On the Maximum Spread of
Planar and Outerplanar Graphs

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Abstract

The spread of a graph G is the difference between the largest and smallest eigenvalue of the adjacency matrix of G. Gotshall, O'Brien and Tait conjectured that for sufficiently large n, the n-vertex outerplanar graph with maximum spread is the graph obtained by joining a vertex to a path on $n-1$ vertices. In this paper, we disprove this conjecture by showing that the extremal graph is the graph obtained by joining a vertex to a path on $\lceil (2n-1)/3 \rceil$ vertices and $\lfloor (n-2)/3 \rfloor$ isolated vertices. For planar graphs, we show that the extremal n-vertex planar graph attaining the maximum spread is the graph obtained by joining two nonadjacent vertices to a path on $\lceil (2n-2)/3 \rceil$ vertices and $\lfloor (n-4)/3 \rfloor$ isolated vertices.

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1   Introduction

Given a square matrix M, the *spread* of M, denoted by $S(M)$, is defined as $S(M) := \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of M. In other words, $S(M)$ is the diameter of the spectrum of M. Given a graph $G = (V, E)$, the *spread* of G, denoted by $S(G)$, is defined as the spread of the adjacency matrix $A(G)$ of G. Let $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$ be the eigenvalues of $A(G)$. Since $A(G)$ is a real symmetric matrix, we have that the λ_is are all real numbers. Thus $S(G) = \lambda_1 - \lambda_n$.

The systematic study of the spread of graphs was initiated by Gregory, Hershkowitz, and Kirkland [14]. One of the central focuses of this area is to find the maximum or minimum spread over a fixed family of graphs and characterize the extremal graphs. Problems of such extremal flavor have been investigated for trees [1], graphs with few cycles [12, 18, 27], the family of all n-vertex graphs [2, 4, 20, 22, 23, 25], the family of bipartite graphs [4], graphs with a given matching number [16], girth [26], or size [15], and very recently for the family of outerplanar graphs [13]. We note that the spreads of other

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matrices associated with a graph have also been extensively studied (see e.g. references in [13, 6, 7]).

A graph G is *planar* if it can be embedded in the plane, *i.e.*, if it can be drawn on the plane in such a way that edges intersect only at their endpoints. A graph is *outerplanar* if it can be embedded in the plane so that all vertices lie on the boundary of its outer face. By Wagner's Theorem, a graph is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a minor. Analogously, a graph is outerplanar if and only if it does not contain K_4 nor $K_{2,3}$ as a minor. In this paper, we call an outerplanar graph G *linear* if it contains a vertex u that is adjacent to all other vertices. Here we call u the *center* vertex of G. Similarly, we call a planar graph G *linear* if it contains two vertices u and w that are each adjacent to all vertices of $V(G)\backslash\{u, w\}$, and call u, w the *center* vertices of G. A graph F is called a *linear forest* if F is a disjoint union of paths.

Given two graphs G and H, the *join* of G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H by connecting every vertex of G with every vertex of H. Let P_k denote the path on k vertices. Given two graphs G and H, let $G \cup H$ denote the disjoint union of G and H. Given a graph G and a positive integer k, we use kG to denote the disjoint union of k copies of G. Given $v \subseteq V(G)$, let $N_G(v)$ denote the set of neighbors of v in G, and let $d_G(v)$ denote the degree of v in G, i.e., $d_G(v) = |N(v)|$. Given $S \subseteq V(G)$, define $N_G(S)$ as $N_G(S) = \{N_G(v) : v \in S\}$. We may ignore the subscript G when there is no ambiguity.

There has been extensive research on finding the maximum spectral radius (i.e. the largest eigenvalue) of planar and outerplanar (hyper)graphs and the corresponding extremal (hyper)graphs; see, for example, [3, 6, 7, 9, 11, 32, 17, 19, 21, 28, 29, 30, 31, 10]. In [13], Gotshall, O'Brien and Tait studied the maximum spread of outerplanar graphs and narrowed down the structure of the extremal graph attaining the maximum spread.

Theorem 1. *[13] For sufficiently large n, any graph which maximizes the spread over the family of outerplanar graphs on n vertices is of the form $K_1 \vee F$, where F is a linear forest with $\Omega(n)$ edges.*

In the same paper, Gotshall, O'Brien and Tait [13] asked whether or not F should be a path on $n - 1$ vertices, and conjectured in the affirmative.

Conjecture 2. [13] For n sufficiently large, the outerplanar graph on n vertices with the maximum spread is given by $K_1 \vee P_{n-1}$.

In this paper, we disprove this conjecture and determine the unique outerplanar graph attaining the maximum spread for sufficiently large n.

Theorem 3. *For n sufficiently large, the outerplanar graph on n vertices with the maximum spread is $K_1 \vee \left(P_{\lceil \frac{2n-1}{3} \rceil} \cup \lfloor \frac{n-2}{3} \rfloor P_1 \right)$.*

We also extend our investigation to planar graphs, and determine the unique extremal n-vertex planar graph attaining the maximum spread.

Theorem 4. *For n sufficiently large, the planar graph on n vertices with the maximum spread is $(K_1 \cup K_1) \vee \left(P_{\lceil \frac{2n-2}{3} \rceil} \cup \lfloor \frac{n-4}{3} \rfloor P_1 \right)$.*

The remainder of the paper is organized as follows. In Section 2, we first reduce the problem for outerplanar graphs to a special family of linear outerplanar graphs with only one non-trivial path (after deleting the center vertex). Theorem 3 is proved by calculating the spread of these linear outerplanar graphs up to the error term $O(n^{-3})$. In Section 3, we first prove that the maximum-spread planar graphs G must contain $K_{2,n-2}$ as a subgraph by carefully estimating the eigenvectors of λ_n and λ_1. From this structural lemma, we prove that G is one of three types of planar graphs: a double wheel, a linear planar graph of the first kind, or a linear planar graph of the second kind (defined later). For each family, we calculate the maximum spread with enough precision to distinguish the maximum-spread planar graph. In the appendix, we prove a lemma on the existence of the Laurent series of the solution of certain equations. We believe that our method could be useful for other problems on determining the maximum or minimum spectral parameters of some family of graphs.

2   Maximum spread over all outerplanar graphs

Let G be a simple graph and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of the adjacency matrix of G. Here λ_1 is called the *spectral radius* of G. Let $A(G)$ denote the adjacency matrix of G. Given a vector $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{x}'$ denote its transpose, and for each $i \in [n]$, let $\mathbf{x}_i$ denote the i-th coordinate of $\mathbf{x}$. Using the Rayleigh quotient of symmetric matrices, we have the following equalities for λ_1 and λ_n:

$$\lambda_1 = \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}' A(G) \mathbf{x}}{\mathbf{x}' \mathbf{x}} = \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{2 \sum_{ij \in E(G)} \mathbf{x}_i \mathbf{x}_j}{\mathbf{x}' \mathbf{x}} \tag{1}$$

$$\lambda_n = \min_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}' A(G) \mathbf{x}}{\mathbf{x}' \mathbf{x}} = \min_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{2 \sum_{ij \in E(G)} \mathbf{x}_i \mathbf{x}_j}{\mathbf{x}' \mathbf{x}} \tag{2}$$

Consider a linear outerplanar graph $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$. Let u be the center vertex of G and $v_1, v_2, \ldots, v_\ell$ be the vertices in order of a path component P_ℓ in $G - u$ where $\ell \in \{\ell_i : i \in [r]\}$ (we consider each path P_ℓ separately, so to simplify notation we reuse v_1, v_2, *etc.* for the first, second, *etc.* vertex along each path). Let α be a normalized eigenvector corresponding to an eigenvalue λ of the adjacency matrix of G so that $\alpha(u) = 1$. Let A_ℓ be the adjacency matrix of P_ℓ, $x_i = \alpha(v_i)$ for $1 \leqslant i \leqslant \ell$, $\mathbf{x} = (x_1, \ldots, x_\ell)' \in \mathbb{R}^\ell$, and $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^\ell$. Let I denote the identity matrix. The following lemma computes the vector $\mathbf{x}$.

Lemma 5. *Let $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear outerplanar graph on n vertices with center vertex u. Suppose λ is an eigenvalue of $A(G)$ with $|\lambda| \geqslant 2$, and α is a normalized eigenvector of $A(G)$ corresponding to λ such that $\alpha(u) = 1$. Let P_ℓ be one*

of the path components of $G - u$ and let $\mathbf{x}$ and A_ℓ be defined as above. Then

$$\mathbf{x} = \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_\ell^k \mathbf{1}. \tag{3}$$

Proof. Each vertex v_i is adjacent to u and $\alpha(u) = 1$. Hence when restricting the coordinates of $A(G)\alpha$ to $v_1, \ldots, v_\ell$, we have that

$$A_\ell \mathbf{x} + \mathbf{1} = \lambda \mathbf{x}. \tag{4}$$

It then follows that

$$\begin{aligned}
\mathbf{x} &= (\lambda I - A_\ell)^{-1} \mathbf{1} \\
&= \lambda^{-1} (I - \lambda^{-1} A_\ell)^{-1} \mathbf{1} \\
&= \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1} A_\ell)^k \mathbf{1} \\
&= \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_\ell^k \mathbf{1}.
\end{aligned} \tag{5}$$

Here we use the assumption that $|\lambda| \geqslant 2 > \lambda_1(A_\ell)$ so that the infinite series converges and the matrix $\lambda I - A_\ell$ is invertible. $\square$

Corollary 6. *Let $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear outerplanar graph on n vertices with center vertex u. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of $A(G)$ and $\{\alpha_i\}_{i \in [n]}$ be a set of pairwise orthogonal eigenvectors of $A(G)$ such that α_i corresponds to λ_i and $\alpha_i(u) = 1$ for each $i \in [n]$. Then the following properties hold:*

(i) If $\lambda_i \geqslant 2$, then all entries of α_i are positive.

(ii) If $\lambda_i \leqslant -2$, then all entries of α_i but the u-entry are negative.

(iii) For $2 \leqslant i \leqslant n-1$, we have $\lambda_i \in (-2, 2)$.

Proof. Let $P_\ell = v_1 \cdots v_\ell$ be one of the path components in $G - u$. When $\lambda_i \geqslant 2$, by Equation (3), since each entry in the vector $A_\ell^k \mathbf{1}$ is non-negative, it is clear that $\alpha_i(v_j) > 0$ for each $j \in [\ell]$. This holds for every path component of $G - u$. Moreover, $\alpha_i(u) = 1$. Hence all entries of α_i are positive.

For (ii), note that for each $j \in [\ell]$, $(A_\ell^k \mathbf{1})_j$ counts the number of walks starting from v_j in P_ℓ. Since the degree of v_j is at most 2 in $G - u$ and $|\lambda| \geqslant 2$, we have that $\{|\lambda|^{-(k+1)}(A_\ell^k \mathbf{1})_j, k \geqslant 0\}$ is a monotone nonincreasing sequence for all k and a monotone decreasing sequence for $k \geqslant j$ (as v_1 only has degree 1 in P_ℓ). Hence, if $\ell \geqslant 2$, then for each $j \in [\ell]$, the alternating series $\sum_{k=0}^{\infty} \lambda^{-(k+1)}(A_\ell^k \mathbf{1})_j$ satisfies

$$\sum_{k=0}^{\infty} \lambda^{-(k+1)}(A_\ell^k \mathbf{1})_j < \sum_{k=0}^{j} \frac{1}{\lambda^{k+1}}(A_\ell^k \mathbf{1})_j \leqslant \frac{1}{\lambda} + \frac{1}{\lambda^2} \cdot 2 + \cdots + \frac{1}{\lambda^{j+1}} \cdot 2^j \leqslant 0,$$

since $\lambda \leqslant -2$ and $(A_\ell^k \mathbf{1})_j > 0$ for each choice of k and j. If $\ell = 1$, then we have $\sum_{k=0}^\infty \lambda^{-(k+1)} (A_\ell^k \mathbf{1})_j = \frac{1}{\lambda}(A_\ell^0)_1 < 0$.

To prove (iii), we will prove that $\lambda_2 < 2$ and $\lambda_{n-1} > -2$. If $\lambda_2 \geqslant 2$, then by (i) the corresponding normalized eigenvectors α_1 and α_2 both have all entries positive, contradicting the spectral theorem that $\alpha_1 \perp \alpha_2$. Similarly, if $\lambda_{n-1} \leqslant -2$, then by (ii) the corresponding normalized eigenvectors α_{n-1} and α_n both have all negative entries except for the u-entry (which is 1), contradicting $\alpha_{n-1} \perp \alpha_n$. $\qquad\square$

One of our main ideas in determining the extremal structure is to merge the paths in the linear forests until there is only one non-trivial path. Consider a linear outerplanar graph $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots P_{\ell_r})$. A merge operation replaces G by $G' = K_1 \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots P_{\ell_r})$, i.e., it merges two non-trivial paths in $G - u$ into one. Note that a merge operation does not change the number of edges in G. In the following two lemmas, we show that a merge operation increases λ_1 of G and decreases λ_n of G, and thus increases the spread of G.

Lemma 7. *Let $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear outerplanar graph on $n \geqslant 5$ vertices with $\ell_1, \ell_2 \geqslant 2$, and let G' be obtained from G by applying a merge operation, i.e., $G' = K_1 \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots \cup P_{\ell_r})$. Then $\lambda_1(G') > \lambda_1(G)$.*

Proof. Let $\lambda_1 = \lambda_1(G)$ be the largest eigenvalue of G and α be an eigenvector of $A(G)$ corresponding to λ_1. Assume that α is normalized so that $\alpha(u) = 1$ at the center vertex u. Moreover, assume $x_1, \ldots, x_{\ell_1}$ are the entries of α at the vertices $v_1, \ldots v_{\ell_1}$ of P_{ℓ_1}, and $y_1, \ldots, y_{\ell_2}$ are the entries of α at the vertices $v_1', \ldots, v_{\ell_2}'$ of P_{ℓ_2}. By the Perron-Frobenius Theorem (or Corollary 6), all x_i's and y_j's are positive. Without loss of generality, we can assume $x_1 \geqslant y_1$. We can then obtain an isomorphic copy of G' from G by adding an edge $v_1 v_2'$ and deleting an edge $v_1' v_2'$. It then follows that

$$
\begin{aligned}
\lambda_1(G') &\geqslant \frac{\alpha' A_{G'} \alpha}{\|\alpha\|^2} \\
&= \frac{\alpha' A_G \alpha + 2x_1 y_2 - 2y_1 y_2}{\|\alpha\|^2} \\
&\geqslant \frac{\alpha' A_G \alpha}{\|\alpha\|^2} \\
&= \lambda_1(G).
\end{aligned}
$$

Note that the equality cannot hold since α is not an eigenvector of $A_{G'}$ by Lemma 5. Thus, the merge operation on linear outerplanar graphs strictly increases the largest eigenvalue λ_1. $\qquad\square$

Lemma 8. *Let $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear outerplanar graph on $n \geqslant 8$ vertices with $\ell_1, \ell_2 \geqslant 2$, and let G' be obtained from G by applying a merge operation, i.e., $G' = K_1 \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots \cup P_{\ell_r})$. Then $\lambda_n(G') < \lambda_n(G)$.*

Proof. Consider a linear outerplanar graph $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots)$ with $\ell_1, \ell_2 \geqslant 2$. Let $G' = K_1 \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots)$ be the graph after merging. Let $\lambda_n = \lambda_n(G)$ be the smallest eigenvalue of G and α be an eigenvector of A_G corresponding to λ_n. Assume that α is normalized so that $\alpha(u) = 1$ at the center vertex u. Moreover, assume $x_1, \ldots, x_{\ell_1}$ are the entries of α at the vertices $v_1, \ldots v_{\ell_1}$ of P_{ℓ_1}, and $y_1, \ldots, y_{\ell_2}$ are the entries of α at the vertices $v'_1, \ldots, v'_{\ell_2}$ of P_{ℓ_2}.

We claim now that $\lambda_n(G) \leqslant -2$ for any $n \geqslant 8$. For $8 \leqslant n \leqslant 10$, it could be easily verified by computer. Suppose now $n \geqslant 10$. Let $\beta \in \mathbb{R}^n$ be a column vector indexed by the vertices of G such that $\beta(u) = 1$ and $\beta(v) = -\frac{1}{\sqrt{n-1}}$ for any $v \neq u$. It follows by Equation (2) that

$$\lambda_n \leqslant \frac{2\sum_{ij \in E(G)} \beta(i)\beta(j)}{\beta'\beta} = 2 \cdot \frac{(n-2)\left(\frac{1}{\sqrt{n-1}}\right)^2 - (n-1)\frac{1}{\sqrt{n-1}}}{(n-1)\left(\frac{1}{\sqrt{n-1}}\right)^2 + 1} \leqslant -2,$$

which holds for $n \geqslant 10$.

Now since $\lambda_n \leqslant -2$, all x_i's and y_j's are negative by Corollary 6. Without loss of generality, we can assume $|x_1| \leqslant |y_1|$. (Here we don't assume any relation between ℓ_1 and ℓ_2.) We can then obtain an isomorphic copy of G' from G by adding an edge $v_1 v'_2$ and deleting an edge $v'_1 v'_2$. It now follows that

$$
\begin{aligned}
\lambda_n(G') &\leqslant \frac{\alpha' A_{G'}\alpha}{\|\alpha\|^2} \\
&= \frac{\alpha' A_G \alpha + 2x_1 y_2 - 2y_1 y_2}{\|\alpha\|^2} \\
&\leqslant \frac{\alpha' A_G \alpha}{\|\alpha\|^2} \\
&= \lambda(G).
\end{aligned}
$$

Here we use the assumption $|x_1| \leqslant |y_1|$ and x_1, y_1, y_2 are all negative. Note that the equality cannot hold since α is not an eigenvector of $A_{G'}$ by Lemma 5. Thus, the merge operation on a linear outerplanar graph strictly decreases the smallest eigenvalue λ_1. □

Hence we have the following corollary.

Corollary 9. *Let $G = K_1 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear outerplanar graph on $n \geqslant 8$ vertices with $\ell_1, \ell_2 \geqslant 2$, and let G' be obtained from G by applying a merge operation, i.e., $G' = K_1 \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots \cup P_{\ell_r})$ Then $S(G') > S(G)$.*

Repeatedly applying Lemma 7 and Lemma 8, we conclude that the maximum spread is reached at a linear outerplanar graph with only one non-trivial path (after deleting the center vertex). Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let G be a graph attaining the maximum spread among all outerplanar graphs on n vertices (for sufficiently large n) and let u be its center vertex.

By Corollary 9, the merge operation strictly increases the spread of a linear outerplanar graph. Hence, we can assume that there is at most one non-trivial path in $G - u$. Thus, $G = G_\ell = K_1 \vee (P_\ell \cup (n - 1 - \ell)K_1)$ for some $\ell \in [n - 1]$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of $A(G_\ell)$. Suppose $\lambda \in \{\lambda_1, \lambda_n\}$ and α is a normalized eigenvector of $A(G_\ell)$ corresponding to λ with $\alpha(u) = 1$. Let $\mathbf{x} = (x_1, \ldots, x_\ell)$ be the vector of α restricted to the vertices of P_ℓ. By the eigen-equation, the entries of α at those vertices not on P_ℓ and not u are equal to $\frac{1}{\lambda}$. The eigen-equation at u is given by

$$\lambda = (n - 1 - \ell)\frac{1}{\lambda} + \sum_{i=1}^{\ell} x_i. \tag{6}$$

Applying Lemma 5, we get

$$\sum_{i=1}^{\ell} x_i = \mathbf{1}' \cdot \mathbf{x}$$

$$= \mathbf{1}' \cdot \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_\ell^k \mathbf{1}$$

$$= \sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_\ell^k \mathbf{1}.$$

Plugging it into Equation (6), we have

$$\lambda = (n - 1 - \ell)\frac{1}{\lambda} + \sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_\ell^k \mathbf{1}. \tag{7}$$

Recall that $P_\ell = v_1 v_2 \cdots v_\ell$ is the only nontrivial path in the neighborhood of u. Given $v_j \in V(P_\ell)$, let $w_k(v_j)$ denote the number of walks of length k starting from v_j in P_ℓ (so that the walks exclude the vertex u). Observe that for each $j \in [\ell]$, $(A_\ell^k \mathbf{1})_j = w_k(v_j)$ and $\mathbf{1}' A_\ell^k \mathbf{1} = \sum_{j \in [\ell]} w_k(v_j)$. We compute $\mathbf{1}' A_\ell^k \mathbf{1}$ for several values of k assuming ℓ is large enough (which we can assume for large n by Theorem 1).

When $k = 1$,
$$\mathbf{1}' A_\ell^k \mathbf{1} = \sum_{j \in [\ell]} d(v_j) = 2(\ell - 2) + 2 = 2(\ell - 1).$$

When $k = 2$, it is not hard to see that $w_2(v_1) = w_2(v_\ell) = 2$; $w_2(v_2) = w_2(v_{\ell-1}) = 3$; and $w_2(v_j) = 4$ for all $j \in [3, \ell - 2]$. Hence

$$\mathbf{1}' A_\ell^2 \mathbf{1} = 4(\ell - 4) + 3 \cdot 2 + 2 \cdot 2 = 4\ell - 6.$$

Similarly, when $k = 3$, it is not hard to see that $w_3(v_1) = w_3(v_\ell) = 3$; $w_3(v_2) = w_3(v_{\ell-1}) = 6$; $w_3(v_3) = w_3(v_{\ell-2}) = 7$; and $w_3(v_j) = 8$ for all $j \in [4, \ell - 3]$. Hence

$$\mathbf{1}' A_\ell^3 \mathbf{1} = 8(\ell - 6) + 7 \cdot 2 + 6 \cdot 2 + 3 \cdot 2 = 8(\ell - 2).$$

Similarly, when $k = 4$, it is not hard to see that $w_4(v_1) = w_4(v_\ell) = 6$; $w_4(v_2) = w_4(v_{\ell-1}) = 10$; $w_4(v_3) = w_4(v_{\ell-2}) = 14$; $w_4(v_4) = w_4(v_{\ell-3}) = 15$; and $w_4(v_j) = 16$ for all $j \in [5, \ell - 4]$. Hence

$$\mathbf{1}'A_\ell^4\mathbf{1} = 16(\ell - 8) + 15 \cdot 2 + 14 \cdot 2 + 10 \cdot 2 + 6 \cdot 2 = 16\ell - 38.$$

Similarly, when $k = 5$, it is not hard to see that $w_5(v_1) = w_5(v_\ell) = 10$; $w_5(v_2) = w_5(v_{\ell-1}) = 20$; $w_5(v_3) = w_5(v_{\ell-2}) = 25$; $w_5(v_4) = w_5(v_{\ell-3}) = 30$; $w_5(v_5) = w_5(v_{\ell-4}) = 31$; and $w_5(v_j) = 32$ for all $j \in [6, \ell - 5]$. Hence

$$\mathbf{1}'A_\ell^5\mathbf{1} = 32(\ell - 10) + 31 \cdot 2 + 30 \cdot 2 + 25 \cdot 2 + 20 \cdot 2 + 10 \cdot 2 = 32\ell - 88.$$

Similarly, for $k \geqslant 6$, we have that

$$\sum_{k=6}^{\infty} \lambda^{-k} \mathbf{1}'A_\ell^k\mathbf{1} \leqslant \sum_{k=6}^{\infty} \frac{2^k\ell}{\lambda^k} = O\left(\frac{\ell}{\lambda^6}\right).$$

Multiplying both sides of (7) by λ and simplifying it, we get

$$\lambda^2 = (n-1) + \frac{2(\ell-1)}{\lambda} + \frac{4\ell - 6}{\lambda^2} + \frac{8(\ell-2)}{\lambda^3} + \frac{16\ell - 38}{\lambda^4} + \frac{32\ell - 88}{\lambda^5} + O\left(\frac{\ell}{\lambda^6}\right). \quad (8)$$

Equation (8) has two real roots, which determines λ_1 and λ_n. λ_1 is near $\sqrt{n-1}$ and λ_n is near $-\sqrt{n-1}$. By Lemma 27 in the appendix, λ_1 has a series expansion

$$\begin{aligned}
\lambda_1 = \sqrt{n-1} + c_1 &+ \frac{c_2}{\sqrt{n-1}} + \frac{c_3}{n-1} + \frac{c_4}{(n-1)^{3/2}} \\
&+ \frac{c_5}{(n-1)^2} + \frac{c_6}{(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right).
\end{aligned} \quad (9)$$

Plugging it into Equation (8), and comparing the terms, using SageMath[1], we compute the values of all c_is as follows:

$$c_1 = \frac{\ell - 1}{n - 1}, \quad (10)$$

$$c_2 = 2\left(\frac{\ell-1}{n-1}\right) - \frac{3}{2}\left(\frac{\ell-1}{n-1}\right)^2, \quad (11)$$

$$c_3 = 4\left(\frac{\ell-1}{n-1}\right) - 8\left(\frac{\ell-1}{n-1}\right)^2 + 4\left(\frac{\ell-1}{n-1}\right)^3, \quad (12)$$

$$c_4 = -1 + 8\left(\frac{\ell-1}{n-1}\right) - 30\left(\frac{\ell-1}{n-1}\right)^2 + 35\left(\frac{\ell-1}{n-1}\right)^3 - \frac{105}{8}\left(\frac{\ell-1}{n-1}\right)^4, \quad (13)$$

$$c_5 = -4 + 20\left(\frac{\ell-1}{n-1}\right) - 96\left(\frac{\ell-1}{n-1}\right)^2 + 192\left(\frac{\ell-1}{n-1}\right)^3 - 160\left(\frac{\ell-1}{n-1}\right)^4$$

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[1]Computation available at:

$$+ 48 \left(\frac{\ell - 1}{n - 1} \right)^5, \tag{14}$$

$$c_6 = -11 + 62 \left(\frac{\ell - 1}{n - 1} \right) - \frac{595}{2} \left(\frac{\ell - 1}{n - 1} \right)^2 + 840 \left(\frac{\ell - 1}{n - 1} \right)^3 - 1155 \left(\frac{\ell - 1}{n - 1} \right)^4$$

$$+ \frac{3003}{4} \left(\frac{\ell - 1}{n - 1} \right)^5 - \frac{3003}{16} \left(\frac{\ell - 1}{n - 1} \right)^6. \tag{15}$$

By a similar calculation, we get the following series expansion of λ_n:

$$\lambda_n = -\sqrt{n - 1} + c_1 - \frac{c_2}{\sqrt{n - 1}} + \frac{c_3}{n - 1} - \frac{c_4}{(n - 1)^{3/2}}$$

$$+ \frac{c_5}{(n - 1)^2} - \frac{c_6}{(n - 1)^{5/2}} + O \left(\frac{1}{(n - 1)^3} \right). \tag{16}$$

Here $c_1, c_2, c_3, c_4, c_5, c_6$ are the same quantities as in Equations (10), (11), (12), (13), (14), and (15). Observe that all c_is are bounded since they are polynomials of $\frac{\ell - 1}{n - 1}$ over the interval $[0, 1]$. Thus, the spread of G_ℓ, written by $f(\ell)$, can be expressed as follows:

$$f(\ell) = \lambda_1 - \lambda_n = 2\sqrt{n - 1} + \frac{2c_2}{\sqrt{n - 1}} + \frac{2c_4}{(n - 1)^{3/2}} + \frac{2c_6}{(n - 1)^{5/2}} + O \left(\frac{1}{(n - 1)^3} \right).$$

Since

$$c_2(\ell) = 2 \left(\frac{\ell - 1}{n - 1} \right) - \frac{3}{2} \left(\frac{\ell - 1}{n - 1} \right)^2 = \frac{2}{3} - \frac{3}{2} \left(\frac{\ell - 1}{n - 1} - \frac{2}{3} \right)^2 \leqslant \frac{2}{3},$$

the function c_2 reaches the maximum at $\ell_1 = \frac{2}{3}(n - 1) + 1$. Let $\ell_0 = \lceil \frac{2n - 1}{3} \rceil$ be the target argument maximum of $f(\ell)$. We have

$$f(\ell_0) = 2\sqrt{n - 1} + \frac{4}{3\sqrt{n - 1}} + O \left(\frac{1}{(n - 1)^{3/2}} \right).$$

Claim 10. *There is a constant C such that all argument maximum point(s) of $f(\ell)$ must belong to the interval*

$$(\ell_1 - C\sqrt{n - 1}, \ell_1 + C\sqrt{n - 1}).$$

Proof. Otherwise, for any ℓ not in this interval, we have

$$c_2(\ell) \leqslant \frac{2}{3} - \frac{3C^2}{2(n - 1)}.$$

This implies

$$f(\ell) \leqslant 2\sqrt{n - 1} + 2\frac{\frac{2}{3} - \frac{3C^2}{2(n - 1)}}{\sqrt{n - 1}} + O \left(\frac{1}{(n - 1)^{3/2}} \right) < f(\ell_0).$$

Here we choose the constant C big enough such that

$$-\frac{3C^2}{(n - 1)^{3/2}} + O \left(\frac{1}{(n - 1)^{3/2}} \right) < 0. \qquad \square$$

From now on, we assume $\ell \in (\ell_1 - C\sqrt{n-1}, \ell_1 + C\sqrt{n-1})$. Let us compute $f(\ell + 1) - f(\ell)$. We have

$$c_2(\ell+1) - c_2(\ell) = \frac{2}{n-1} - \frac{3}{2}\frac{(2\ell-1)}{(n-1)^2} = \frac{2(n-1) - \frac{3}{2}(2\ell-1)}{(n-1)^2},$$

$$c_4(\ell+1) - c_4(\ell) = \frac{8}{n-1} - 30\frac{(2\ell-1)}{(n-1)^2} + 35\frac{(3\ell^2 - 3\ell + 1)}{(n-1)^3}$$
$$- \frac{105}{8}\frac{(4\ell^3 - 6\ell^2 + 4\ell - 1)}{(n-1)^4},$$

$$c_6(\ell+1) - c_6(\ell) = O\left(\frac{1}{n-1}\right).$$

Plugging $\ell = \ell_1 \cdot \left(1 + O\left(\frac{1}{\sqrt{n-1}}\right)\right)$ into $c_4(\ell+1) - c_4(\ell)$, we have

$$c_4(\ell+1) - c_4(\ell) = \frac{1}{n-1}\left(8 - 30 \cdot 2 \cdot \frac{2}{3} + 35 \cdot 3 \cdot \left(\frac{2}{3}\right)^2 - \frac{105}{8} \cdot 4 \cdot \left(\frac{2}{3}\right)^3\right)$$
$$+ O\left(\frac{1}{n-1)^{3/2}}\right)$$
$$= -\frac{8}{9(n-1)} + O\left(\frac{1}{(n-1)^{3/2}}\right).$$

Therefore, we have

$$f(\ell+1) - f(\ell) = 2\frac{c_2(\ell+1) - c_2(\ell)}{\sqrt{n-1}} + 2\frac{c_4(\ell+1) - c_4(\ell)}{(n-1)^{3/2}}$$
$$+ 2\frac{c_6(\ell+1) - c_6(\ell)}{(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right)$$
$$= \frac{4(n-1) - 3(2\ell-1)}{(n-1)^{5/2}} - \frac{16}{9(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right)$$
$$= \frac{4n - 6\ell - \frac{25}{9}}{(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right). \tag{17}$$

When $\ell \geqslant \ell_0$, we have

$$4n - 6\ell - \frac{25}{9} \leqslant 4n - 6\ell_0 - \frac{25}{9} \leqslant 4n - 6 \cdot \frac{2n-1}{3} - \frac{25}{9} = -\frac{7}{9} < 0.$$

Plugging it into Equation (17), we have

$$f(\ell+1) - f(\ell) \leqslant \frac{-\frac{7}{9}}{(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) < 0.$$

When $\ell \leqslant \ell_0 - 1$, we have

$$4n - 6\ell - \frac{25}{9} \geqslant 4n - 6(\ell_0 - 1) - \frac{25}{9} \geqslant 4n - 6 \cdot \left(\frac{2n-1}{3} - \frac{1}{3}\right) - \frac{25}{9} = \frac{11}{9} > 0.$$

Thus,

$$f(\ell + 1) - f(\ell) \geqslant \frac{\frac{11}{9}}{(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) > 0.$$

Therefore, $f(\ell)$ reaches the unique maximum at ℓ_0 for n sufficiently large. This completes the proof of the theorem. □

3 Maximum spread over all planar graphs

Let G be the planar graph on n vertices with maximum spread, and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of the adjacency matrix of G. Note that we may assume G is connected, as adding an edge between two connected components of G can neither decrease λ_1 (by the Perron-Frobenius Theorem) nor increase λ_n. Indeed, if C_1 and C_2 are two components of G, and H is the graph obtained from G by adding an edge between C_1 and C_2, then by the eigenvalue interlacing theorem we have $\lambda_n(H) \leqslant \lambda_{|V(C_1)|}(C_1)$ and $\lambda_n(H) \leqslant \lambda_{|V(C_2)|}(C_2)$; so $\lambda_n(H) \leqslant \lambda_n(G)$ as the spectrum of a disconnected graph is the union of the spectra of its connected components.

We briefly outline the proof of Theorem 4. First, in Section 3.1 we show that G must contain $K_{2,n-2}$ as a subgraph. We accomplish this by showing that the maximum-spread planar graph must contain two vertices v_1 and v_2 with large entries in the eigenvectors $\mathbf{x}$ and $\mathbf{z}$ corresponding to λ_1 and λ_n, respectively, while the entries for the other vertices are small. This will imply that v_1 and v_2 have large degrees and we boost this result by using the Rayleigh principle to show that v_1 and v_2 are adjacent to every vertex in $V(G) \setminus \{v_1, v_2\}$. From this structural characterization, we show in Corollary 20 that G must be one of three possible types of graphs: (i) the double wheel $(K_1 \cup K_1) \vee C_{n-2}$; (ii) the graph $(K_1 \cup K_1) \vee T_{n-2}$, where T_{n-2} is a disjoint union of paths on $n-2$ vertices; or (iii) the graph $K_2 \vee T_{n-2}$. Sections 3.2–3.4 are devoted to calculating the spread of each of these three types of graphs. The spread of the double wheel can be determined exactly. The spreads of $(K_1 \cup K_1) \vee T_{n-2}$ and $K_2 \vee T_{n-2}$ can be calculated using similar methods as in the proof of Theorem 3 determining the maximum-spread outerplanar graph. In Section 3.5, we complete the proof of Theorem 4 by using our spread calculations from Sections 3.2–3.4 to conclude that the maximum spread graph is of the form $(K_1 \cup K_1) \vee T_{n-2}$, and we further show that $T_{n-2} = P_{\lceil \frac{2n-2}{3} \rceil} \cup \lfloor \frac{n-4}{3} \rfloor P_1$.

3.1 Structure of maximum spread planar graphs

We show in this section that G must contain $K_{2,n-2}$ as subgraph. We recall the result of Tait and Tobin [24] on the maximum spectral radius of planar graphs.

Theorem 11. *[24] For n sufficiently large, the planar graph on n vertices with maximal spectral radius is $P_2 \vee P_{n-2}$.*

We first give some upper and lower bounds on $\lambda_1(G)$ and $|\lambda_n(G)|$ when n is sufficiently large. We use known expressions for the eigenvalues of a join of two regular graphs [5, pg.19].

Lemma 12. *[5] Let G and H be regular graphs with degrees k and ℓ respectively. Suppose that $|V(G)| = m$ and $|V(H)| = n$. Then, the characteristic polynomial of $G \vee H$ is $p_{G \vee H}(t) = ((t-k)(t-\ell) - mn)\frac{p_G(t)p_H(t)}{(t-k)(t-\ell)}$. In particular, if the eigenvalues of G are $k = \lambda_1 \geqslant \cdots \geqslant \lambda_m$ and the eigenvalues of H are $\ell = \mu_1 \geqslant \cdots \geqslant \mu_n$, then the eigenvalues of $G \vee H$ are $\{\lambda_i : 2 \leqslant i \leqslant m\} \cup \{\mu_j : 2 \leqslant j \leqslant n\} \cup \{x : (x-k)(x-\ell) - mn = 0\}$.*

We will apply Lemma 12 to the graphs $(2K_1) \vee C_{n-2}$ and $K_2 \vee C_{n-2}$.

Lemma 13.

$$\sqrt{2n-4} - \frac{3}{2} - O\left(\frac{1}{\sqrt{n}}\right) \leqslant |\lambda_n| \leqslant \lambda_1 \leqslant \sqrt{2n-4} + \frac{3}{2} + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. By Lemma 12, $\lambda_1(K_2 \vee C_{n-2})$ is the largest root of $(x-1)(x-2) - 2(n-2) = 0$, which is $\frac{3}{2} + \frac{1}{2}\sqrt{8n-15}$. Now by Theorem 11, we have $\lambda_1 \leqslant \lambda_1(K_2 \vee P_{n-2}) \leqslant \lambda_1(K_2 \vee C_{n-2}) = \frac{3}{2} + \frac{1}{2}\sqrt{8n-15}$. Note that $\frac{3}{2} + \frac{1}{2}\sqrt{8n-15} = \frac{3}{2} + \sqrt{2n-4} + O\left(\frac{1}{\sqrt{n}}\right)$. Hence, $-\lambda_n \leqslant \lambda_1 \leqslant \frac{3}{2} + \sqrt{2n-4} + O\left(\frac{1}{\sqrt{n}}\right)$. Since G is the planar graph with maximal spread and $\lambda_1(K_{2,n-2}) = -\lambda_n(K_{2,n-2}) = \sqrt{2(n-2)}$, we have that

$$\lambda_1 - \lambda_n \geqslant S(K_{2,n-2}) = \sqrt{2(n-2)} - (-\sqrt{2(n-2)}) = 2\sqrt{2n-4}.$$

Hence, $-\lambda_n \geqslant \sqrt{2n-4} - \frac{3}{2} - O\left(\frac{1}{\sqrt{n}}\right)$. $\qquad \square$

For the rest of this section, let $\mathbf{x}$ and $\mathbf{z}$ be the eigenvectors of $A(G)$ corresponding to the eigenvalues λ_1 and λ_n respectively. For convenience, let $\mathbf{x}$ and $\mathbf{z}$ be indexed by the vertices of G. By the Perron-Frobenius theorem, we may assume that all entries of $\mathbf{x}$ are positive. We also assume that $\mathbf{x}$ and $\mathbf{z}$ are normalized so that the maximum absolute values of the entries of $\mathbf{x}$ and $\mathbf{z}$ are equal to 1, and so there are vertices u_0 and w_0 with $\mathbf{x}_{u_0} = |\mathbf{z}_{w_0}| = 1$.

Let $V_+ = \{v : \mathbf{z}_v > 0\}$, $V_0 = \{v : \mathbf{z}_v = 0\}$, and $V_- = \{v : \mathbf{z}_v < 0\}$. Since $\mathbf{z}$ is a non-zero vector, at least one of V_+ and V_- is non-empty. By considering the eigen-equations of $\lambda_n \sum_{v \in V_+} \mathbf{z}_v$ or $\lambda_n \sum_{v \in V_-} \mathbf{z}_v$, or by using the fact that $\mathbf{x}$ and $\mathbf{z}$ are orthogonal, both V_+ and V_- are non-empty. For any vertex subset S, we define the *volume* of S, denoted by $\mathrm{Vol}(S)$, as $\mathrm{Vol}(S) = \sum_{v \in S} |\mathbf{z}_v|$. In the following lemmas, we use the bounds of λ_n to deduce some information on V_+, V_- and V_0.

Lemma 14. $|V_0| \leqslant \left(\frac{3}{2} + o(1)\right)\sqrt{2n-4}$.

Proof. For any $v \in V_+$,

$$|\lambda_n|\mathbf{z}_v = -\lambda_n \mathbf{z}_v = -\sum_{u \in N(v)} \mathbf{z}_u \leqslant -\sum_{u \in N(v) \cap V_-} \mathbf{z}_u = \sum_{u \in N(v) \cap V_-} |\mathbf{z}_u|. \qquad (18)$$

Similarly for any $u \in V_-$, we have

$$|\lambda_n||\mathbf{z}_u| \leqslant \sum_{v \in N(u) \cap V_+} |\mathbf{z}_v|. \qquad (19)$$

Summing over all $v \in V_+$ of Equation (18) and multiplying by $|\lambda_n|$, we get

$$
|\lambda_n|^2 \sum_{v \in V_+} |\mathbf{z}_v| \leqslant \sum_{v \in V_+} \sum_{u \in N(v) \cap V_-} |\lambda_n| |\mathbf{z}_u|
$$

$$
\leqslant \sum_{v \in V_+} \sum_{u \in N(v) \cap V_-} \sum_{y \in N(u) \cap V_+} |\mathbf{z}_y|
$$

$$
= \sum_{y \in V_+} |\mathbf{z}_y| \cdot \sum_{u \in N(y) \cap V_-} |N(u) \cap V_+|
$$

$$
= \sum_{y \in V_+} |\mathbf{z}_y| |E(N(y) \cap V_-, V_+)|
$$

$$
\leqslant \sum_{y \in V_+} |\mathbf{z}_y| (2(|N(y) \cap V_-| + |V_+|) - 4). \tag{20}
$$

In the last step, we use the fact that a bipartite planar graph on m vertices can have at most $2m - 4$ edges. We use the trivial bound

$$
2(|N(y) \cap V_-| + |V_+|) - 4 \leqslant 2(|V_-| + |V_+|) - 4 = 2(n - |V_0|) - 4. \tag{21}
$$

It then follows from (20) and (21) that

$$
|\lambda_n|^2 \mathrm{Vol}(V_+) \leqslant (2(n - |V_0|) - 4) \mathrm{Vol}(V_+). \tag{22}
$$

Applying the lower bound of $|\lambda_n| \geqslant \sqrt{2n - 4} - \frac{3}{2} - O(\frac{1}{\sqrt{n}})$, and simplifying (22), we then obtain that $|V_0| \leqslant \left(\frac{3}{2} + o(1) \right) \sqrt{2n - 4}$. This completes the proof of the lemma. $\square$

Since $|V_-| + |V_+| = n - |V_0| = n - O(\sqrt{n})$, without loss of generality, we can assume $|V_-| > \frac{n}{2} - O(\sqrt{n}) > 20\sqrt{2n - 4}$ (note that we do not know at this point whether $w_0 \in V_-$ or $w_0 \in V_+$). The following lemma bounds the volume of V_+.

Lemma 15. *Let $V_+' = \{v \in V_+ : |N(v) \cap V_-| \geqslant |V_-| - 5\sqrt{2n - 4}\}$ and $V_+'' = V_+ \setminus V_+'$. Then we have*

$$
\mathrm{Vol}(V_+'') \leqslant \left(\frac{3}{7} + O\left(\frac{1}{\sqrt{n}} \right) \right) \mathrm{Vol}(V_+'), \tag{23}
$$

and thus $\mathrm{Vol}(V_+) \leqslant \left(\frac{10}{7} + O\left(\frac{1}{\sqrt{n}} \right) \right) \mathrm{Vol}(V_+')$.

Proof. By Inequality (20) of Lemma 14, we have

$$
|\lambda_n|^2 \sum_{v \in V_+} |\mathbf{z}_v| \leqslant \sum_{y \in V_+} |\mathbf{z}_y| (2(|N(y) \cap V_-| + |V_+|) - 4). \tag{24}
$$

For $y \in V_+'$, we use the trivial bound

$$
2(|N(y) \cap V_-| + |V_+|) - 4 \leqslant 2(|V_-| + |V_+|) - 4 = 2(n - |V_0|) - 4. \tag{25}
$$

For $y \in V''_+$, we use a better bound

$$2(|N(y) \cap V_-| + |V_+|) - 4 \leqslant 2((|V_-| - 5\sqrt{2n-4}) + |V_+|) - 4 \leqslant 2n - 4 - 10\sqrt{2n-4}. \quad (26)$$

Plugging Equations (25) and (26) into Equation (24), we get

$$\begin{aligned}
|\lambda_n|^2 \sum_{v \in V_+} |\mathbf{z}_v| &\leqslant \sum_{y \in V_+} |\mathbf{z}_y|(2(|N(y) \cap V_-| + |V_+|) - 4) \\
&\leqslant \sum_{y \in V'_+} |\mathbf{z}_y|(2n - 4) + \sum_{y \in V''_+} |\mathbf{z}_y|(2n - 4 - 10\sqrt{2n-4}) \\
&= (2n - 4) \sum_{y \in V_+} |\mathbf{z}_y| - 10\sqrt{2n-4} \sum_{y \in V''_+} |\mathbf{z}_y|. \quad (27)
\end{aligned}$$

Now we apply again the lower bound of $|\lambda_n| \geqslant \sqrt{2n-4} - \frac{3}{2} - O(\frac{1}{\sqrt{n}})$. Simplifying Equation (27), we get

$$\left(3\sqrt{2n-4} + O(1)\right) \sum_{v \in V_+} |\mathbf{z}_v| \geqslant 10\sqrt{2n-4} \sum_{v \in V''_+} |\mathbf{z}_v|.$$

Thus, we have

$$\mathrm{Vol}(V''_+) \leqslant \left(\frac{3}{10} + O\left(\frac{1}{\sqrt{n}}\right)\right) \mathrm{Vol}(V_+).$$

Equivalently,

$$\mathrm{Vol}(V''_+) \leqslant \left(\frac{3}{7} + O\left(\frac{1}{\sqrt{n}}\right)\right) \mathrm{Vol}(V'_+). \quad (28)$$

As a corollary, we have

$$\mathrm{Vol}(V_+) = \mathrm{Vol}(V'_+) + \mathrm{Vol}(V''_+) \leqslant \left(\frac{10}{7} + O\left(\frac{1}{\sqrt{n}}\right)\right) \mathrm{Vol}(V'_+). \qquad \square$$

Using Lemma 15, we deduce some important information on the structure of the extremal graph.

Lemma 16. *We have*

(i) *There exist $v_1, v_2 \in V_+$ with $\min\{d(v_1), d(v_2)\} \geqslant n - 5\sqrt{2n-4}$.*

(ii) *$w_0 \in \{v_1, v_2\}$.*

(iii) *For all $v \in V(G) \backslash \{v_1, v_2\}$, $d(v) \leqslant 10\sqrt{2n-4} + 4$.*

(iv) *For all $v \in V(G) \backslash \{v_1, v_2\}$, $|\mathbf{z}_v| = O(\frac{1}{\sqrt{n}})$.*

(v) *Assume $w_0 = v_1$. Then $\mathbf{z}_{v_2} \geqslant 1 - O(\frac{1}{\sqrt{n}})$.*

Proof. Observe that $|V'_+| \leqslant 2$. Otherwise, any three vertices in $|V'_+|$ have common neighbors of size at least $|V_-| - 15\sqrt{2n-4} \geqslant 5\sqrt{2n-4} > 3$. Thus G contains a subgraph $K_{3,3}$, contradicting that G is planar. Thus, we have that for sufficiently large n,

$$\mathrm{Vol}(V'_+) \leqslant |V'_+| \leqslant 2.$$

Hence by Lemma 15, we have

$$\mathrm{Vol}(V''_+) \leqslant \left(\frac{3}{7} + O\left(\frac{1}{\sqrt{n}}\right)\right) \cdot 2 < 1.$$

This implies that $w_0 \notin V''_+$ since $|\mathbf{z}_{w_0}| = 1$. Moreover $w_0 \notin V_-$, as otherwise

$$\sqrt{2n-4} - \frac{3}{2} - O\left(\frac{1}{\sqrt{n}}\right) \leqslant |\lambda_n| = \lambda_n \mathbf{z}_{w_0} \leqslant \sum_{u \in V_+ \cap N(w_0)} \mathbf{z}_u \leqslant \mathrm{Vol}(V_+) < 3,$$

giving a contradiction. Thus $w_0 \in V'_+$. In particular, $z_{w_0} = 1$.

Now we show V'_+ has exactly two vertices. If not, assume w_0 is the only vertex in V'_+. We have

$$
\begin{aligned}
|\lambda_n|^2 = |\lambda_n|^2 \mathbf{z}_{w_0} &\leqslant |\lambda_n| \sum_{u \in N(w_0) \cap V_-} |\mathbf{z}_u| \\
&\leqslant \sum_{u \in N(w_0) \cap V_-} \sum_{y \in N(u) \cap V_+} \mathbf{z}_y \\
&\leqslant n + (2n-4) \sum_{y \in V''_+} \mathbf{z}_y \\
&\leqslant n + (2n-4)\left(\frac{3}{7} + O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&\leqslant (2 - \frac{1}{7})n + O(\sqrt{n}),
\end{aligned}
$$

contradicting the lower bound of $|\lambda_n|$. Hence $|V'_+| = 2$. Let $V'_+ = \{v_1, v_2\}$. Notice that $d(v) \leqslant 10\sqrt{2n-4} + 4$ for any $v \neq v_1, v_2$. Otherwise v, v_1, v_2 have a common neighborhood of size at least 3, contradicting that G is $K_{3,3}$-free.

Now we will show that for any $v \notin \{v_1, v_2\}$, $|\mathbf{z}_v| = O\left(\frac{1}{\sqrt{n}}\right)$. For any $v \in V''_+$, we have

$$
\begin{aligned}
|\lambda_n|^2 \mathbf{z}_v &\leqslant |\lambda_n| \sum_{u \in N(v) \cap V_-} |\mathbf{z}_u| \\
&\leqslant \sum_{u \in N(v) \cap V_-} \sum_{y \in N(u) \cap V_+} \mathbf{z}_y \\
&= \sum_{y \in V_+} \mathbf{z}_y \cdot |N(v) \cap N(y) \cap V_-|
\end{aligned}
$$

$$\leqslant \left(10\sqrt{2n-4}+4\right) \sum_{y \in V_+} \mathbf{z}_y$$

$$\leqslant \left(10\sqrt{2n-4}+4\right) 2\left(\frac{10}{7}+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Thus, $\mathbf{z}_v = O\left(\frac{1}{\sqrt{n}}\right)$.

For $u \in V_-$, applying Equation (19), we have

$$|\lambda_n|\mathbf{z}_u \leqslant \sum_{v \in N(v) \cap V_+} \mathbf{z}_v$$

$$\leqslant \sum_{v \in V_+} \mathbf{z}_v$$

$$\leqslant 2\left(\frac{10}{7}+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Therefore, $\mathbf{z}_u = O\left(\frac{1}{\sqrt{n}}\right)$.

Finally, we estimate $\mathbf{z}_{v_2}$. From the eigen-equations, we get

$$|\lambda_n|(\mathbf{z}_w - \mathbf{z}_{v_2}) = -\sum_{u \in N(w) \setminus N(v_2)} \mathbf{z}_u + \sum_{u \in N(v_2) \setminus N(w)} \mathbf{z}_u \tag{29}$$

$$\leqslant \sum_{u \in (N(w) \setminus N(v_2)) \cap V_-} |\mathbf{z}_u| + \sum_{u \in (N(v_2) \setminus N(w)) \cap V_+} \mathbf{z}_u \tag{30}$$

$$\leqslant \sum_{u \in (N(w) \setminus N(v_2)) \cap V_-} |\mathbf{z}_u| + \sum_{u \in V_+''} \mathbf{z}_u \tag{31}$$

$$\leqslant 10\sqrt{2n-4} \cdot O\left(\frac{1}{\sqrt{n}}\right) + \frac{6}{7} + O\left(\frac{1}{\sqrt{n}}\right) \tag{32}$$

$$= O(1). \tag{33}$$

Therefore, we have $\mathbf{z}_{v_2} \geqslant 1 - O\left(\frac{1}{\sqrt{n}}\right)$. $\qquad\square$

For $i \in \{0,1,2\}$, let $V_i = \{v \in V(G) \setminus \{v_1, v_2\} : |N(v) \cap \{v_1, v_2\}| = i\}$. We have the following lemma on the structure of G.

Lemma 17. *We have the following properties.*

(i) $|V_2| \geqslant n - 10\sqrt{2n-4}$.

(ii) *For any $v \in V_0 \cup V_1 \cup V_2$, $|N(v) \cap V_2| \leqslant 2$.*

(iii) *In $H = G[V_0 \cup V_1 \cup V_2]$, for any vertex $v \in V(H)$, $|N_H(N_H(v)) \cap V_2| \leqslant 4$.*

Proof. By Lemma 16, $\min\{d(v_1), d(v_2)\} \geqslant n - 5\sqrt{2n-4}$. It follows that $|V_2| \geqslant n - 10\sqrt{2n-4}$. For any $v \in V_0 \cup V_1 \cup V_2$, v has at most two neighbors in V_2, otherwise, v, v_1, v_2 and three of their common neighbors would form a $K_{3,3}$ in G.

Now for any $v \in V(G[V_0 \cup V_1 \cup V_2])$, we claim that $|N_H(N_H(v)) \cap V_2| \leqslant 4$. Indeed, suppose not, then by (ii) and the Pigeonhole principle, there exist three vertex-disjoint 2-vertex paths $u_1 w_1, u_2 w_2, u_3 w_3$ in H such that v is adjacent to u_1, u_2, u_3, and $w_1, w_2, w_3 \in N(N(v)) \cap V_2$. We then have a $K_{3,3}$ minor in G, contradicting that G is planar. $\square$

Using Lemma 17, we can obtain bounds on the entries of $\mathbf{x}$.

Lemma 18. *Let u_0 be the vertex such that $\mathbf{x}_{u_0} = 1$.*

(i) $u_0 \in \{v_1, v_2\}$.

(ii) $\min\{\mathbf{x}_{v_1}, \mathbf{x}_{v_2}\} \geqslant 1 - O(\frac{1}{\sqrt{n}})$.

(iii) For any other vertex $v \notin \{v_1, v_2\}$, $\mathbf{x}_v = O(\frac{1}{\sqrt{n}})$.

Proof. Let prove (iii) first. For any vertex $v \notin \{v_1, v_2\}$, we have

$$\lambda_1^2 \mathbf{x}_v = \lambda_1 \sum_{s \in N(v)} \mathbf{x}_s$$

$$= \lambda_1 \left(\sum_{s \in N(v) \cap V_2} \mathbf{x}_s + \sum_{s \in N(v) \cap \{v_1, v_2\}} \mathbf{x}_s + \sum_{s \in N(v) \cap (V_0 \cup V_1)} \mathbf{x}_s \right)$$

$$\leqslant 4\lambda_1 + \sum_{s \in N(v) \cap (V_0 \cup V_1)} \lambda_1 \mathbf{x}_s$$

$$= 4\lambda_1 + \sum_{s \in N(v) \cap (V_0 \cup V_1)} \sum_{t \in N(s)} \mathbf{x}_t$$

$$= 4\lambda_1 + \sum_{s \in N(v) \cap (V_0 \cup V_1)} \left(\sum_{t \in N(s) \cap \{v_1, v_2\}} \mathbf{x}_t + \sum_{t \in N(s) \cap V_2} \mathbf{x}_t + \sum_{t \in N(s) \cap (V_0 \cup V_1)} \mathbf{x}_t \right)$$

$$\leqslant 4\lambda_1 + 4|N(v) \cap (V_0 \cup V_1)| + 4 + \sum_{s \in N(v) \cap (V_0 \cup V_1)} \sum_{t \in N(s) \cap (V_0 \cup V_1)} \mathbf{x}_t$$

$$\leqslant 4\lambda_1 + 4|V_0 \cup V_1| + 4 + 2|E(G[V_0 \cup V_1])|$$

$$\leqslant 4\lambda_1 + 4|V_0 \cup V_1| + 4 + 2(3|V_0 \cup V_1| - 6)$$

$$\leqslant 4\lambda_1 + 10|V_0 \cup V_1|$$

$$= O(\sqrt{n}).$$

We conclude that

$$\mathbf{x}_v = O\left(\frac{1}{\sqrt{n}}\right).$$

Thus, u_0 must be one of v_1 or v_2.

If v_1v_2 is not an edge of G, then we have

$$\lambda_1|\mathbf{x}_{v_1} - \mathbf{x}_{v_2}| \leqslant \sum_{v \in V_1} \mathbf{x}_v$$

$$\leqslant |V_1| \cdot O\left(\frac{1}{\sqrt{n}}\right)$$

$$= O(1).$$

If v_1v_2 is an edge of G, we have

$$(\lambda_1 - 1)|\mathbf{x}_{v_1} - \mathbf{x}_{v_2}| \leqslant \sum_{v \in V_1} \mathbf{x}_v$$

$$\leqslant |V_1| \cdot O\left(\frac{1}{\sqrt{n}}\right)$$

$$= O(1).$$

In both cases, we have

$$|\mathbf{x}_{v_1} - \mathbf{x}_{v_2}| = O\left(\frac{1}{\sqrt{n}}\right).$$

It follows that $\min\{\mathbf{x}_{v_1}, \mathbf{x}_{v_2}\} \geqslant 1 - O(\frac{1}{\sqrt{n}})$. $\qquad\square$

In the next lemma, we show that the extremal planar graph attaining the maximum spread must contain $K_{2,n-2}$ as a subgraph.

Lemma 19. *Let G be a graph obtaining the maximum spread among all n-vertex planar graphs. Then there exist two vertices v_1, v_2 in G such that each of v_1, v_2 is adjacent to all vertices in $V \backslash \{v_1, v_2\}$.*

Proof. Let $\mathbf{x}$ and $\mathbf{z}$ be the eigenvectors associated with λ_1 and λ_n respectively. Assume that $\mathbf{x}$ and $\mathbf{z}$ are both normalized such that their largest entries in absolute value are 1. By Lemma 16, there exist two vertices $v_1, v_2 \in V_+$ such that $\min\{d_{v_1}, d_{v_2}\} \geqslant n - 5\sqrt{2n-4}$. Recall that for $i \in \{0, 1, 2\}$, $V_i = \{v \in V(G) \backslash \{v_1, v_2\} : N(v) \cap \{v_1, v_2\} = i\}$.

It suffices to show that $V_0 \cup V_1$ is empty. Suppose otherwise that $V_0 \cup V_1$ is not empty. Since $V_0 \cup V_1$ induces a planar graph, there exists some vertex $v \in V_0 \cup V_1$ such that $|N(v) \cap (V_0 \cup V_1)| \leqslant 5$. Moreover, observe that v has at most two neighbors in V_2, as otherwise v, v_1, v_2 and three of their common neighbors would form a $K_{3,3}$ in G. Let G' be obtained from G by removing all the edges of G incident with v and adding the edges vv_1, vv_2, so that $E(G') = E(G - v) \cup \{vv_1, vv_2\}$. Observe G' is still planar.

We claim that $\lambda_n(G') < \lambda_n(G)$. Indeed, consider the vector $\tilde{\mathbf{z}}$ such that $\tilde{\mathbf{z}}_u = \mathbf{z}_u$ for $u \neq v$ and $\tilde{\mathbf{z}}_v = -|\mathbf{z}_v|$. Then

$$\tilde{\mathbf{z}}'A(G')\tilde{\mathbf{z}} \leqslant \mathbf{z}'A(G)\mathbf{z} + 2\sum_{y \sim v}|\mathbf{z}_y\mathbf{z}_v| - 2|\mathbf{z}_v|(\mathbf{z}_{v_1} + \mathbf{z}_{v_2})$$

$$\leqslant \mathbf{z}'A(G)\mathbf{z} + 2 \cdot (2+5) \cdot O\left(\frac{1}{\sqrt{n}}\right) \cdot |\mathbf{z}_v| - \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right)|\mathbf{z}_v|$$

$$< \mathbf{z}'A(G)\mathbf{z}.$$

Similarly, we claim that $\lambda_1(G') > \lambda_1(G)$. Indeed,

$$\mathbf{x}'\mathbf{x}\lambda_n(G') = \mathbf{x}'A(G')\mathbf{x}$$
$$= \mathbf{x}'A(G)\mathbf{x} - 2\sum_{y \sim v}\mathbf{x}_y\mathbf{x}_v + 2\mathbf{x}_v(\mathbf{x}_{v_1} + \mathbf{x}_{v_2})$$
$$\geqslant \mathbf{x}'\mathbf{x}\lambda_n(G) - 2 \cdot (2 + 5) \cdot O\left(\frac{1}{\sqrt{n}}\right) \cdot \mathbf{x}_v + \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right)\mathbf{x}_v$$
$$> \mathbf{x}'\mathbf{x}\lambda_n(G).$$

Hence we have $S(G') = \lambda_1(G') - \lambda_n(G') > \lambda_1(G) - \lambda_n(G) = S(G)$, giving a contradiction. $\square$

Corollary 20. *For sufficiently large n, the planar graph on n vertices attaining the maximum spread must belong to one of the three families below.*

(i) *G is a double wheel $(K_1 \cup K_1) \vee C_{n-2}$.*

(ii) *G is $(K_1 \cup K_1) \vee T_{n-2}$, where T_{n-2} is a linear forest on $n-2$ vertices.*

(iii) *G is $K_2 \vee T_{n-2}$, where T_{n-2} is a linear forest on $n-2$ vertices.*

Proof. By Lemma 19, for sufficiently large n, the maximum-spread planar graph G on n vertices has two vertices v_1 and v_2 which are adjacent to every vertex in $A = V(G) \setminus \{v_1, v_2\}$. If there is a vertex $u \in A$ with $|N(u) \cap A| \geqslant 3$, then G has $K_{3,3}$ as a subgraph. It follows that the $G[A]$ has maximum degree at most 2 and hence is a disjoint union of paths and cycles. If $v_1v_2 \in E(G)$, and $G[A]$ contains a cycle C, then $G[\{v_1, v_2\} \cup C]$ contains a K_5 minor, contradicting that G is planar. Hence, $G[A]$ must be a linear forest on $n-2$ vertices, proving (iii). Now suppose $v_1v_2 \notin E(G)$. If $G[A]$ is a cycle, then G is the double wheel $(K_1 \cup K_1) \vee C_{n-2}$, proving (i). Otherwise, if $G[A]$ contains a cycle C which does not span all the vertices of A, then by contracting edges and deleting vertices in the subgraph induced by $A \setminus C$, we obtain that G contains $K_2 \vee C$ as a minor, and therefore has K_5 as a minor, giving a contradiction. Hence, if $G[A]$ is not C_{n-2}, it must be a linear forest, proving (ii). $\square$

We call $(K_1 \cup K_1) \vee T_{n-2}$ a *linear planar graph of the first kind* and $K_2 \vee T_{n-2}$ a *linear planar graph of the second kind*. By Corollary 20, the maximum spread on planar graphs is achieved by either a double wheel, a linear planar graph of the first kind, or a linear planar graph of the second kind. In the next few subsections, we will compute the maximum spread of graphs in these three families respectively, and then obtain the extremal graph attaining the maximum spread among all planar graphs.

3.2 Double wheel graph

We first treat the first case of Corollary 20, the double wheel $(K_1 \cup K_1) \vee C_{n-2}$.

Lemma 21. *The spread of the double wheel graph $(K_1 \cup K_1) \vee C_{n-2}$ is*

$$\sqrt{8n-12} = 2\sqrt{2(n-2)} + \frac{1}{\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{\frac{3}{2}}}\right).$$

Proof. By Lemma 12, for n sufficiently large, the largest and smallest eigenvalue of the graph $(K_1 \cup K_1) \vee C_{n-2}$ are $1 + \frac{\sqrt{8n-12}}{2}$ and $1 - \frac{\sqrt{8n-12}}{2}$ respectively. Thus the spread is $\sqrt{8n-12}$. We now give an asymptotic expansion for $\sqrt{8n-12}$ using the Taylor expansion for $(1+x)^{\frac{1}{2}}$. We have

$$
\begin{aligned}
\sqrt{8n-12} &= \sqrt{(8n-16)+4} \\
&= \sqrt{8n-16}\left(\sqrt{1+\frac{4}{8n-16}}\right) \\
&= \sqrt{8n-16}\left(1 + \frac{1}{2}\left(\frac{4}{8n-16}\right) - \frac{1}{8}\left(\frac{4}{8n-16}\right)^2 + O\left(\frac{1}{(8n-16)^3}\right)\right) \\
&= 2\sqrt{2(n-2)} + \frac{1}{\sqrt{2(n-2)}} - \frac{1}{4(2(n-2))^{\frac{3}{2}}} + O\left(\frac{1}{(8n-16)^{\frac{5}{2}}}\right). \quad \square
\end{aligned}
$$

3.3 Linear planar graphs of the first kind

Consider a linear planar graph of the first kind $G = (K_1 \cup K_1) \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$. Let u, w be the center vertices of G and $v_1, \ldots, v_\ell$ be the vertices in order of a path component in $G - u - w$. Note that there is a graph automorphism ϕ that maps u to w. Let α be a normalized eigenvector (invariant under ϕ) corresponding to an eigenvalue λ of the adjacency matrix of G so that $\alpha(u) = \alpha(w) = 1$. Let A_ℓ be the adjacency matrix of P_ℓ, $x_i = \alpha(v_i)$ for $1 \leqslant i \leqslant \ell$, and $\mathbf{x} = (x_1, \ldots, x_\ell)' \in \mathbb{R}^\ell$.

Following along the same lines as the outerplanar case, we have the following lemmas and corollaries.

Lemma 22. *Let $G = (K_1 \cup K_1) \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear planar graph on n vertices with center vertices u and w. Suppose λ is an eigenvalue of $A(G)$ with $|\lambda| \geqslant 2$, and α is a normalized eigenvector of $A(G)$ corresponding to λ such that $\alpha(u) = \alpha(w) = 1$. Let P_ℓ be one of the path components of $G - u - w$ and let $\mathbf{x}$ and A_ℓ be defined as above. Then*

$$\mathbf{x} = 2\sum_{k=0}^{\infty} \lambda^{-(k+1)} A_\ell^k \mathbf{1}. \tag{34}$$

The proof is similar to the proof for Lemma 5, but instead of $A_\ell \mathbf{x} + \mathbf{1} = \lambda \mathbf{x}$ in equation 14, we have $A_\ell \mathbf{x} + 2 \cdot \mathbf{1} = \lambda \mathbf{x}$ in the case of two center vertices. The results below follow correspondingly.

Corollary 23. *The following properties hold for any linear planar graph.*

1. *If $\lambda \geqslant 2$, all entries of α are positive.*

2. *If $\lambda \leqslant -2$, all entries of α but the u-entry are negative.*

Consider a linear planar graph $G = H \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ with $\ell_1, \ell_2 \geqslant 2$, and $H \in \{K_1 \cup K_1, K_2\}$. Similar to the case for outerplanar graphs, the merge operation replaces G by $G' = H \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots \cup P_{\ell_r})$. Following along the same lines of the outerplanar case, we have that the merge operation increases the spread of a linear planar graph.

Lemma 24. *Let $G = H \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$ be a linear planar graph on $n \geqslant 8$ vertices with $\ell_1, \ell_2 \geqslant 2$ and $H \in \{K_1 \cup K_1, K_2\}$. Let G' be obtained from G by applying a merge operation, i.e., $G' = H \vee (P_{\ell_1+\ell_2-1} \cup P_1 \cup \cdots \cup P_{\ell_r})$ Then $S(G') > S(G)$.*

The proofs are nearly identical to the proofs for Lemma 7 and 8. We proceed to the main theorem for linear planar graphs of the first kind. By Lemma 24, we only need to consider the linear planar graph with only one non-trivial path. Let $G'_\ell = (K_1 \cup K_1) \vee (P_\ell \cap (n - 2 - \ell)K_1)$.

Lemma 25. *For sufficiently large n, the spread of G'_ℓ is given by*

$$S(G'_\ell) = 2\sqrt{2(n-2)} + \frac{2c'_2}{\sqrt{2(n-2)}} + \frac{2c'_4}{(2(n-2))^{\frac{3}{2}}} + \frac{2c'_6}{(2(n-2))^{\frac{5}{2}}} + O\left(\frac{1}{(2(n-2))^3}\right).$$

Here,

$$c'_2 = -\frac{3}{2}\left(\frac{\ell-2}{n-2}\right)^2 + 2\left(\frac{\ell-2}{n-2}\right),$$

$$c'_4 = -\frac{105}{8}\left(\frac{\ell-2}{n-2}\right)^4 + 35\left(\frac{\ell-2}{n-2}\right)^3 - 30\left(\frac{\ell-2}{n-2}\right)^2 + 2\left(\frac{\ell-2}{n-2}\right) + 2,$$

$$c'_6 = -\frac{3003}{16}\left(\frac{\ell-2}{n-2}\right)^6 + \frac{3003}{4}\left(\frac{\ell-2}{n-2}\right)^5 - 1155\left(\frac{\ell-2}{n-2}\right)^4 + 735\left(\frac{\ell-2}{n-2}\right)^3,$$

$$- 105\left(\frac{\ell-2}{n-2}\right)^2 - 28\left(\frac{\ell-2}{n-2}\right) - 12.$$

Proof. Let λ be either λ_1 or λ_n. Let α be the normalized eigenvector associated with λ such that $\alpha(u) = \alpha(w) = 1$, where u, w are the center vertices in G'_ℓ. Let P_ℓ be the unique non-trivial path in $G - u - w$ and $\mathbf{x} = (x_1, \ldots, x_\ell)$ be the vector of α restricted to the vertices of P_ℓ. Let v be a vertex in G'_ℓ that is not on P_ℓ and is not u or w. The eigen-equation on v gives that $\lambda\alpha(v) = \sum_{y \sim v} \alpha(y) = \alpha(u) + \alpha(w) = 2$, so $\alpha(v) = \frac{2}{\lambda}$. The eigen-equation at u (and w) gives

$$\lambda = \sum_{i=1}^{\ell} x_i + (n - 2 - \ell)\frac{2}{\lambda}. \tag{35}$$

Applying Lemma 22, we get

$$\sum_{i=1}^{\ell} x_i = \mathbf{1}' \cdot \mathbf{x}$$

$$= \mathbf{1}' \cdot 2 \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_\ell^k \mathbf{1}$$

$$= 2 \sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_\ell^k \mathbf{1}.$$

Plugging into Equation (35), we have

$$\lambda = (n-2-\ell)\frac{2}{\lambda} + 2 \sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_\ell^k \mathbf{1}. \tag{36}$$

By a similar argument in the proof of Theorem 3, we get

$$\lambda^2 = 2n - 4 + \frac{4\ell - 4}{\lambda} + \frac{8\ell - 12}{\lambda^2} + \frac{16\ell - 32}{\lambda^3} + \frac{32\ell - 76}{\lambda^4} + \frac{64\ell - 176}{\lambda^5} + O\left(\frac{\ell}{\lambda^6}\right). \tag{37}$$

This equation has two real roots which determine λ_1 and λ_n. λ_1 is near $\sqrt{2(n-2)}$ and λ_n is near $-\sqrt{2(n-2)}$. By Lemma 27 in the appendix, λ has the following series expansion:

$$\lambda_1 = \sqrt{2(n-2)} + c_1' + \frac{c_2'}{\sqrt{2(n-2)}} + \frac{c_3'}{2(n-2)} + \frac{c_4'}{(2(n-2))^{\frac{3}{2}}}$$
$$+ \frac{c_5'}{(2(n-2))^2} + \frac{c_6'}{(2(n-2))^{\frac{5}{2}}} + O\left(\frac{1}{(2(n-2))^3}\right). \tag{38}$$

Calculating this out in SageMath[2], we get

$$c_1' = \frac{\ell - 2}{n - 2},$$

$$c_2' = -\frac{3}{2}\left(\frac{\ell - 2}{n - 2}\right)^2 + 2\left(\frac{\ell - 2}{n - 2}\right),$$

$$c_3' = 4\left(\frac{\ell - 2}{n - 2}\right)^3 - 8\left(\frac{\ell - 2}{n - 2}\right)^2 + 4\left(\frac{\ell - 2}{n - 2}\right) + 2,$$

$$c_4' = -\frac{105}{8}\left(\frac{\ell - 2}{n - 2}\right)^4 + 35\left(\frac{\ell - 2}{n - 2}\right)^3 - 30\left(\frac{\ell - 2}{n - 2}\right)^2 + 2\left(\frac{\ell - 2}{n - 2}\right) + 2,$$

$$c_5' = 48\left(\frac{\ell - 2}{n - 2}\right)^5 - 160\left(\frac{\ell - 2}{n - 2}\right)^4 + 192\left(\frac{\ell - 2}{n - 2}\right)^3 - 72\left(\frac{\ell - 2}{n - 2}\right)^2 - 8\left(\frac{\ell - 2}{n - 2}\right),$$

---

[2]Computation available at https://github.com/wzy3210/Spread-of-outerplanar-and-planar-graphs

$$c_6' = -\frac{3003}{16}\left(\frac{\ell-2}{n-2}\right)^6 + \frac{3003}{4}\left(\frac{\ell-2}{n-2}\right)^5 - 1155\left(\frac{\ell-2}{n-2}\right)^4 + 735\left(\frac{\ell-2}{n-2}\right)^3$$
$$- 105\left(\frac{\ell-2}{n-2}\right)^2 - 28\left(\frac{\ell-2}{n-2}\right) - 12.$$

Similarly, by Lemma 27 in the appendix, we get the following series expansion of λ_n:

$$\lambda_n = -\sqrt{2(n-2)} + c_1' - \frac{c_2'}{\sqrt{2(n-2)}} + \frac{c_3'}{2(n-2)} - \frac{c_4'}{(2(n-2))^{\frac{3}{2}}}$$
$$+ \frac{c_5'}{(2(n-2))^2} - \frac{c_6'}{(2(n-2))^{\frac{5}{2}}} + O\left(\frac{1}{(2(n-2))^3}\right). \tag{39}$$

Here, $c_1', c_2', c_3', c_4', c_5', c_6'$ are the same as before. Observe that all c_i's are bounded since they are polynomials of $\frac{\ell-2}{n-2}$, which is contained in the interval $[0,1]$. Thus, the spread of G_ℓ', denoted by $S(G_\ell')$, can be expressed as:

$$S(G_\ell') = \lambda_1 - \lambda_n = 2\sqrt{2(n-2)} + \frac{2c_2'}{\sqrt{2(n-2)}} + \frac{2c_4'}{(2(n-2))^{\frac{3}{2}}}$$
$$+ \frac{2c_6'}{(2(n-2))^{\frac{5}{2}}} + O\left(\frac{1}{(2(n-2))^3}\right). \qquad \square$$

3.4 Linear planar graphs of the second kind

Consider the linear planar graphs of the second kind $G = K_2 \vee (P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r})$. By Lemma 24, the maximum spread among all linear planar graphs of the second kind can only be achieved by $G_\ell'' = K_2 \vee (P_\ell \cup (n-\ell-2)P_1)$. Using the same method from Theorem 3 and Lemma 25, we obtain that the spread of G_ℓ'' is

$$S(G_\ell'') = 2\sqrt{2(n-2)} + \frac{2c_2''}{\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{\frac{3}{2}}}\right),$$

where $c_2'' = -\frac{3}{2}\left(\frac{l-2}{n-2}\right)^2 + \frac{3}{2}\left(\frac{l-2}{n-2}\right) + \frac{1}{8}$.

3.5 Proof of the main theorem

Lemma 26. *For sufficiently large n, a maximum-spread planar graph on n vertices is a linear planar graph of the first kind.*

Proof. Let G_ℓ' and G_ℓ'' denote the linear planar graph of the first kind and second kind, respectively, such that the non-center vertices induce a linear forest with a unique non-trivial path P_ℓ. For linear planar graphs of the first kind, we have $c_2' \leqslant \frac{2}{3}$, since

$$c_2'(\ell) = -\frac{3}{2}\left(\frac{\ell-2}{n-2}\right)^2 + 2\left(\frac{\ell-2}{n-2}\right)$$

$$= \frac{2}{3} - \frac{3}{2} \left(\frac{\ell - 2}{n-2} - \frac{2}{3} \right)^2$$

$$\leqslant \frac{2}{3}.$$

Furthermore, c_2' reaches its maximum at $\ell_1 = \frac{2(n-2)}{3} + 2 = \frac{2n-1}{3} + 1$. Let $\ell_0 = \lceil \frac{2n-2}{3} \rceil$ be the target maximum point of $S(G_\ell')$. Then, by Lemma 25, we have

$$S(G_{\ell_0}') = 2\sqrt{2(n-2)} + \frac{4}{3\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{3/2}} \right). \tag{40}$$

For linear planar graphs of the second kind, observe that $c_2'' \leqslant \frac{1}{2}$. Thus for sufficiently large n and for any $\ell \in [n]$,

$$S(G_\ell'') \leqslant 2\sqrt{2(n-2)} + \frac{1}{\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{\frac{3}{2}}} \right) < S(G_{\ell_o}'). \tag{41}$$

Hence for sufficiently large n, the maximum spread of linear planar graphs of the second kind is less than the maximum spread of linear planar graphs of the first kind. Similarly, for the double wheel graph, by Lemma 21, we have

$$S((K_1 \cup K_1) \vee C_{n-2}) = 2\sqrt{2(n-2)} + \frac{1}{\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{\frac{3}{2}}} \right) < S(G_{\ell_0}').$$

This completes the proof of the lemma. $\qquad\square$

We proceed with the proof of the main theorem for planar graphs.

Proof of Theorem 4. By Lemma 26, the maximum-spread planar graph is a linear planar graph of the first kind with at most one non-trivial path. Recall from the proof of Lemma 26 that since

$$c_2'(\ell) = \frac{2}{3} - \frac{3}{2} \left(\frac{\ell - 2}{n-2} - \frac{2}{3} \right)^2,$$

the function c_2' reaches its maximum at $\ell_1 = \frac{2(n-2)}{3} + 2 = \frac{2n-1}{3} + 1$. Let $\ell_0 = \lceil \frac{2n-2}{3} \rceil$ be the target maximum point of $f(\ell)$.

We claim that there is a constant C such that all possible maximal points must be contained in the interval $(\ell_1 - C\sqrt{2(n-2)}, \ell_1 + C\sqrt{2(n-2)})$. Otherwise, for any ℓ not in this interval, we have

$$c_2'(\ell) \leqslant \frac{2}{3} - \frac{3C^2}{2(n-1)}.$$

This implies

$$S(G_\ell') \leqslant 2\sqrt{2(n-2)} + 2\frac{\frac{2}{3} - \frac{3C^2}{2(n-1)}}{\sqrt{2(n-2)}} + O\left(\frac{1}{(2(n-2))^{3/2}} \right) < S(G_{\ell_0}').$$

Here, we choose C big enough such that

$$-\frac{3C^2}{\sqrt{2}(n-2)^{3/2}} + O\left(\frac{1}{(2(n-2))^{3/2}}\right) < 0,$$

which proves the claim.

From now on, we can assume $\ell \in (\ell_1 - C\sqrt{2(n-2)}, \ell_1 + C\sqrt{2(n-2)})$. Next, we compute $S(G'_{\ell+1}) - S(G'_\ell)$. We have

$$c'_2(\ell+1) - c'_2(\ell) = -\frac{3}{2}\left(\frac{2\ell - 3}{(n-2)^2}\right) + 2\left(\frac{1}{n-2}\right) = \frac{2(n-2) - \frac{3}{2}(2\ell - 3)}{(n-2)^2},$$

$$c'_4(\ell+1) - c'_4(\ell) = -\frac{105}{8}\left(\frac{4\ell^3 - 18\ell^2 + 28\ell - 15}{(n-2)^4}\right) + 35\left(\frac{3\ell^2 - 9\ell + 7}{(n-2)^3}\right)$$
$$- 30\left(\frac{2\ell - 3}{(n-2)^2}\right) + 2\left(\frac{1}{n-2}\right),$$

$$c'_6(\ell+1) - c'_6(\ell) = O\left(\frac{1}{n-2}\right).$$

Plugging in $\ell = \ell_1 \cdot \left(1 + O\left(\frac{1}{\sqrt{2(n-2)}}\right)\right)$ into $c'_4(\ell+1) - c'_4(\ell)$, we have

$$c'_4(\ell+1) - c'_4(\ell) = \frac{1}{n-2}\left(-\frac{105}{8} \cdot 4 \cdot \left(\frac{2}{3}\right)^3 + 35 \cdot 3\left(\frac{2}{3}\right)^2 - 30 \cdot 2 \cdot \left(\frac{2}{3}\right) + 2\right)$$
$$+ O\left(\frac{1}{\sqrt{2}(n-2)^{3/2}}\right)$$
$$= -\frac{62}{9(n-2)} + O\left(\frac{1}{\sqrt{2}(n-2)^{3/2}}\right).$$

Therefore, we have

$$S(G'_{\ell+1}) - S(G'_\ell) = \frac{2(c'_2(\ell+1) - c'_2(\ell))}{\sqrt{2(n-2)}} + \frac{2(c'_4(\ell+1) - c'_4(\ell))}{(2(n-2))^{\frac{3}{2}}}$$
$$+ \frac{2(c'_6(\ell+1) - c'_6(\ell))}{(2(n-2))^{\frac{5}{2}}} + O\left(\frac{1}{(2(n-2))^3}\right)$$
$$= \frac{4(n-2) - 3(2\ell - 3)}{\sqrt{2}(n-2)^{5/2}} - \frac{62}{9\sqrt{2}(n-2)^{5/2}} + O\left(\frac{1}{(n-2)^3}\right)$$
$$= \frac{4n - 6\ell - \frac{53}{9}}{\sqrt{2}((n-2))^{5/2}} + O\left(\frac{1}{(n-2)^3}\right).$$

When $\ell \geqslant \ell_0$, we have

$$4n - 6\ell - \frac{53}{9} \leqslant 4n - 6\ell_0 - \frac{53}{9} \leqslant 4n - 6 \cdot \frac{2n-2}{3} - \frac{53}{9} = -\frac{17}{9} < 0.$$

It follows that $S(G'_{\ell+1}) - S(G'_\ell) < 0$.

When $\ell \leqslant \ell_0 - 1$, we have

$$4n - 6\ell - \frac{53}{9} \geqslant 4n - 6(\ell_0 - 1) - \frac{53}{9} \geqslant 4n - 6 \cdot \left(\frac{2n-2}{3} - \frac{1}{3} \right) - \frac{53}{9} = \frac{1}{9} > 0.$$

It follows that $S(G'_{\ell+1}) - S(G'_\ell) > 0$. This shows that ℓ_0 is the unique maximal point for $S(G'_\ell)$. This completes the proof of the theorem. $\qquad\square$

Acknowledgements

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4 Appendix

Note that Equations (8) and (37) are quite similar. We have the following lemma on the existence of Laurent series of their solutions. Here we let $N = n - 1$ for Equation (8) and $N = 2n - 4$ for Equation (37).

Lemma 27. *Suppose that λ satisfies the equation*

$$\lambda^2 = N + \sum_{i=1}^{\infty} \frac{a_i}{\lambda^i}, \tag{42}$$

where $a_i = p_i \ell + q_i$'s are a linear function of ℓ for some constants p_i and q_i for each $i \in \mathbb{N}$. Then λ has the following Laurent series expansion in term of $\frac{1}{\sqrt{N}}$ near $\sqrt{N}$:

$$\lambda = \sqrt{N} + \sum_{i=1}^{\infty} c_i N^{-(i-1)/2}, \tag{43}$$

and the following Laurent series expansion in term of $\frac{1}{\sqrt{N}}$ near $-\sqrt{N}$:

$$\lambda = -\sqrt{N} + \sum_{i=1}^{\infty} (-1)^{i-1} c_i N^{-(i-1)/2}, \tag{44}$$

where the c_i's are polynomials of $\frac{\ell}{N}$ of degree at most i.

Proof. Let $x = \frac{1}{\lambda}$ and $z^2 = \frac{1}{N}$. Equation (42) can be rewritten as

$$\frac{1}{z^2} = \frac{1}{x^2} - \sum_{i=1}^{\infty} a_i x^i. \tag{45}$$

Thus

$$
\begin{aligned}
z &= \pm \left(\frac{1}{x^2} - \sum_{i=1}^{\infty} a_i x^i \right)^{-\frac{1}{2}} \\
&= \pm x \left(1 - \sum_{i=1}^{\infty} a_i x^{i+2} \right)^{-\frac{1}{2}} \\
&= \pm \frac{x}{\phi(x)}.
\end{aligned}
$$

Here $\phi(x) = \sqrt{1 - \sum_{i=1}^{\infty} a_i x^{i+2}}$. We have the following Taylor expansion of $\phi(x)$.

$$\phi(x) = \sqrt{1 - \sum_{i=1}^{\infty} a_i x^{i+2}} \tag{46}$$

$$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \left(-\sum_{i=1}^{\infty} a_i x^{i+2} \right)^k \tag{47}$$

$$= 1 + \sum_{j=3}^{\infty} b_j x^j, \tag{48}$$

where b_j are multi-variable polynomials of the a_i's.

Consider the case $z > 0$ first. Let $x = g(z) = \sum_{i=1} d_i z^i$ be the Taylor series of the inverse function in the equation $z = \frac{x}{\phi(x)}$. The Lagrange-Bürmann formula states

$$d_n = [z^n]g(z) = \frac{1}{n}[x^{n-1}]\phi(x)^n.$$

Here $[x^r]$ is an operator which extracts the coefficient of x^r in the Taylor series of a function of x. Note that $d_0 = 0$ and $d_1 = 1$.

Now taking the reciprocal of both sides, we have

$$\lambda = \frac{1}{x} \tag{49}$$

$$= \frac{1}{z} \left(1 + \sum_{i=1}^{\infty} d_{i+1} z^i \right)^{-1} \tag{50}$$

$$= \frac{1}{z} \left(1 + \sum_{j=1}^{\infty} (-1)^j \left(\sum_{i=1}^{\infty} d_{i+1} z^i \right)^j \right) \tag{51}$$

$$= \frac{1}{z} \left(1 + \sum_{j=1}^{\infty} f_j z^j \right) \tag{52}$$

$$= \sqrt{N} + \sum_{i=1}^{\infty} f_i N^{-(i-1)/2}, \tag{53}$$

where in (53), the f_is are multi-variable polynomials of the d_is; in (50) the d_is are multi-variable polynomials of the b_is; and the b_is are multi-variable polynomials of the a_is. Hence, the f_is are multi-variable polynomials of the a_is. Consider a typical monomial $a_{i_1} a_{i_2} \cdots a_{i_s}$ (with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$). By (47) and (48), the monomial $a_{i_1} a_{i_2} \cdots a_{i_s}$ appears in $f_j z^j$ of (52) only if

$$j \geqslant \sum_{t=1}^{s} (a_{i_t} + 2) = 2s + \sum_{t=1}^{s} i_t$$

since $b_1 = b_2 = 0$. We have

$$a_{i_1} a_{i_2} \cdots a_{i_s} N^{-(j-1)/2} = N^{-(j-1-2s)/2} \prod_{t=1}^{s} \frac{a_{i_t}}{N} = N^{-(j-1-2s)/2} \prod_{t=1}^{s} (p_{i_t} \frac{\ell}{N} + \frac{q_{i_t}}{N}). \tag{54}$$

Expanding and grouping all items in $f_j N^{-(j-1)/2}$ in the Laurent series of λ, we can rewrite it as

$$\lambda = \sqrt{N} + \sum_{i=1}^{\infty} c_i N^{-(i-1)/2} \tag{55}$$

where c_i is a polynomial of $\frac{\ell}{N}$. Note that in (54), since $(j - 1 - 2s) \geqslant \sum_{t=1}^{s} i_t - 1 \geqslant s - 1$, a term $\left(\frac{\ell}{N}\right)^s$ can only appear in c_j for $j \geqslant s$. Thus, the degree of c_i is at most i.

By the symmetry between z and $-z$ in (45), we get the Laurent series expansion in term of $\frac{1}{\sqrt{N}}$ near $-\sqrt{N}$:

$$\lambda = \frac{1}{-z} \left(1 + \sum_{j=1}^{\infty} c_j (-z)^j \right)$$

$$= -\sqrt{N} + \sum_{i=1}^{\infty} (-1)^{i-1} c_i N^{-(i-1)/2}. \qquad \square$$