_1^3 - 3\zeta_1(\zeta_2 - 2\zeta_1) + 2(\zeta_3 - 3\zeta_2 + 3\zeta_1) \\ &= (\zeta_1\zeta_1\zeta_1 - 3\zeta_1\zeta_2 + 2\zeta_3) + (6\zeta_1\zeta_1 - 6\zeta_2) + (6\zeta_1). \end{aligned}$$

So G has $1 - 3 + 2 = 0$ acyclic orientations with 3 sinks, $6 - 6 = 0$ acyclic orientations with 2 sinks, and 6 acyclic orientations with 1 sink.

Lastly, Theorem 7 also gives an equivalent form of Zaslavsky's result [9, Corollary 4.1]: the sum of the absolute value of the coefficients of X_Σ written in the p-basis is equal to the total number of acyclic orientations of Σ. Again, see §8 for a proof.

Example 12. Let Σ be the signed graph 

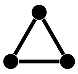
.

Then

$$X_\Sigma = p_{1,0}^3 - p_{1,0}p_{1,1} - 2p_{1,0}p_{2,0} + 2p_{2,1} + p_{3,0} - x_0^3.$$

So Σ has $|1| + |-1| + |-2| + |2| + |1| + |-1| = 8$ acyclic orientations.

We can also see that this holds for unsigned graphs as well:

Example 13. Let G be the unsigned graph .

Then

$$X_G = p_1^3 - 3p_1p_2 + 2p_3.$$

So G has $|1| + |-3| + |2| = 6$ acyclic orientations.

4 Weighted deletion–contraction

Now we will take a moment to consider how we might be able to calculate X_Σ.

The classical chromatic polynomial satisfies a relation known as "deletion–contraction" [7], which turns out to be immensely useful for both computation and theoretical considerations. Stanley's chromatic symmetric function satisfies a similar relation, but this time involving weighted graphs. This work appears in [1, 5] and was recently rediscovered in [3]. Due to unpublished work of James Enouen, Eric Fawcett, Rushil Raghavan, and Ishaan Shah (see [2]) we have a generalization of this for the chromatic B-symmetric function, namely a "doubly weighted" deletion–contraction rule for signed graphs.

For a signed graph Σ, a *double weight function*, $w = (w_+, w_-)$, will mean a pair of functions $w_+, w_- \colon V(\Sigma) \to \mathbb{N}$. For the doubly weighted signed graph (Σ, w), we define

$$X_{(\Sigma,w)} := \sum_{\kappa \in \mathcal{P}(G)} x_{\kappa(v_1)}^{w_+(v_1)} x_{-\kappa(v_1)}^{w_-(v_1)} \cdots x_{\kappa(v_n)}^{w_+(v_n)} x_{-\kappa(v_n)}^{w_-(v_n)}.$$

We observe that this function is B-symmetric and that this extends the chromatic B-symmetric function defined above if we treat an unweighted signed graph Σ as having a double weight function satisfying $w(v) = (1, 0)$ for all $v \in V(\Sigma)$.

To define the weighted deletion–contraction rule, we must have a notion of both deletion and contraction on signed graphs. The former is easy: given a (possibly doubly weighted) signed graph Σ and an edge e in Σ, we write $\Sigma \setminus e$ for the graph obtained by deleting the edge e. If $e\colon uv$ is a positive edge, the *contraction* of Σ along e is the graph Σ/e whose vertex set is $V(\Sigma)$ modulo the relation $u \sim v$, and whose edges are obtained from $E(\Sigma) \setminus \{e\}$ by replacing all endpoints u and v by the equivalence class $\{u, v\}$. If Σ has a double weight function w, then we consider the induced double weight $\widetilde{w}$ on G/e, which only differs from w by $\widetilde{w}(\{u, v\}) = w(u) + w(v)$ (where the addition of pairs is component-wise).

Theorem 14. *Let (Σ, w) be a doubly weighted signed graph, and suppose $e_0 \in E(G)$ is a positive edge, i.e.* $\operatorname{sgn}(e_0) = +$. *Then $X_{(\Sigma,w)} = X_{(\Sigma\setminus e_0,w)} - X_{(\Sigma/e_0,\widetilde{w})}$.*

Proof. We write the chromatic B-symmetric function as a sum over all (not necessarily proper) colorings $\kappa\colon V(\Sigma) \to \mathbb{Z}$ as follows:

$$X_{(\Sigma,w)} = \sum_{\kappa: V(\Sigma)\to\mathbb{Z}} \left(\prod_{v\in V(\Sigma)} x_{\kappa(v)}^{w_+(v)} x_{-\kappa(v)}^{w_-(v)} \prod_{\substack{e\in E(\Sigma)\\ e:uv}} \left(1 - \delta_{\kappa(u)}^{\operatorname{sgn}(e)\kappa(v)}\right) \right),$$

where δ_a^b is the Kronecker delta function, given by $\delta_a^b = 1$ if $a = b$ and $\delta_a^b = 0$ otherwise. Write $e_0\colon u_0v_0$ and expand across the factor $1 - \delta_{\kappa(u_0)}^{\kappa(v_0)}$ to get

$$\begin{aligned} X_{(\Sigma,w)} = &\sum_{\kappa: V(\Sigma)\to\mathbb{Z}} \left(\prod_{v\in V(\Sigma)} x_{\kappa(v)}^{w_+(v)} x_{-\kappa(v)}^{w_-(v)} \prod_{\substack{e\in E(\Sigma)\setminus\{e_0\}\\ e:uv}} \left(1 - \delta_{\kappa(u)}^{\operatorname{sgn}(e)\kappa(v)}\right) \right) \\ &- \sum_{\substack{\kappa:\Sigma\to\mathbb{Z}\\ \kappa(u_0)=\kappa(v_0)}} \left(\prod_{v\in V(\Sigma)} x_{\kappa(v)}^{w_+(v)} x_{-\kappa(v)}^{w_-(v)} \prod_{\substack{e\in E(\Sigma)\setminus\{e_0\}\\ e:uv}} \left(1 - \delta_{\kappa(u)}^{\operatorname{sgn}(e)\kappa(v)}\right) \right). \end{aligned}$$

But this is just the deletion–contraction rule we wanted to prove, since proper colorings graph after $e_0\colon u_0v_0$ has been contracted can be identified with colorings of the original graph that assign the same color to u_0 and v_0. ☐

Since this theorem deals only with positive edges, we will now consider a method of turning negative edges into positive edges under which the chromatic B-symmetric function is invariant. Given a doubly weighted signed graph (Σ, w) and a vertex $v \in V(\Sigma)$, the graph obtained from *switching* at v is the graph (Σ^v, w^v), where w^v differs only from w in that if $w(v) = (a, b)$ then $w^v(v) = (b, a)$, and Σ^v is the signed graph Σ except that all

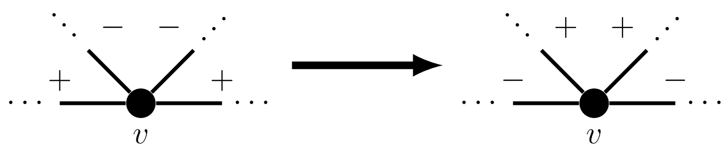


Figure 6: Switching a vertex v.

positive non-loop edges connected to v are now negative and all negative non-loop edges connected to v are now positive.

We now justify our claim above that the chromatic B-symmetric function is invariant under switching.

Lemma 15. *Let (Σ, w) be a doubly weighted signed graph, and let $v_0 \in V(\Sigma)$. Write (Σ^{v_0}, w^{v_0}) for the graph obtained by switching at (Σ, w) at v_0. Then $X_{(\Sigma,w)} = X_{(\Sigma^{v_0},w^{v_0})}$.*

Proof. We write

$$X_{(\Sigma^{v_0},w^{v_0})} = \sum_{\kappa:\Sigma\to\mathbb{Z}} \left(\prod_{v\in V(\Sigma)\setminus\{v_0\}} x_{\kappa(v)}^{w_+(v)} x_{-\kappa(v)}^{w_-(v)} \right) \left(x_{-\kappa(v_0)}^{w_+(v_0)} x_{\kappa(v_0)}^{w_-(v_0)} \right) \cdot \prod_{\substack{e\in E(G) \\ e:uv \\ v_0\notin\{u,v\}}} \left(1 - \delta_{\kappa(u)}^{\operatorname{sgn}(e)\kappa(v)}\right) \prod_{\substack{e\in E(G) \\ e:uv_0}} \left(1 - \delta_{\kappa(u)}^{-\operatorname{sgn}(e)\kappa(v_0)}\right).$$

If we reindex the sum by replacing each κ with the function $\kappa^{v_0} \colon \Sigma \to \mathbb{Z}$ differing from κ only by $\kappa^{v_0}(v_0) = -\kappa(v_0)$, we see that the whole expression is just X_G. □

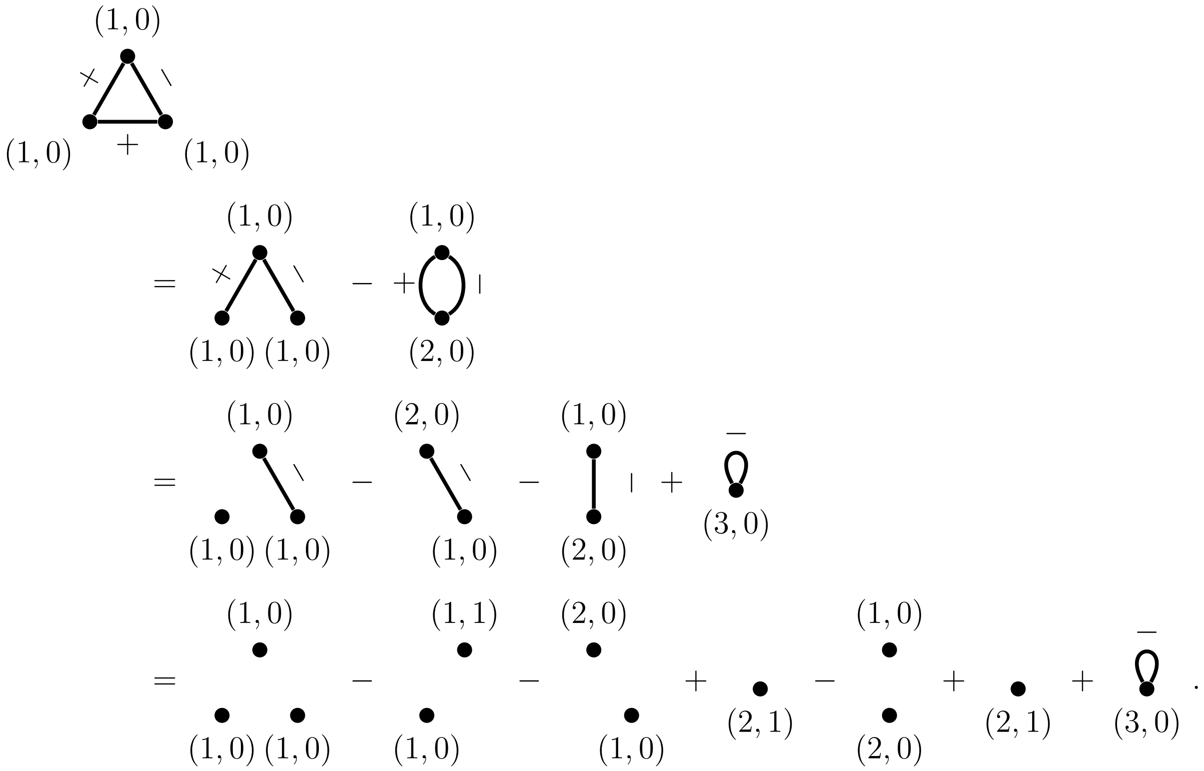
Now we can perform deletion–contraction on a graph, switching negative edges to positive when necessary, until we end up with disjoint vertices, some of which may have a negative loop. The chromatic B-symmetric function of isolated vertices is the product of the chromatic B-symmetric function of each of the individual vertices, and if (Σ, w) is a doubly weighted signed graph which consists of a single vertex of weight (a, b), then $X_{(\Sigma,w)} = \sum_{i\in\mathbb{Z}} x_i^a x_{-i}^b = p_{a,b}$. If this vertex has a negative loop, then we have

$$X_{(\Sigma,w)} = \sum_{i\in\mathbb{Z}\setminus\{0\}} x_i^a x_{-i}^b = \left(\sum_{i\in\mathbb{Z}} x_i^a x_{-i}^b \right) - x_0^{a+b} = p_{a,b} - x_0^{a+b}.$$

This gives a simple algorithm to write $X_{(\Sigma,w)}$, and hence X_Σ, in terms of the p-basis that was discussed in §3.

Example 16. Take Σ to be the triangle with one negative edge and two positive edges. For convenience we will write just a doubly weighted signed graph in place of writing the graph and its double weight function in the subscript of X.

Now, switching vertices as necessary, we have



And so

$$\begin{aligned} X_\Sigma &= p_{1,0}^3 - p_{1,0}p_{1,1} - p_{1,0}p_{2,0} + p_{2,1} - p_{1,0}p_{2,0} + p_{2,1} + p_{3,0} - x_0^3 \\ &= p_{1,0}^3 - p_{1,0}p_{1,1} - 2p_{1,0}p_{2,0} + 2p_{2,1} + p_{3,0} - x_0^3. \end{aligned}$$

5 Signed posets

5.1 The covering graph

In working with signed graphs, it is exceptionally helpful to consider a construction due to Zaslavsky called the *covering graph* (called "*signed* covering graph" in [9, Theorem 3.1]). Given a signed graph Σ, the covering graph of Σ, denoted $\overline{\Sigma}$, is an unsigned graph such that for every vertex v in Σ, there are vertices $+v$ and $-v$ in $\overline{\Sigma}$. Also, for every edge $e\colon uv$ in Σ, there are two edges, one between $+v$ and $\mathrm{sgn}(e)u$ and another one between $-v$ and $-\mathrm{sgn}(e)u$ in $\overline{\Sigma}$.

Additionally, given an orientation on Σ, we may induce an orientation on $\overline{\Sigma}$ such that for every arrow incident with a vertex v in Σ, the corresponding arrow is incident with $+v$ in $\overline{\Sigma}$ and the reverse arrow is incident with $-v$. It will be helpful to note that distinct orientations on Σ will induce distinct orientations on $\overline{\Sigma}$.

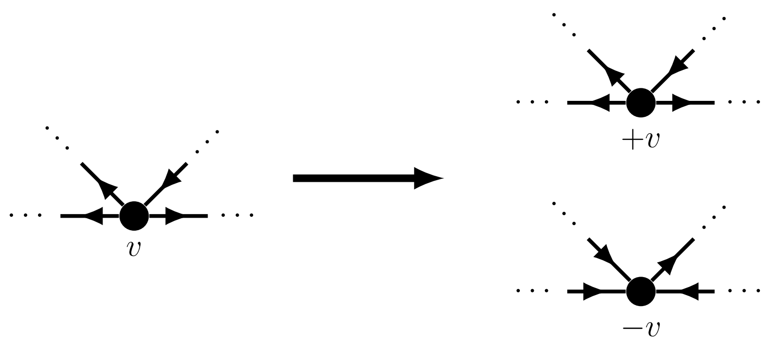


Figure 7: Creating an oriented covering graph.

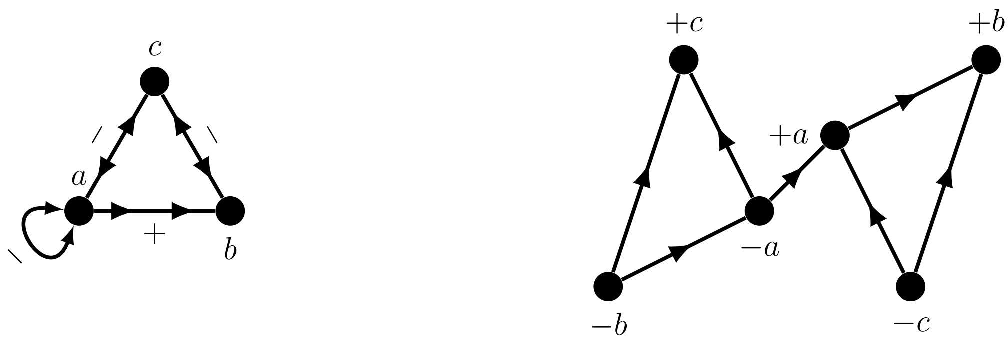


Figure 8: An oriented signed graph Σ on the left and its covering graph $\overline{\Sigma}$ with the induced orientation on the right.

On any given edge in $\overline{\Sigma}$ and for any orientation induced this way, both arrows on the edge will be pointing in the same direction, and so we will replace the two arrows on an edge with a single arrow pointing in their shared direction. See Figure 8.

5.2 Orientation-preserving colorings

Following Zaslavsky's lead [9], we will consider the connection between colorings and orientations. Each proper coloring of a signed graph Σ induces (or *preserves*) a unique acyclic orientation, and moreover each acyclic orientation can be induced from (or equivalently, is *preserved by*) infinitely many proper colorings. Given a proper coloring of Σ, we can construct an acyclic orientation of Σ by first constructing a coloring on the covering graph $\overline{\Sigma}$ such that for every vertex v in Σ, the vertex $+v$ in $\overline{\Sigma}$ has the same color as v, and the vertex $-v$ in $\overline{\Sigma}$ has the negative of the color of v. With this new coloring on $\overline{\Sigma}$, we create an orientation on $\overline{\Sigma}$ such that lower colors always point to higher colors, and we use this to give us an orientation on Σ. Proof that an orientation constructed this way is acyclic is given in [9].

For an orientation P of a signed graph Σ, we will we use $\mathcal{C}(P)$ to denote the set of all

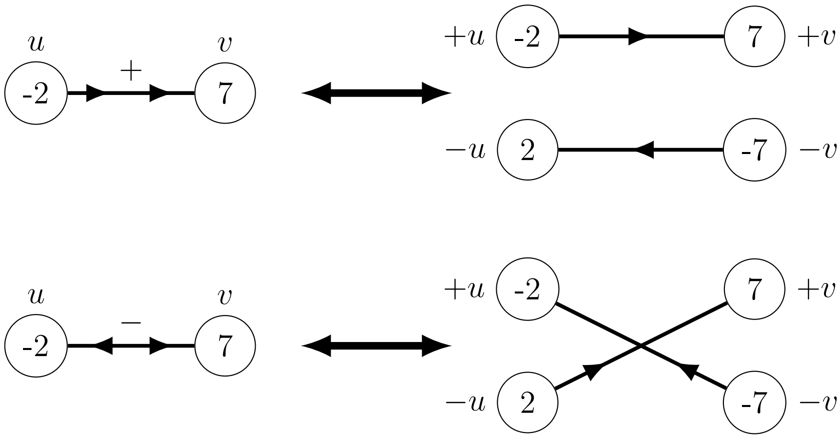


Figure 9: Two examples of colorings preserving orientations.

proper colorings of Σ which preserve P. Additionally, we will use $\mathcal{A}(\Sigma)$ to denote the set of acyclic orientations of Σ.

Now it is easy to see that

$$\mathcal{P}(\Sigma) = \bigsqcup_{P \in \mathcal{A}(\Sigma)} \mathcal{C}(P).$$

In other words, we can partition the set of all proper colorings of Σ into sets corresponding to the acyclic orientations of Σ.

If, for an acyclic orientation P of Σ, we put $Y_P \coloneqq \sum_{\kappa \in \mathcal{C}(P)} x^\kappa$, then this gives us that

$$X_\Sigma = \sum_{\kappa \in \mathcal{P}(\Sigma)} x^\kappa = \sum_{P \in \mathcal{A}(\Sigma)} \left(\sum_{\kappa \in \mathcal{C}(P)} x^\kappa \right) = \sum_{P \in \mathcal{A}(\Sigma)} Y_P.$$

5.3 Linear extensions of signed posets

It will be very useful to partition the proper colorings of Σ even further. In Stanley's treatment of the unsigned case [8], he does this by viewing acyclic oriented graphs as posets of their vertices and considering *linear extensions* (total orderings which contain all relations in the poset) of these posets. In analogy with this, we will consider a linear extension of an orientation of a signed graph to be a linear extension (in the unsigned sense) of its oriented covering graph such that if a vertex $+v$ is kth from the top of the total order, then $-v$ is kth from the bottom of the total order.

If P is an acyclic orientation of some signed graph, then we will call P a *signed poset*. For a signed poset P, we will use the notation $\varepsilon_1 v <_P \varepsilon_2 u$ to mean that $\varepsilon_1 v$ points to $\varepsilon_2 u$ in the oriented covering graph, where $\varepsilon_1, \varepsilon_2 \in \{+, -\}$. We will retain the convention that $\varepsilon_1, \varepsilon_2 \in \{+, -\}$ for the rest of the paper.

We will find it very convenient to treat linear extensions as functions. For a signed poset P on a graph Σ which has d vertices, we define a linear extension of P to be a

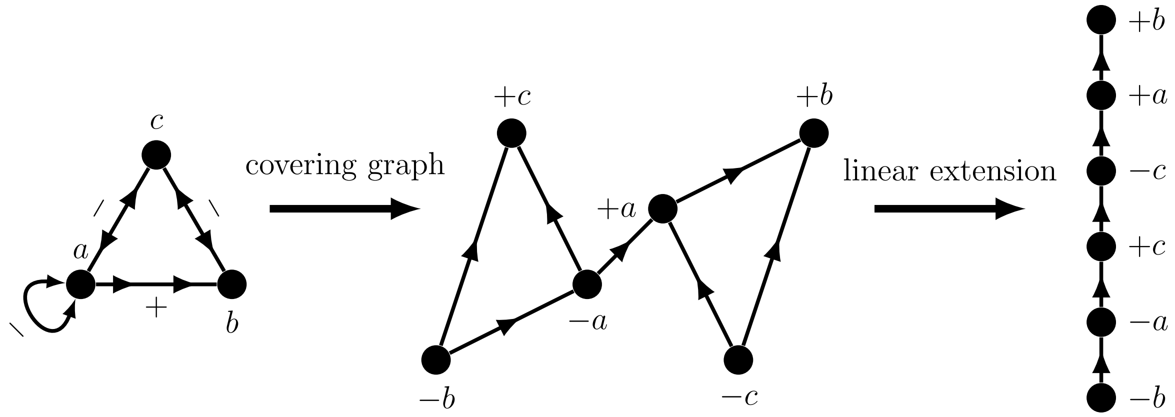


Figure 10: A signed graph, its corresponding covering graph, and one possible linear extension of the covering graph.

function α from the vertex set of $\overline{\Sigma}$ to the set $\{-d, -d+1, \ldots, -1, 1, \ldots, d-1, d\}$ such that $\varepsilon_1 u >_P \varepsilon_2 v \implies \alpha(\varepsilon_1 u) > \alpha(\varepsilon_2 v)$. Additionally we require that $\alpha(\varepsilon v) = \varepsilon\alpha(+v)$ for all v. We will also put $\operatorname{sgn}_\alpha(v) := \frac{|\alpha(+v)|}{\alpha(+v)}$.

Definition 17. For a signed poset P, let $\mathcal{L}(P)$ denote the set of all linear extensions of P.

Definition 18. For a proper coloring κ and a vertex v, let $\operatorname{sgn}_\kappa(v) := \frac{|\kappa(v)|}{\kappa(v)}$ when $\kappa(v) \neq 0$.

Definition 19. Given a signed poset P and $\alpha, \omega \in \mathcal{L}(P)$, define $\mathcal{K}(\alpha, \omega)$ to be the set of all colorings κ such that for all vertices u, v:

(4.6.1) If $|\alpha(\varepsilon_1 u)| < |\alpha(\varepsilon_2 v)|$, then $|\kappa(u)| \leqslant |\kappa(v)|$.

(4.6.2) $\operatorname{sgn}_\alpha(v) = \operatorname{sgn}_\kappa(v)$ or $\kappa(v) = 0$.

(4.6.3) If both $\alpha(\varepsilon_1 u) < \alpha(\varepsilon_2 v)$ and $\omega(\varepsilon_1 u) < \omega(\varepsilon_2 v)$ are true, then $\varepsilon_1\kappa(u) < \varepsilon_2\kappa(v)$.

Essentially, for $\kappa \in \mathcal{K}(\alpha, \omega)$, if $\alpha(\varepsilon_1 u) > \alpha(\varepsilon_2 v)$, then $\varepsilon_1\kappa(u) \geqslant \varepsilon_2\kappa(v)$, and this inequality is strict if ω agrees with α here. See Example 20.

It will be useful later to know that if (u_i) is a labeling of the vertices such that $|\alpha(u_1)| < |\alpha(u_2)| < \cdots < |\alpha(u_n)|$, then $\mathcal{K}(\alpha, \omega)$ is the set of all colorings such that

(1) If $i < j$, then $|\kappa(u_i)| \leqslant |\kappa(u_j)|$.

(2) $\operatorname{sgn}_\kappa(u_i) = \operatorname{sgn}_\alpha(u_i)$ or $\kappa(u_i) = 0$ for all i.

(3) If $i < j$ and $\omega(\operatorname{sgn}_\alpha(u_i)u_i) < \omega(\operatorname{sgn}_\alpha(u_j)u_j)$, then $\operatorname{sgn}_\alpha(u_i)\kappa(u_i) < \operatorname{sgn}_\alpha(u_j)\kappa(u_j)$.

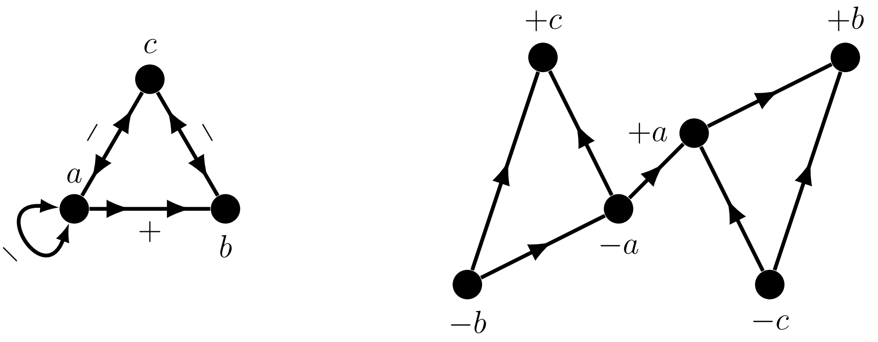
We can easily verify that for any $\omega \in \mathcal{L}(P)$,

$$\mathcal{C}(P) = \bigsqcup_{\alpha \in \mathcal{L}(P)} \mathcal{K}(\alpha, \omega).$$

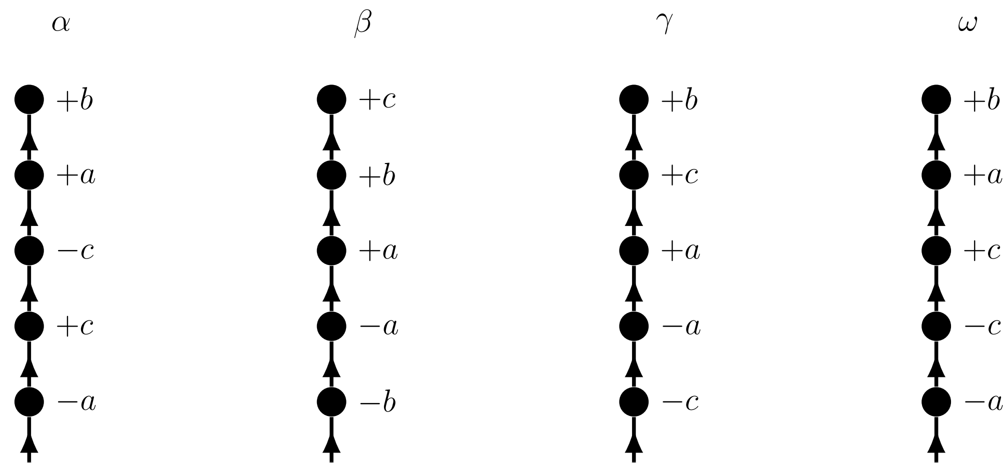
First we show $\mathcal{K}(\alpha_1, \omega)$ and $\mathcal{K}(\alpha_2, \omega)$ must be disjoint when $\alpha_1 \neq \alpha_2$. When $\alpha_1 \neq \alpha_2$, there must be some $\varepsilon_1 u$ and $\varepsilon_2 v$ in $\overline{\Sigma}$ such that $\alpha_1(\varepsilon_1 u) > \alpha_1(\varepsilon_2 v)$ but $\alpha_2(\varepsilon_1 u) < \alpha_2(\varepsilon_2 v)$. Without loss of generality, let $\omega(\varepsilon_1 u) > \omega(\varepsilon_2 v)$. Then any $\kappa \in \mathcal{K}(\alpha_1, \omega) \cap \mathcal{K}(\alpha_2, \omega)$ must satisfy both $\kappa(\varepsilon_1 u) > \kappa(\varepsilon_2 v)$ by (4.6.3) and $\kappa(\varepsilon_1 u) \leqslant \kappa(\varepsilon_2 v)$ by (4.6.1) and (4.6.2).

Now we show that for any $\omega \in \mathcal{L}(P)$ and for any κ which preserves P, there exists $\alpha \in \mathcal{L}(P)$ such that κ is contained in $\mathcal{K}(\alpha, \omega)$. Simply construct α such that for all u and v in Σ, if $|\kappa(u)| < |\kappa(v)|$, then $|\alpha(+u)| < |\alpha(+v)|$, and if $|\kappa(u)| = |\kappa(v)|$, then α does the opposite of what ω does, i.e. $|\alpha(+u)| > |\alpha(+v)|$ iff $|\omega(+u)| < |\omega(+v)|$. Also put $\mathrm{sgn}_\alpha(v) = \mathrm{sgn}_\kappa(v)$ when $\kappa(v) \neq 0$ and if $\kappa(v) = 0$ then put $\mathrm{sgn}_\alpha(v) = -\mathrm{sgn}_\omega(v)$.

Example 20. Consider the following signed poset P and its (oriented) covering graph:



This oriented covering graph has four possible linear extensions, which we will call α, β, γ, and ω:



Then we can calculate

- $\mathcal{K}(\alpha, \omega)$ is the set of all colorings κ with $|\kappa(b)| \geqslant |\kappa(a)| \geqslant |\kappa(c)|$, $\kappa(b), \kappa(a) \geqslant 0$, $\kappa(c) \leqslant 0$, and also the following relations:

$$\begin{array}{lllll} -\kappa(b) < -\kappa(a), & -\kappa(b) < \kappa(c), & -\kappa(b) < -\kappa(c), & -\kappa(b) < \kappa(a), & -\kappa(b) < \kappa(b), \\ -\kappa(a) < \kappa(c), & -\kappa(a) < -\kappa(c), & -\kappa(a) < \kappa(a), & -\kappa(a) < \kappa(b), & \\ \kappa(c) < -\kappa(c), & \kappa(c) < \kappa(a), & \kappa(c) < \kappa(b), & & \\ -\kappa(c) < \kappa(a), & -\kappa(c) < \kappa(b), & & & \\ \kappa(a) < \kappa(b). & & & & \end{array}$$

  It turns out that most of these relations are redundant, and the only relations which give us more information are $\kappa(a) < \kappa(b)$ and $-\kappa(c) < \kappa(a)$.

  This means that $\mathcal{K}(\alpha, \omega) = \{\kappa \colon \kappa(b) > \kappa(a) > -\kappa(c) \geqslant 0\}$.

- We can note that the colorings of $\mathcal{K}(\beta, \omega)$ are those with $|\kappa(c)| \geqslant |\kappa(b)| \geqslant |\kappa(a)|$, $\kappa(b), \kappa(b), \kappa(a) \geqslant 0$, and the extra relations that $\kappa(a) < \kappa(b)$ and $-\kappa(a) < \kappa(a)$ (since all of the others are redundant). So $\mathcal{K}(\beta, \omega) = \{\kappa \colon \kappa(c) \geqslant \kappa(b) > \kappa(a) > 0\}$.

- We may similarly calculate that $\mathcal{K}(\gamma, \omega) = \{\kappa \colon \kappa(b) > \kappa(c) \geqslant \kappa(a) > 0\}$.

- Lastly, $\mathcal{K}(\omega, \omega) = \{\kappa \colon \kappa(b) > \kappa(a) > \kappa(c) > 0\}$

We can note that these sets are pairwise disjoint, each coloring contained in one of these sets preserves P, and every coloring which preserves P is contained in one of these sets.

For $\alpha, \omega \in \mathcal{L}(P)$, let us put $F_{\alpha,\omega} := \sum_{\kappa \in \mathcal{K}(\alpha,\omega)} x^\kappa$, so that for any fixed $\omega \in \mathcal{L}(P)$, we have $Y_P = \sum_{\alpha \in \mathcal{L}(P)} F_{\alpha,\omega}$. Then for any fixed $\omega \in \mathcal{L}(P)$, we have that

$$X_\Sigma = \sum_{P \in \mathcal{A}(\Sigma)} Y_P = \sum_{P \in \mathcal{A}(\Sigma)} \left(\sum_{\alpha \in \mathcal{L}(P)} F_{\alpha,\omega} \right).$$

6 A sink-counting function

We are considering the $F_{\alpha,\omega}$ so that we may define a linear function φ with the property that $\varphi(Y_P) = t^{\mathrm{sink}(P)}$. Here $\mathrm{sink}(P)$ denotes the number of vertices which are sinks under the orientation P.

6.1 Preliminary facts

First we need some preliminary facts about the $F_{\alpha,\omega}$. Let $\varnothing_n$ denote the trivial orientation on the signed graph with n vertices and no edges, so that $\mathcal{L}(\varnothing_n)$ contains the linear extensions of any signed graph with n vertices. For fixed $\omega \in \mathcal{L}(\varnothing_n)$, the set $\{F_{\alpha,\omega} \colon \alpha \in \mathcal{L}(\varnothing_n)\}$ is linearly independent. Before this fact is proven, it should be noted that for fixed ω, there may exist $\alpha \neq \beta$ such that $F_{\alpha,\omega} = F_{\beta,\omega}$.

Example 21. Let $\omega \in \mathcal{L}(\varnothing_2)$ be such that $\omega(+v_1) = 1$ and $\omega(+v_2) = 2$, take α such that $\alpha(+v_1) = -1$ and $\alpha(+v_2) = 2$, and take β such that $\beta(+v_2) = -1$ and $\beta(+v_1) = 2$. Then it can be computed that $F_{\alpha,\omega} = \sum_{0 \leqslant i_1 < i_2} x_{-i_1} x_{i_2} = F_{\beta,\omega}$.

We will also require an algebraic fact about the $F_{\alpha,\omega}$. We say that a power series with non-negative coefficients A is a *sub-sum* of a power series with non-negative coefficients B if $B - A$ has non-negative coefficients. Equivalently, this means that every term which appears in A also appears in B with an equal or larger coefficient. It is easily seen that if A is a sub-sum of B and B is a sub-sum of C, then A is a sub-sum of C, and additionally if A is a sub-sum of B and B is a sub-sum of A, then $A = B$.

Lemma 22. *If $F_{\alpha,\omega}$ is a sub-sum of $\sum_{i=1}^{k} F_{\beta_i,\omega}$, then $F_{\alpha,\omega}$ is a sub-sum of $F_{\beta_i,\omega}$ for some i.*

Proof. To see this, consider a term $x_{a_1}^{b_1} x_{a_2}^{b_2} \dots x_{a_k}^{b_k}$ in $F_{\alpha,\omega}$ which has minimal length k and with $0 < a_1 < a_2 < \cdots < a_k$. This term corresponds to a coloring in $\mathcal{K}(\alpha,\omega)$, call it κ, for which the least number of colors are used. This term must also exist in $\sum_{i=1}^{k} F_{\beta_i,w}$, and so there is some $F_{\beta_i,\omega}$ which contains $x_{a_1}^{b_1} x_{a_2}^{b_2} \dots x_{a_k}^{b_k}$, coming from a coloring $\kappa' \in \mathcal{K}(\beta_i,\omega)$.

Let (v_j) be a labeling of the vertices such that $|\alpha(v_1)| < \cdots < |\alpha(v_n)|$, and let (u_j) be a labeling of the vertices such that $|\beta_i(u_1)| < \cdots < |\beta_i(u_n)|$. Let θ be the function such that $\theta(v_j) = u_j$ for all j. Then $\kappa' = \kappa \circ \theta$, and moreover, by considering condition (3) in the definition of $\mathcal{K}(\alpha,\omega)$, we can see that any other coloring in $\mathcal{K}(\alpha,\omega)$ will become a coloring in $\mathcal{K}(\beta_i,\omega)$ when pre-composed with θ. This is because any other coloring in $\mathcal{K}(\alpha,\omega)$ will use at least as many colors as κ does, and so when another coloring is pre-composed with θ, it will satisfy the conclusion of condition (3) in the definition of $\mathcal{K}(\beta_i,\omega)$ whenever $\kappa \circ \theta$ does. Therefore $F_{\alpha,\omega}$ is a sub-sum of $F_{\beta_i,\omega}$. □

Lemma 23. *For fixed $\omega \in \mathcal{L}(\varnothing_n)$, the set $\{F_{\alpha,\omega} \colon \alpha \in \mathcal{L}(\varnothing_n)\}$ is linearly independent over $\mathbb{Q}$.*

Proof. Suppose that for some ω, this set is linearly dependent. Then some non-trivial linear combination of elements of this set is equal to zero. After multiplying by constants, rearranging terms, and writing terms of the form $n \cdot F_{\alpha,\omega}$ as $\sum_{k=1}^{n} F_{\alpha,\omega}$, the linear dependence equation becomes $\sum_{i=1}^{m} F_{\gamma_i,\omega} = \sum_{i=1}^{\ell} F_{\beta_i,\omega}$. We may also suppose that this linear dependence is minimal, i.e. all equal terms have been canceled, and so $F_{\beta_i,\omega} \neq F_{\gamma_j,\omega}$ for all i, j.

Pick a term on the right hand side which is not a sub-sum of any other term on the right hand side, except for terms it is equal to. There is a term like this because there are only finitely many terms. Without loss of generality, $F_{\beta_1,\omega}$ is such a term. Since $F_{\beta_1,\omega}$ is a sub-sum of the left hand side, it is a sub-sum of a particular term. Without loss of generality, $F_{\beta_1,\omega}$ is a sub-sum of $F_{\gamma_1,\omega}$. Since $F_{\gamma_1,\omega}$ is a sub-sum of the right hand side, it is a sub-sum of a particular term. Without loss of generality, $F_{\gamma_1,\omega}$ is a sub-sum of $F_{\beta_2,\omega}$. But this means that $F_{\beta_1,\omega}$ is also a sub-sum of $F_{\beta_2,\omega}$ and therefore $F_{\beta_1,\omega} = F_{\beta_2,\omega}$ by assumption. Then it must be that $F_{\beta_1,\omega} = F_{\gamma_1,\omega}$ since each is a sub-sum of the other. This is a contradiction since we assumed that this linear dependence was minimal. Therefore $\{F_{\alpha,\omega} \colon \alpha \in \mathcal{L}(\varnothing_n)\}$ is linearly dependent for each $\omega \in \mathcal{L}(\varnothing_n)$. □

6.2 An auxiliary function

Define $V_\omega \coloneqq \mathrm{Span}(\{F_{\alpha,\omega} \colon \alpha \in \mathcal{L}(\varnothing_n)\})$. For fixed ω, we will define a linear function $\varphi_\omega \colon V_\omega \to \mathbb{Q}[t]$. Given $\alpha \in \mathcal{L}(\varnothing_n)$, let (v_i) be a relabeling of the vertices such that $|\alpha(+v_1)| < \cdots < |\alpha(+v_n)|$. Also, let $\varepsilon_i \coloneqq \mathrm{sgn}_\alpha(v_i)$. Then define φ_ω such that

$$\varphi_\omega(F_{\alpha,\omega}) \coloneqq \begin{cases} t(t-1)^k & \text{if } \omega(\varepsilon_i v_i) > \omega(\varepsilon_j v_j) \text{ for all } i,j \text{ with } n-k \leqslant i < j \leqslant n, \\ & 0 < \omega(\varepsilon_i v_i) < \omega(\varepsilon_j v_j) \text{ for all } i,j \text{ with } 1 \leqslant i < j \leqslant n-k, \\ & \text{ and } \varepsilon_i = + \text{ for } n-k \leqslant i \leqslant n, \text{ for } 0 \leqslant k < n. \\ (t-1)^k & \text{if } \omega(\varepsilon_i v_i) > \omega(\varepsilon_j v_j) \text{ for all } i,j \text{ with } n-k \leqslant i < j \leqslant n, \\ & 0 < \omega(\varepsilon_i v_i) < \omega(\varepsilon_j v_j) \text{ for all } i,j \text{ with } 1 \leqslant i < j \leqslant n-k, \\ & \varepsilon_i = + \text{ for } n-k < i \leqslant n \text{ and } \varepsilon_{n-k} = -, \text{ for } 0 \leqslant k < n. \\ (t-1)^n & \text{if } 0 > \omega(+v_1) > \omega(+v_2) > \cdots > \omega(+v_n) \text{ and } \varepsilon_i = + \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

It should be noted that the first case corresponds to the situation where largest element under ω is positive, namely it is $+v_{n-k}$, where α places $k-1$ positive elements above $+v_{n-k}$ and where the ordering from $\varepsilon_1 v_1$ to $+v_{n-k}$ agrees with ω, i.e. $\alpha(\varepsilon_1 v_1) < \alpha(\varepsilon_2 v_2)$ and $\omega(\varepsilon_1 v_1) < \alpha(\varepsilon_2 v_2)$ and so on.

The second case corresponds to an almost identical situation to the first case, except that the maximal element under ω is negative. The third case is the natural interpretation of the second case (i.e. $v_0 = 0$) with $k = n$.

Currently it is not clear that φ_ω is well defined. To show that it is, it suffices to show that whenever we have $F_{\alpha,\omega} = F_{\beta,\omega}$, we also have $\varphi_\omega(F_{\alpha,\omega}) = \varphi_\omega(F_{\beta,\omega})$.

Suppose we have $F_{\alpha,\omega} = F_{\beta,\omega}$ for some $\alpha, \beta \in \mathcal{L}(\varnothing_n)$. Let (v_i) be a labeling of the vertices such that $|\alpha(+v_1)| < \cdots < |\alpha(+v_n)|$, and let $\varepsilon_i \coloneqq \mathrm{sgn}_\alpha(v_i)$ for each i.

Let (u_i) be a analogous relabeling of the vertices such that $|\beta(+u_1)| < \cdots < |\beta(+u_n)|$. Since $F_{\alpha,\omega} = F_{\beta,\omega}$, we have that $\mathrm{sgn}_\beta(u_i) = \mathrm{sgn}_\alpha(v_i) = \varepsilon_i$ for all i.

It must also be that any coloring in $\mathcal{K}(\alpha,\omega)$ when pre-composed with the map $u_j \mapsto v_j$ becomes a coloring in $\mathcal{K}(\alpha,\omega)$, and moreover, this defines a bijection between $\mathcal{K}(\alpha,\omega)$ and $\mathcal{K}(\beta,\omega)$.

Pick any i and j with $i < j$. We want to show that $\omega(\varepsilon_i v_i) > \omega(\varepsilon_j v_j)$ iff $\omega(\varepsilon_i u_i) > \omega(\varepsilon_j u_j)$.

Suppose that $\omega(\varepsilon_i v_i) > \omega(\varepsilon_j v_j)$. Then there exists $\kappa \in \mathcal{K}(\alpha,\omega)$ with $\varepsilon_i \kappa(v_i) = \varepsilon_j \kappa(v_j)$. This also means that there exists $\kappa' \in \mathcal{K}(\beta,\omega)$ with $\varepsilon_i \kappa'(v_i) = \varepsilon_j \kappa'(v_j)$, and hence that $\omega(\varepsilon_i u_i) > \omega(\varepsilon_j u_j)$. By symmetry, we have the other direction, so that $\omega(\varepsilon_i v_i) > \omega(\varepsilon_j v_j)$ iff $\omega(\varepsilon_i u_i) > \omega(\varepsilon_j u_j)$, and therefore $\omega(\varepsilon_i v_i) < \omega(\varepsilon_j v_j)$ iff $\omega(\varepsilon_i u_i) < \omega(\varepsilon_j u_j)$. This fact makes it clear that $\varphi_\omega(F_{\alpha,\omega}) = \varphi_\omega(F_{\beta,\omega})$.

6.3 Sink-counting

Lemma 24. *If P is a signed poset and ω any linear extension of P, then $\varphi_\omega(Y_P) = t^{\mathrm{sink}(P)}$.*

Proof. This proof is largely based on the proof of Theorem 3.3 in [8].

We will prove the lemma in each of the cases in the definition of φ. We may do this since if α is a linear extension of P such that $F_{\alpha,\omega}$ falls into the first case of φ_ω, then $F_{\beta,\omega}$ will fall into the first case or the zero case of φ_ω for all other β which are linear extensions of P.

First, consider the case where the largest element under ω is positive, say $+s$, and so the vertex s is a sink in P. Now select any k of the remaining sinks of P other than s (of which there are $\text{sink}(P) - 1$ to choose from), and call these vertices $u_1, u_2, \ldots, u_k$.

The remaining $n - k - 1$ vertices will be labeled $v_1, v_2, \ldots, v_{n-k-1}$ in such a way that $\big|\omega(v_i)\big| < \big|\omega(v_j)\big|$ iff $i < j$.

Now consider the linear extension α which satisfies

(1) $\alpha(u_i) = n - i + 1$,

(2) $\alpha(s) = n - k$,

(3) for $i < n - k$, $\big|\alpha(v_i)\big| = i$, and $\text{sgn}_\alpha(v_i) = \text{sgn}_\omega(v_1)$.

Hence this definition places $+u_1$ as the largest element under α, $+u_2$ as the second largest, and so on. Then $+s$ is the greatest element below $+u_k$, and after $+s$, the vertices are arranged as their ordering in ω and each with the same sign as in ω.

Note that after initially choosing k sinks, no more choices are made, meaning that any α constructed this way is unique when given a choice of k sinks.

To see that α is a linear extension of P, we will examine every possible pair of vertices, and determine that α respects the relation between them, if present in P.

For any two vertices $v_i, v_j \in \{v_1, \ldots, v_{n-k-1}\}$, it follows that $\big|\alpha(v_i)\big| < \big|\alpha(v_j)\big|$ iff $\big|\omega(v_i)\big| < \big|\omega(v_j)\big|$, with the signs of these vertices under α being the same as under ω. Since ω respects all relations of P, this means α respects any relations between v_i and v_j present in P.

So too α respects relations between any $p, q \in \{u_1, \ldots, u_k\} \cup \{s\}$. Since they are all sinks of P, there is no directed positive edge or inward facing directed negative edge between any two of them. If there is an outward facing negative edge between two of them, this translates to the relation $p >_P -q$. But this relation also holds in α since $\alpha(+p) > 0$ for any $p \in \{u_1, \ldots, u_k\} \cup \{s\}$. This logic holds for any two vertices that were sinks in P, not just elements $u_1, \ldots, u_k, s$. So the only consideration left is that of a sink and non-sink.

Call the sink p and the non-sink r. Then the edges possibly present in P are a directed positive edge from r to p or an inward facing directed negative edge. These directed edges invoke the relations $p >_P r$ and $p >_P -r$, respectively. In either case, we can see that α satisfies these relations as well since $\big|\alpha(+p)\big| > \big|\alpha(+r)\big|$ by construction.

Therefore, α is a linear extension of P.

So for any $k < \text{sink}(P)$, and for each choice of k sinks of P (other than s), there is exactly one linear extension α which satisfies the first case of φ. This means that there are $N = \binom{\text{sink}(P)-1}{k}$ linear extensions $\alpha_1, \ldots, \alpha_N$ for which $\varphi(F_{\alpha_i,\omega}) = t(t-1)^k$. This holds for all $k \leqslant \text{sink}(P) - 1$, and any α not of this form must have $\varphi(F_{\alpha,\omega}) = 0$ by uniqueness of α.

Hence we obtain

$$\varphi(Y_P) = \varphi\left(\sum_{\alpha \in \mathcal{L}(P)} F_{\alpha,\omega}\right) = \sum_{k=0}^{\text{sink}(P)-1} \binom{\text{sink}(P) - 1}{k} t(t-1)^k$$
$$= t \cdot \sum_{k=0}^{\text{sink}(P)-1} \binom{\text{sink}(P) - 1}{k} (t-1)^k = t \cdot t^{\text{sink}(P)-1} = t^{\text{sink}(P)}.$$

This proves the theorem in the case where the maximal element under ω is positive.

In the case where the maximal element under ω is negative the argument is almost identical, except that there are $\text{sink}(P)$ vertices to choose from when selecting $u_1, \dots, u_k$, since s is a source in P. The verification that the previous construction still works is straightforward. The third case of φ occurs when $\text{sink}(P) = n$ and can be included in the construction.

So for any $k \leqslant \text{sink}(P)$, and for each choice of k sinks of P, there is exactly one linear extension α which satisfies the second or third case of φ. Therefore there are $N = \binom{\text{sink}(P)}{k}$ linear extensions $\alpha_1, \dots, \alpha_N$ for which $\varphi(F_{\alpha_i,\omega}) = (t-1)^k$. This holds for all $k \leqslant \text{sink}(P)$, and any α not of this form must have $\varphi(F_{\alpha,\omega}) = 0$ by uniqueness of α.

Hence we obtain

$$\varphi(Y_P) = \varphi\left(\sum_{\alpha \in \mathcal{L}(P)} F_{\alpha,\omega}\right)$$
$$= \sum_{k=0}^{\text{sink}(P)} \binom{\text{sink}(P)}{k} (t-1)^k = t^{\text{sink}(P)}.$$

This completes the proof. □

6.4 Constructing the sink-counting function

First we will define

$$\mathbb{Y} := \text{Span}(\{Y_P \colon P \text{ is a signed poset}\})$$

and note that $\mathbb{Y}$ is a subspace of $\bigoplus_{n=1}^{\infty} \sum_{\omega \in \mathcal{L}(\varnothing_n)} V_\omega$ since every Y_P can be written as a sum of the $F_{\alpha,\omega}$.

For fixed $n \in \mathbb{N}$ and for each $\omega \in \mathcal{L}(\varnothing_n)$, we have a linear function φ_ω, and moreover, these functions have the property that $\varphi_{\omega_1}|_{V_{\omega_1} \cap V_{\omega_2} \cap \mathbb{Y}} = \varphi_{\omega_2}|_{V_{\omega_1} \cap V_{\omega_2} \cap \mathbb{Y}}$ for any ω_1, ω_2. So we may define our desired function $\varphi \colon \mathbb{Y} \to \mathbb{Q}[t]$ to agree with the φ_ω. More specifically, let B be a basis for $\mathbb{Y}$ with the property that any element of B is an element of V_ω for some ω. Then for any $b \in B$, pick ω such that $b \in V_\omega$, and define $\varphi(b) = \varphi_\omega(b)$. Our choice of ω is not unique, but φ does not depend on choices of ω since φ_{ω_1} and φ_{ω_2} agree on $\mathbb{Y}$ when applicable. Therefore we have a well defined linear function $\varphi \colon \mathbb{Y} \to \mathbb{Q}[t]$ such that $\varphi(Y_P) = t^{\text{sink}(P)}$ for any signed poset P.

7 Proofs of the main theorems

We begin with several auxiliary calculations (Lemmas 25–33).

Lemma 25. *Let Σ be a signed graph, and let $\operatorname{acyc}_\Sigma(k)$ denote the number of acyclic orientations of Σ which have k sinks. Then $\varphi(X_\Sigma) = \sum_{k=0}^{\infty} \operatorname{acyc}_\Sigma(k) t^k$.*

Proof. We know that $X_\Sigma = \sum_{P\in\mathcal{A}(\Sigma)} Y_P$, so we may apply φ to arrive at

$$\varphi(X_\Sigma) = \sum_{P\in\mathcal{A}(\Sigma)} \varphi(Y_P) = \sum_{P\in\mathcal{A}(\Sigma)} t^{\operatorname{sink}(P)}.$$

After counting terms, we have $\sum_{P\in\mathcal{A}(\Sigma)} t^{\operatorname{sink}(P)} = \sum_{k=0}^{\infty} \operatorname{acyc}_\Sigma(k) t^k$, as desired. □

Lemma 26. *If $f, g \in \mathbb{Y}$, then $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$.*

Proof. If P_1 and P_2 are disjoint signed posets, then $Y_{P_1 \sqcup P_2} = Y_{P_1} \cdot Y_{P_2}$. Since also $\operatorname{sink}(P_1 \sqcup P_2) = \operatorname{sink}(P_1) + \operatorname{sink}(P_2)$, we can see that

$$\varphi(Y_{P_1}) \cdot \varphi(Y_{P_2}) = t^{\operatorname{sink}(P_1)} \cdot t^{\operatorname{sink}(P_2)} = t^{\operatorname{sink}(P_1)+\operatorname{sink}(P_2)} = \varphi(Y_{P_1\sqcup P_2}) = \varphi(Y_{P_1} \cdot Y_{P_2}).$$

Since $\mathbb{Y}$ is spanned by the Y_P, and φ is linear, this means that for arbitrary $f, g \in \mathbb{Y}$, we have $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$. □

Definition 27. For $k \geqslant 0$, let S_k be the star graph which has k positive edges and $k+1$ vertices.

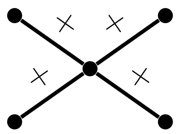


Figure 11: The signed graph S_4.

Lemma 28. *We have $X_{S_k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} p_{1,0}^{k-i} p_{i+1,0}$ for any $k \geqslant 0$.*

Proof. Consider the terms obtained by contracting i of the k total edges and deleting the rest. This term will have a factor of $(-1)^i$ since the contracted term is always subtracted. When i of the edges are contracted, there are $k+1-i$ vertices left. The one in the center has weight $(i+1, 0)$, and the vertices around it have weight $(1, 0)$. The term this corresponds to is $(-1)^i p_{1,0}^{k-i} p_{i+1,0}$. Finally, there are $\binom{k}{i}$ ways to contract i edges, and we can do this for any $i = 0, 1, \ldots, k$, demonstrating equality. □

Lemma 29. *We have $\varphi(p_{a,0}) = (t-1)^a - (-1)^a$ for any $a \geqslant 1$.*

Proof. We first compute $\varphi(X_{S_k})$. Consider the orientation of S_k with i edges pointing away from the center vertex. If $i \neq 0$, then there are i sinks, and if $i = 0$ then the vertex in the center is a sink. There are $\binom{k}{i}$ orientations of this form, and so by Lemma 25,

$$\varphi(X_{S_k}) = \sum_{i=0}^{k} \mathrm{acyc}_\Sigma(k)t^i = t + \sum_{i=1}^{k} \binom{k}{i} t^i = (t-1) + \sum_{i=0}^{k} \binom{k}{i} t^i = (t-1) + (t+1)^k.$$

We now compute $\varphi(p_{a,0})$ by induction on a. First, note that $\varphi(p_{1,0}) = t$, since $p_{1,0}$ is the chromatic B-symmetric polynomial of a single vertex with no edges. Suppose that $\varphi(p_{a,0}) = (t-1)^a - (-1)^a$ for all positive $a \leqslant k$. Using Lemmas 26 and 28, we see that

$$\begin{aligned}
\varphi(X_{S_k}) &= \sum_{i=0}^{k} (-1)^i \varphi(p_{1,0}^{k-i}) \cdot \varphi(p_{i+1,0}) \\
&= (-1)^k \varphi(p_{k+1,0}) + \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} t^{k-i} \cdot \left((t-1)^{i+1} - (-1)^{i+1}\right) \\
&= (-1)^k \varphi(p_{k+1,0}) + \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} t^{k-i}(t-1)^{i+1} - \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} t^{k-i}(-1)^{i+1} \\
&= (-1)^k \varphi(p_{k+1,0}) + (t-1) \sum_{i=0}^{k-1} \binom{k}{i} t^{k-i}(1-t)^i + \sum_{i=0}^{k-1} \binom{k}{i} t^{k-i} \\
&= (-1)^k \varphi(p_{k+1,0}) + (t-1)\left(1 - (-1)^k (t-1)^k\right) + \left((t+1)^k - 1\right).
\end{aligned}$$

We know that $\varphi(X_{S_k}) = (t-1) + (t+1)^k$, so we have that $(-1)^k \varphi(p_{k+1,0}) - (-1)^k (t-1)^{k+1} - 1 = 0$, and hence $\varphi(p_{k+1,0}) = (t-1)^{k+1} - (-1)^{k+1}$. So by induction, we are done. □

Lemma 30. *Let $\S_{n,m}$ denote the signed graph created by connecting the center vertices of S_n and S_m with a negative edge. Then, for any $m, n \geqslant 0$,*

$$X_{\S_{n,m}} = \sum_{i=0}^{n} \left(\sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} (-1)^{i+j} p_{1,0}^{n+m-i-j} (p_{i+1,0} p_{j+1,0} - p_{i+1,j+1}) \right).$$

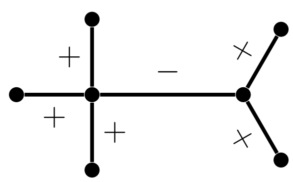


Figure 12: The graph $\S_{2,3} = \S_{3,2}$.

Proof. We will perform weighted deletion–contraction. Suppose that i of the positive edges in the S_n portion of the graph (of which there are $\binom{n}{i}$ combinations) and j of the positive edges in the S_m portion of the graph (of which there are $\binom{m}{j}$ combinations) are contracted, and the rest of the positive edges are deleted. This term will have a factor of $(-1)^{i+j}$ since the contraction terms are subtracted from the deletion terms. Furthermore, we are left with $(n-i)+(m-j)$ disjoint vertices of weight $(1,0)$ and two vertices of weights $(i+1,0)$ and $(j+1,0)$ which are connected by a negative edge. Performing weighted deletion–contraction on this last edge gives us the term

$$(-1)^{i+j}\left(p_{1,0}^{n+m-i-j}p_{i+1,0}p_{j+1,0} - p_{1,0}^{n+m-i-j}p_{i+1,j+1}\right)$$
$$= (-1)^{i+j}p_{1,0}^{n+m-i-j}(p_{i+1,0}p_{j+1,0} - p_{i+1,j+1}).$$

There are $\binom{n}{i}\binom{m}{j}$ ways to arrive at this term, and so

$$X_{\S_{n,m}} = \sum_{i=0}^{n}\left(\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(-1)^{i+j}p_{1,0}^{n+m-i-j}(p_{i+1,0}p_{j+1,0} - p_{i+1,j+1})\right),$$

as desired. $\square$

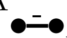
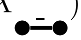
Lemma 31. *We have* $\varphi(p_{a,b}) = -(-1)^{a+b}$ *for any* $a,b \geqslant 1$.

Proof. We will begin by noting that

$$X_{\S_{n,m}} - X_{S_n \sqcup S_m} = X_{\S_{n,m}} - X_{S_n}X_{S_m} = -\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(-1)^{i+j}p_{1,0}^{n+m-i-j}p_{i+1,j+1}.$$

We are able to calculate $\varphi(X_{\S_{n,m}} - X_{S_n \sqcup S_m})$ by noting that the graph $S_n \sqcup S_m$ is almost identical to the graph $\S_{n,m}$, but $\S_{n,m}$ has a negative edge between the center vertices of the two star graphs. Any orientation of the graph $S_n \sqcup S_m$ will have the same number of sinks as the corresponding orientation of $\S_{n,m}$ for which the negative edge points outward. All orientations of both $S_n \sqcup S_m$ and $\S_{n,m}$ are acyclic, and so it suffices to consider the orientations of $\S_{n,m}$ for which the negative edge points inward. Consider an orientation of $\S_{n,m}$ such the the negative edge points inward, with i of the edges belonging to S_n and j of the edges belonging to S_m pointing away from their center vertex. This orientation will have $i+j$ sinks, and there are $\binom{n}{i}\binom{m}{j}$ orientations of this form. Therefore

$$\varphi(X_{\S_{n,m}} - X_{S_n \sqcup S_m}) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}t^{i+j} = (t+1)^{n+m}.$$

Now we will proceed by induction. We know that $\varphi(p_{1,1}) = -1$ because X $=$ $p_{1,0}^2 - p_{1,1}$ and $\varphi(X$$) = \varphi(p_{1,0}^2) - \varphi(p_{1,1}) = t^2+1$. Now suppose that $\varphi(p_{a,b}) = (-1)^{a+b+1}$

for all $a, b \geqslant 1$ with $a + b \leqslant n + m + 2$. Then by the induction hypothesis, we have that

$$\varphi(X_{\S_{n,m}} - X_{S_n \sqcup S_m}) = \varphi\left(-\sum_{i}^{n}\sum_{j}^{m}\binom{n}{i}\binom{m}{j}(-1)^{i+j}p_{1,0}^{n+m-i-j}p_{i+1,j+1}\right)$$
$$= -\varphi(p_{n+1,m+1})(-1)^{n+m} - 1 - \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(-1)^{i+j}t^{n+m-i-j}(-1)^{(i+1)+(j+1)+1}$$
$$= -\varphi(p_{n+1,m+1})(-1)^{n+m} - 1 + (t+1)^{n+m}.$$

But we know that $\varphi(X_{\S_{n,m}} - X_{S_n \sqcup S_m}) = (t+1)^{n+m}$, and so it must be that

$$-\varphi(p_{n+1,m+1})(-1)^{n+m} - 1 = 0,$$

and so $\varphi(p_{n+1,m+1}) = (-1)^{n+m+1} = (-1)^{(n+1)+(m+1)+1}$, which completes the proof. □

Lemma 32. *We have $\varphi(x_0) = -1$.*

Proof. Let Σ be the signed graph  . Then $X_\Sigma = p_{1,0} - x_0$, and Σ has two orientations, one with 1 sink and one with no sinks, and so $\varphi(X_\Sigma) = t + 1$. Now $\varphi(p_{1,0}) = t$ by Lemma 29, and so it must be that $\varphi(x_0) = -1$. □

Recall that we defined the *elementary symmetric functions* in the variables $\{x_i\}_{i \in \mathbb{Z}}$ to be (as usual)

$$e_n \coloneqq \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the sum ranges over all integer valued increasing sequences of length n.

Lemma 33. *We have $\varphi(e_n) = t$ for any $n > 0$.*

Proof. We will proceed by induction. Newton's identities state that

$$n \cdot e_n = (-1)^{n+1} p_{n,0} + \sum_{i=1}^{k-1} (-1)^{i+1} e_{k-i} p_{i,0}.$$

We have $\varphi(e_1) = t$ by Lemma 29, since $e_1 = p_{1,0}$. Suppose that for all $k \leqslant n - 1$, we have that $\varphi(e_k) = t$. Then

$$\varphi(n \cdot e_n) = (-1)^{n+1}\varphi(p_{n,0}) + \sum_{i=1}^{n-1}(-1)^{i+1}\varphi(e_{k-i}) \cdot \varphi(p_{i,0})$$
$$= (-1)^{n+1}((t-1)^n - (-1)^n) + \sum_{i=1}^{n-1}(-1)^{i+1} t \cdot ((t-1)^i - (-1)^i)$$
$$= 1 - (1-t)^n + t\sum_{i=1}^{n-1}(1 - (1-t)^i)$$

$$
\begin{aligned}
&= 1-(1-t)^n+(n-1)t-t\sum_{i=1}^{n-1}(1-t)^i \\
&= 1-(1-t)^n+(n-1)t-t(1-t)\frac{1-(1-t)^{n-1}}{1-(1-t)} \\
&= 1-(1-t)^n+(n-1)t-(1-t)+(1-t)^n \\
&= nt,
\end{aligned}
$$

as desired. □

Now we finally prove the main theorems stated in §3. For the reader's convenience, we give again the relevant statements and definitions.

Definition 34. Put $q_{a,b} := (-1)^{a+b+1}p_{a,b}$ and $z := -x_0$. We will call the set $\{e_n : n \geqslant 1\} \cup \{q_{a,b} : a, b \geqslant 1\} \cup \{z\}$ the *augmented elementary B-symmetric basis*.

Theorem 35. *If the chromatic B-symmetric function X_Σ of some signed graph Σ is written in terms of sums and products from the augmented elementary B-symmetric basis, then the number of acyclic orientations of Σ with k sinks is the sum of the coefficients of terms having k elementary symmetric function factors.*

Proof. We know that $\varphi(q_{a,b}) = \varphi(z) = 1$ for all $a, b, n \geqslant 1$ and that $\varphi(e_n) = t$ for all $n \geqslant 1$. We also know that the coefficient of t^k in $\varphi(X_\Sigma)$ is the number of acyclic orientations of Σ with k sinks. The terms which φ sends to a multiple of t^k are precisely those terms which have exactly k elementary symmetric function factors. □

See Example 5.

Definition 36. Let $\xi_n := \sum_{a=1}^{n}\binom{n}{a}p_{a,0}$ for $n \geqslant 1$. Alternatively, we may write this as $p_{n,0} = \sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}\xi_i$.

Lemma 37. *We have $\varphi(\xi_n) = t^n$ for any $n \geqslant 1$.*

Proof. Indeed,

$$
\begin{aligned}
\varphi(\xi_n) &= \sum_{a=1}^{n}\binom{n}{a}\varphi(p_{a,0}) \\
&= \sum_{a=1}^{n}\binom{n}{a}((t-1)^a-(-1)^a) \\
&= \left(1+\sum_{a=1}^{n}\binom{n}{a}(t-1)^a\right)-\left(1+\sum_{a=1}^{n}\binom{n}{a}(-1)^a\right) \\
&= t^n-0,
\end{aligned}
$$

as desired. □

Theorem 38. *If the chromatic B-symmetric function X_Σ of some signed graph Σ is written in terms of sums and products from the set $\{\xi_n : n \geqslant 1\} \cup \{q_{a,b} : a, b \geqslant 1\} \cup \{z\}$, then the number of acyclic orientations of Σ with k sinks is the sum of the coefficients of terms such that the sum of the indices of each ξ_n factor is equal to k.*

This follows from Lemma 37 by the same argument as given for Theorem 35. See Example 8.

8 Corollaries of the main theorems

As mentioned previously, a signed graph with all positive edges can be considered as an unsigned graph, and vice versa. Suppose that Σ is a signed graph with all positive edges. Let $|\Sigma|$ denote the unsigned graph which corresponds to Σ, i.e. $|\Sigma|$ would become Σ if we added plus signs to each of its edges. We can note that the proper colorings of $|\Sigma|$ are precisely the proper colorings of Σ which only use positive colors. So we may define $\mathrm{Proj}_{>0}$ to be a linear and multiplicative function such that $\mathrm{Proj}_{>0}(x_i) = x_i$ for $i \geqslant 1$ and $\mathrm{Proj}_{>0}(x_i) = 0$ for $i \leqslant 0$. It is clear that the chromatic B-symmetric function of Σ becomes the chromatic symmetric function of $|\Sigma|$ when each x_i which has a non-positive index is replaced by zero, i.e. $\mathrm{Proj}_{>0}(X_\Sigma) = X_{|\Sigma|}$. It should be noted that due to the process of weighted deletion–contraction, X_Σ will not contain any terms of the form $q_{a,b}$ or z since Σ has no negative edges.

It is easy to see that $\mathrm{Proj}_{>0}$ sends the elementary symmetric functions to the elementary symmetric functions. This means that Stanley's result about the elementary symmetric basis [8, Theorem 3.3] follows immediately from the result about the augmented elementary B-symmetric basis (Theorem 35).

Additionally, since $\mathrm{Proj}_{>0}(\xi_n) = \sum_{a=1}^{n} \binom{n}{a} p_a$, where $p_a := \sum_{i\geqslant 0} x_i^a$, we may define $\zeta_n := \sum_{a=1}^{n} \binom{n}{a} p_a$, so that we have the following result.

Corollary 39. *If the chromatic symmetric function X_G of some graph G is written in terms of sums and products from the set $\{\zeta_n : n \geqslant 1\}$, then the number of acyclic orientations of G with k sinks is the sum of the coefficients of terms such that the sum of the indices of each ζ_n factor is equal to k.*

See Example 11.

We may also use φ to recover Zaslavsky's result [9, Corollary 4.1]. Specifically, if χ_Σ is the chromatic polynomial of a signed graph Σ, i.e. $\chi_\Sigma(\lambda)$ is the number of proper colorings of Σ in the colors $-\lambda, \ldots, -1, 0, 1, \ldots, \lambda$, then we may recall that the p-basis is algebraically independent over $\mathbb{Q}$, which allows us to define a linear and multiplicative function $f : \mathbb{Y} \to \mathbb{Q}[\lambda]$ with $f(p_{a,b}) = 2\lambda+1$ for all $a, b \geqslant 0$ and $f(x_0) = 1$. Then f has the property that $f(X_\Sigma) = \chi_\Sigma$. Next, note that $\varphi(X_\Sigma)|_{t=1}$ is equal to the total number of acyclic orientations of Σ and that $\varphi(p_{a,b})|_{t=1} = (-1)^{a+b+1}$ for all $a \geqslant 1$, $b \geqslant 0$. To recover Zaslavsky's result, we must show that $f(X_\Sigma)|_{\lambda=-1} = (-1)^n \cdot (\#\text{ of acyclic orientations of } \Sigma)$ when Σ is a signed graph with n vertices.

To do this, note that $\varphi(p_{a_1,b_1}\cdots p_{a_k,b_k}x_0^c)|_{t=1} = (-1)^{n+k}$ where $n = a_1+b_1+\cdots+a_k+b_k+c$ and k is the number of $p_{a,b}$ terms. Also, we have that $f(p_{a_1,b_1}\cdots p_{a_k,b_k}x_0^c)|_{\lambda=-1} = (-1)^k$ since $f(p_{a,b})|_{\lambda=-1} = -1$ and $f(x_0)|_{\lambda=-1} = 1$. Therefore if Σ is a signed graph with n vertices, then we have

$$\chi_\Sigma(-1) = f(X_\Sigma)|_{\lambda=-1} = (-1)^n\varphi(X_\Sigma)|_{t=1} = (-1)^n \cdot (\#\text{ of acyclic orientations of }\Sigma)$$

which is Zaslavsky's result.

An equivalent way of stating this is that the sum of the absolute value of the coefficients of X_Σ written in the p-basis is equal to the total number of acyclic orientations of Σ. See Examples 12 and 13.

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References

[1] S. V. Chmutov, S. V. Duzhin, S. K. Lando, *Vassiliev knot invariants III. Forest algebra and weighted graphs*, Adv. Soviet Math. 21 (1994) 135-145.

[2] S. Chmutov, *B-symmetric chromatic function of signed graphs*. https://people.math.osu.edu/chmutov.1/talks/2020/slides-Moscow.pdf. Video available at https://www.youtube.com/watch?v=khA7rP84sYY.

[3] L. Crew, S. Spirkl, *A deletion–contraction relation for the chromatic symmetric function*. European J. Combin. 89 (2020) 103-143.

[4] E. S. Egge, *A chromatic symmetric function for signed graphs*. https://www.ericegge.net/slides/athens_slides.pdf.

[5] S. D. Noble, D. J. A. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Ann. Inst. Fourier (Grenoble) 49(3) (1999) 1057-1087.

[6] V. Reiner, *Signed posets*, J. Combin. Theory Ser. A 62 (1993), 324-360.

[7] R. P. Stanley, *Acyclic orientations of graphs*. Discrete Math. 5 (1973) 171-178.

[8] R. P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. 111(1) (1995) 166-194.

[9] T. Zaslavsky, *Signed graph coloring*, Discrete Math. 39(2) (1982) 215-228.