New Equidistributions on Plane Trees and Decompositions of 132-Avoiding Permutations

Zi-Wei Bai Ricky X. F. Chen

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Abstract

Our main results in this paper are new equidistributions on plane trees and 132-avoiding permutations, two closely related and ubiquitous objects. As for the former, we discover a characteristic for vertices of plane trees that is equally distributed as the height for vertices. The latter is concerned with four distinct ways of decomposing a 132-avoiding permutation into subsequences. We show combinatorially that the subsequence length distributions of the four decompositions are mutually equal, and there is a way to group the four into two groups such that each group is symmetric and the joint length distribution of one group is the same as that of the other. Some consequences are discussed. For instance, we provide a new refinement of the fundamental equidistribution of internal vertices and leaves, and present new sets of 132-avoiding permutations that are counted by the Motzkin numbers and their refinements.

Mathematics Subject Classifications: 05C05, 05A19, 05A15

1 Introduction

Permutations with or without certain patterns have been extensively studied since Knuth's work [13]. Let $[n] = \{1, 2, \dots, n\}$ and $\mathfrak{S}_n$ be the symmetric group of permutations on $[n]$. Let $\tau = \tau_1\tau_2 \cdots \tau_m \in \mathfrak{S}_m$ with $m \leqslant n$. A permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$ is said to have a pattern τ if there exists a subsequence $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ of π such that $\pi_{i_j} < \pi_{i_k}$ if and only if $\tau_j < \tau_k$. If π does not have the pattern τ, π is called τ-avoiding. The permutation π is said to have a consecutive pattern τ if there exists a subsequence $\pi_i\pi_{i+1} \cdots \pi_{i+m-1}$ of π that provides an occurrence of the pattern τ.

It is well understood that the number of permutations on $[n]$ avoiding a pattern τ of length three is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ for any $\tau \in \mathfrak{S}_3$. It is also very well known that C_n counts plane trees of n edges and respectively Dyck paths of semilength n. A bijection between 132-avoiding permutations and plane trees was given in

School of Mathematics, Hefei University of Technology, 485 Danxia Road, Hefei, Anhui 230601, P. R. China (`2021111458@mail.hfut.edu.cn`, `xiaofengchen@hfut.edu.cn`).

Jani and Rieper [11], while a bijection between 132-avoiding permutations and Dyck paths was given in Krattenthaler [14]. We refer to Claesson and Kitaev [5] and references therein for more detailed discussion on bijections related to permutations avoiding a length three pattern.

This paper is mainly concerned with the set $\mathfrak{S}_n(132)$ of 132-avoiding permutations on $[n]$ and plane trees. We examine four types of decompositions of permutations into subsequences. When restricted to 132-avoiding permutations, two of the four decompositions refine ascents and others refine descents. In fact, two of them are respectively increasing run and decreasing run decompositions which have been studied, for instance, in Zhuang [19], and Elizalde and Noy [8]. One of our main results states that the length distributions of the subsequences of the four decompositions are mutually equivalent, and there is a way to group the four into two groups such that each group is symmetric and the joint length distribution of one group is the same as that of the other. We prove this combinatorially, connecting several bijections, some are well-known and some are recently discovered or new. As a consequence, we are able to enumerate 132-avoiding permutations according to a variety of filtrations. For instance, taking advantage of a result of Elizalde and Mansour [7], we present four sets of 132-avoiding permutations that are respectively counted by the Motzkin numbers. We additionally carry out some refined enumeration of these sets.

One employed new bijection between plane trees and 132-avoiding permutations also allows us to derive a new equidistribution result on plane trees concerning height for vertices. As a corollary, we immediately recover the fundamental fact that internal vertices and leaves of plane trees are equidistributed.

The paper is organized as follows. In Section 2, we introduce the four types of decompositions and present some basic properties. In Section 3, several relevant bijections are reviewed. In Section 4, we present a bijection between plane trees and 132-avoiding permutations that appears new. As a consequence, we introduce the right spanning width of vertices and show this new characteristic is equally distributed as the height of vertices. Finally, we prove the equidistribution result of the four decompositions and provide a number of enumerative results as applications in Section 5. For instance, we present four sets of permutations that are all counted by the Motzkin numbers.

2 Decompositions of permutations

For a permutation treated as a sequence, we have many ways to decompose it into different subsequences. Here we are interested in four distinct decompositions which will be introduced in order. The four decompositions can be viewed as refinements of the well studied statistics ascents and descents of permutations as we shall see shortly. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. An ascent of π is an index $1 \leqslant i < n$ such that $\pi_i < \pi_{i+1}$. The rest are called descents of π. Note that here we always treat n as a descent.

2.1 Increasing and decreasing run decompositions

Definition 1. Suppose $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. A subsequence $\pi_i\pi_{i+1}\cdots\pi_{i+k-1}$ is called an *increasing run* (IR[1]) if $\pi_i < \pi_{i+1} < \cdots < \pi_{i+k-1}$, and it is not contained in a longer such subsequence.

Obviously, a permutation π can be uniquely decomposed into IR. This decomposition will be simply referred to as IRD. For example, we can decompose a permutation $\pi = 5346127$ into three IR:

$$\tau_1 = 5, \quad \tau_2 = 346, \quad \tau_3 = 127.$$

An integer partition λ of n, denoted by $\lambda \vdash n$, is a sequence of non-increasing positive integers $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ such that $\sum_i \lambda_i = n$. The lengths of the IR of π give the length distribution of π w.r.t. IRD. We will encode the length distribution by an integer partition of n. For example, the length distribution of $\pi = 5346127$ w.r.t. IRD is given by $\lambda = 331$.

Note that if i is a descent of π, then π_i is the rightmost (or last) element of an IR while π_{i+1} (if $i < n$) starts a new IR of π. Thus, there is an obvious one-to-one correspondence between IR and descents. Consequently, the set of permutations with k descents can be further refined into subsets by the length distribution of the corresponding k IR.

Decreasing runs (DR) and decreasing run decomposition (DRD) are defined analogously. It is also apparent that the number of ascents of π plus one is the same as the number of segments from the DRD of π. Consequently, DR with the associated length distribution may be viewed as a refinement of ascents.

2.2 Value-consecutive increasing subsequences

An increasing run can be alternatively interpreted as a maximal position-consecutive increasing subsequence. Then, it suggests a natural counterpart which may be called maximal *value-consecutive increasing subsequences* (v-CIS).

Definition 2. A v-CIS of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is a subsequence of the form $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} = j(j+1)\cdots(j+k-1)$ for some $j \geqslant 1$ and $k \geqslant 1$.

By abuse of notation, the decomposition of a permutation π into its maximal v-CIS is also referred to as v-CIS (of π). Taking $\pi = 5346127$ as an example, its v-CIS decomposition gives subsequences:

$$567, \quad 34, \quad 12,$$

the length distribution of which is given by the partition 322.

Lemma 3. *The number of descents of $\pi \in \mathfrak{S}_n(132)$ equals the number of subsequences from its v-CIS. In particular, if $i \neq n$ is a descent of π, then π_{i+1} starts a maximal v-CIS of π.*

[1]We will write IR (DR and v-CIS defined later) in singular as well as plural form.

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$. It suffices to show that there is a one-to-one correspondence between the descents $i \neq n$ of π and the maximal v-CIS of π which do not start with π_1. First, if $i \neq n$ is a descent of π, then we claim that π_{i+1} starts a maximal v-CIS of π. Otherwise, $\pi_j = \pi_{i+1} - 1$ for some $j < i$. In this case, $\pi_j\pi_i\pi_{i+1}$ yields a 132 pattern, a contradiction. Conversely, suppose π_{i+1} $(i > 0)$ starts a maximal v-CIS. If $\pi_{i+1} = 1$, then obviously i is a descent; If $\pi_{i+1} \neq 1$, then $\pi_j = \pi_{i+1} - 1$ for some $j > i + 1$ due to the maximality of the v-CIS containing π_{i+1}. Consequently, $\pi_i < \pi_{i+1}$ implies $\pi_i < \pi_{i+1} - 1 = \pi_j$, which again yields a 132 pattern $\pi_i\pi_{i+1}\pi_j$. Thus, we always have $\pi_i > \pi_{i+1}$ whence i is a descent of π. In view that there is a maximal v-CIS starting with π_1 and n is a descent of π, the lemma follows. □

We remark that the above relation is not true for a general permutation. For instance, $\pi = 153642$ clearly has a 132 pattern. We can check that π has four descents but only three subsequences in its v-CIS. This is actually interesting as the other three decompositions studied in this paper are directly related to ascents and descents for general permutations, and may deserve future investigations.

Lemma 4. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a 132-avoiding permutation. Then, there do not exist $1 \leqslant i < j < k < l \leqslant n$ such that π_i and π_k are contained in the same v-CIS τ_1 while π_j and π_l are contained in the same v-CIS τ_2 where $\tau_1 \neq \tau_2$.*

Proof. Suppose such i, j, k, l exist. By construction, if $\pi_j > \pi_i$, then $\pi_j > \pi_k$ as well since otherwise the three must be contained in the same v-CIS. Thus, $\pi_i\pi_j\pi_k$ provides a 132 pattern, a contradiction. Analogously, if $\pi_j < \pi_i$ (so $\pi_j < \pi_k$), we then have $\pi_l < \pi_k$, which implies $\pi_j\pi_k\pi_l$ being a 132 pattern. Either way yields a contradiction and the lemma follows. □

Lemma 5. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$. Suppose τ and τ' are two distinct v-CIS of π. Then, either all elements in τ lie between two consecutive elements in τ', or all elements in τ are to the left of the starting element of τ'. Moreover, in the former case, the maximal element in τ is smaller than the minimal element of τ', while in the latter case, the maximal element in τ' is smaller than the minimal element in τ.*

Proof. The first statement follows from Lemma 4. In the remaining part, the former case is true because the minimal (i.e., starting) element of τ is the image of an element determining a descent in view of Lemma 3; the latter case is true since otherwise an element from τ, π_j and an element from τ' form a 132 pattern, where π_j determines the descent corresponding to τ' in the light of Lemma 3. This completes the proof. □

2.3 Layered decreasing envelops

Definition 6. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. A right-to-left maximum w.r.t. a position m (or starting with π_m) is an entry π_i such that $\pi_i > \pi_j$ for all $i < j \leqslant m$, and π_m is always viewed as a right-to-left maximum.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. Starting with π_n, we search all right-to-left maxima in π which will give us a decreasing subsequence of π (i.e., a decreasing path). See an example for $\pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2$ in Figure 1. The obtained decreasing path from this step is $12, 2$. Next starting with the position that precedes the leftmost element of the last found path, we search all right-to-left maxima which will give us a new decreasing path (i.e., $11, 6, 5$ for the example); Continue doing this until we get a path starting with π_1 ($10, 9$ for the example). At this point, we have found the "outermost" layer of decreasing paths. Imagine we have connected adjacent elements in the same decreasing path by edges. Repeat the procedure of obtaining decreasing paths with respect to the segment of entries covered by the same edge in the existing paths until all entries in π have been placed into a path. This decomposition of permutation elements into decreasing subsequences will be called layered decreasing envelop decomposition, simply referred to as LDE.

Apparently, if $i \neq n$ is an ascent of π, then π_i is the rightmost element of a decreasing path. Conversely, if $i \neq n$ and π_i is the rightmost element of a decreasing path, then i is an ascent. Taking into account the decreasing path ending with π_n, the number of decreasing paths is one greater than the number of ascents of π. We will be interested in the length distribution of these decreasing paths. For example, the LDE length distribution of π in Figure 1 is the partition 322221.

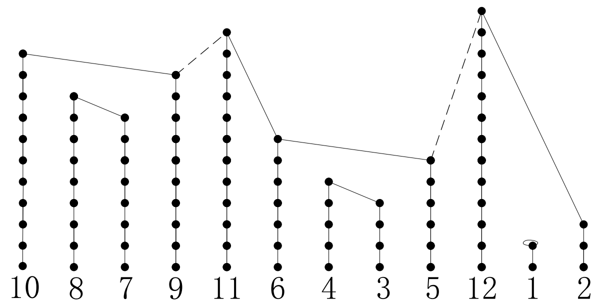

Figure 1: The LDE of a 132-avoiding permutation, where elements belonging to the same decreasing subsequence are connected by solid lines.

It is also worth noting that by construction any two decreasing paths are either in a left-right position or in a covering (or nesting) relation, i.e., crossing free.

3 Relevant existing bijections

In this section, we review several bijections involving plane trees and 132-avoiding permutations which can be found in the literature and will be used later.

3.1 A bijection φ on plane trees

A *plane tree* T can be recursively defined as an unlabeled tree with one distinguished vertex called the *root* of T, where the unlabeled trees resulted from the removal of the root as well as its incident edges from T are linearly ordered, and are plane trees with the vertices adjacent to the root of T as their respective roots. In a plane tree T, the number of edges in the unique path from a vertex v to the root of T is called the *height* (or *level*) of v, and the vertices adjacent to v but having greater heights are called the *children* of v. The number of children of v is called the *outdegree* of v. The vertices on level $2i$ (resp. $2i+1$) for $i \geqslant 0$ are called even-level (resp. odd-level) vertices. A *leaf* is a non-root vertex with no children, and a non-leaf vertex is called an *internal vertex*. The root is always treated as an internal vertex. We will draw a plane tree with its root on the top level, i.e., level 0, and with the children of a level i vertex arranged on level $i+1$ (below level i) left-to-right following their linear order.

A plane tree can be decomposed into a set of paths where each path has a leaf as a terminal vertex. There are two ways to achieve that: left path decomposition and right path decomposition.

Definition 7 (Left path decomposition)**.** Suppose all leaves in a plane tree T are ordered by their relative order in the depth-first search of T from left to right. The first path is the path from the first leaf to the root, and for $t > 1$, the t-th path goes from the t-th leaf up to the first vertex that is already in a path that has been obtained.

We will call the multiset consisting of the lengths of the obtained paths the *left path distribution* of the given tree. The right path decomposition is analogous. That is, the paths are successively obtained from right to left.

There is a bijection φ from plane trees to plane trees obtained by connecting two bijections between plane trees and RNA secondary structures, one being the Schmitt-Waterman bijection [17] and the other being the new bijection recently discovered by Chen [3]. We refer to [17] and [3] for details about the bijections, and to Smith and Waterman [18] as well as Schmitt and Waterman [17] for the definition and discussion on RNA secondary structures. What one really needs in this paper is the following property of the bijection φ.

Theorem 8 (Chen [3])**.** *Let A be the set of plane trees of $n > 0$ edges with x_q internal vertices of outdegree q and y_l (left) paths of length l in its left path decomposition. Let B be the set of plane trees of n edges with x_q odd-level vertices of outdegree $q-1$ and y_l even-level vertices of degree l. Then, the bijection φ induces a bijection between A and B.*

Furthermore, the length of the first (leftmost) path in $T \in A$ equals the degree of the root of $\varphi(T) \in B$.

3.2 The Jani-Rieper bijection

An explicit bijection between plane trees and 132-avoiding permutations was given by Jani and Rieper [11]. The following is how it works. Let T be a plane tree of n edges.

We use a preorder traversal of T (from left to right) to label the non-root vertices in decreasing order with the integers $n, n-1, \ldots, 1$. As such, the first vertex visited gets the label n and the last receives 1. A permutation written as a word is next obtained by reading the labeled tree in postorder, that is, traverse the tree from left to right and record the label of a vertex when it is last visited.

The reverse from a 132-avoiding permutation to a plane tree was not explicitly presented in Jani and Rieper [11]. Here we present a procedure and we leave it to the reader to verify its effectiveness. Let π be a 132-avoiding permutation. Suppose the IR of π from left to right are $\tau_1, \tau_2, \ldots, \tau_k$. For example, for $\pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2$, we have

$$\tau_1 = 10,\ \tau_2 = 8,\ \tau_3 = 7, 9, 11,\ \tau_4 = 6,\ \tau_5 = 4,\ \tau_6 = 3, 5, 12,\ \tau_7 = 1, 2.$$

For $1 \leqslant i \leqslant k$, make τ_i into a path (graph) with the vertex labels increasing along the path (still referred to as τ_i). We next begin with a plane tree with only one vertex r (i.e., the root) and will "integrate" the paths into the tree, one path at a time. Start with the path τ_k and connect the maximal element in τ_k to the vertex r. For our exemplary π, we obtain a partial tree which is the path from vertex 1 to the root of the left tree in Figure 2 at this point. After τ_i has been integrated into the tree, we find the minimal element u in the leftmost path (i.e., the one from the leftmost leaf to the root) in the current partial tree that is larger than the maximal element x in the path τ_{i-1}, and connect u and x; if no such a u exists, we connect x to the vertex r. In addition, make sure that the minimal element in τ_{i-1} is the leftmost leaf of the resulting partial tree, i.e., τ_{i-1} is integrated into the existing partial tree from the lefthand side. Eventually, we obtain a plane tree $JR(\pi)$. In the following, we will regard the corresponding plane trees of 132-avoiding permutations as plane trees with vertex labels, although the vertex labels are uniquely determined (by the underlying bijections) and can be omitted. Moreover, we may sometimes use the label of a vertex to refer to the vertex.

Lemma 9 (Jani-Rieper [11])**.** *The longest increasing subsequence in $\pi \in \mathfrak{S}_n(132)$ starting with an entry is the height of the vertex corresponding to the entry in $JR(\pi)$.*

Lemma 10. *For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$, the outdegree distribution of the internal vertices of $JR(\pi)$ is the same as the LDE length distribution of π, while the right path distribution of $JR(\pi)$ is the same as the IR length distribution of π.*

Proof. The "while" part succinctly follows from our reverse procedure from 132-avoiding permutations to plane trees. We next focus on the correspondence between the outdegree and LDE length distributions. First, we have the following observations from our reverse procedure:

(i) The left-to-right order of the children of an internal vertex in $JR(\pi)$ is the same as the left-to-right order of these children in π, and these children form a decreasing subsequence of π;

(ii) The descendants of a vertex v are smaller than the vertex (in terms of their labels). In particular, if $v = \pi_i$, then the rightmost child of v is π_{i-1} and $\pi_{i-1} < \pi_i$;

(iii) Let v be a non-root vertex in $JR(\pi)$. Suppose v' is the first vertex on the path directing from v (including v) to the root that has left siblings. Let u be the immediate left sibling of v' in $JR(\pi)$. Then, the entries to the left of v and to the right of u in π are descendants of v. If no such v' exists, then all entries to the left of v in π are descendants of v.

Note that the latter case of (iii) only happens when v is on the path from the leftmost leaf to the root of $JR(\pi)$.

By construction, the rightmost child of the root of $JR(\pi)$ is π_n. Based on (i), (ii) and (iii), the children of the root are the right-to-left maxima starting with π_n whence an LDE decreasing path in π. Next, for the children of a non-root vertex $v = \pi_i$ for some i, they are easily seen to be contained in the same LDE decreasing path $\mathcal{L}$ since they are right-to-left maxima starting with π_{i-1}. (Recall π_{i-1} corresponds to an ascent in π whence the rightmost element of an LDE.) It remains to show there does not exist a right-to-left maximum starting with π_{i-1} which is to the left of the leftmost child of v in π and is also contained in $\mathcal{L}$. If the desired u in (iii) does not exist, obviously, no such a right-to-left maximum exists. If the desired u exists, u is certainly a right-to-left maximum starting with π_{i-1}. However, u and v' as children of the same vertex are in the same LDE decreasing path. In addition, (ii) and (iii) imply that a vertex cannot be contained in the same LDE with any of its descendant. Thus, u and v' are not contained in $\mathcal{L}$. No other elements to the left of u in π are contained in $\mathcal{L}$ either, otherwise $\mathcal{L}$ and the LDE containing u and v' cross which is impossible as analyzed before (see the discussion right before Section 3). This completes the proof. □

3.3 Odd-even level switching of plane trees

Given a plane tree T, we obtain a new plane tree T' by taking the leftmost child v of the root of T as the root of the new tree T', i.e., lifting v to the top level such that the even-level vertices in T become odd-level vertices in T' and vice versa. This is clearly a bijection. In addition, it is not difficult to verify that the degree distribution of the even-level (resp. odd-level) vertices of T becomes the degree distribution of the odd-level (resp. even-level) vertices of T'.

4 A new bijection and consequences

In this section, we first present a bijection between plane trees and 132-avoiding permutations which appears to be new. Then, we discuss several applications of the bijection, in particular, a new equidistribution result on plane trees.

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$. Suppose there are k subsequences in its v-CIS: $\tau_1 = \pi_1^1\pi_2^1\cdots\pi_{i_1}^1, \ldots, \tau_k = \pi_1^k\pi_2^k\cdots\pi_{i_k}^k$, and suppose for $1 \leqslant i \leqslant k$,

$$\pi_1^i = \max\{\pi_1^i, \pi_1^{i+1}, \ldots, \pi_1^k\}.$$

We then construct a plane tree $T = \phi(\pi)$ recursively with the following procedure.

- First, start with a single vertex and arrange $\pi_1^1, \ldots, \pi_{i_1}^1$ from left to right as the children of the vertex.
- For $j = 2$ to k, find vertex π_t in the constructed partial tree that is immediately to the left of π_1^j in π; then let $\pi_1^j, \ldots, \pi_{i_j}^j$ be the left-to-right children of vertex π_t.

Due to Lemma 3 and Lemma 5, the above desired π_t must be contained in τ_i for some $i < j$. So, π_t is indeed contained in the partial tree. Hence, the above procedure eventually yields a labeled plane tree T. It is also easy to observe that: (i) there are k internal vertices in T, and (ii) the descendants (if any) of any non-root vertex π_i have smaller labels and their positions in π are to the right of π_i and to the left of the immediate right sibling of π_i if any due to Lemma 5.

For example, $\pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2$ is a 132-avoiding permutation, and its v-CIS are

$$\tau_1 = 10, 11, 12, \quad \tau_2 = 89, \quad \tau_3 = 7, \quad \tau_4 = 6, \quad \tau_5 = 45, \quad \tau_6 = 3, \quad \tau_7 = 12.$$

Then, its corresponding labeled plane tree is depicted in Figure 2 (right).

The labels of the vertices are uniquely determined by the underlying unlabeled plane tree as follows: if we travel the internal vertices of the plane tree in the left-to-right depth-first manner, then the k left-to-right children of the current internal vertex carry the remaining k largest elements from $[n]$ in increasing order.

The above claim can be seen in the following manner. If this is not the case, then there will be two internal vertices u and v such that u is first visited in the left-to-right depth-first travel but the children of v carry larger labels than that of the children of u. Note that there are only two situations when u gets visited first: either v is a descendant of u, or v is a descendant of a vertex x (other than u) that is on the path from u to the root of the tree and v is located on the right-hand side of the path. The former case is impossible, since we construct the tree in a top-down manner and the children of a vertex must carry smaller labels (we may assume the label of the root is ∞). As for the latter case, we further assume the child of x that is on the path from u to x is u' (could be u itself) , and the child of x that is on the path from v to x is v' (could be v itself). By assumption, v' is a right-hand side sibling of u'. By construction, the labels of u, u', v, v' satisfy $u < u' < v'$ and $v < v'$. Pick a child u'' of u and a child v'' of v. Obviously, $u'' < u < v'$ and $v'' < v < v'$. Moreover, in π, the left-to-right order of u'', v', v'' is exactly u'', v', v''. Now if v'' has a larger label than u'', then $u''v'v''$ gives a 132 pattern in π, a contradiction. Therefore, the vertex labels of the tree are uniquely determined as claimed.

So, in theory, we can remove the vertex labels and only consider the underlying unlabeled plane tree. But, we prefer to keeping the vertex labels as it is more convenient to refer to the vertices.

As for the reverse, from a plane tree that is uniquely labeled in the manner just described right above, it is not difficult to see that the left-to-right depth-first search (or preorder) gives us the desired 132-avoiding permutation. We leave the proofs of this and the following lemma to the reader.

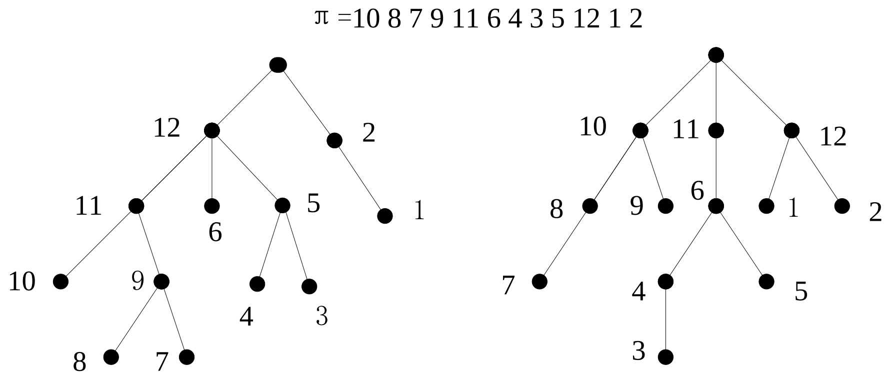

Figure 2: The corresponding plane trees under the Jani-Rieper bijection (left) and the bijection ϕ (right) of the same 132-avoiding permutation π.

Lemma 11. *Given $\pi \in \mathfrak{S}_n(132)$, let $T = \phi(\pi)$. Then, the outdegree distribution of internal vertices of T is the same as the v-CIS length distribution of π, while the left path distribution of T equals the DR length distribution of π. In particular, the length of the DR starting with π_1 in π is the same as the length of the left first path in T.*

As the first application of the bijection ϕ, we obtain a new equidistribution result on plane trees which involves a new quantity associated to vertices.

Definition 12 (Right spanning width)**.** Let $\{v, v_1, v_2, \ldots, v_k\}$ be the set of vertices on the path from v to the root of a plane tree T. Then, the right spanning width of v equals the sum of the number of children of v and the number $pr(v)$ of edges incident to v_i's that are on the right-hand side of the path.

Let $rsw(v)$ denote the right spanning width of v. See an illustration in Figure 4. Note that the right spanning width of the root is just its outdegree, and $rsw(v) = pr(v)$ if v is a leaf. In addition, we define

$$rsw(T) = \max\{rsw(v) : v \text{ is an internal vertex in } T\}.$$

Lemma 13. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$ and $T = \phi(\pi)$. Then, the length of the longest increasing subsequence starting with π_i in π equals $pr(\pi_i) + 1$ for vertex π_i in T.*

Proof. From previous analyses, the entries larger than π_i and to the right of π_i in π correspond to some vertices in T that are located on the right-hand side of the path from π_i to the root of T. Since these "some vertices" cannot be the children of any internal vertices that are on the right-hand side of the path due to the previous unique vertex labeling argument, they must be adjacent to the vertices different from π_i on the path. By definition, there are $pr(\pi_i)$ such vertices which also obviously form an increasing subsequence in π. Taking into account π_i itself, the lemma follows. $\square$

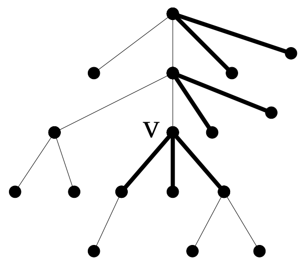

Figure 3: The right spanning width of the vertex v is the number of bold edges.

Our new equidistribution result can be stated symbolically as: over all plane trees of n edges, we have

$$\text{height} \sim rsw. \tag{1}$$

Theorem 14. *Suppose $\mathcal{M}$ is a multiset consisting of $n+1$ integers. Let $ht(\mathcal{M})$ be the number of plane trees of n edges the heights of whose vertices constitute $\mathcal{M}$, and let $rsw(\mathcal{M})$ be the number of plane trees of n edges the right spanning widths of whose vertices constitute $\mathcal{M}$. Then, we have*

$$ht(\mathcal{M}) = rsw(\mathcal{M}). \tag{2}$$

Moreover, if $\mathcal{M}'$ is a multiset consisting of k integers, then the number of plane trees of n edges and k leaves whose heights constitute $\mathcal{M}'$ is the same as the number of plane trees of n edges and k internal vertices whose right spanning widths constitute $\mathcal{M}'$.

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$. Suppose $T = JR(\pi)$ and $T' = \phi(\pi)$ (with vertex labels). Assume π_i is a leaf in T. Then, π_i starts an IR of π and thus starts a v-CIS of π. Thus, π_i is the label of the leftmost child of an internal vertex v in T' by construction. Consider the longest increasing subsequence τ in π starting with π_i. According to Lemma 9, the length of τ equals the height of π_i in T, while the length of τ equals $pr(\pi_i)+1$ in T' due to Lemma 13. Since π_i is the leftmost child of v, we have $rsw(v) = pr(\pi_i) + 1$. Thus, the height of a leaf π_i in T is the same as $rsw(v)$ for v being internal in T'.

Next, assume π_i is an internal vertex in T. The longest increasing subsequence starting with π_i in π also equals the height of π_i in T. By construction, π_i being of an internal vertex in T implies that $i \neq 1$ and $\pi_{i-1} < \pi_i$, i.e., $i-1$ is an ascent of π. However, in T', π_{i-1} must be a leaf. Note that the longest increasing subsequence starting with π_{i-1} has length exactly $pr(\pi_{i-1}) + 1 = rsw(\pi_{i-1}) + 1$ in T'. Accordingly, the length of the longest increasing subsequence starting with π_i equals $rsw(\pi_{i-1})$. The last part of the theorem on the correspondence between heights of leaves and rsw of internal vertices (and vice versa) is also implied here, and the theorem follows. □

Remark 15. Recall the height of a plane tree is the maximum height of leaves in the tree. While average height of various trees and a single leaf there were examined in a plethora of work, see for instance, de Bruijin, Knuth and Rice [6], Kemp [12], and Prodinger [15], to the best of our knowledge, no statistics have been found to be equidistributed as height.

Remark 16 (Open problem[2]). It is natural to define left spanning width (lsw) of a vertex analogously. We believe there exists another statistic A associated to vertices such that the pair (lsw, rsw) is in equidistribution with the pair of A and height of vertices. However, we failed to find it.

Corollary 17. *The number of plane trees of n edges and height k equals the number of plane trees T of n edges with $rsw(T) = k$.*

Apparently, the latter part of Theorem 14 refines the following well-known fact.

Corollary 18. *The number of plane trees of n edges with k internal vertices is the same as the number of plane trees of n edges with k leaves.*

It is well known that the number of Dyck paths of semilength n with k peaks is given by the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ with $N(0,0) = 1$. It is worthy of pointing out that the number of plane trees of n edges and k leaves is also counted by $N(n,k)$, in which Corollary 18 is implied. In Callan [2], it was proved via a novel combinatorial argument that the number of Dyck paths of semilength n with i returns to ground level and j peaks equals the generalized Narayana number

$$N_i(n,j) = \frac{i}{n}\binom{n}{j}\binom{n-i-1}{j-i}.$$

Callan's result can be easily translated into a result in the world of plane trees: the number of plane trees of n edges where there are j leaves and the outdegree of the root is i is $N_i(n,j)$.

Theorem 19. *The number of 132-avoiding permutations on $[n]$ starting with i and having k descents is given by $\frac{n+1-i}{n}\binom{n}{k-1}\binom{i-2}{i-k}$. Furthermore, the number of 132-avoiding permutations on $[n]$ starting with i, ending with j and having k descents is*

$$\begin{cases} N_{n-i}(n-1, n-k), & \text{if } i < j, \\ \sum_{m=1}^{k-1} N_{n+1-i}(n-j, n-m+1)N(j-1, j+m-k+1), & \text{else.} \end{cases} \tag{3}$$

Proof. Let $\pi \in \mathfrak{S}_n(132)$. If π starts with i, then its v-CIS containing the first element i is evidently $i(i+1)\cdots n$. According to the bijection ϕ, this implies the outdegree of the root of the tree $\phi(\pi)$ equals $n+1-i$. Recall that the number of descents of π is equal to the number of v-CIS of π which equals the number of internal vertices in $\phi(\pi)$. So, the number of leaves in $\phi(\pi)$ is $n+1-k$. Hence, the desired number in the first part of the theorem equals the number of plane trees with n edges and $n+1-k$ leaves, the outdegree of whose root is $n+1-i$. As stated, the number of such trees is given by

$$N_{n+1-i}(n, n+1-k) = \frac{n+1-i}{n}\binom{n}{k-1}\binom{i-2}{i-k}.$$

[2]Zhicong Lin recently told us that he and his coauthors have solved the open problem.

As for the remaining part of the theorem, we need the following *claim*: In a 132-avoiding permutation ending with j, the last j entries are from the set $[j]$. Otherwise, there is no difficulty to show there exists a 132 pattern.

The first consequence of the above claim is that if $i < j$, then $j = n$. Then, the considered set is equivalent to the set of 132-avoiding permutations on $[n-1]$ starting with i and having k descents which is counted by $N_{n-i}(n-1, n-k)$ as just proved in the first part of the theorem.

Next, assume $i > j$ and $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$ where $\pi_1 = i$ and $\pi_n = j$. We distinguish two cases: $j = 1$ and $j > 1$. For $j = 1$, the index $n-1$ clearly gives a descent. If π has k descents, then there are $k-2$ descents contained in $[n-2]$. It follows that the uniquely induced $\pi' = (\pi_1 - 1)(\pi_2 - 1)\cdots(\pi_{n-1} - 1) \in \mathfrak{S}_{n-1}(132)$ starts with $i-1$ and has $k-1$ descents. There are obviously $N_{n+1-i}(n-1, n-k+1)$ such π'.

If $i > j > 1$, from the above claim, any 132-avoiding permutation in this case is a concatenation of two 132-avoiding permutations: $\pi_1\cdots\pi_{n-j}$ is on $[n]\setminus[j]$ and starts with i as well as has m descents, and $\pi_{n-j+1}\cdots\pi_n$ is on $[j]$, ending with j and having $k-m$ descents, for some $1 \leqslant m < k$. The former permutations on $[n]\setminus[j]$ are equivalent to 132-avoiding permutations on $[n-j]$ which start with $i-j$ and have m descents, while the latter are equivalent to 132-avoiding permutations on $[j-1]$ having $k-m$ descents. The former permutations are counted by $N_{n+1-i}(n-j, n-j-m+1)$. Recall the latter permutations are in one-to-one correspondence with plane trees with j vertices $k-m$ of which are internal via ϕ. Summing over all possible m, the desired number is

$$\sum_{m=1}^{k-1} N_{n+1-i}(n-j, n-j-m+1)N(j-1, j+m-k).$$

Note that $N(0,0) = 1$ and $N(0,x) = 0$ for $x > 0$. For $j = 1$, the last quantity equals $N_{n+1-i}(n-1, n-k+1)$, agreeing to the number for the case $i > j = 1$, and the proof follows. □

We remark that 132-avoiding permutations with k descents are known to be counted by the Narayana numbers, see e.g. [16].

5 Equidistributions on 132-avoiding permutations

Our main goal in this section is to prove the following new equidistribution result on 132-avoiding permutations. We point out that common techniques such as taking the inverse, taking the reverse (i.e., reading from right to left) or taking the complement (i.e., replacing i with $n+1-i$) will not work since the resulting permutations may not be 132-avoiding permutations anymore. Instead, we will connect several bijections carefully in order for proving the theorem.

Theorem 20. *Given two partitions $\lambda, \mu \vdash n$, the following four sets are of equal size:*

(1) $\pi \in \mathfrak{S}_n(132)$ whose IRD and LDE length distributions are resp. λ and μ.

(2) $\pi \in \mathfrak{S}_n(132)$ *whose IRD and LDE length distributions are resp.* μ *and* λ.

(3) $\pi \in \mathfrak{S}_n(132)$ *whose v-CIS and DRD length distributions are resp.* λ *and* μ.

(4) $\pi \in \mathfrak{S}_n(132)$ *whose v-CIS and DRD length distributions are resp.* μ *and* λ.

Proof. We first prove the sets of (1) and (2) have the same size. Let $\pi \in \mathfrak{S}_n(132)$ and suppose its IRD and LDE length distributions are resp. λ and μ. Let $T_1 = JR(\pi)$, i.e., the corresponding tree of π under the Jani-Rieper bijection. Denote by T_2 the mirror image of T_1 (i.e., horizontal flipping). Clearly, in view of Lemma 10, the internal vertex outdegree distribution of T_2 is μ and the left path length distribution of T_2 is λ. Suppose $\mu = \mu_1\mu_2\cdots\mu_k$. Accordingly, following from Chen's bijection φ (Theorem 8), the odd-level vertex outdegree distribution of $T_3 = \varphi(T_2)$ is $\mu - 1$, that is, $(\mu_1 - 1)(\mu_2 - 1)\cdots(\mu_k - 1)$, while the even-level vertex degree distribution of T_3 is λ.

Let T_4 be the resulted tree from the odd-even level switching transform from T_3. Then, it is easily seen that the odd-level vertex outdegree distribution of T_4 is $\lambda-1$ while the even-level vertex degree distribution of T_4 is μ. According to Theorem 3.1, the internal vertex outdegree distribution of $T_5 = \varphi^{-1}(T_4)$ is thus λ and the left path length distribution of T_5 is μ. Consequently, the mirror image T_6 of T_5 has the internal vertex outdegree distribution λ and the right path length distribution μ. Thus, the corresponding permutation π' of T_6 under the Jani-Rieper bijection has the IRD and LDE length distributions resp. μ and λ.

See Figure 4 for an illustration. Obviously, the correspondence between π and π' is a bijection as it is essentially the composition of a number of bijections. Hence, the sets of (1) and (2) are of equal size.

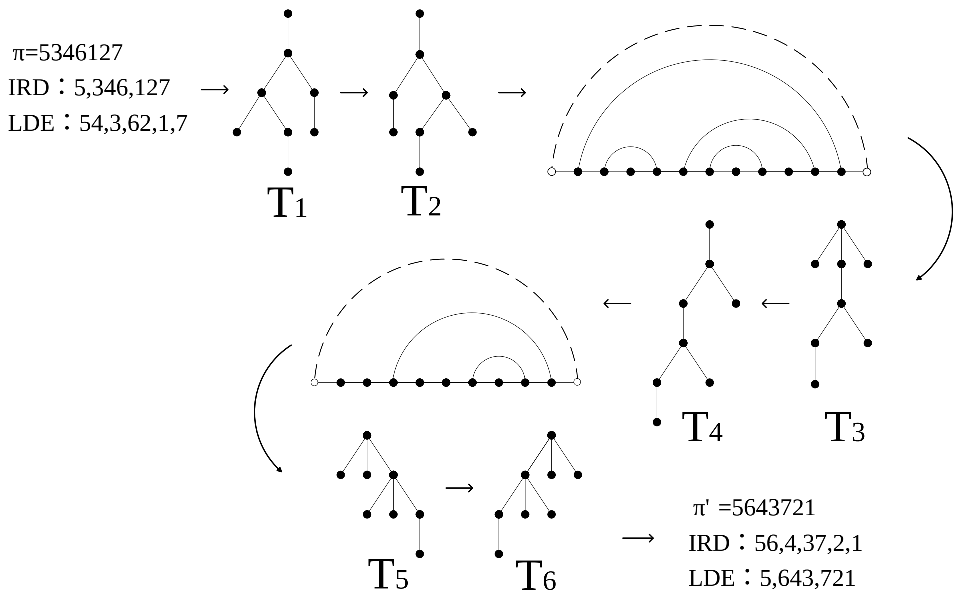

Figure 4: The correspondence between set of (1) and set of (2).

Next we prove that the sets of (1) and (3) contain the same number of 132-avoiding permutations. Again, let $\pi \in \mathfrak{S}_n(132)$ and suppose its IRD and LDE length distributions are respectively λ and μ. Let $T_1 = JR(\pi)$, and denote by T_2 the mirror image of T_1. Then, the internal vertex outdegree distribution of T_2 is μ and the left path length distribution of T_2 is λ. Suppose π' is the corresponding permutation of T_2 under the bijection ϕ, i.e., $\pi' = \phi^{-1}(T_2)$. According to Lemma 11, the v-CIS and DRD length distributions of π' are resp. λ and μ. The correspondence is apparently reversible, so the sets of (1) and (3) are of equal size. Other pairs from the four sets can be dealt with analogously, completing the proof. □

In the rest of the paper, we present some applications of Theorem 20. Recall that the Motzkin numbers M_n can be defined by their generating function:

$$\sum_{n\geqslant 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

Elizalde and Mansour [7] showed that the number of 132-avoiding permutations in $\mathfrak{S}_n(132)$ that also avoid the consecutive pattern 123 is the Motzkin number M_n. Note that avoiding the consecutive pattern 123 is tantamount to requiring all IR have length at most two. As an immediate corollary of Theroem 20, we present more sets of permutations which are counted by the Motzkin numbers.

Corollary 21 (Motzkin family)**.** *The following four sets of permutations are all counted by the Motzkin number M_n:*

(1) $\pi \in \mathfrak{S}_n(132)$ *whose IR have length at most two;*

(2) $\pi \in \mathfrak{S}_n(132)$ *whose LDE decreasing paths have length at most two;*

(3) $\pi \in \mathfrak{S}_n(132)$ *whose v-CIS have length at most two;*

(4) $\pi \in \mathfrak{S}_n(132)$ *whose DR have length at most two.*

We proceed to present more enumerative results. Some notation are needed first. An *unlabeled set-alternating E-tree (resp. O-tree)* is a plane tree where the even-level vertices carry indistinguishable labels from a set E (resp. O) and the odd-level vertices carry indistinguishable labels from a set O (resp. E).

Let t be a nonnegative integer and $\kappa_t(n,m)$ denote the number of weak compositions of n into $m \geqslant 0$ parts each of which is no larger than t, i.e., $a_1 + a_2 + \cdots + a_m = n$ and a_i is an integer satisfying $0 \leqslant a_i \leqslant t$. We make the convention that $\kappa_t(0,0) = 1$. Obviously, $\kappa_t(n,m) = 0$ if $n < 0$.

Theorem 22. *Suppose $p + q = n + 1$. Let $\xi_{h,l}(p,q)$ be the number of $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$ whose DRD and v-CIS respectively give p DR and q v-CIS, where the DR starting with π_1 has a length at most h, each of the remaining $p-1$ DR is of length no longer than $h+1$, and each v-CIS is of length no longer than l. Then,*

$$\xi_{h,l}(p,q) = \kappa_l(p-1,q)\kappa_h(q,p) - \Big\{\sum_{i=0}^{l-1}(i+1)\kappa_l(p-i-2,q-1)\Big\}\Big\{\sum_{j=0}^{h-1}(j+1)\kappa_h(q-j-1,p-1)\Big\}. \quad (4)$$

Proof. Combining Theorem 8 and Lemma 11, the set of 132-avoiding permutations under consideration has the same size as the set of plane trees of n edges where there are p even-level vertices with outdegree at most h, and q odd-level vertices with outdegree at most l. We shall enumerate the latter set by using generating functions and the Lagrange inversion formula.

Let $\mathfrak{F}_E$ (resp. $\mathfrak{F}_O$) denote the set of unlabeled set-alternating E-trees (resp. O-trees) with every E-vertex (i.e., vertex with a label from the set E) having at most h children, and every O-vertex (i.e., vertex with a label from the set O) having at most l children.

Let

$$\omega_1(t_1,t_2) = \sum_{T\in\mathfrak{F}_E} t_1^{\#\text{vertices in } E \text{ in } T} t_2^{\#\text{vertices in } O \text{ in } T},$$

$$\omega_2(t_1,t_2) = \sum_{T\in\mathfrak{F}_O} t_1^{\#\text{vertices in } E \text{ in } T} t_2^{\#\text{vertices in } O \text{ in } T}.$$

Clearly, the number in question is $[t_1^p t_2^q]\omega_1$, i.e., the coefficient of the term $t_1^p t_2^q$ in the power series expansion of ω_1.

Note that the subtrees of an E-tree in $\mathfrak{F}_E$ are O-trees in $\mathfrak{F}_O$ and the subtrees of an O-tree in $\mathfrak{F}_O$ are E-trees in $\mathfrak{F}_E$. Then, the following relation is obvious

$$\omega_1 = t_1(1+\omega_2+\omega_2^2+\cdots+\omega_2^h) = t_1\frac{1-\omega_2^{h+1}}{1-\omega_2},$$

$$\omega_2 = t_2(1+\omega_1+\omega_1^2+\cdots+\omega_1^l) = t_2\frac{1-\omega_1^{l+1}}{1-\omega_1}.$$

In order for obtaining $[t_1^p t_2^q]\omega_1$, let us first recall the following bivariate Lagrange inversion formula [1, 10]. Suppose $g(x_1,x_2),\ f_1(x_1,x_2),\ f_2(x_1,x_2)$ are formal power series in $x_1,\ x_2$ such that $f_i(0,0)\neq 0$. Then the set of equations $w_i = t_i f_i(t_1,t_2)$ for $1\leqslant i\leqslant 2$ uniquely determine w_i as formal power series in $t_1,\ t_2$, and

$$[t_1^p t_2^q]g(w_1,w_2) = [x_1^p x_2^q]g(x_1,x_2)f_1^p(x_1,x_2)f_2^q(x_1,x_2)\det\begin{Bmatrix} 1-\frac{x_1}{f_1}\frac{\partial f_1}{\partial x_1} & -\frac{x_1}{f_2}\frac{\partial f_2}{\partial x_1} \\ -\frac{x_2}{f_1}\frac{\partial f_1}{\partial x_2} & 1-\frac{x_2}{f_2}\frac{\partial f_2}{\partial x_2}\end{Bmatrix},$$

where det is for taking determinant.

In our case, we have

$$g(x_1,x_2)=x_1, \quad f_1(x_1,x_2)=\frac{1-x_2^{h+1}}{1-x_2}, \quad f_2(x_1,x_2)=\frac{1-x_1^{l+1}}{1-x_1}.$$

Therefore, we compute

$$\begin{aligned}&[t_1^p t_2^q]\omega_1 = [x_1^p x_2^q]\, g\cdot f_1^p\cdot f_2^q\cdot \det\begin{Bmatrix} 1-\frac{x_1}{f_1}\frac{\partial f_1}{\partial x_1} & -\frac{x_1}{f_2}\frac{\partial f_2}{\partial x_1} \\ -\frac{x_2}{f_1}\frac{\partial f_1}{\partial x_2} & 1-\frac{x_2}{f_2}\frac{\partial f_2}{\partial x_2}\end{Bmatrix}\\ =&[x_1^p x_2^q]\frac{x_1(1-x_1^{l+1})^q}{(1-x_1)^q}\frac{(1-x_2^{h+1})^p}{(1-x_2)^p}\end{aligned}$$

$$\times \left[1 - \frac{x_1x_2}{(1-x_1^{l+1})(1-x_2^{h+1})}\Big(\frac{1-x_1^l}{1-x_1} - lx_1^l\Big)\Big(\frac{1-x_2^h}{1-x_2} - hx_2^h\Big)\right]$$

$$=[x_1^p x_2^q]\frac{x_1(1-x_1^{l+1})^q}{(1-x_1)^q}\frac{(1-x_2^{h+1})^p}{(1-x_2)^p} - \Bigg\{[x_1^p]\frac{x_1(1-x_1^{l+1})^q}{(1-x_1)^q}\frac{x_1}{(1-x_1^{l+1})}\Big(\frac{1-x_1^l}{1-x_1} - lx_1^l\Big)$$

$$\times [x_2^q]\frac{(1-x_2^{h+1})^p}{(1-x_2)^p}\frac{x_2}{(1-x_2^{h+1})}\Big(\frac{1-x_2^h}{1-x_2} - hx_2^h\Big)\Bigg\}$$

$$=\kappa_l(p-1,q)\kappa_h(q,p) - \sum_{i=0}^{l-1}(i+1)\kappa_l(p-i-2,q-1)\sum_{j=0}^{h-1}(j+1)\kappa_h(q-j-1,p-1)$$

where the last simplification follows from Lemma 25 in the Appendix. □

Let h go to infinity, we obtain

Corollary 23. *The number of* 132*-avoiding permutations* $\pi = \mathfrak{S}_n(132)$ *with* q *IR each of which has a length at most* l *is given by*

$$\sum_{i=0}^{l}\left\{\binom{n}{q} - i\binom{n}{q-1}\right\}\kappa_l(n-q-i,q-1). \tag{5}$$

Proof. When h is large enough, we obviously have

$$\kappa_h(n,m) = \binom{n+m-1}{m-1}.$$

Then, eq. (4) reduces to

$$\binom{q+p-1}{p-1}\kappa_l(p-1,q) - \Big\{\sum_{j=0}^{q-1}(j+1)\binom{q+p-3-j}{p-2}\Big\}\Big\{\sum_{i=0}^{l-1}(i+1)\kappa_l(p-i-2,q-1)\Big\}.$$

Next, it is not hard to show by generating functions that

$$\sum_{k=r}^{n-s}\binom{k}{r}\binom{n-k}{s} = \binom{n+1}{r+s+1}.$$

Then, we have

$$\sum_{j=0}^{q-1}(j+1)\binom{q+p-3-j}{p-2} = \sum_{j=0}^{q-1}\binom{j+1}{1}\binom{q+p-2-j-1}{p-2} = \binom{n}{p}.$$

Consequently, eq. (4) equals

$$\binom{q+p-1}{p-1}\kappa_l(p-1,q) - \binom{n}{p}\Big\{\sum_{i=0}^{l-1}(i+1)\kappa_l(p-i-2,q-1)\Big\}$$

$$= \sum_{i=0}^{l} \left\{ \binom{n}{p-1} - i\binom{n}{p} \right\} \kappa_l(p-1-i, q-1),$$

which is the number of $\pi = \mathfrak{S}_n(132)$ with q v-CIS each of which has a length at most l. Since IR and v-CIS are equidistributed from Theorem 20, the proof follows. □

Obviously, the numbers obtained by setting $l = 2$ provide a refinement of the Motzkin numbers. To be specific, we have the following result.

Corollary 24. *The number of* 132*-avoiding permutations in* $\mathfrak{S}_n(132)$ *that avoid the consecutive pattern* 123 *and have* q *descents is given by*

$$\sum_{i=0}^{2} \left\{ \binom{n}{q} - i\binom{n}{q-1} \right\} \kappa_2(n-q-i, q-1). \tag{6}$$

Moreover, we have the Motzkin number

$$M_n = \sum_{q} \sum_{i=0}^{2} \left\{ \binom{n}{q} - i\binom{n}{q-1} \right\} \kappa_2(n-q-i, q-1). \tag{7}$$

Finally, we leave it to the interested reader to show that $\kappa_2(n, m)$ can be explicitly computed as follows:

$$\kappa_2(n, m) = \sum_{i=0}^{m} \binom{m}{i} \binom{m-i}{2m-2i-n}. \tag{8}$$

We remark that more enumerative results can be obtained and some of them may be found in the arXiv versions of the work. IR and DR can be formulated into consecutive monotone patterns. Which patterns correspond to v-CIS and LDE paths? IR and LDE (resp. DR and v-CIS) are somehow intertwined, and what their joint length distribution really reveals w.r.t. the structure of 132-avoiding permutations remains not entirely clear. These problems are interesting and left for future investigations.

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A Computation in Theorem 22

Lemma 25. *For $l > 0$, $q > 0$ and p being any integer, we have*

$$[x_1^p]x_1 \cdot \frac{(1-x_1^{l+1})^q}{(1-x_1)^q} \cdot \frac{x_1}{(1-x_1^{l+1})} \left(\frac{1-x_1^l}{1-x_1} - lx_1^l \right) = \sum_{i=0}^{l-1} (i+1)\kappa_l(p-i-2, q-1).$$

Proof. First, for $t \geqslant 0$ and $m \geqslant 0$, it is easy to see

$$\sum_{n \geqslant 0} \kappa_t(n,m)x^n = (1 + x + \cdots + x^t)^m = \left(\frac{1-x^{t+1}}{1-x} \right)^m.$$

Then, we have

$$\begin{aligned}
&[x_1^p]x_1 \cdot \frac{(1-x_1^{l+1})^q}{(1-x_1)^q} \cdot \frac{x_1}{(1-x_1^{l+1})} \left(\frac{1-x_1^l}{1-x_1} - lx_1^l \right) \\
=&[x_1^{p-2}] \frac{(1-x_1^{l+1})^q}{(1-x_1)^q} \cdot \frac{1}{(1-x_1^{l+1})} \left(\frac{1-x_1^l}{1-x_1} - lx_1^l \right) \\
=&[x_1^{p-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^{q+1}} - [x_1^{p-l-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^{q+1}} - l[x_1^{p-l-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^q} \\
=&[x_1^{p-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^{q-1}(1-x_1)^2} - [x_1^{p-l-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^{q-1}(1-x_1)^2} - l[x_1^{p-l-2}] \frac{(1-x_1^{l+1})^{q-1}}{(1-x_1)^q} \\
=&\sum_{i=0}^{p-2} \binom{i+1}{i} \kappa_l(p-i-2, q-1) - \sum_{i=0}^{p-l-2} \binom{i+1}{i} \kappa_l(p-l-i-2, q-1) \\
&\quad - l \sum_{i=0}^{p-l-2} \kappa_l(p-l-i-2, q-1) \\
=&\sum_{i=0}^{p-2} (i+1)\kappa_l(p-i-2, q-1) - \sum_{i=0}^{p-l-2} (i+l+1)\kappa_l(p-l-i-2, q-1) \\
=&\sum_{i=0}^{p-2} (i+1)\kappa_l(p-i-2, q-1) - \sum_{i=l}^{p-2} (i+1)\kappa_l(p-i-2, q-1) \\
=&\sum_{i=0}^{l-1} (i+1)\kappa_l(p-i-2, q-1),
\end{aligned}$$

and the proof follows. □