The Geometry and Combinatorics of Some Hessenberg Varieties Related to the Permutohedral Variety

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Abstract

We construct a concrete isomorphism from the permutohedral variety to the regular semisimple Hessenberg variety associated to the Hessenberg function $h_+(i) = i+1$, $1 \leqslant i \leqslant n-1$. In the process of defining the isomorphism, we introduce a sequence of varieties which we call the prepermutohedral varieties. We first determine the toric structure of these varieties and compute the Euler characteristics and the Betti numbers using the theory of toric varieties. Then, we describe the cohomology of these varieties. We also find a natural way to encode the one-dimensional components of the cohomology using the codes defined by Stembridge [15]. Applying the isomorphisms we constructed, we are also able to describe the geometric structure of regular semisimple Hessenberg varieties associated to the Hessenberg function represented by $h_k = (2, 3, \cdots, k+1, n, \ldots, n)$, $1 \leqslant k \leqslant n-3$. In particular, we are able to write down the cohomology ring of the variety. Finally, we determine the dot representation of the permutation group $\mathfrak{S}_n$ on these varieties.

Mathematics Subject Classifications: 05E14

1 Introduction

There are several different ways to describe the permutohedral variety $\mathcal{X}$ of dimension $n-1$, for instances:

1. It is the toric variety associated to the normal fan of the permutohedron.

2. It is the graph of the Cremona involution

$$J : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$$
$$[z_1 : \cdots : z_n] \longmapsto [z_1^{-1} : \cdots : z_n^{-1}]$$

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3. It is an iterated blowup of $\mathbb{P}^{n-1}$ at all (strict transforms of) coordinate linear subspaces in a certain order, as follows.

$$\mathcal{X}_{n-2} \xrightarrow{\pi_{n-2}} \mathcal{X}_{n-3} \xrightarrow{\pi_{n-3}} \cdots \xrightarrow{\pi_2} \mathcal{X}_1 \xrightarrow{\pi_1} \mathcal{X}_0 = \mathbb{P}^{n-1}$$

   Each $\mathcal{X}_{k+1}$ is the blowup of $\mathcal{X}_k$ at all the strict transform of the k-dimensional coordinate spaces $(0 \leqslant k \leqslant n-3)$. Further details are in Section 2.

4. It is the regular semisimple Hessenberg variety $\mathcal{H}ess(\mathbf{S}, h_+)$ associated to the Hessenberg function h_+ defined by $h_+(i) = i+1$ for $1 \leqslant i \leqslant n-1$ and $h_+(n) = n$.

The isomorphisms among the first three descriptions are well-known. On the other hand, to the author's knowledge, the isomorphism between the permutohedral variety and the Hessenberg variety is proved by general theory of toric varieties [6, Lemma 10 and Theorem 11]. The first goal of this article is to construct a concrete isomorphism from the iterated blowups on $\mathbb{P}^{n-1}$ to the Hessenberg variety. In the process, we further obtain isomorphisms from $\mathcal{X}_k$, the k-th step in the iterated blowups, to a "Hessenberg type" subvariety of a partial flag variety which is denoted by $\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$ (for details, see Section 3).

Next, in Section 4, we apply the theory of toric varieties to explore certain geometric properties of the varieties $\mathcal{X}_k$. In particular, we show that the Euler characteristic of $\mathcal{X}_k$ is equal to the permutation number $P(n, k+1) = \frac{n!}{(n-k-1)!}$. In fact, there is a basis for the homology $H_*(\mathcal{X}_k)$ whose elements are in one-to-one correspondence with permutations of $(k+1)$ different numbers chosen from $[n] := \{1, \ldots, n\}$. As a consequence, the question of finding Betti numbers of $\mathcal{X}_k$ turns into a counting question of permutations with certain property. More precisely, if we make the following definition:

Definition. For a permutation $a_1, \ldots, a_{k+1}$ of $k+1$ different numbers in $[n]$, set $\alpha_0 = [n] \setminus \{a_1, \ldots, a_{k+1}\}$, a *descent* (resp. *ascent*) for the permutation $a_1, \ldots, a_{k+1}$ is either $a_j > a_{j+1}$ (resp. $a_j < a_{j+1}$) for some $j = 1, \ldots, k$, or $a > a_1$ (resp. $a < a_1$) for some $a \in \alpha_0$.

Then we can calculate the even Betti numbers of $\mathcal{X}_k$ as follows (note that the odd Betti numbers of $\mathcal{X}_k$ are all 0.)

Proposition 1.1 (Proposition 4.2)**.** *The $2i$-th Betti number of $\mathcal{X}_k$ is given by*

$$\begin{aligned}\beta_{2i}(\mathcal{X}_k) &= \#(\text{permutations of } k+1 \text{ different numbers in } [n] \text{ with } i \text{ descents}) \\ &= \#(\text{permutations of } k+1 \text{ different numbers in } [n] \text{ with } n-1-i \text{ ascents}).\end{aligned}$$

These Betti numbers are quite natural generalization of the Eulerian numbers. In fact, for $k = n-2$, $\mathcal{X}_{n-2}$ is the permutohedral variety and it is well known that $\beta_{2i}(\mathcal{X}_{n-2}) = A(n, i+1)$ are the Eulerian numbers. However, the author's cannot find information about them in the literature. It might be interesting to study further properties of these numbers.

In Section 5, we first describe $H^*(\mathcal{X}_k)$ using the blowup structure (Proposition 5.2). There is a system of codes defined by Stembridge in [15]. He also proved in the article that the representation of the symmetric group $\mathfrak{S}_n$ on $H^*(\mathcal{X})$ is isomorphic to the permutation representation induced by the action on codes [15, Proposition 4.1]. In Proposition 5.3, we realize the isomorphism by defining a natural one-to-one correspondence between codes of length n and one-dimensional components of $H^*(\mathcal{X}^{n-1})$. The correspondence can also be restricted to $H^*(\mathcal{X}_k^{n-1})$, and is compatible with the $\mathfrak{S}_n$ action. This immediately gives us a way to concretely construct permutation basis for the $\mathfrak{S}_n$ representation on each $H^*(\mathcal{X}_k^{n-1})$, $0 \leqslant k \leqslant n-2$.

The question of finding a permutation basis for the dot action on the cohomology of Hessenberg varieties has drawn people's attention recently [1, 3, 4, 11] because of it's relation with the Stanley-Stembridge conjecture. The relation was observed by Shareshian and Wahcs [14]. They also announced an important conjecture that was later proved by Brosnan and Chow [2], and independently by Guay-Paquet [10]. In the case of the permutohedral variety, the Stanley-Stembridge conjecture is known to be true, and a permutation basis is also known. The known basis was first conjectured by Chow [4] and then proved by Cho, Hong, and Lee [3]. It is based on the theory on equivariant cohomology, then pass it on to the usual cohomology. The basis constructed in this paper is based on the geometric structure of $\mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$ and the combinatorics of the codes, and is for the usual cohomology. It might be interesting to compare these two kinds of basis.

In Section 6, we use the isomorphisms at each step of the blowups to investigate semisimple Hessenberg varieties $\mathcal{H}ess(\mathbf{S}, h_k)$ associated to the Hessenberg function $h_k = (2, 3, \ldots, k+1, n, \ldots, n)$, $1 \leqslant k \leqslant n-3$. We observe that $\mathcal{H}ess(\mathbf{S}, h_k)$ has a fiber bundle structure over $\mathcal{X}_k$ with fibers isomorphic to the flag variety $\mathcal{F}lag(\mathbb{C}^{n-k-1})$. We use this structure to obtain a description of the cohomology of $\mathcal{H}ess(\mathbf{S}, h_k)$.

Proposition 1.2 (Proposition 6.1). *The cohomology ring $H^*(\mathcal{H}ess(\mathbf{S}, h_k))$ is generated over $H^*(\mathcal{X}_k)$ as*

$$H^*(\mathcal{H}ess(\mathbf{S}, h_k)) \cong H^*(\mathcal{X}_k)[X_{k+2}, \ldots, X_n]/(e_1(X_{k+2}, \ldots, X_n), \ldots, e_{n-k-1}(X_{k+2}, \ldots, X_n)),$$

where e_j is the j-th elementary symmetric polynomials. In addition, for the images x_i's of the X_i's, the classes $x_{k+2}^{i_{k+2}} \cdots x_n^{i_n}$, with exponents $0 \leqslant i_j \leqslant n-j$, form a basis for $H^*(\mathcal{Y})$ over $H^*(\mathcal{X}_k)$.

Finally, we determine the dot representation on the cohomologies of $\mathcal{H}ess(\mathbf{S}, h_k)$.

Proposition 1.3 (Proposition 6.2). *The dot representation on $\mathcal{H}ess(\mathbf{S}, h_k)$ is isomorphic to the representation on*

$$H^*(\mathcal{X}_k)[X_{k+2}, \ldots, X_n]/(e_1(X_{k+2}, \ldots, X_n), \ldots, e_{n-k}(X_{k+2}, \ldots, X_n))$$

which acts on $H^*(\mathcal{X}_k)$ as described in Section 5, and acts trivially on the $H^*(\mathcal{X}_k)$-basis $x_{k+2}^{i_{k+2}} \cdots x_n^{i_n}$, $0 \leqslant i_j \leqslant n-j$.

Our proof is based on the characteristic series of the representation, and the conclusion is up to an isomorphism. It would be interesting to compute the $\mathfrak{S}_n$ action on the basis elements; presumably first on the equivariant cohomology then pass to the usual cohomology, as was done in [17].

2 The permutohedral variety as iterated blowups of $\mathbb{P}^{n-1}$

One can obtain the permutohedral variety by performing a sequence of blowups on $\mathbb{P}^{n-1}$, as follows.

1. First, we blowup the n points $Z_1 = [1 : 0 : \cdots : 0]$, $Z_2 = [0 : 1 : 0 \cdots : 0]$, $\cdots$, $Z_n = [0 : \cdots : 0 : n]$. We denote the resulting variety and the projection map by $\pi_1 : \mathcal{X}_1 \to \mathbb{P}^{n-1} := \mathcal{X}_0$. We also have the exceptional divisors $E_i = \pi_1^{-1}(Z_i) \subset \mathcal{X}_1$.

2. Next, we blowup the strict transforms (in $\mathcal{X}_1$) of all the lines $Z_{\{i,j\}} \subset \mathbb{P}^{n-1}$ connecting Z_i and Z_j for $1 \leqslant i < j \leqslant n$. This gives us the second level space $\pi_2 : \mathcal{X}_2 \to \mathcal{X}_1$ and the exceptional divisors $E_{\{i,j\}} \subset \mathcal{X}_2$. Notice that, although the lines $Z_{\{i,j\}}$ and $Z_{\{i,k\}}$ intersect at Z_i in $\mathbb{P}^{n-1}$, the blowups in step 1 would separate their strict transforms. Therefore, the resulting space $\mathcal{X}_2$ is independent of the order of blowups.

3. We repeat the above process until we reach codimension 2. More precisely, in the k-th step ($1 \leqslant k \leqslant n-2$) we do the following. For $\alpha \subset [n]$ a subset of k elements, let Z_α denote the linear subvariety of $\mathbb{P}^{n-1}$ generated by $\{Z_i | i \in \alpha\}$, and $\overline{Z}_\alpha$ the strict transform of Z_α in $\mathcal{X}_{k-1}$. The blowups in the previous steps have the effect of blowing up all coordinate linear subspaces on Z_α, thus $\overline{Z}_\alpha$ is a permutohedral variety of dimension $k-1$. We blow up all the $\overline{Z}_\alpha$ in this step. The $\overline{Z}_\alpha$'s intersect with each other along coordinate subspaces of lower dimensions, hence are separated by previous blowups. Thus, the order to perform blowups in this step does not matter. This produces the space $\mathcal{X}_k$, the map $\pi_k : \mathcal{X}_k \to \mathcal{X}_{k-1}$, as well as the exceptional divisors $E_\alpha \subset \mathcal{X}_k$

The end result is a sequence of spaces and projection maps:

$$\mathcal{X}_{n-2} \xrightarrow{\pi_{n-2}} \mathcal{X}_{n-3} \xrightarrow{\pi_{n-3}} \cdots \xrightarrow{\pi_2} \mathcal{X}_1 \xrightarrow{\pi_1} \mathcal{X}_0 = \mathbb{P}^{n-1}$$

The variety $\mathcal{X} = \mathcal{X}_{n-2}$ is the permutohedral variety.

Definition. We call the varieties $\mathcal{X}_k$ ($0 \leqslant k \leqslant n-2$) the *prepermutohedral variety* of order k.

3 Permutohedral varieties as Hessenberg varieties

We consider regular semisimple Hessenberg varieties of type A. To set the notations, let $\mathbf{S}$ be an $n \times n$ complex diagonal matrix with different diagonal entries $s_1, \ldots, s_n$ and

$h : [n] \to [n]$ be a Hessenberg function, i.e., $h(i) \geqslant h(j)$ for all $n \geqslant i > j \geqslant 1$ (h is non-decreasing) and $h(i) \geqslant i$ for $i = 1, \ldots, n$.

$$\mathcal{H}ess(\mathbf{S}, h) = \left\{ V_\bullet = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \mathbf{S}(V_i) \subset V_{h(i)} \text{ for } i = 1, \ldots, n \right\}.$$

We also use the list notation $h = (h(1), \ldots, h(n))$ to denote a Hessenberg function. In this section, we focus on the specific Hessenberg function $h_+ = (2, 3, \ldots, n-1, n, n)$, i.e. $h_+(i) = \min(i+1, n)$ for $1 \leqslant i \leqslant n$. We start with an observation in linear algebra.

Lemma 3.1. *Let* $\mathbf{S}$ *be an* $n \times n$ *complex diagonal matrix with different diagonal entries* $s_1, \ldots, s_n$ *and* $\mathbf{v} = {}^t(z_1, \cdots, z_n) \in \mathbb{C}^n$ *be a (column) vector. If at least* k *of the* z_i*'s are nonzero (i.e. there are no more than* $n-k$ *of the* $z_i = 0$*), then the vectors*

$$\mathbf{v},\ \mathbf{S}\mathbf{v},\ \cdots,\ \mathbf{S}^{k-1}\mathbf{v}$$

are linearly independent. If exactly k *of the* z_i*'s are nonzero, then the vectors*

$$\mathbf{v},\ \mathbf{S}\mathbf{v},\ \cdots,\ \mathbf{S}^{k}\mathbf{v}$$

are linearly dependent.

Proof. Assuming $z_i \neq 0$, then a linear relation

$$a_0\mathbf{v} + a_1\mathbf{S}\mathbf{v} + \cdots + a_{k-1}\mathbf{S}^{k-1}\mathbf{v} = 0$$

implies the relation (on the first coordinate)

$$a_0 + a_1 s_i + \cdots + a_{k-1} s_i^{k-1} = 0.$$

That is, s_i is a root of the polynomial $\sum_{i=0}^{k-1} a_i z^i$. The first part of the lemma is then a consequence of the fact that a polynomial equation of degree $k-1$ cannot have k or more distinct roots.

For the second part of the lemma, assume that $z_1, \ldots, z_k$ are the non-zero z_i's. Then the coefficients of the polynomial $(z - s_1) \cdots (z - s_k)$ give a non-trivial linear relation among $\mathbf{v},\ \mathbf{S}\mathbf{v},\ \cdots,\ \mathbf{S}^k\mathbf{v}$. ☐

In particular, if $\mathbf{0} = (0, \ldots, 0)$ denotes the origin, and $\mathbf{v} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then $\mathbf{v}$ gives rise to a point $[\mathbf{v}] \in \mathbb{P}^{n-1}$. We further denote

$$\mathcal{I}nd := \bigcup_{1 \leqslant i < j \leqslant n} (z_i = z_j = 0).$$

The set $\mathcal{I}nd$ is the indeterminate set of the Cremona involution J defined in the introduction. If $\mathbf{v} \in \mathbb{P}^{n-1} \setminus \mathcal{I}nd$, then at least $n-1$ of the x_i are nonzero. Thus, by the lemma, the vectors

$$\mathbf{v},\ \mathbf{S}\mathbf{v},\ \cdots,\ \mathbf{S}^{n-1}\mathbf{v}$$

are linearly independent, and the following is a well-defined flag (i.e. the dimension of the vector spaces are correct).

$$V_\bullet = \left(\langle \mathbf{0} \rangle \subset \langle \mathbf{v} \rangle \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v} \rangle \subset \cdots \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v}, \ldots \mathbf{S}^{n-2}\mathbf{v} \rangle \subset \mathbb{C}^n\right).$$

Moreover, it is obvious that $V_\bullet \in \mathcal{H}ess(\mathbf{S}, h_+)$. Conversely, suppose that

$$V_\bullet = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n)$$

is a flag in $\mathcal{H}ess(\mathbf{S}, h_+)$, and the one-dimensional space V_1, as an element of $\mathbb{P}^{n-1}$, satisfies $V_1 \notin \mathcal{I}nd$, then $V_\bullet$ must be in the form

$$V_\bullet = \left(\langle \mathbf{0} \rangle \subset \langle \mathbf{v} \rangle \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v} \rangle \subset \cdots \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v}, \ldots \mathbf{S}^{n-2}\mathbf{v} \rangle \subset \mathbb{C}^n\right)$$

for any nonzero $\mathbf{v} \in V_1$. This defines an isomorphism

$$\begin{aligned} \mathbb{P}^{n-1} \setminus \mathcal{I}nd &\longrightarrow \mathcal{U} \subset \mathcal{H}ess(\mathbf{S}, h_+) \\ \mathbf{v} &\longmapsto \left(\langle \mathbf{0} \rangle \subset \langle \mathbf{v} \rangle \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v} \rangle \subset \cdots \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v}, \ldots \mathbf{S}^{n-2}\mathbf{v} \rangle \subset \mathbb{C}^n\right), \end{aligned}$$

where $\mathcal{U}$ is the open subset of $\mathcal{H}ess(\mathbf{S}, h_+)$ consists of all flags

$$V_\bullet = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n)$$

such that $V_1 \notin \mathcal{I}nd$. We will extend this isomorphism to isomorphisms between blowups of $\mathbb{P}^{n-1}$ and Hessenberg type varieties defined in the next paragraph.

In order to do so, we introduce a type of partial flag variety. For $0 \leqslant k \leqslant n-2$, define

$$\mathcal{F}lag^{(k+1)}(\mathbb{C}^n) := \{V_\bullet = (V_0 := \langle \mathbf{0} \rangle \subset V_1 \subset \cdots \subset V_{k+1}) | \dim_{\mathbb{C}}(V_i) = i \text{ for } i = 1, \ldots, k+1\}.$$

We also define the corresponding Hessenberg type variety

$$\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+) := \left\{V_\bullet \in \mathcal{F}lag^{(k+1)}(\mathbb{C}^n) \mid \mathbf{S}V_i \subset V_{i+1} \text{ for } i = 0, \ldots, k\right\}.$$

Notice that $\mathcal{F}lag^{(n-1)}(\mathbb{C}^n) = \mathcal{F}lag(\mathbb{C}^n)$ and $\mathcal{H}ess^{(n-1)}(\mathbf{S}, h_+) = \mathcal{H}ess(\mathbf{S}, h_+)$. The main goal of this section is to show the following.

Proposition 3.2. *There is a natural isomorphism*

$$\mathcal{X}_k \xrightarrow{\cong} \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$$

for $k = 0, \ldots, n-2$, where $\mathcal{X}_k$ is the prepermutohedral variety.

Proof. It is clear that $\mathcal{X}_0 := \mathbb{P}^{n-1} \cong \mathcal{H}ess^{(1)}(\mathbf{S}, h_+)$. For $V_\bullet = (V_0 := \langle \mathbf{0} \rangle \subset V_1 \subset V_2) \in \mathcal{H}ess^{(2)}(\mathbf{S}, h_+)$, suppose that $\mathbf{0} \neq \mathbf{v} = (z_1, \ldots, z_n) \in V_1$. By lemma 3.1, if at least two of the x_i's are non-zero, then $\mathbf{v}$ and $\mathbf{S}\mathbf{v}$ are linearly independent, and $V_2 = \langle \mathbf{v}, \mathbf{S}\mathbf{v} \rangle$ is determined. If only one of the $z_i \neq 0$, then $V_1 = Z_i$ (see Section 2 for the definition of Z_i) for some $i = 1, \ldots, n$ and thus $\mathbf{S}V_1 = V_1$. To determine V_2, we need an extra piece of information, which is the direction that give us the second dimension of V_2. This can be specified as a point in $\mathbb{P}(\mathbb{C}^n/V_1)$, which is canonically isomorphic to the exceptional divisor E_i when we blowup $\mathbb{P}^{n-1}$ at Z_i.

More concretely, if we look at the instance $i = n$, then $Z_n = [0 : \cdots : 0 : 1]$ and $V_1 = \operatorname{span}\{e_n\}$. The blowup space $\tilde{\mathbb{P}}^{n-1}_{Z_n}$ is defined as

$$\tilde{\mathbb{P}}^{n-1}_{Z_n} = \left\{([z_1 : \cdots : z_n], [w_1 : \cdots : w_{n-1}]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-2} \,|\, z_i w_j = z_j w_i \ \forall 1 \leqslant i, j \leqslant n-1\right\}.$$

The projection $\pi_1 : \tilde{\mathbb{P}}^{n-1}_{Z_n} \to \mathbb{P}^{n-1}$ is one-to-one except over the point Z_n. The preimage $\pi_1^{-1}(Z_n)$ is the set of all points of the form $(Z_n, [w_1 : \cdots : w_{n-1}])$, where $[w_1 : \cdots : w_{n-1}]$ can be any point in $\mathbb{P}^{n-2}$. The point $(Z_n, [w_1 : \cdots : w_{n-1}])$ in the blowup space then corresponds to the vector space $V_2 = \operatorname{span}\{e_n, (w_1, \ldots, w_{n-1}, 0)\}$. Note that V_2 is well defined: although the ways to write $w_1, \ldots, w_{n-1}$ is not unique, the span is unique.

Therefore, after we blowup all the Z_i's, then for those V_1 such that $\mathbf{S}V_1 = V_1$, we also know what V_2 is. This gives the isomorphism $\mathcal{X}_1 \to \mathcal{H}ess^{(2)}(\mathbf{S}, h_+)$.

We continue the blowup process inductively as follows. First, we introduce the "forgetful" morphism

$$f^{(k)} : \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+) \to \mathcal{H}ess^{(k)}(\mathbf{S}, h_+) \text{ for } k = 1, \ldots, n-3,$$

which sends a flag of $k+1$ vector spaces to the first k of them, and "forgets" the last vector space.

Claim 1. *The morphism $f^{(k)}$ is a birational map.*

Proof. Given $V_\bullet = (V_0 := \langle \mathbf{0} \rangle \subset V_1 \subset \cdots \subset V_k) \in \mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$, since $\mathbf{S}V_{k-1} \subset V_k$, we know from linear algebra that $\dim_{\mathbb{C}}(\langle V_k \cup \mathbf{S}V_k \rangle) = k$ or $k+1$. Moreover, $\dim_{\mathbb{C}}(\langle V_k \cup \mathbf{S}V_k \rangle) = k$ if and only if $\mathbf{S}V_k = V_k$. Since $\mathbf{S}$ is diagonal, $\mathbf{S}V_k = V_k$ means that V_k is spanned by k standard basis elements, and there are only finitely many of such V_k. Therefore, $\dim_{\mathbb{C}}(\langle V_k \cup \mathbf{S}V_k \rangle) = k+1$ is the generic situation in $\mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$. Moreover, the set of all $V_\bullet$ with $\dim_{\mathbb{C}}(\langle V_{k-1} \cup \mathbf{S}V_k \rangle) = k$ consists all $V_\bullet$ such that V_k is one of the finitely many coordinate subspaces, thus is a closed subset of $\mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$. If $\dim_{\mathbb{C}}(\langle V_k \cup \mathbf{S}V_k \rangle) = k+1$, then sending

$$(V_0 \subset V_1 \subset \cdots \subset V_k) \longmapsto (V_0 \subset V_1 \subset \cdots \subset V_k \subset \langle V_k \cup \mathbf{S}V_k \rangle)$$

gives the inverse of $f^{(k)}$ for a generic flag in $\mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$. □

If $\dim_{\mathbb{C}}(\langle V_k \cup \mathbf{S}V_k \rangle) = k$, then $\mathbf{S}V_k = V_k$, and some thinking about Lemma 3.1 tells us that, via the isomorphism $\mathcal{X}_k \cong \mathcal{H}ess^{(k)}(\mathbf{S}, h_+)$, $V_\bullet$ lies in the strict transform of Z_α, denoted by $\overline{Z}_\alpha$, for some $\alpha \subset [n]$ with k elements.

If we blowup $\mathcal{X}_{k-1}$ along $\overline{Z}_\alpha$, then the exceptional divisor E_α is canonically identified with the projective normal bundle $\mathbb{P}(N_{Z_\alpha \subset X_{k-1}})$. A point on E_α carries the information of $V_\bullet$, together with a (projective) normal direction of Z_α in $\mathbb{C}^n$, i.e. an element in $\mathbb{P}(\mathbb{C}^n/Z_\alpha)$. This assigns a unique flag in $\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$. More precisely, if $\mathbf{v} \in \mathbb{C}^n/Z_\alpha$ represents the direction in $\mathbb{P}(\mathbb{C}^n/Z_\alpha)$, then one sets $V_{k+1} = \langle V_k, \mathbf{v} \rangle$. Once we blowup $\mathcal{X}_{k-1}$ along the strict transforms of all Z_α for $\alpha \subset [n]$ with k elements, we obtain the isomorphism $\mathcal{X}_k \cong \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$. □

Example 3.1. Suppose we have $\mathbf{v}_1 = (1,1,0,0,0) \in \mathbb{C}^5$, then $V_1 = \langle \mathbf{v}_1 \rangle \subset V_2 = \langle \mathbf{v}_1, \mathbf{S}\mathbf{v}_1 \rangle$ form the first two spaces in the flag but $\mathbf{S}V_2 = V_2$. Thus we need to blowup $Z_{\{1,2\}}$. Notice that $V_2 = Z_{\{1,2\}} \subset \mathbb{C}^5$ and elements in $E_{\{1,2\}}$ over $\mathbf{v}_1$ are in one-to-one correspondence to projectivized normal vectors for the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{C}^5$ at $\mathbf{v}_1$, i.e., $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^2)$. Suppose we pick $\mathbf{v}_2 = (1,1,1,0,0)$ as a representative for an element in $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^2)$, then that gives us $V_3 = \langle V_2, \mathbf{v}_2 \rangle$ (one can check that V_3 is independent of the choice of $\mathbf{v}_2$), but then we would have $\mathbf{S}V_3 = V_3$ again. This means that the element represented by $\mathbf{v}_2$ in $E_{\{1,2\}}$ is in the strict transform of the set $Z_{\{1,2,3\}}$. When we blowup $Z_{\{1,2,3\}}$, elements of $E_{\{1,2,3\}}$ over $\mathbf{v}_2$ will then corresponds to $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^3)$. Picking a representative, say $\mathbf{v}_3 = (1,1,1,1,1)$, we will have $V_4 = \langle V_3, \mathbf{v}_3 \rangle$ and $V_5 = \mathbb{C}^5$. This gives a flag $V_\bullet = (\langle \mathbf{0} \rangle \subset V_1 \subset \cdots \subset V_5) \in \mathcal{H}ess(\mathbf{S}, h_+)$.

4 Prepermutohedral varieties as toric varieties

From the isomorphism constructed in the previous section, we can discover how the torus $(\mathbb{C}^*)^{n-1}$ sits inside $\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$, as follows. The point $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in (\mathbb{C}^*)^{n-1}$ corresponds to the flag

$$V_\bullet := \big(\langle \mathbf{0} \rangle \subset \langle \mathbf{v} \rangle \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v} \rangle \subset \cdots \subset \langle \mathbf{v}, \mathbf{S}\mathbf{v}, \ldots \mathbf{S}^k \mathbf{v} \rangle \subset \mathbb{C}^n\big)$$

where $\mathbf{v} = (1, z_1, \ldots, z_{n-1})$. We denote this correspondence as an injective map $\phi_{k+1} : (\mathbb{C}^*)^{n-1} \hookrightarrow \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$ by $\phi(\mathbf{z}) = V_\bullet$.

One can also observe the algebraic group structure of $(\mathbb{C}^*)^{n-1} \subset \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$. Given $(z'_1, \ldots, z'_{n-1}) \in (\mathbb{C}^*)^{n-1}$ and $\mathbf{v}' = (1, z'_1, \ldots, z'_{n-1})$, the product of $(z_1, \ldots, z_{n-1})$ and $(z'_1, \ldots, z'_{n-1})$ corresponds to the flag

$$\big(\langle \mathbf{0} \rangle \subset \langle \mathbf{v}\mathbf{v}' \rangle \subset \langle \mathbf{v}\mathbf{v}', \mathbf{S}(\mathbf{v}\mathbf{v}') \rangle \subset \cdots \subset \langle \mathbf{v}\mathbf{v}', \mathbf{S}(\mathbf{v}\mathbf{v}'), \ldots, \mathbf{S}^k(\mathbf{v}\mathbf{v}') \rangle \subset \mathbb{C}^n\big),$$

where $\mathbf{v}\mathbf{v}'$ is the coordinate-wise product of $\mathbf{v}$ and $\mathbf{v}'$. That is, $\phi(\mathbf{z})\phi(\mathbf{z}') := \phi(\mathbf{z}\mathbf{z}')$. Moreover, one can discover the action of $(\mathbb{C}^*)^{n-1}$ on $\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$ similarly.

4.1 Fan structure for the prepermutohedral varieties

Given that $\mathcal{X}_k \cong \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$ is a toric variety, we would like to know the structure of the corresponding fan, and the geometry properties we can conclude from the fan structure. The standard reference for this part is [7, 5].

First, recall the structure of the fan $\Delta_{\mathbb{P}^{n-1}}$ corresponding to $\mathbb{P}^{n-1}$. Let $e_1, \ldots, e_{n-1}$ be the standard basis of $\mathbb{Z}^{n-1}$ and $e_n = -(e_1 + \cdots + e_{n-1})$. The cones in $\Delta_{\mathbb{P}^{n-1}}$ are generated by proper subsets of $\{e_1, \ldots, e_n\}$. More precisely, for any proper subset $\alpha \subset [n]$, we denote σ_α to be the cone generated by $\{e_i | i \in \alpha\}$, i.e., $\sigma_\alpha = \{\sum_{i \in \alpha} a_i e_i | a_i \geqslant 0\}$. Then

$$\Delta_{\mathbb{P}^{n-1}} = \{\sigma_\alpha \mid \alpha \subset [n]\},$$

with the convention that $\sigma_\emptyset = \{\mathbf{0}\}$.

As we showed earlier, the prepermutohedral variety $\mathcal{X}_k$ $(1 \leqslant k \leqslant n-2)$ is obtained from $\mathbb{P}^{n-1}$ by blowing up the torus invariant subvarieties (in the order of dimensions) up to dimensions $k-1$. For the fan structure, this means we perform the star subdivisions (see [5, Definition 3.3.17]) on all cones of codimensions $0, 1$, and so on, up to cones of codimension $k-1$. We will describe the fan $\Delta_{\mathcal{X}_k}$ corresponds to the prepermutohedral variety $\mathcal{X}_k$ after setting up some notations.

For a non-empty proper subset $\alpha \subset [n]$, we define the vector $e_\alpha := \sum_{i\in\alpha} e_i$. The rays (i.e. one-dimensional cones) of $\Delta_{\mathcal{X}_k}$ are generated by e_α for non-empty proper subset $\alpha \subset [n]$ such that $|\alpha| \geqslant n-k$ or $|\alpha| = 1$. To simplify notation, we write e_i instead of $e_{\{i\}}$ for $i \in [n]$.

A chain of subsets of $[n]$ is a sequence of strict inclusions

$$\mathcal{C} = (\alpha_0 \subset \cdots \subset \alpha_p)$$

of proper subsets of $[n]$. It is also allowed that $p = 0$, i.e., $\mathcal{C} = (\alpha_0)$ is a chain consisting of only one set. We will also use an alternative list notation to denote a chain. We use special bracket symbols $[\![\ ]\!]$ to enclose the list: first list the numbers in α_0, then list the numbers in $\alpha_1 \setminus \alpha_0$ (separated by a bar), and so on, all the way to $[n] \setminus \alpha_p$. For example, suppose $n = 9$ and we have

$$\mathcal{C} = (\{1,4\} \subset \{1,2,3,4\} \subset \{1,2,3,4,6,7,9\}),$$

then we use $L(\mathcal{C}) = [\![1,4|2,3|6,7,9|5,8]\!]$ to denote that $[\![1,4|2,3|6,7,9|5,8]\!]$ is the list notation for $\mathcal{C}$.

Cones in $\Delta_{\mathcal{X}_k}$ are in one-to-one correspondence with chains $\mathcal{C}$ such that $|\alpha_0| \leqslant n-k-1$ and $|\alpha_j| \geqslant n-k$ for $j > 0$. The correspondence is given as follows. For each such chain $\mathcal{C}$, we associate it with the cone $\sigma_{\mathcal{C}}$ generated by the vectors e_i, $i \in \alpha_0$ and e_{α_j}, $j \geqslant 1$. The dimension of the cone $\sigma_{\mathcal{C}}$ is equal to $|\alpha_0| + p$.

In particular, the top dimensional cones of $\Delta_{\mathcal{X}_k}$ are in one-to-one correspondence with chains of the form

$$\mathcal{C} = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$$

such that $|\alpha_j| = n+j-k-1$ for all $j = 0, \ldots, k$. (For $k = 0$, this means $\mathcal{C} = (\alpha_0)$ with $|\alpha_0| = n-1$). For $k \geqslant 2$, such chains are determined by the sequence of numbers

$$\begin{aligned} a_{k+1} &:= [n] \setminus \alpha_k, \quad \text{(note that } |\alpha_k| = n-1) \\ a_j &:= \alpha_j \setminus \alpha_{j-1}, \quad j = 2, \ldots, k \\ a_1 &:= \alpha_1 \setminus \alpha_0. \end{aligned}$$

For $k = 0$, such chains are determined by $a_1 = [n] \setminus \alpha_0$; and for $k = 1$ the chains are determined by $a_2, a_1 = \alpha_{n-1} \setminus \alpha_0$. In the list notation, by a slightly abuse of notation, we would write $L(\mathcal{C}) = [\![\alpha_0|a_1|\cdots|a_{k+1}]\!]$.

Notice that, since α_0 can be written as $\alpha_0 = [n] \setminus \{a_j | 1 \leqslant j \leqslant k+1\}$, the information on the list of numbers $a_1, \ldots, a_{k+1}$ is sufficient to determine the chain $\mathcal{C}$. Therefore, the top dimensional cones in $\Delta_{\mathcal{X}_k}$ are in one-to-one correspondence with permutations of $k+1$

different numbers from $[n]$. The number of top dimensional cones in $\Delta_{\mathcal{X}_k}$ is then given by the permutation number

$$P(n, k+1) = \frac{n!}{(n-k-1)!}.$$

A result in toric varieties [7, Section 3.2] states that the Euler characteristic of a toric variety is equal to the number of top dimensional cones in its associated fan. Therefore, we conclude the following.

Proposition 4.1. *The Euler characteristic of the variety $\mathcal{X}_k$ is given by*

$$\chi(\mathcal{X}_k) = P(n, k+1).$$

We remark here that one can also calculate the Euler characteristics using the structure of the cohomology groups in the next section.

4.2 Betti numbers of the prepermutohedral varieties

In order to calculate the Betti numbers for $\mathcal{X}_k$, we need to study more on the cone structure of the fan associated to $\mathcal{X}_k$.

The intersection of two cones is given by the following operation on chains: Given $\mathcal{C} = (\alpha_0 \subset \cdots \subset \alpha_p)$ and $\mathcal{C}' = (\alpha'_0 \subset \cdots \subset \alpha'_{p'})$, define $\mathcal{C} \cap \mathcal{C}'$ to be the following.

- The first set in the chain is $\alpha_0 \cap \alpha'_0$.
- The rest of the chain is the greatest common subchain of the chains of sets $\alpha_1 \subset \cdots \subset \alpha_p$ and $\alpha'_1 \subset \cdots \subset \alpha'_{p'}$

For instance, if $n = 10$, $L(\mathcal{C}) = [\![1, 4, 9|2, 3|6, 7|5|8, 10]\!]$ and $L(\mathcal{C}') = [\![1, 4, 6|7, 2, 3|5, 9|10|8]\!]$, then $L(\mathcal{C} \cap \mathcal{C}') = [\![1, 4|2, 3, 5, 6, 7, 9|8, 10]\!]$. Recall from the previous section that the cones in $\Delta_{\mathcal{X}_k}$ correspond to chains with certain property. This definition on the intersection of chains is compatible with the intersection of the corresponding cones, in the following sense. Suppose $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{C}'}$ are two cones in the fan $\Delta_{\mathcal{X}_k}$, then we have $\sigma_{\mathcal{C}} \cap \sigma_{\mathcal{C}'} = \sigma_{\mathcal{C} \cap \mathcal{C}'}$.

Also, with the same notations as above, for the chains $\mathcal{C}, \mathcal{C}'$, the corresponding cones $\sigma_{\mathcal{C}} \subset \sigma_{\mathcal{C}'}$ if $\alpha_0 \subset \alpha'_0$ and for all $j > 0$, $\alpha_j = \alpha'_{j'}$ for some j'.

We would like to assign a total order '$>$' on the top dimensional cones of $\Delta_{\mathcal{X}_k}$ with some desired property. Equivalently, we will define the order on the set of chains. Set $\tau_{\mathcal{C}}$ to be the intersection of $\sigma_{\mathcal{C}}$ with all $\sigma_{\mathcal{C}'}$ that comes after $\sigma_{\mathcal{C}}$ (i.e. $\mathcal{C}' > \mathcal{C}$) and such that $\dim(\sigma_{\mathcal{C}} \cap \sigma_{\mathcal{C}'}) = n - 2$. The desired property is that

$$\text{If } \tau_{\mathcal{C}} \subset \sigma_{\mathcal{C}'}, \text{ then } \mathcal{C}' \geqslant \mathcal{C} \tag{$*$}$$

This is also the condition $(*)$ in [7, Section 5.2]. It implies that the classes $[V(\tau_{\mathcal{C}})]$ form a basis for $H_*(X; \mathbb{Z})$ (see [7, Theorem on p.102]). We claim that the reversed lexicographic order in the list notation for chains will satisfy $(*)$.

To define the order, suppose $L(\mathcal{C}) = [\![\alpha_0|a_1|\cdots|a_{k+1}]\!]$ and $L(\mathcal{C}') = [\![\alpha'_0|a'_1|\cdots|a'_{k+1}]\!]$. We define $\mathcal{C} < \mathcal{C}'$ if for the greatest j such that $a_j \neq a'_j$, we have $a_j < a'_j$. We need the following definition to describe the intersection $\tau_{\mathcal{C}}$.

Definition. Given a chain $L(\mathcal{C}) = [\![\alpha_0|a_1|\cdots|a_{k+1}]\!]$ in the list notation, a *descent* (resp. *ascent*) in $\mathcal{C}$ is either $a_j > a_{j+1}$ (resp. $a_j < a_{j+1}$) for some $j \in \{1, \dots, k\}$, or $a > a_1$ (resp. $a < a_1$) for some $a \in \alpha_0$.

We can now calculate $\tau_{\mathcal{C}}$. Recall that $\tau_{\mathcal{C}}$ to be the intersection of $\sigma_{\mathcal{C}}$ with all $\sigma_{\mathcal{C}'}$ such that $\mathcal{C}' > \mathcal{C}$ and $\dim(\sigma_{\mathcal{C}} \cap \sigma_{\mathcal{C}'}) = n - 2$. Such $\mathcal{C}'$ are in one-to-one correspondence with descents in $\mathcal{C}$, as described in the following. Suppose $L(\mathcal{C}) = [\![\alpha_0|a_1|\cdots|a_{k+1}]\!]$, there are two types of descents in $\mathcal{C}$:

Type 1: $a_j > a_{j+1}$ for some $j \in \{1, \dots, k\}$. In this case,

$$L(\mathcal{C}') = [\![\alpha_0|a_1|\cdots|a_{j+1}|a_j|\cdots|a_{k+1}]\!],$$

and we have

$$L(\mathcal{C} \cap \mathcal{C}') = [\![\alpha_0|a_1|\cdots|a_j, a_{j+1}|\cdots|a_{k+1}]\!].$$

Type 2: $a > a_1$ for some $a \in \alpha_0$ In this case,

$$L(\mathcal{C}') = [\![\alpha_0 \cup \{a_1\} \setminus \{a\}|a|\cdots|a_{j+1}|a_j|\cdots|a_{k+1}]\!],$$

and we have

$$L(\mathcal{C} \cap \mathcal{C}') = [\![\alpha_0 \setminus \{a\}|a, a_1|a_2|\cdots|a_{k+1}]\!]$$

If we repeat the process of taking intersections for all descents in $\mathcal{C}$, we combine all the blocks $[\![\cdots|a_j|a_{j+1}|\cdots]\!]$ where type 1 of descents happens; and we move all the elements in α_0 that involve in a descent of type 2 to the next block. The result is the chain corresponds to the cone $\tau_{\mathcal{C}}$.

For instance, suppose $n = 9, k = 4$ and $L(\mathcal{C}) = [\![1, 2, 5, 8|4|3|6|9|7]\!]$, then there are 4 descents in $\mathcal{C}$: $5 > 4, 8 > 4, 4 > 3$, and $9 > 7$. The cone of intersections $L(\tau_{\mathcal{C}}) = [\![1, 2|5, 8, 4, 3|6|9, 7]\!]$, and $\dim(\tau_{\mathcal{C}}) = 5$.

In terms of chains of subsets of $[n]$, $\mathcal{C} = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$, $a_i = \alpha_i \setminus \alpha_{i-1}$ $(1 \leqslant i \leqslant k)$ and $a_{k+1} = [n] \setminus \alpha_k$, $\tau_{\mathcal{C}}$ is obtained from $\sigma_{\mathcal{C}}$ through the following process.

- Whenever there is a descent $a_j > a_{j+1}$, one removes the set α_j in the chain $\mathcal{C}$.
- Whenever there is a descent $a > a_1$ for some $a \in \alpha_0$, one removes the number a from the set α_0. All other α_i, $i \neq 0$, remain the same.

Recall that for a chain $(\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_p)$, the dimension of its corresponding cone is $|\alpha_0| + p$. One observes that every time there is an descent in $\mathcal{C}$, the dimension of $\tau_{\mathcal{C}}$ would go down by 1. Therefore, the dimension of $\tau_{\mathcal{C}}$ is given by

$$\dim(\tau_{\mathcal{C}}) = n - 1 - \#(\text{descents in } \mathcal{C}) = \#(\text{ascents in } \mathcal{C}).$$

The complex dimension of corresponding orbit closure $V[\tau_{\mathcal{C}}]$ will then be

$$\begin{aligned}\dim_{\mathbb{C}}(V[\tau_{\mathcal{C}}]) &= n - 1 - \dim(\tau_{\mathcal{C}}) \\ &= \#(\text{descents in } \mathcal{C}) = n - 1 - \#(\text{ascents in } \mathcal{C}).\end{aligned}$$

Notice that $\mathcal{C}$ is the chain of a top dimensional cone with the property that for each block (numbers between two bars) in the chain corresponds to $\tau_{\mathcal{C}}$, numbers in the block are descending in $\mathcal{C}$. Thus, it is the smallest in the reverse lexicographic order among the maximal chains that refine the chain for $\tau_{\mathcal{C}}$, and we can conclude that $\tau_{\mathcal{C}} \subset \sigma_{\mathcal{C}'}$ implies $\mathcal{C}' \geqslant \mathcal{C}$. Therefore, by [7, Theorem on p.102], the homology classes of orbit closures $V[\tau_{\mathcal{C}}] \in H_{2d}(X;\mathbb{Z})$ where $d = \dim(\tau_{\mathcal{C}})$, as $\mathcal{C}$ runs through all top dimensional cones, will form a basis of $H_*(X;\mathbb{Z})$. This immediately implies the following.

Proposition 4.2. *The $2i$-th Betti number of $\mathcal{X}_k$ is given by*

$$\begin{aligned}\beta_{2i}(\mathcal{X}_k) &= \#(\textit{permutations of } k+1 \textit{ different numbers in } [n] \textit{ with } i \textit{ descents}) \\ &= \#(\textit{permutations of } k+1 \textit{ different numbers in } [n] \textit{ with } n-1-i \textit{ ascents}).\end{aligned}$$

We remark here that it is possible to find some recursive relations among these Betti numbers using the recursive structure of the cohomology groups discuss in the next section (2).

5 The dot action on cohomology groups

In this section, we will switch our perspective from toric varieties to iterated blowups. We use the iterated blowup structure for pre-permutohedral varieties to describe its cohomology. Furthermore, we investigate the dot action on the cohomology groups of the pre-permutohedral varieties. We work on the usual cohomology groups. In this section and the next, we sometimes need to specify the dimension of a prepermutohedral variety. In such situation, we will use $\mathcal{X}_k^{n-1}$ to denote a prepermutohedral variety of dimension $n-1$ and of order k.

5.1 The cohomology groups of prepermutohedral varieties

First, we recall the standard result on the cohomology of blowup spaces. For more details, see [18, §7.3.3]. Let X be a Kähler manifold of dimension n, $Z \subset X$ be a submanifold, and $\tilde{X}_Z \xrightarrow{\tau} X$ be the blowup of X along Z. Then $\tilde{X}_Z$ is still a Kähler manifold. Let $E = \tau^{-1}(Z)$ be the exceptional divisor, then $E = \mathbb{P}(N_{Z\subset X})$ is the projective bundle of the normal bundle of Z in X. Thus E is of rank $r-1$, where $r = \operatorname{codim}(Z)$, and $E \overset{j}{\hookrightarrow} X$ is a hypersurface.

Theorem 5.1. *[18, Theorem 7.31] Let $h = c_1(\mathcal{O}_E(1)) \in H^2(E;\mathbb{Z})$, we have isomorphism*

$$H^p(X;\mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{p-2i-2}(Z;\mathbb{Z})\right) \xrightarrow{\tau^* + \sum_i j_* \circ (\cup h^i) \circ \tau|_E^*} H^p(\tilde{X}_Z;\mathbb{Z}).$$

The map on the second component is decomposed as follows.

$$H^{p-2i-2}(Z;\mathbb{Z}) \xrightarrow{\tau|_E^*} H^{p-2i-2}(E;\mathbb{Z}) \xrightarrow{\cup h^i} H^{p-2}(E;\mathbb{Z}) \xrightarrow{j_*} H^p(\tilde{X}_Z;\mathbb{Z}).$$

Here j_* is the Gysin morphism which is defined as the Poincare dual of the map

$$j_* : H_{2n-p}(E;\mathbb{Z}) \to H_{2n-p}(\tilde{X}_Z;\mathbb{Z}).$$

Some remarks:

- The theorem is also valid for cohomology groups with $\mathbb{C}$ coefficients. In this paper, we always use complex coefficients, thus we will write $H^p(X)$ for $H^p(X;\mathbb{C})$ from now on.

- We denote $H^*(X) = \oplus_p H^p(X)$ as a graded complex vector space. Moreover, define $H^p_q(X) := H^{p+q}(X)$, i.e. the degree p part of $H^*_q(X)$ is equal to the degree $p+q$ part of $H^*(X)$. The theorem can be written in term of the graded $\mathbb{C}$ vector spaces as

$$H^*(\tilde{X}_Z) \cong H^*(X) \oplus \left(\bigoplus_{i=1}^{r-1} H^*_{-2i}(Z) \right). \tag{1}$$

In the process of obtaining $\mathcal{X}_k$, we blowup strict transforms of coordinate linear subvarieties Z_α. The strict transforms $\overline{Z}_\alpha$ are themselves permutohedral varieties of lower dimension. We will use $\mathcal{X}^d$ to denote the permutohedral variety of dimension d. A more compact form of the following description is in [13].

Theorem 5.2. *The cohomology, as a graded complex vector space, of the pre-permutohedral variety $\mathcal{X}^{n-1}_k$, $1 \leqslant k \leqslant n-2$, is given by*

$$H^*(\mathcal{X}^{n-1}_k) \cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{k} \bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\mathcal{X}^{j-1}) \right). \tag{2}$$

Here $\mathcal{X}^{j-1} = \mathcal{X}^{j-1}_{j-2}$ is the permutohedral variety of dimension $j-1$.

Proof. We apply Theorem 5.1 with our setting:

- $\mathcal{X}^{n-1}_0 = \mathbb{P}^{n-1}$. (Recall that $H^*(\mathcal{X}^{n-1}_0) \cong H^*(\mathbb{P}^{n-1}) \cong \mathbb{C}[\xi]/(\xi^n)$ where $\xi \in H^2(\mathbb{P}^{n-1})$ is the first Chern class of the hyperplane bundle.)

- For $1 \leqslant k \leqslant n-2$, $\mathcal{X}^{n-1}_k$ is the blowup of $\mathcal{X}^{n-1}_{k-1}$ along all $\overline{Z}_\alpha$ with $\alpha \subset [n]$ and $|\alpha| = k$.

- $\overline{Z}_\alpha$ is isomorphic to $\mathcal{X}^{k-1}$, thus the codimension of $\overline{Z}_\alpha$ is $n-k$.

This gives us

$$H^*(\mathcal{X}^{n-1}_k) \cong H^*(\mathcal{X}^{n-1}_{k-1}) \oplus \left(\bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=k}} \bigoplus_{i=1}^{n-k-1} H^*_{-2i}(\overline{Z}_\alpha) \right) \cong H^*(\mathcal{X}^{n-1}_{k-1}) \oplus \left(\bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=k}} \bigoplus_{i=1}^{n-k-1} H^*_{-2i}(\mathcal{X}^{k-1}) \right).$$

We then prove the theorem by induction. For the base case $j = 1$, we consider $\mathcal{X}_1^{n-1}$. This is the space obtained by blowing up the points $Z_1, \ldots, Z_n$. The points are of dimension 0, thus the codimension $r = n - 1$. By (1), the cohomology becomes

$$H^p(\mathcal{X}_1^{n-1}) \cong H^p(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{\ell=1}^{n} \bigoplus_{i=1}^{n-2} H^p_{-2i}(Z_\ell) \right) \cong H^p(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{\ell=1}^{n} \bigoplus_{i=1}^{n-2} H^{p-2i}(Z_\ell) \right).$$

Again, since $Z_1, \ldots, Z_n$ are points, the direct sum $\oplus_{i=1}^{n-2} H^{p-2i}(Z_\ell)$ has only one term $H^0(Z_\ell)$, which appears when $p = 2i$, $i = 1, \ldots, n-2$. All other cohomologies in the direct sum vanish. This is exactly the form in (2) with $k = 1$ and $\alpha = \{\ell\}$. One can summarize the cohomologies of $\mathcal{X}_1^{n-1}$ as follows:

$$\begin{aligned} H^0(\mathcal{X}_1^{n-1}) &\cong H^0(\mathbb{P}^{n-1}) \cong \mathbb{C} \\ H^{2i}(\mathcal{X}_1^{n-1}) &\cong H^{2i}(\mathbb{P}^{n-1}) \oplus \bigoplus_{\ell=1}^{n} H^0(Z_\ell) \cong \mathbb{C} \oplus \mathbb{C}^n, \ 1 \leqslant i \leqslant n-2 \\ H^{2n-2}(\mathcal{X}_1^{n-1}) &\cong H^{2n-2}(\mathbb{P}^{n-1}) \cong \mathbb{C}. \end{aligned}$$

Next, for the inductive step, assume that

$$H^*(\mathcal{X}_{k-1}^{n-1}) \cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{k-1} \bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\mathcal{X}^{j-1}) \right),$$

then we have

$$\begin{aligned} H^*(\mathcal{X}_k^{n-1}) &\cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{k-1} \bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\mathcal{X}^{j-1}) \right) \oplus \left(\bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = k}} \bigoplus_{i=1}^{n-k-1} H^*_{-2i}(\overline{Z}_\alpha) \right) \\ &\cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{k-1} \bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\mathcal{X}^{j-1}) \right) \oplus \left(\bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = k}} \bigoplus_{i=1}^{n-k-1} H^*_{-2i}(\mathcal{X}^{k-1}) \right) \\ &\cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{k} \bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\mathcal{X}^{j-1}) \right). \end{aligned}$$

Therefore, the isomorphism in the theorem is proved by induction. □

In the following, we compute some examples in low dimensions.

Example 5.1. We investigate the cohomology of $\mathcal{X}_2^{n-1}$, that is, when $j = 2$. In order to obtain the space, we blowup $\mathcal{X}_1^{n-1}$ along all the lines Z_α where $\alpha \subset [n]$ and $|\alpha| = 2$. There are $\binom{n}{2}$ such lines. Notice Z_α is isomorphic to its strict transform $\overline{Z}_\alpha$. Since its dimension is one, we have $H^0(Z_{\{i,j\}}) \cong \mathbb{C}$ and $H^2(Z_\alpha) \cong \mathbb{C}$. Therefore, by Theorem 5.2, we have

$$
\begin{aligned}
H^0(\mathcal{X}_2^{n-1}) &\cong H^0(\mathbb{P}^{n-1}) \cong \mathbb{C} \\
H^2(\mathcal{X}_2^{n-1}) &\cong H^2(\mathbb{P}^{n-1}) \oplus \bigoplus_{\ell=1}^{n} H^0(Z_\ell) \oplus \bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=2}} H^0(Z_\alpha) \cong \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{C}^{\binom{n}{2}} \\
H^{2i}(\mathcal{X}_2^{n-1}) &\cong H^{2i}(\mathbb{P}^{n-1}) \oplus \bigoplus_{\ell=1}^{n} H^0(Z_i) \oplus \bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=2}} \big(H^0(Z_\alpha) \oplus H^2(Z_\alpha)\big) \\
&\cong \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{C}^{\binom{n}{2}} \oplus \mathbb{C}^{\binom{n}{2}}, \quad 2 \leqslant i \leqslant n-3. \\
H^{2n-4}(\mathcal{X}_2^{n-1}) &\cong H^{2n-4}(\mathbb{P}^{n-1}) \oplus \bigoplus_{\ell=1}^{n} H^0(Z_\ell) \oplus \bigoplus_{\substack{\alpha\subset[n],\\ |\alpha|=2}} H^2(Z_\alpha) \cong \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{C}^{\binom{n}{2}} \\
H^{2n-2}(\mathcal{X}_2^{n-1}) &\cong H^{2n-2}(\mathbb{P}^{n-1}) \cong \mathbb{C}.
\end{aligned}
$$

5.2 Encoding the cohomology groups and the $\mathfrak{S}_n$ representation

Unwinding the recursion in Theorem 5.2 gives a description of $H(\mathcal{X}^{n-1})$ as a direct sum of one dimensional summands. There is a nice way to encode these summands by the codes defined by Stembridge [15]. We recall the notations first. For a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of nonnegative integers, we will call n the *length* of $\mathbf{a}$. Let $S^+(\mathbf{a}) = \{a_i | 1 \leqslant i \leqslant n, a_i > 0\}$ denote the set of positive integers in $\mathbf{a}$. For a positive integer k, the sequence $\mathbf{a}$ is called k-*admissible* if $S^+(\mathbf{a}) = \{1, \ldots, k\}$; $\mathbf{a}$ is 0-*admissible* if $S^+(\mathbf{a}) = \emptyset$, i.e. $\mathbf{a}$ consists of all 0's. The sequence $\mathbf{a}$ is called *admissible* if it is k-admissible for some $k \geqslant 0$.

Let $m_j(\mathbf{a})$ denote the number of occurrences of the integer j in the sequence $\mathbf{a}$. A *marked sequence* is a pair $(\mathbf{a}, f)$ where $\mathbf{a}$ is a sequence of nonnegative integers and $f : S^+(\mathbf{a}) \to \mathbb{N}$ be a map such that $1 \leqslant f(j) < m_j(\mathbf{a})$ for all $j \in S^+(\mathbf{a})$. We will adapt the notation in [15] and represent a marked sequence $(\mathbf{a}, f)$ by putting a hat notation on top of j at the $[f(j) + 1]$-st occurrence of j. The *index* of a marked sequence is defined as $\operatorname{ind}(\mathbf{a}, f) := \sum_{j \in S^+(\mathbf{a})} f(j)$.

A *code* is defined to be an admissible marked sequence, i.e. a marked sequence $(\mathbf{a}, f)$ such that $\mathbf{a}$ is admissible. There is one code $((0, \ldots, 0), \emptyset)$ consists of all 0's and the empty function as the marking function. We define the index of this code to be 0.

For a code $\mathbf{a}$, let $\max(\mathbf{a}) = \max\{a_i | i = 1, \ldots, n\}$ be the maximum number in $\mathbf{a}$, and let $\mu(\mathbf{a}) := m_{\max(\mathbf{a})}(\mathbf{a})$ be the number of occurrences of $\max(\mathbf{a})$ in $\mathbf{a}$. For example, if $\mathbf{a} = 1201\hat{2}\hat{1}2$, then $\max(\mathbf{a}) = 2$ and $\mu(\mathbf{a}) = 3$. An immediate consequence of this definition is that $\mu(\mathbf{a}) \geqslant 2$ for all codes $\mathbf{a}$ as long as $n \geqslant 2$. We further define $\mathbf{a}'$ to be the sequence obtained after removing all the $\max(\mathbf{a})$ from $\mathbf{a}$, but keep everything else unchanged. For the example $\mathbf{a} = 1201\hat{2}\hat{1}2$ above, one would get $\mathbf{a}' = 101\hat{1}$. It is possible to have $\mathbf{a}'$ equals to the empty sequence, in which case $\mathbf{a}$ consists all 0's or all 1's. Notice that if $\mathbf{a}$ is a code, then $\mathbf{a}'$ is either a code or an empty sequence.

Recall, from Theorem 5.2, that we have the decomposition of the cohomology of the permutohedral variety, by setting $k = n - 2$ in (2), as follows.

$$H^*(\mathfrak{X}^{n-1}) \cong H^*(\mathbb{P}^{n-1}) \oplus \left(\bigoplus_{j=1}^{n-2} \bigoplus_{\substack{\alpha \subset [n], \\ |\alpha| = j}} \bigoplus_{i=1}^{n-j-1} H^*_{-2i}(\overline{Z}_\alpha) \right). \tag{3}$$

The main result of this section is the following.

Proposition 5.3. *Let $n \geqslant 2$ be an integer. There is a natural one-to-one correspondence between codes of length n and one-dimensional summands of $H^*(\mathfrak{X}^{n-1})$.*

More precisely, one-dimensional summands of $H^{2j}(\mathfrak{X}^{n-1})$ are in one-to-one correspondence with codes $(\mathbf{a}, f)$ of index j, i.e. $\operatorname{ind}(\mathbf{a}, f) = j$.

Proof. We will construct the correspondence inductively. The base case is a code $\mathbf{a}$ with $\mu(\mathbf{a}) = n$. That means, $\mathbf{a}$ consists of either all 0's or all 1's. In this case, suppose $\operatorname{ind}(\mathbf{a}, f) = i$, we assign $(\mathbf{a}, f)$ to the $2i$-th cohomology of $H^*(\mathbb{P}^{n-1})$ in the above decomposition.

Next, suppose that $\mu(\mathbf{a}) < n$. Then we must have $\max(\mathbf{a}) > 0$ and thus $2 \leqslant \mu(\mathbf{a}) \leqslant n - 1$. We first set the integers i, j and the subset $\alpha \subset [n]$ that are used as indices in the above decomposition.

- $j = n - \mu(\mathbf{a})$,
- $\alpha = \{i \mid a_i < \max(\mathbf{a})\}$,
- $i = f(\max(\mathbf{a}))$.

Notice that, since $\max(\mathbf{a}) > 0$, $i = f(\max(\mathbf{a}))$ is well-defined; moreover, we have $1 \leqslant i \leqslant \mu(\mathbf{a}) - 1 = n - j - 1$ by the definition of f. Concretely, what these indices record are the following:

- $\mu(\mathbf{a})$ record the codimension of the subspace which is blown up in the process of constructing $\mathfrak{X}^{n-1}$.
- $\overline{Z}_\alpha$ is the subspace getting blown up in the process of constructing $\mathfrak{X}^{n-1}$. Equivalently, the set
$$[n] \setminus \alpha = \{i \mid a_i = \max(\mathbf{a})\}$$
encodes the coordinates of the subspace that are set to be 0. For example, suppose $n = 6$ and $\mathbf{a} = 12\hat{1}01\hat{2}$, then $\mu(\mathbf{a}) = 2$, $\alpha = \{1, 3, 4, 5\}$, and the subspace getting blown up is defined by the equations $z_2 = z_6 = 0$ in $\mathbb{P}^5$ (then taking the strict transform to the proper space).

- $i = f(\max(\mathbf{a}))$ denotes the shifting of degrees when we map $H^*(\overline{Z}_\alpha)$ into $H^*(\mathcal{X}^{n-1})$.

Next, $\mathbf{a}'$ is a code of length $j = n - \mu(\mathbf{a})$. Therefore, by induction hypothesis, it corresponds to a unique component in $H^*_{-2i}(\overline{Z}_\alpha) \cong H^*_{-2i}(\mathcal{X}^{j-1})$.

Finally, since one can recover $\mathbf{a}$ uniquely from $\mathbf{a}'$, α, and i, this correspondence is one-to-one. One can also observe that the ranges of j and i we defined from codes of length n are the same as the range for the indices in the decomposition (3). Therefore the correspondence is surjective. □

Recall that, for integers $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$, the prepermutohedral variety $\mathcal{X}^{n-1}_k$ is obtained from $\mathcal{X}^{n-1}_0 = \mathbb{P}^{n-1}$ by blowing up points, lines, ..., all the way to the coordinate subspaces of dimension $k-1$. Since $\mu(\mathbf{a})$ records the codimension of the subspace getting blown up, one can conclude the following coding for the cohomologies of $\mathcal{X}^{n-1}_k$ as well.

Corollary 5.4. *For integers $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$, there is a natural one-to-one correspondence between one-dimensional summands of $H^*(\mathcal{X}^{n-1}_k)$ and the set of codes $\mathbf{a}$ of length n such that $\mu(\mathbf{a}) \geqslant n-k$.*

More precisely, one-dimensional summands of $H^{2j}(\mathcal{X}^{n-1}_k)$ are in one-to-one correspondence with codes $(\mathbf{a}, f)$ such that the index of $(\mathbf{a}, f)$ is j and the maximum number in $\mathbf{a}$ occurs exactly $n-k$ times.

Example 5.2. For $n = 4$, we can read the cohomologies of $\mathcal{X}^3_k$, $k = 0, 1, 2$, and the corresponding codes, from the following table. The codes recorded below are the representatives for all codes in the same $\mathfrak{S}_4$-orbit. For instance, the code $01\hat{1}1$ represents the four codes: $01\hat{1}1, 10\hat{1}1, 1\hat{1}01$, and $1\hat{1}10$. We pick the representative to be the one that is increasing in numbers.

	$\mathcal{X}^3_0$		$\mathcal{X}^3_1$		$\mathcal{X}^3_2$	
		codes		codes		codes
$H^0(\mathcal{X}^3_\bullet) \cong$	$H^0(\mathbb{P}^3)$	0000				
$H^2(\mathcal{X}^3_\bullet) \cong$	$H^2(\mathbb{P}^3)$	$1\hat{1}11$	$\oplus_{j=1}^4 H^0(Z_j)$	$01\hat{1}1$	$\oplus_{i,j\in[n], i\neq j} H^0(Z_{\{i,j\}})$	$001\hat{1}$
$H^4(\mathcal{X}^3_\bullet) \cong$	$H^4(\mathbb{P}^3)$	$11\hat{1}1$	$\oplus_{j=1}^4 H^0(Z_j)$	$011\hat{1}$	$\oplus_{i,j\in[n], i\neq j} H^2(Z_{\{i,j\}})$	$1\hat{1}2\hat{2}$
$H^6(\mathcal{X}^3_\bullet) \cong$	$H^6(\mathbb{P}^3)$	$111\hat{1}$				

Entries in the table record the direct summands needed to obtain the cohomology of the corresponding spaces. For example, one can read from the table that $H^2(\mathcal{X}^3_1) \cong H^2(\mathbb{P}^3) \oplus_{j=1}^4 H^0(Z_j)$ and $H^4(\mathcal{X}^3_2) \cong H^4(\mathbb{P}^3) \oplus_{j=1}^4 H^0(Z_j) \oplus_{i,j\in[n], i\neq j} H^2(Z_{\{i,j\}})$. Notice that Z_j here are points and $Z_{\{i,j\}}$ are lines.

In Proposition 5.3, we use the code to encode the one dimensional summands of $H^*(\mathcal{X}^{n-1};\mathbb{C})$. Remember that in Section 4.2, we found a basis for $H_*(\mathcal{X}^{n-1};\mathbb{Z})$, therefore also a basis for $H_*(\mathcal{X}^{n-1};\mathbb{C})$, by assigning a good total order on the top dimensional cones of the fan $\Delta_{\mathcal{X}^{n-1}}$. Moreover, the top-dimensional cones of $\Delta_{\mathcal{X}^{n-1}}$ are in one-to-one correspondence with permutations of $1,\dots,n$, (i.e. elements of $\mathfrak{S}_n$) and each basis element spans a one-dimensional summand. The homology $H_*(\mathcal{X}^{n-1};\mathbb{C})$ and the cohomology $H^*(\mathcal{X}^{n-1};\mathbb{C})$ are isomorphic by Poincaré duality. Therefore, one can ask the following natural question.

Question. What is the correspondence, via Poincaré duality and the encodings of homology and cohomology, between the basis of $H_*(\mathcal{X}^{n-1};\mathbb{C})$ represented by permutations $w\in\mathfrak{S}_n$ and the basis of $H^*(\mathcal{X}^{n-1};\mathbb{C})$ represented by codes (α,f) of length n? Can one write down the transition matrix between the two basis?

The convenience of using the codes to encode one-dimensional summands of cohomology comes in two ways. First, it give a nice compact way to write $H^*(\mathcal{X}_k^{n-1})$. Let $\mathcal{C}ode(n)$ denote the set of all codes of length n, then

$$H^*(\mathcal{X}_k^{n-1}) = \bigoplus_{\substack{\mathbf{a}\in\mathcal{C}ode(n)\\ \mu(\mathbf{a})\geqslant n-k}} \mathbb{C}_{\mathbf{a}}.$$

Second, the $\mathfrak{S}_n$-representation on $H^*(\mathcal{X}_k^{n-1})$ is compatible with the $\mathfrak{S}_n$ action on $\mathcal{C}ode(n)$, in the sense that for any $w\in\mathfrak{S}_n$, we have $w\,\mathbb{C}_{\mathbf{a}}=\mathbb{C}_{w\mathbf{a}}$ in the above decomposition of $H^*(\mathcal{X}_k^{n-1})$. We explain this assertion in the next paragraph.

For $w\in\mathfrak{S}_n$, the action of $\mathfrak{S}_n$ on $\mathcal{C}ode(n)$ is given by $w\cdot(\mathbf{a},f)=(w\mathbf{a},f)$ where $(w\mathbf{a})_i=a_{w^{-1}(i)}$, $1\leqslant i\leqslant n$. Notice that the action does not change f. Therefore, in the hat notation, the hat for a number j is still at the the $[f(j)+1]$-st occurrence of j in $w\mathbf{a}$. In general, the $\mathfrak{S}_n$ action on the flag varieties *does not* induce actions on Hessenberg varieties. For the special Hessenberg function h_+, we can define an $\mathfrak{S}_n$ action (not the one induced by the action on flag varieties) as follows. The action on $\mathcal{X}_k^{n-1}$ is induced from the action on $\mathbb{P}^{n-1}$ given by permuting the coordinates, i.e. $w\cdot[z_1:\cdots:z_n]=[z_{w(1)}:\cdots:z_{w(n)}]$. Then the action on the cohomology is given by the pullback of the action on $\mathcal{X}_k^{n-1}$. For the coordinate hyperplane $(z_i=0)$, its pullbak is $(z_{w^{-1}(i)}=0)$. Recall that, in the encoding, the locations of the maximal numbers corresponds to coordinates to set to be 0. Moreover, permuting the coordinates also results in permuting the normal directions to the coordinate linear subspaces, and the exceptional divisor corresponds to the projective bundle associated to the normal bundle. Hence we know that w maps $H^*(\overline{Z}_\alpha)$ to $H^*(\overline{Z}_{w^{-1}\alpha})$. Then, we can conclude $w\,\mathbb{C}_{\mathbf{a}}=\mathbb{C}_{w\mathbf{a}}$ by induction on n.

Therefore, the $\mathfrak{S}_n$ representation on $\mathcal{X}_k^{n-1}$ is the permutation representation induced by the permutation on codes of length n with $\mu(\mathbf{a})\geqslant n-k$. We remark here that the $\mathfrak{S}_n$ representation on cohomology described above is the same as the dot representation on T-equivariant cohomology ([17]), passing to the usual cohomology. This is because the above described $\mathfrak{S}_n$ representation is induced by $\mathfrak{S}_n$ actions on the corresponding T-spaces $\mathcal{X}_k^{n-1}$ as the pullback on cohomologies. The definition of the dot action on equivariant cohomology is exactly the pullback for the $\mathfrak{S}_n$ action of permuting the coordinates of the ambient space $\mathbb{C}^n$ in the definition of flags.

An immediate consequence of this fact is the following.

Proposition 5.5. *There is a permutation basis for the dot representation on the cohomology of the prepermutohedral varieties $\mathcal{X}_k^{n-1}$, $1 \leqslant k \leqslant n-2$.*

Proof. For each $\mathfrak{S}_n$ orbit of codes under the encoding, pick a representative $\mathbf{a}$ and pick an element $\xi \in \mathbb{C}_{\mathbf{a}}$. Setting all other components to be 0 gives us an element in $H^*(\mathcal{X}_k)$. Then the $w\xi$'s, as w runs through $\mathfrak{S}_n$, form a linearly independent set. As we pick these elements for all orbits, we obtain a permutation basis. □

5.3 The characteristic series for the dot action on prepermutohedral varieties

We follow the notation in [13] and denote

$$A_{n-1}(t) = \sum_{j=0}^{n-1} \operatorname{ch} H^{2j}(\mathcal{X}^{n-1}) t^j,$$

where $H^{2j}(\mathcal{X}^{n-1})$ denotes the $\mathfrak{S}_n$-representation, and 'ch' is the Frobenius characteristic map. In [13], Procesi observed the following recursive formula

$$A_{n-1}(t) = s_n \sum_{i=0}^{n-1} t^i + \sum_{i=0}^{n-3} s_{n-1-i} A_i(t) \left(\sum_{l=1}^{n-i-2} t^l \right)$$

from the iterated blowup structure of $\mathcal{X}^{n-1}$. Here s_n and s_{n-1-i} are the Schur symmetric functions. The first term in the recursive relation corresponds to the representation on base space $\mathbb{P}^{n-1}$, and for each $i = 0, \ldots, n-3$, the term $s_{n-1-i} A_i(t) \sum_{l=1}^{n-i-2} t^l$ corresponds to the representation on the blowup of i-dimensional coordinate space.

For a prepermutohedral variety $\mathcal{X}_k^{n-1}$, we denote

$$A_{n-1,k}(t) := \sum_{j=0}^{n-1} \operatorname{ch} H^{2j}(\mathcal{X}_k^{n-1}) t^j.$$

Since we also have the iterated blowup structure for $\mathcal{X}_k^{n-1}$, we have a similar recursive formula for $A_{n-1,k}(t)$, with the second summation only goes from 0 to $k-1$. With the more compact notation $[n]_t = \sum_{i=0}^{n-1} t^i$, and with the identity $s_j = h_j$ (the complete homogeneous symmetric functions), the recursive formula can be written as

$$A_{n-1,k}(t) = h_n [n]_t + \sum_{i=0}^{k-1} h_{n-1-i} A_i(t) t [n-i-2]_t. \tag{4}$$

Notice that for $k = 0$, the second term disappears. This recursive formula is sufficient for our purpose, but it might be interesting to derive a close form for $A_{n-1,k}(t)$.

6 The geometry of certain Hessenberg varieties

In this section, we consider Hessenberg varieties associated with Hessenberg functions of the type $h_k = (2, 3, \ldots, k+1, n, \ldots, n)$, for some $k \leqslant n-3$. That is,

$$h_k(j) = \begin{cases} j+1, & j = 1, \ldots, k \\ n, & j = k+1, \ldots, n. \end{cases}$$

We denote the corresponding Hessenberg variety $\mathcal{Y} = \mathcal{H}ess(\mathbf{S}, h_k)$. It is a smooth complex variety of dimension $k + \frac{1}{2}(n-k)(n-k-1)$. There is a morphism from $\mathcal{Y}$ to the Hessenberg type variety $\mathcal{X}_k \cong \mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$ which remembers only the flags up to dimension $k+1$. More precisely, we define

$$f : \mathcal{Y} \to \mathcal{X}_k$$

as follows:

$$(V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots \subset V_n) \longmapsto (V_0 \subset V_1 \subset \cdots \subset V_{k+1}).$$

Since $h_k(j) = n$ for $j \geqslant k+2$, there is no condition imposed from h_k on V_j for $j \geqslant k+2$. (Notice that $h_k(k) = k+1$ implies $\mathbf{S}V_k \subset V_{k+1}$, so there is still condition from h_k for V_{k+1}.) Therefore, the fiber of f over a partial flag $(V_0 \subset \cdots \subset V_{k+1})$ can be identified with the flag variety $\mathcal{F}lag(\mathbb{C}^n/V_{k+1}) \cong \mathcal{F}lag(\mathbb{C}^{n-k-1})$. Hence $\mathcal{Y}$ has a fiber bundle structure over $\mathcal{X}_k$.

Over $\mathcal{Y}$, there is a tautological filtration

$$\mathcal{V}_{k+1} \subset \mathcal{V}_{k+2} \subset \cdots \subset \mathcal{V}_n \cong \mathcal{Y} \times \mathbb{C}^n$$

of vector subbundles of the trivial bundle over $\mathcal{Y}$: the fiber of $\mathcal{V}_j$ over a flag $(V_0 \subset \cdots \subset V_n)$ is the vector space V_j. We then have the line bundles $\mathcal{L}_j := \mathcal{V}_j/\mathcal{V}_{j-1}$, $k+2 \leqslant j \leqslant n$. Set $x_j = -c_1(\mathcal{L}_j)$ to be the negative of the first Chern class of the line bundle $\mathcal{L}_j$, then we have the following description of the cohomology ring $H^*(\mathcal{Y})$.

Proposition 6.1. *The cohomology ring $H^*(\mathcal{Y}) = H^*(\mathcal{Y}; \mathbb{C})$ is generated over $H^*(\mathcal{X}_k)$ by the classes $x_{k+2}, \ldots, x_n$, subject to the relations $e_i(x_{k+2}, \ldots, x_n) = 0$ for $1 \leqslant i \leqslant n-k-1$. That is,*

$$H^*(\mathcal{Y}) \cong H^*(\mathcal{X}_k)[X_{k+2}, \ldots, X_n]/(e_1(X_{k+2}, \ldots, X_n), \ldots, e_{n-k-1}(X_{k+2}, \ldots, X_n)),$$

and the classes $x_{k+2}, \ldots, x_n$ can be identified with the images of $X_{k+2}, \ldots, X_n$ under the quotient. In addition, the classes $x_{k+2}^{i_{k+2}} \cdots x_n^{i_n}$, with exponents $0 \leqslant i_j \leqslant n-j$, form a basis for $H^*(\mathcal{Y})$ over $H^*(\mathcal{X}_k)$.

Proof. The proof mimics the proof for the cohomology of the flag variety [7, Proposition 10.2.3] and is based on basic facts about projective bundles. For more details, see [9, p. 606] or [8, Appendix B.4]. For a vector bundle $\mathcal{V}$ over a variety $\mathcal{X}$, let $\rho : \mathbb{P}(\mathcal{V}) \to \mathcal{X}$

denote the corresponding projective bundle. There is a tautological bundle $\mathcal{L} \subset \rho^*(\mathcal{V})$. Set $\xi = -c_1(\mathcal{L})$, then

$$H^*(\mathbb{P}(\mathcal{V})) \cong H^*(\mathcal{X})[\xi]/(\xi^r + a_1\xi^{r-1} + \cdots + a_r), \tag{5}$$

where $a_i = c_i(\mathcal{V}) \in H^{2i}(\mathcal{X})$.

In this proof we suppress all the notions of pullbacks of bundles. One can construct $\mathcal{Y}$ from $\mathcal{X}_k$ as a sequence of projective bundles. First, over $\mathcal{X}_k$ there is a bundle $\mathcal{U} \to \mathcal{X}_k$ of rank $n-k-1$ whose fiber over the flag $(V_0 \subset \cdots \subset V_{k+1})$ is the vector space $\mathbb{C}^n/V_{k+1}$. The projective bundle $\mathbb{P}(\mathcal{U})$ is the first bundle in the sequence. It gives the direction of the extra dimension of V_{k+2} over the flag $(V_0 \subset \cdots \subset V_{k+1})$. The tautological bundle $\mathcal{U}_1$ of $\mathbb{P}(\mathcal{U})$ pulls back to the line bundle $\mathcal{L}_{k+2}$ on $\mathcal{Y}$.

Next, over $\mathbb{P}(\mathcal{U})$, we have the bundle $\mathcal{U}/\mathcal{U}_1$ of rank $n-k-2$, and we construct the second projective bundle $\mathbb{P}(\mathcal{U}/\mathcal{U}_1) \to \mathbb{P}(\mathcal{U})$. The tautological bundle of $\mathbb{P}(\mathcal{U}/\mathcal{U}_1)$ is of the form $\mathcal{U}_2/\mathcal{U}_1$ for some vector bundle $\mathcal{U}_2$ of rank 2 and $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}$ as bundles over $\mathbb{P}(\mathcal{U}/\mathcal{U}_1)$. Moreover, the tautological bundle of $\mathbb{P}(\mathcal{U}/\mathcal{U}_1)$ pulls back to $\mathcal{L}_{k+3}$ on $\mathcal{Y}$. One can continue this process and construct $\mathbb{P}(\mathcal{U}/\mathcal{U}_2)$ as a projective bundle over $\mathbb{P}(\mathcal{U}/\mathcal{U}_1)$, with tautological bundle of the form $\mathcal{U}_3/\mathcal{U}_2$, and so on. At the end, one arrive at the space $\mathbb{P}(\mathcal{U}/\mathcal{U}_{n-k-1})$ which is isomorphic to the Hessenberg variety $\mathcal{Y}$. Therefore, by the formula 5 and the fact that the tautological bundle of $\mathbb{P}(\mathcal{U}/\mathcal{U}_j)$ pulls back to the line bundle $\mathcal{L}_{k+j+1}$ on $\mathcal{Y}$, we obtain the conclusion that $H^*(\mathcal{Y})$ is generated by $x_{k+2}, \ldots, x_n$ over $H^*(\mathcal{X}_k)$.

The rest of the proof is also very similar to the proof of [7, Proposition 10.2.3]. The reason for $e_i(x_{k+2}, \ldots, x_n) = 0$ is because it is the i-th Chern class of the trivial bundle $\mathcal{V}_n$ defined above; and the isomorphism in the proposition is based on the same algebraic fact stated on [7, p.163]. ☐

The dot action on $H^*(\mathcal{X}_k)$ was described in Section 5. Motivated by the fact that the dot action acts trivially on the usual cohomology of the flag variety, it is natural to conjecture that the action on the basis classes $x_{k+2}^{i_{k+2}} \cdots x_n^{i_n}$ are trivial. While we can not prove this result for the classes, we can prove the result on the isomorphism level with an argument using characteristics series of the representation. The author would like to thank John Shareshian for bringing the article [12] to his attention.

Proposition 6.2. *The dot representation on $\mathcal{Y}$ is isomorphic to the representation on*

$$H^*(\mathcal{X}_k)[X_{k+2}, \ldots, X_n]/(e_1(X_{k+2}, \ldots, X_n), \ldots, e_{n-k}(X_{k+2}, \ldots, X_n))$$

which acts on $H^*(\mathcal{X}_k)$ as described in Section 5, and acts trivially on the $H^*(\mathcal{X}_k)$-basis $x_{k+2}^{i_{k+2}} \cdots x_n^{i_n}$, $0 \leqslant i_j \leqslant n-j$, where the x_i's are the images of the X_i's under the quotient map.

Proof. We have the recursive relation for the characteristic series of $\mathcal{X}_k$ (4), with some terms rearranged:

$$A_{n-1,k}(t) = h_n[n]_t + \sum_{i=0}^{k-1} t[n-i-2]_t A_i(t) h_{n-1-i}. \tag{6}$$

Suppose that S_n acts trivially on $X_{k+2}^{i_{k+2}} \cdots X_n^{i_n}$, $0 \leqslant i_j \leqslant n-j$, then on the characteristic series, what happen would be a degree shifting for each basis element, i.e. multiplying by a power of t. For $X_{k+2}^{i_{k+2}}$, $0 \leqslant i_{k+2} \leqslant n-k-2$, it corresponds to multiplying $1+t+\cdots+t^{n-k-2} = [n-k-1]_t$. Finally, the effect of taking the basis $X_{k+1}^{i_{k+2}} \cdots X_n^{i_n}$, $0 \leqslant i_j \leqslant n-j$ in to consideration on the characteristic series, assuming S_n acts trivially on them, would be multiplying by

$$[n-k-1]_t[n-k-2]_t \cdots [1]_t = [n-k-1]_t!$$

The incomparability graph of the Hessenberg function h_k is the lollipop graph $L_{n-k,k}$. For the chromatic quasisymmetric function $X_{Ln-k,k}(\mathbf{x},t)$, there is a recursive formula [12, Proposition 4.4]

$$X_{L_{n-k,k}}(\mathbf{x},t) = [n-k-1]_t! \left([n]_t e_n + \sum_{i=0}^{k-1} t[n-k+i-1]_t X_{P_{k-i}}(\mathbf{x},t) e_{n-k+i} \right) \tag{7}$$

Setting $i' = k-1-i$ in the sum, we have

$$X_{L_{n-k,k}}(\mathbf{x},t) = [n-k-1]_t! \left([n]_t e_n + \sum_{i'=0}^{k-1} t[n-i'-2]_t X_{P_{i'+1}}(\mathbf{x},t) e_{n-1-i'} \right) \tag{8}$$

Notice that the incomparability graph of the Hessenberg function h_+ for $\mathcal{X}^i$ is P_{i+1}, and it is known that $\omega X_{P_{i+1}}(\mathbf{x},t) = A_i(t)$, where ω is the involution on the ring of symmetric functions. By definition, $\omega e_i = h_i$, therefore, comparing the right sides of (6) and (8), we obtain the following.

$$\omega X_{L_{n-k,k}}(\mathbf{x},t) = [n-k-1]_t! A_{n-1,k}(t)$$

Finally, by a theorem that was originally conjectured by Shareshian and Wachs [14, Conjecture 1.4] and proved independently by Brosnan and Chow [2], and by Guay-Paquet [10], we have the identity.

$$\sum_{j=0}^{\dim(\mathcal{Y})} \operatorname{ch} H^{2j}(\mathcal{Y}) t^j = \omega X_{L_{n-k,k}}(\mathbf{x},t)$$

Therefore we can conclude that $\sum_{j=0}^{\dim(\mathcal{Y})} \operatorname{ch} H^{2j}(\mathcal{Y}) t^j = [n-k-1]_t! A_{n-1,k}(t)$. Since the representation is determined by the characteristic series up to isomorphism, this concludes the proof. $\square$

We remark that the proposition also implies the dot representation is a permutation representation on $H^*(\mathcal{Y})$.

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