On the number of partitions of n into exactly m parts whose even parts are distinct

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Abstract

Let $ped(n)$ be the number of partitions of n whose even parts are distinct and whose odd parts are unrestricted. For a positive integer m, let $ped(n, m)$ be the number of all possible partitions of the number n into exactly m parts whose even parts are distinct and whose odd parts are unrestricted. In this paper, we give new recurrence formulas for $ped(n, m)$ as well as explicit formulas for $ped(n, m)$, when $m = 2, 3$ and $m = 4$. For a positive integer q and $j \in \{0, 1, 2, \dots, q-1\}$, we also give a recurrence formula for $p_{q,j}(n, m)$ the number of partitions of n into m parts such that the parts congruent to $-j$ modulo q are distinct where other parts are unrestricted.

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1 Introduction and results

A partition of a positive integer n [2, Chapter 14] is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = n.$$

The λ_i's are called the parts of the partition. The number of parts is unrestricted, repetition is allowed, and the order of the parts is not taken into account (nevertheless, the usual assumption is that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m$).

Let $ped(n)$ be the number of partitions of n with distinct even parts (while odd parts are unrestricted). The function $ped(n)$ has been studied by many authors, see [1, 4, 5, 7, 9]. In 2008, Fink, Guy and Krusemeyer [6] gave a recurrence relation for $ped(n)$ as

$$\sum_{j=-\infty}^{\infty} (-1)^j ped\left(n - \frac{j(3j+1)}{2}\right) = \begin{cases} (-1)^k, & \text{if } n = 2k(3k+1),\ k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

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Recently, Merca [8] provided the following new recurrence relations for $ped(n)$ that involve the triangular numbers $T_k = \frac{k(k+1)}{2}$, where $k \in \mathbb{N}_0$,

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} ped(n - T_j) = \begin{cases} 1, & \text{if } n = 2T_k,\ k \in \mathbb{N}_0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{j=-\infty}^{\infty} (-1)^j ped(n - 2j^2) = \begin{cases} 1, & \text{if } n = T_k,\ k \in \mathbb{N}_0; \\ 0, & \text{otherwise.} \end{cases}$$

Finding a new formula for the numbers of partitions of n with distinct even parts is a problem of current interest. For a positive integer m, let $p(n,m)$ be the number of partitions of the number n into exactly m parts, whereas $ped(n,m)$ is the number of partitions of the number n into exactly m parts with distinct even parts (while odd parts are unrestricted).

Let us determine the values $ped(n,1), ped(n,2), \ldots, ped(n,n)$. Some of these are trivial, such as

$$ped(n,1) = 1 \text{ and } ped(n,n) = 1 \text{ for all } n \in \mathbb{N};\ ped(n,n-1) = 1 \text{ for all } n > 1.$$

It is obvious that, for $n \geqslant 1$,

$$ped(n) = ped(n,1) + ped(n,2) + \cdots + ped(n,n),$$

which is similar to the formula

$$p(n) = p(n,1) + p(n,2) + \cdots + p(n,n).$$

In particular, the recurrence formula for $p(n,m)$ is

$$p(n,m) = p(n-1,m-1) + p(n-m,m).$$

In this paper, we will give a recurrence formula for $ped(n,m)$. We obtain the following result.

Theorem 1. *For any positive integers $n \geqslant 1$ and $1 \leqslant m \leqslant n$, we have*

$$ped(n,m) = ped(n-1,m-1) + ped(n-2m,m-1) + ped(n-2m,m).$$

Here, $ped(n,m) = 0$ when $n \leqslant 0$ and $m > 0$.

Moreover, we also give explicit formulas for $ped(n,m)$, when $m = 2,3$ and $m = 4$. For integers n and q with $q > 0$, define

$$\chi_q(n) = \begin{cases} 1, & \text{if } q \nmid n; \\ 0, & \text{if } q \mid n. \end{cases}$$

Using an elementary method as in [3], we obtain the following theorems.

Theorem 2. *For any positive integers $n \geqslant 1$, we have*

$$ped(n,2) = \left\lfloor \frac{n}{2} \right\rfloor - \chi_4(n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } 4 \nmid n; \\ \left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } \ 4 \mid n. \end{cases}$$

Theorem 3. *For any positive integer n, we have*

$$ped(n,3) = \left\lfloor \frac{n^2+6}{12} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

More precisely, if $n = 12k + i$ for some integers $k \geqslant 0$ and $i \in \{0, 1, 2, \ldots, 11\}$, then we have

$$ped(12k+i,3) = \begin{cases} 12k^2 - 3k + 1, & \text{if } i = 0; \\ 12k^2 - k, & \text{if } \ i = 1; \\ 12k^2 + k, & \text{if } \ i = 2; \\ 12k^2 + 3k + 1, & \text{if } \ i = 3; \\ 12k^2 + 5k + 1, & \text{if } \ i = 4; \\ 12k^2 + 7k + 1, & \text{if } \ i = 5; \\ 12k^2 + 9k + 2, & \text{if } \ i = 6; \\ 12k^2 + 11k + 3, & \text{if } \ i = 7; \\ 12k^2 + 13k + 4, & \text{if } i = 8; \\ 12k^2 + 15k + 5, & \text{if } \ i = 9; \\ 12k^2 + 17k + 6, & \text{if } \ i = 10; \\ 12k^2 + 19k + 8, & \text{if } \ i = 11. \end{cases}$$

We can easily deduce Theorem 3 in the following form.

Corollary 4. *For $n \geqslant 1$, we have*

$$ped(n,3) = \begin{cases} \frac{n^2}{12} - \frac{n}{4} + 1, & \text{if } 12 \mid n; \\ 12 \left\lfloor \frac{n}{12} \right\rfloor^2 + \Big(2(n \bmod 12) - 3\Big) \left\lfloor \frac{n}{12} \right\rfloor + ped((n \bmod 12), 3), & \text{if } 12 \nmid n. \end{cases}$$

However, the value of $ped(n,3)$ can be obtained easily in the form of $p(n,3)$. This gives us a new explicit formula for $p(n,3)$ in the form of $ped(n,3)$. The number of partitions of n into exactly 3 parts is equal to the sum of the number of partitions of n into exactly 3 parts with distinct even parts and the number of partitions of n into exactly 3 parts with at least two parts being even. We note that a partition of n into exactly 3 parts with at least two parts being even is of the form $(n-4k, 2k, 2k)$, for $1 \leqslant k \leqslant \left\lfloor \frac{n-1}{4} \right\rfloor$. Thus, we obtain the following corollary.

Corollary 5. *For $n \geqslant 1$, we have*

$$p(n,3) = \left\lfloor \frac{n-1}{4} \right\rfloor + \begin{cases} \frac{n^2}{12} - \frac{n}{4} + 1, & \text{if } 12 \mid n; \\ 12 \left\lfloor \frac{n}{12} \right\rfloor^2 + \Big(2(n \bmod 12) - 3\Big) \left\lfloor \frac{n}{12} \right\rfloor + ped((n \bmod 12), 3), & \text{if } 12 \nmid n. \end{cases}$$

More generally, let q be a positive integer and $j \in \{0, 1, 2, \dots, q-1\}$. Denote by $p_{q,j}(n, m)$ the number of partitions of n into m parts such that the parts congruent to $-j$ modulo q are distinct (other parts are unrestricted). In particular, $ped(n, m)$ is simply $p_{2,0}(n, m)$. We have the following relationship between $p(n, m)$ and $p_{q,j}(n, m)$.

Theorem 6. *For any positive integers n and m such that $m \leqslant n$, we have*

$$p_{q,j}(n,m) = p(n,m) - \sum_{\ell=1}^{\infty} \sum_{r=1}^{\infty} p(r,\ell)\, p_{q,j}(n + 2j\ell - 2qr, m - 2\ell).$$

Here, $p_{q,j}(n, m) = 0$ when $n \leqslant 0$ and $m > 0$, or when $m < 0$. Additionally, $p_{q,j}(0, 0) = 1$ and $p_{q,j}(n, 0) = 0$ for $n \neq 0$.

The following corollary follows immediately from Theorem 6.

Corollary 7. *For any positive integers n and m such that $m \leqslant n$, we have*

$$ped(n,m) = p(n,m) - \sum_{\ell=1}^{\infty} \sum_{r=1}^{\infty} p(r,\ell)\, ped(n - 4r, m - 2\ell).$$

For $m = 4$, we obtain the following corollary.

Corollary 8. *For every positive integer n, we have*

$$ped(n,4) = \left\lfloor \frac{n^3}{144} + \frac{n^2}{48} - \frac{n}{16}\,(n \bmod 2) + \frac{1}{2} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{n-2}{2} \right\rfloor^2 \right\rfloor + \left\lfloor \frac{n-4}{8} \right\rfloor \chi_4(n).$$

More precisely, if $n = 12k + i$ for some integers $k \geqslant 0$ and $i \in \{0, 1, 2, \dots, 11\}$, then we have

$$ped(12k+i,4) = \begin{cases} 12k^3 - 6k^2 + 5k - \left\lfloor \frac{k}{2} \right\rfloor - 1, & \text{if } i = 0; \\ 12k^3 - 3k^2 + 3k, & \text{if } i = 1; \\ 12k^3 + 2k, & \text{if } i = 2; \\ 12k^3 + 3k^2 + 3k, & \text{if } i = 3; \\ 12k^3 + 6k^2 + 4k + \left\lfloor \frac{k}{2} \right\rfloor + 1, & \text{if } i = 4; \\ 12k^3 + 9k^2 + 5k + 1, & \text{if } i = 5; \\ 12k^3 + 12k^2 + 6k + 1, & \text{if } i = 6; \\ 12k^3 + 15k^2 + 9k + 2, & \text{if } i = 7; \\ 12k^3 + 18k^2 + 13k - \left\lfloor \frac{k}{2} \right\rfloor + 3, & \text{if } i = 8; \\ 12k^3 + 21k^2 + 15k + 4, & \text{if } i = 9; \\ 12k^3 + 24k^2 + 18k + 5, & \text{if } i = 10; \\ 12k^3 + 27k^2 + 23k + 7, & \text{if } i = 11. \end{cases}$$

2 Proofs

Proof of Theorem 1. Let $\mathcal{A}$ be the set of all partitions of positive integers with distinct even parts and unrestricted odd parts. Let $\mathcal{A}(n,m)$ be the set of all partitions of the number n in $\mathcal{A}$ into exactly m parts. Note that $|\mathcal{A}(n,m)| = ped(n,m)$. Consider the following two sets P_1 and P_2, where P_1 is the set of partitions of the number n in $\mathcal{A}$ into exactly m parts with at least one part being 1 and P_2 is the set of partitions of the number n in $\mathcal{A}$ into exactly m parts containing parts greater than 1. By definition, we have

$$|\mathcal{A}(n,m)| = P_1 \cup P_2 \text{ and } P_1 \cap P_2 = \emptyset.$$

Thus,

$$ped(n,m) = |P_1| + |P_2|.$$

Since the last part of all elements in P_1 is 1, then, $|P_1| = ped(n-1, m-1)$. To compute the number of elements in P_2, we write $P_2 = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$, where T_1 is the set of partitions of the number n in P_2 into exactly m parts with at least one part being 2 and T_2 is the set of partitions of the number n in P_2 into exactly m parts containing parts greater than 2.

To compute the numbers of element in T_1, we first add 2 to all m parts. Since the partition has distinct even parts, we fix only the last part to be 2. Then there is a one-to-one function from the set T_1 to the set $\mathcal{A}(n-2m, m-1)$, namely $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, 2) \to (\lambda_1 - 2, \lambda_2 - 2, \ldots, \lambda_{m-1} - 2)$. Thus, $|T_1| = ped(n-2m, m-1)$. Similarly, there is a one-to-one function from the set T_2 to the set $ped(n-2m, m)$, namely

$$(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, 2) \to (\lambda_1 - 2, \lambda_2 - 2, \ldots, \lambda_{m-1} - 2, \lambda_m - 2).$$

Thus, $|T_2| = ped(n-2m, m)$. Then, we have

$$ped(n,m) = ped(n-1, m-1) + ped(n-2m, m-1) + ped(n-2m, m).$$

□

Proof of Theorem 2. Let $n \geqslant 2$. We consider the following partitions:

$$\big((n-1),1\big), \big((n-2),2\big), \big((n-3),3\big), \ldots, \left(n - \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right).$$

These partitions have distinct even parts, if $4 \nmid n$. If $4 \mid n$, then the partition $\left(n - \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right)$ has repeated even parts. Thus, we have

$$ped(n,2) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } 4 \nmid n; \\ \left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } 4 \mid n. \end{cases}$$

□

Proof of Theorem 3. Let $n \geqslant 3$. We start with those partitions with distinct even parts and having 3 parts, where the last part is j for some positive integer $j \leqslant \lfloor \frac{n}{3} \rfloor$. Such a partition λ takes the form $\lambda = (n-i-j, i, j)$, where i is a positive integer such that $j \leqslant i \leqslant \lfloor \frac{n-j}{2} \rfloor$. We consider two cases: j is odd or j is even.

Suppose first that j is odd. Then, i can be any integer from $j, j+1, j+2, \ldots, \lfloor \frac{n-j}{2} \rfloor$, with the possible exception when $4 \mid n-j$, that is, if $4 \mid n-j$, i cannot be equal to $\lfloor \frac{n-j}{2} \rfloor = \frac{n-j}{2}$, as the even number $\frac{n-j}{2}$ is repeated in the partition λ. Hence, the number N_j of possible partitions in this case is

$$N_j = \left\lfloor \frac{n-j}{2} \right\rfloor - j + 1 - \chi_4(n-j) = \begin{cases} \lfloor \frac{n-j}{2} \rfloor - j + 1, & \text{if } 4 \nmid n-j; \\ \lfloor \frac{n-j}{2} \rfloor - j, & \text{if } 4 \mid n-j. \end{cases}$$

Assume now that j is even. Then, $i \neq j$. In the subcase $n \geqslant 3j+1$, we note that i can take any value from $j+1, j+2, \ldots, \lfloor \frac{n-j}{2} \rfloor$, with the possible exception when $4 \mid n-j$. As before, if $4 \mid n-j$, then i cannot be equal to $\lfloor \frac{n-j}{2} \rfloor = \frac{n-j}{2}$. Hence, the number of possible partitions N_j in this scenario is

$$N_j = \left\lfloor \frac{n-j}{2} \right\rfloor - j - \chi_4(n-j) = \begin{cases} \lfloor \frac{n-j}{2} \rfloor - j, & \text{if } 4 \nmid n-j; \\ \lfloor \frac{n-j}{2} \rfloor - j - 1, & \text{if } 4 \mid n-j. \end{cases} \tag{1}$$

However, if $n = 3j$, then $N_j = 0$.

From the work above, we obtain

$$N_j = \begin{cases} \lfloor \frac{n-j}{2} \rfloor - j + 1 - \chi_2(j) - \chi_4(n-j), & \text{if } 2 \nmid j \text{ or } n \neq 3j; \\ 0, & \text{if } 2 \mid j \text{ and } n = 3j. \end{cases}$$

We have $ped(n,3) = \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} N_j$. We shall now consider two cases:

- $n \not\equiv 0 \pmod 6$;
- $n \equiv 0 \pmod 6$.

We can easily prove by induction that, for every positive integer p, $\sum_{\ell=1}^{p} \lfloor \frac{\ell}{2} \rfloor = \left\lfloor \frac{p^2}{4} \right\rfloor$.

Case 1. $\boldsymbol{n \not\equiv 0 \pmod 6}$

We get

$$\begin{aligned} ped(n,3) &= \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} N_j = \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} \left(\left\lfloor \frac{n-j}{2} \right\rfloor - j + 1 - \chi_2(j) - \chi_4(n-j) \right) \\ &= \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n-j}{2} \right\rfloor - \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} j + \left\lfloor \frac{n}{3} \right\rfloor - \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} \chi_2(j) - \sum_{j=1}^{\lfloor \frac{n}{3} \rfloor} \chi_4(n-j) \\ &= \sum_{j=n-\lfloor \frac{n}{3} \rfloor}^{n-1} \left\lfloor \frac{j}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \right\rfloor - \sum_{j=n-\lfloor \frac{n}{3} \rfloor}^{n-1} \chi_4(j). \end{aligned}$$

Note that $n - \left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{2n+2}{3} \right\rfloor = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1$ and that $\left\lfloor \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \right\rfloor = \left\lfloor \frac{n}{6} \right\rfloor$. Thus,

$$\begin{aligned} ped(n,3) &= \sum_{j=\left\lfloor \frac{2n-1}{3} \right\rfloor+1}^{n-1} \left\lfloor \frac{j}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{6} \right\rfloor - \sum_{j=\left\lfloor \frac{2n-1}{3} \right\rfloor+1}^{n-1} \chi_4(j) \\ &= \left(\left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor \right) - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) \\ &\quad - \left\lfloor \frac{n}{6} \right\rfloor - \left(\left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor \right\rfloor \right). \end{aligned}$$

Therefore,

$$ped(n,3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{2n-1}{12} \right\rfloor \quad (2)$$

Case 2. $n \equiv 0 \pmod 6$

The only difference between Case 1 and Case 2 comes from the fact that $N_{\left\lfloor \frac{n}{3} \right\rfloor} = N_{\frac{n}{3}} = 0$, but if we were to use Equation (1), we would get $N_{\frac{n}{3}} = -1$. As a result, Equation (2) underestimates the actual value of $ped(n,3)$ by 1. In other words, the correct value of $ped(n,3)$ in this case is given by

$$\begin{aligned} ped(n,3) &= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor \\ &\quad + \left\lfloor \frac{2n-1}{12} \right\rfloor + 1. \end{aligned}$$

Combining Case 1 and Case 2 yields

$$\begin{aligned} ped(n,3) &= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor \\ &\quad + \left\lfloor \frac{2n-1}{12} \right\rfloor + \chi_6(n). \end{aligned}$$

Since $\left\lfloor \frac{n}{6} \right\rfloor - \chi_6(n) = \left\lfloor \frac{n-1}{6} \right\rfloor$ and $\left\lfloor \frac{2n-1}{12} \right\rfloor = \left\lfloor \frac{1}{6} \left\lfloor \frac{2n-1}{2} \right\rfloor \right\rfloor = \left\lfloor \frac{n-1}{6} \right\rfloor$, it follows that

$$ped(n,3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Let now

$$F(n) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4} \left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor - \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right).$$

Observe that

$$
\begin{aligned}
F(n+12)-F(n) &= \left\lfloor \frac{(n+11)^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{1}{4}\left\lfloor \frac{2n+23}{3} \right\rfloor^2 \right\rfloor + \left\lfloor \frac{1}{4}\left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor \\
&\quad - \frac{1}{2}\left\lfloor \frac{n+12}{3} \right\rfloor \left(\left\lfloor \frac{n+12}{3} \right\rfloor - 1 \right) + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) \\
&= \left\lfloor \frac{(n-1)^2}{4} + 6n + 30 \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \\
&\quad - \left\lfloor \frac{1}{4}\left\lfloor \frac{2n-1}{3} \right\rfloor^2 + 4\left\lfloor \frac{2n-1}{3} \right\rfloor + 16 \right\rfloor + \left\lfloor \frac{1}{4}\left\lfloor \frac{2n-1}{3} \right\rfloor^2 \right\rfloor \\
&\quad - \frac{1}{2}\left(\left\lfloor \frac{n}{3} \right\rfloor + 4 \right)\left(\left\lfloor \frac{n}{3} \right\rfloor + 3 \right) + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F(n+12)-F(n) &= (6n+30) - \left(4\left\lfloor \frac{2n-1}{3} \right\rfloor + 16\right) - \left(4\left\lfloor \frac{n}{3} \right\rfloor + 6\right) \\
&= 6n - 4\left\lfloor \frac{n}{3} \right\rfloor - 4\left\lfloor \frac{2n-1}{3} \right\rfloor + 8.
\end{aligned}
$$

Recall that $n - \lfloor \frac{n}{3} \rfloor = \lfloor \frac{2n+2}{3} \rfloor = \lfloor \frac{2n-1}{3} \rfloor + 1$, which means $n-1 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{2n-1}{3} \rfloor$. As a result,

$$F(n+12)-F(n) = 6n - 4(n-1) + 8 = 2n+12.$$

As $F(1) = F(2) = 0$, $F(3) = F(4) = 1$, $F(5) = 2$, $F(6) = 3$, $F(7) = 4$, $F(8) = 5$, $F(9) = 7$, $F(10) = 8$, $F(11) = 10$, and $F(12) = 12$, we can easily prove by induction that $F(n) = \left\lfloor \frac{n^2+6}{12} \right\rfloor$. Since $ped(n,3) = F(n) - \lfloor \frac{n-1}{4} \rfloor$, we obtain

$$ped(n,3) = \left\lfloor \frac{n^2+6}{12} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

□

Proof of Theorem 6. For convenience, let $P_{q,j}(n,m)$ be the set of partitions of n into m parts such that the parts congruent to $-j$ modulo q are distinct. Thus, $p_{q,j}(n,m) = |P_{q,j}(n,m)|$.

In contrast, let $\bar{P}_{q,j}(n,m)$ be the set of partitions of n into m parts such that at least one part congruent to $-j$ modulo q is repeated. Write $\bar{p}_{q,j}(n,m) = \left|\bar{P}_{q,j}(n,m)\right|$. We prove the following equality:

$$\bar{p}_{q,j}(n,m) = \sum_{\ell=1}^{\infty} \sum_{r=1}^{\infty} p(r,\ell)\, p_{q,j}(n+2j\ell-2qr, m-2\ell). \tag{3}$$

Fix $\lambda \in \bar{P}_{q,j}(n,m)$. Suppose that $t_1 > t_2 > \ldots > t_k$ are all the positive integers such that, for each $i = 1, 2, \ldots, k$, there are $s_i (> 0)$ parts in λ of size $qt_i - j$. Define $\ell = \sum_{i=1}^{k} \left\lfloor \frac{s_i}{2} \right\rfloor$ and $r = \sum_{i=1}^{k} t_i \left\lfloor \frac{s_i}{2} \right\rfloor$. Since $\lambda \in \bar{P}_{q,j}(n,m)$, we conclude that $s_i \geqslant 2$ for some $i = 1, 2, \ldots, k$, whence $\ell \geqslant 1$ and $r \geqslant 1$.

The ℓ-term sequence $f(\lambda)$ consisting of $\left\lfloor \frac{s_i}{2} \right\rfloor$ copies of t_i for $i = 1, 2, \ldots, k$ is clearly a partition of r into ℓ parts. Now, we define $g(\lambda)$ to be the sequence consisting of the following terms:

- all terms of λ that are not congruent to $-j$ modulo q, and
- one copy of $qt_i - j$ for all $i = 1, 2, \ldots, k$ such that s_i is an odd integer.

Observe that $g(\lambda)$ is a partition into $m - 2\ell$ parts of

$$n - \sum_{i=1}^{k} 2 \left\lfloor \frac{s_i}{2} \right\rfloor (qt_i - j) = n + 2j \sum_{i=1}^{k} \left\lfloor \frac{s_i}{2} \right\rfloor - 2q \sum_{i=1}^{k} t_i \left\lfloor \frac{s_i}{2} \right\rfloor = n + 2j\ell - 2qr.$$

Note that the parts congruent to $-j$ modulo q of $g(\lambda)$ do not repeat. Therefore, $g(\lambda) \in P_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$.

We have established a map $\lambda \mapsto \big(f(\lambda), g(\lambda)\big)$, where $f(\lambda)$ is a partition of $r \geqslant 1$ into $\ell \geqslant 1$ parts and $g(\lambda)$ is an element of $P_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$. Conversely, for a given partition ϕ of $r (\geqslant 1)$ into $\ell (\geqslant 1)$ parts and for a given $\gamma \in P_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$, we can create an element $\Lambda(\phi, \gamma) \in \bar{P}_{q,j}(n,m)$ as follows: assume that ϕ consists of σ_μ copies of τ_μ for $\mu = 1, 2, \ldots, \nu$. For all $\mu = 1, 2, \ldots, \nu$, we add $2\sigma_\mu$ copies of $q\tau_\mu - j$ to γ. Let $\Lambda(\phi, \gamma)$ be the resulting sequence. Obviously, $\Lambda(\phi, \gamma)$ is an element of $\bar{P}_{q,j}(n,m)$.

Clearly, the maps $\lambda \mapsto \big(f(\lambda), g(\lambda)\big)$ and $(\phi, \gamma) \mapsto \Lambda(\phi, \gamma)$ are inverses of each other. Hence, $\bar{p}_{q,j}(n,m) = \left|\bar{P}_{q,j}(n,m)\right|$ is the number of pairs (ϕ, γ), where ϕ is a partition of r into ℓ parts for some positive integers r and ℓ, and γ is an element of $P_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$. For fixed values of r and ℓ, there are $p(r, \ell)$ choices of ϕ, whilst there are $p_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$ ways to choose γ. Consequently, there are in total $p(r,\ell)\, p_{q,j}(n + 2j\ell - 2qr, m - 2\ell)$ ways to pick (ϕ, γ) for any positive integers r and ℓ. Equation (3) is now evident. □

Proof of Corollary 8. From Corollary 7, we have

$$ped(n,4) = p(n,4) - \sum_{r=1}^{\infty} p(r,1)\, ped(n-4r,2) - \sum_{r=1}^{\infty} p(r,2)\, ped(n-4r,0).$$

According to Theorem 2, $ped(n-4r,2) = \left\lfloor \frac{n-4r}{2} \right\rfloor - \chi_4(n-4r) = \left\lfloor \frac{n}{2} \right\rfloor - 2r - \chi_4(n)$ for $r = 1, 2, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor$, whereas $ped(n-4r,2) = 0$ for $r > \left\lfloor \frac{n-1}{4} \right\rfloor$. Furthermore, the sum $\sum_{r=1}^{\infty} p(r,2)\, ped(n-4r,0)$ is nonzero only when $4 \mid n$, in which case $\sum_{r=1}^{\infty} p(r,2)\, ped(n-$

$4r, 0) = p\left(\frac{n}{4}, 2\right) = \left\lfloor \frac{1}{2}\left(\frac{n}{4}\right)\right\rfloor = \left\lfloor \frac{n}{8} \right\rfloor$. Consequently,

$$\begin{aligned} ped(n,4) &= p(n,4) - \sum_{r=1}^{\left\lfloor \frac{n-1}{4} \right\rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor - 2r - \chi_4(n) \right) - \left\lfloor \frac{n}{8} \right\rfloor \chi_4(n) \\ &= p(n,4) - \sum_{r=1}^{\left\lfloor \frac{n-1}{4} \right\rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor - 2r \right) + \left\lfloor \frac{n-1}{4} \right\rfloor \chi_4(n) - \left\lfloor \frac{n}{8} \right\rfloor \chi_4(n) \\ &= p(n,4) - \sum_{r=1}^{\left\lfloor \frac{n-1}{4} \right\rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor - 2r \right) + \left(\left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor \right) \chi_4(n). \end{aligned}$$

Observe that $\left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor = \left\lfloor \frac{n-4}{8} \right\rfloor$ for every positive integer n such that $4 \mid n$. Moreover, we can easily prove by induction that, for any positive integer p, the sum of all positive integers $i < p$ such that $i \equiv p \pmod 2$ is given by $\left\lfloor \frac{(p-1)^2}{4} \right\rfloor$. Therefore,

$$\begin{aligned} ped(n,4) &= p(n,4) - \left\lfloor \frac{1}{4} \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right)^2 \right\rfloor + \left\lfloor \frac{n-4}{8} \right\rfloor \chi_4(n) \\ &= p(n,4) - \left\lfloor \frac{1}{4} \left\lfloor \frac{n-2}{2} \right\rfloor^2 \right\rfloor + \left\lfloor \frac{n-4}{8} \right\rfloor \chi_4(n). \end{aligned}$$

It remains to show that

$$p(n,4) = \left\lfloor \frac{n^3}{144} + \frac{n^2}{48} - \frac{n}{16} (n \bmod 2) + \frac{1}{2} \right\rfloor \tag{4}$$

For convenience, write $a_n = \left\lfloor \frac{n^3}{144} + \frac{n^2}{48} - \frac{n}{16} (n \bmod 2) + \frac{1}{2} \right\rfloor$. Then

$$a_{n+12} - a_n = \begin{cases} \frac{(n-1)^2}{4} + 4n + 14, & \text{if } 2 \nmid n; \\ \frac{n^2}{4} + \frac{7n}{2} + 15, & \text{if } 2 \mid n. \end{cases}$$

Note that $a_1 = a_2 = a_3 = 0$, $a_4 = a_5 = 1$, $a_6 = 2$, $a_7 = 3$, $a_8 = 5$, $a_9 = 6$, $a_{10} = 9$, $a_{11} = 11$, and $a_{12} = 15$.

To calculate $p(n,4)$, we shall employ Burnside's lemma. Consider the set

$$T_n = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \,\middle|\, x_1, x_2, x_3, x_4 > 0 \text{ and } x_1 + x_2 + x_3 + x_4 = n \right\}.$$

The symmetric group on 4 symbols, or S_4, acts on T_n in the natural way. We want to find the number of orbits of T_n under S_4. Now, S_4 has 4 conjugacy classes:

- C_1: the identity class (1 element),
- C_2: the class of transpositions (6 elements),

- C_3: the class of 3-cycles (8 elements),
- C_4: the class of products of two disjoint transpositions (3 elements), and
- C_5: the class of 4-cycles (6 elements).

For $g \in S_4$, let T_n^g denote the number of elements of T_n fixed by the action of g.

The number of elements of T_n fixed by an element of the class C_1 (the identity element e of S_4) is T_n. It is easily seen that

$$|T_n^e| = |T_n| = \binom{n-1}{3}.$$

The number of elements of T_n fixed by an element $g \in C_2$ is the same as the number of tuples $(x_1, x_2, x_3, x_4) \in T_n$ such that $x_1 = x_2$. If $x_1 = x_2 = j$, then $x_3 + x_4 = n - 2j$. Since $x_3 \geqslant 1$ and $x_4 \geqslant 1$, it holds that $n - 2j \geqslant 2$, whence $j \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor$. Because we can choose x_3 in $n - 2j - 1$ ways, we conclude that

$$|T_n^g| = \sum_{j=1}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (n - 2j - 1) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor.$$

The number of elements of T_n fixed by an element $g \in C_3$ is the same as the number of tuples $(x_1, x_2, x_3, x_4) \in T_n$ such that $x_1 = x_2 = x_3$. If $x_1 = x_2 = x_3 = j$, then from $x_1 + x_2 + x_3 + x_4 = n$ and $x_4 \geqslant 1$, we deduce that $j \leqslant \frac{n-1}{3}$. Hence,

$$|T_n^g| = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

The number of elements of T_n fixed by an element $g \in C_4$ is the same as the number of tuples $(x_1, x_2, x_3, x_4) \in T_n$ such that $x_1 = x_2$ and $x_3 = x_4$. If $x_1 = x_2 = j$ and $x_3 = x_4 = k$, then $n = 2j + 2k$ or $j + k = \frac{n}{2}$. Thus, n must be even, in which case $|T_n^g| = \frac{n}{2} - 1$. In general,

$$|T_n^g| = \left(\frac{n}{2} - 1\right) \chi_2(n).$$

Finally, the number of elements of T_n fixed by an element $g \in C_4$ is the same as the number of tuples $(x_1, x_2, x_3, x_4) \in T_n$ such that $x_1 = x_2 = x_3 = x_4$. Since $x_1 + x_2 + x_3 + x_4 = n$, it follows that $4 \mid n$ and $x_1 = x_2 = x_3 = x_4 = \frac{n}{4}$ is the only solution. In general,

$$|T_n^g| = \chi_4(n).$$

Note that $p(n, 4)$ is precisely the number of orbits of T_n under S_4. According to Burnside's lemma,

$$p(n, 4) = \frac{1}{|S_4|} \sum_{g \in S_4} |T_n^g|.$$

Then,

$$p(n,4) = \frac{1}{24}\left(1\cdot\binom{n-1}{3} + 6\cdot\left\lfloor\frac{(n-2)^2}{4}\right\rfloor + 8\cdot\left\lfloor\frac{n-1}{3}\right\rfloor + 3\cdot\left(\frac{n}{2}-1\right)\chi_2(n) + 6\cdot\chi_4(n)\right).$$

Evidently, $p(n,4) = a_n$ for all $n = 1, 2, \ldots, 12$.

We can easily verify that

$$p(n+12,4) - p(n,4) = \begin{cases} \frac{(n-1)^2}{4} + 4n + 14, & \text{if } 2 \nmid n; \\ \frac{n^2}{4} + \frac{7n}{2} + 15, & \text{if } 2 \mid n. \end{cases}$$

Hence, the sequences $\big(p(n,4)\big)_{n=1}^{\infty}$ and $\big(a_n\big)_{n=1}^{\infty}$ satisfy the same recurrence relation, and share the same initial values. In conclusion, $p(n,4) = a_n$ for every $n = 1, 2, 3, \ldots$. The assertion is now proven. □

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