On the Hopf Algebra of Noncommutative Symmetric Functions in Superspace

Diego Arcis[a] Camilo González[b] Sebastián Márquez[c]

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Abstract

We study the combinatorial Hopf algebra of noncommutative symmetric functions in superspace **sNSym**, introduced by Fishel, Lapointe and Pinto. We introduce a family of primitive elements of **sNSym** and extend the noncommutative elementary and power sum functions to superspace. Then, we give formulas relating these families of functions. Also, we introduce noncommutative ribbon Schur functions in superspace and provide an explicit formula for their product. We show that the dual basis of these functions is given by a family of the so–called fundamental quasisymmetric functions in superspace. This allows us to obtain an explicit formula for the coproduct of fundamental quasisymmetric functions in superspace. Additionally, by projecting the noncommutative ribbon Schur functions in superspace, we define a new basis for the algebra of symmetric functions in superspace.

Mathematics Subject Classifications: 05E05, 16T05, 16T30

1 Introduction

The classical ring of symmetric functions **Sym** is a very important object in algebraic combinatorics. This has been widely studied due to its rich properties and multiple applications in several areas such as representations theory, algebraic geometry and Lie algebras [28, 32]. Symmetric functions are important not only in mathematics, but also in connection with integrable models in physics [30]. Due to this connection, and motivated by a supersymmetric generalization of the Calogero–Moser–Sutherland model [7, 10], a new class of functions called *symmetric functions in superspace* was developed in [8, 9], generalizing the classical symmetric functions.

[a]Departamento de Matemáticas, Universidad de La Serena, Cisternas 1200 – 1700000 La Serena, Chile (diego.arcis@userena.cl).

[b]Departamento de Matemáticas, Universidad de Concepción, Casilla 160–C – 4030000 Concepción, Chile (camgonzalezp@udec.cl).

[c]Departamento de Matemáticas, Universidad Autónoma de Chile – Sede Talca, 4 Norte 95 – 3460000 Talca, Chile (sebastian.marquez@uautonoma.cl) – corresponding author.

A *symmetric function in superspace* is obtained by considering the classical infinite collection of commutative variables $x = (x_1, x_2, \ldots)$ together with an infinite collection of anticommutative variables $\theta = (\theta_1, \theta_2, \ldots)$, that is, $\theta_i\theta_j = -\theta_j\theta_i$. In particular, $\theta_i^2 = 0$. As it was shown in [11], the classical bases of **Sym**, as the monomials, power–sum, elementary and homogeneous symmetric functions, can be naturally extended to superspace. More complex bases as Schur, Jack and Macdonald polynomial were also extended and studied in [4, 5, 12, 27, 13, 22, 19]. The combinatorial tools used to study these objects are the so–called *superpartitions*, which generalize the classical partitions of numbers.

A *superpartition* is a pair $\Lambda = (\Lambda^a; \Lambda^s)$, where Λ^a is a strictly decreasing partition, possibly including a zero, and Λ^s is a usual partition. The *degree* $|\Lambda|$ of Λ is the sum of the components of Λ^a with the ones of Λ^s. The *fermionic degree* $\mathrm{df}(\Lambda)$ of Λ is the length of Λ^a. In particular, every usual partition can be regarded as a superpartition with null fermionic degree.

Thus, given a superpartition $\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_n)$, some of the classical symmetric functions in superspace [11] are defined as follows:

- *Monomial symmetric functions in superspace*:

$$m_\Lambda(x,\theta) = \frac{1}{n_\Lambda!}\sum_{\sigma\in\mathfrak{S}_n} \theta_{\sigma(1)}\cdots\theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1}\cdots x_{\sigma(n)}^{\Lambda_n}, \quad \text{with} \quad n_\Lambda! = n_{\Lambda^s}(0)! n_{\Lambda^s}(1)!\cdots$$

  where $n_{\Lambda^s}(i)!$ is the number of i's in Λ^s.

- *Power–sum symmetric functions in superspace*: $p_\Lambda = \tilde{p}_{\Lambda_1}\cdots\tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}}\cdots p_{\Lambda_n}$, where

$$\tilde{p}_k = \sum_{i=1}^{n} \theta_i x_i^k \quad \text{and} \quad p_r = \sum_{i=1}^{n} x_i^r, \quad \text{for} \quad k \geqslant 0, r \geqslant 1.$$

- *Elementary symmetric functions in superspace*: $e_\Lambda = \tilde{e}_{\Lambda_1}\cdots\tilde{e}_{\Lambda_m} e_{\Lambda_{m+1}}\cdots e_{\Lambda_n}$, where

$$\tilde{e}_k = m_{(0;1^k)} \quad \text{and} \quad e_r = m_{(\emptyset;1^r)}, \quad \text{for} \quad k \geqslant 0, r \geqslant 1.$$

- *Complete homogeneous symmetric functions in superspace*:
  $h_\Lambda = \tilde{h}_{\Lambda_1}\ldots\tilde{h}_{\Lambda_m} h_{\Lambda_{m+1}}\cdots h_{\Lambda_n}$, where

$$\tilde{h}_k = \sum_{|\Lambda|=k,\, \mathrm{df}(\Lambda)=1} (\Lambda_1+1) m_\Lambda \quad \text{and} \quad h_r = \sum_{|\Lambda|=r,\, \mathrm{df}(\Lambda)=0} m_\Lambda, \quad \text{for} \quad k \geqslant 0, r \geqslant 1.$$

Note that if $\Lambda^a = \emptyset$, we obtain the corresponding classical symmetric function.

Recall that classical Schur functions can be obtained from Macdonald polynomials $P_\lambda(q,t)$ by taking $q = t$. *Schur functions in superspace* are defined by means of the *Macdonald polynomials in superspace* $P_\Lambda(q,t)$ [4, Section 4]. A first kind of these functions is obtained by taking $q = t = 0$ [4, Subsection 7.1] and a second kind is obtained by taking $q = t = \infty$ [4, Proposition 28]. On the other hand, the classical ribbon Schur functions are a special case of the so–called *skew Schur functions.* In superspace, these functions were defined by taking the adjoint of the two kinds of Schur functions in superspace mentioned above [27, Section 9]. A suitable definition of ribbon Schur functions in superspace has not yet been provided.

It is well known that **Sym** has a Hopf algebra structure [15]. In [16], the set of symmetric functions in superspace was also given with a Hopf algebra structure **sSym**, which has explicit formulas for the product, coproduct and antipode in terms of the bases given above. This Hopf algebra is commutative, cocommutative (up a sign), self dual [16, Proposition 4.8] and contains **Sym** as a Hopf subalgebra.

Another important ring related to symmetric functions is the one of *quasisymmetric functions* **QSym** [21]. The Hopf structure of **QSym** has several applications in the theory of symmetric functions such as an expansion of the Macdonald polynomial in terms of the so–called fundamental quasisymmetric functions [21]. The fundamental quasisymmetric functions provide an important basis of **QSym**, since they share similar properties with Schur functions. The product, coproduct and antipode of these type of functions in superspace is an open problem. Here, we lead with this problem.

The dual structure of **QSym**, defined in [20], is the Hopf algebra of *noncommutative symmetric functions* **NSym**. It was shown in [20] that there is a connection of **NSym** with the Solomon's descent algebra, which is applied to the study of formal power series with coefficients in a noncommutative algebra. The usual bases of **QSym** and **NSym** are indexed by compositions of numbers.

It was shown in [16] that both **QSym** and its dual **NSym** can be extended to superspace as the Hopf algebras of *quasisymmetric function in superspace* **sQSym** and of *noncommutative symmetric functions in superspace* **sNSym**, respectively. These structures can be $\mathbb{Z}_2$–graded by means of the fermionic degree, which implies that they can be regarded as Hopf superalgebras. In superspace, the classical bases are indexed by the so–called *dotted compositions*, that is, tuples of nonnegative integers in which some of their components are labelled by a dot. In particular, classical compositions can be regarded as dotted compositions with no dotted components. The vector space **sQSym** has a basis formed by the *monomial quasisymmetric functions in superspace* M_α, with $\alpha = (\alpha_1, \ldots, \alpha_k)$ a dotted composition, defined by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} \theta_{i_1}^{\eta_1} \cdots \theta_{i_k}^{\eta_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k},$$

where each η_i is either 0 if α_i is labelled by a dot or 1 otherwise. The product of these monomials in **sQSym** can be described by means of *overlapping shuffles.* The coproduct of a monomial is obtained by deconcatenating its components. Since monomial symmetric functions in superspace m_Λ can be expanded in terms of the M_α's, then **sSym** is a Hopf subalgebra of **sQSym**. See [16] for details.

The Hopf algebra **sNSym** has a basis $\{H_\alpha \mid \alpha \text{ is a dotted composition}\}$ that is dual to the monomial quasisymmetric functions in superspace, that is, $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$.

In this paper, we study the structure of **sNSym**. The results obtained here extend the classical theory of noncommutative symmetric functions.

It is known that **sNSym** is isomorphic to the free algebra generated by the family of elements $H_n := H_{(n)}$ and $\tilde{H}_n := H_{(\dot{n})}$. Moreover, the coproducts of H_n and $\tilde{H}_n$ are similar to the ones of h_n and $\tilde{h}_n$ in **sSym**, respectively. See [16, Proposition 6.2]. Note that **NSym** is the Hopf subalgebra generated by the H_n's.

In Section 2, we recall basic notions and results about Hopf algebras and Lie superalgebras. In Section 3, we study dotted compositions and characterize one of their lattice structures. The combinatorics of these objects will be used to describe most of the results in this paper.

Section 4 is dedicated to study in detail the Hopf structure of **sNSym**. We show that **sNSym** admits a grading via the degree of a dotted composition. In particular, we deduce that the dimension of its nth homogeneous component $\mathbf{sNSym}_n$ is $2 \cdot 3^{n-1}$. By means of this grading, we show in Proposition 10 that the antipode $S : \mathbf{sNSym} \to \mathbf{sNSym}$ can be described by the formula:

$$S(H_\alpha) = (-1)^{\binom{\mathrm{df}(\alpha)}{2}} \sum_{\beta \preceq \mathrm{rev}(\alpha)} (-1)^{\ell(\beta)} H_\beta.$$

In this section we also present two Hopf subalgebras of **sNSym** that can be realized as polynomial structures (Subsection 4.1).

Recall that the classical noncommutative power sum function P_n is a primitive element of **NSym**, and is therefore also of **sNSym**. In Subsection 4.3, by direct study of the structure of **sNSym**, we deduce the existence of a second family of primitive elements $\{\Psi_n \mid n \geqslant 0\}$, different from the analogous power functions in superspace, which can be described recursively by means of the following relation:

$$\Psi_n = \tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k} \Psi_k.$$

We show that these functions are indeed primitive (Proposition 11), and we write these elements in terms of the generators (Proposition 12). We conclude this section by proving that the primitive part of **sNSym** is the free Lie superalgebra generated by $\{P_n \mid n \in \mathbb{N}\} \cup \{\Psi_n \mid n \in \mathbb{N}_0\}$ (Proposition 15).

Noncommutative versions of the elementary and power sum symmetric functions were introduced in [20] by means of generating functions. These functions, S_n and P_n respectively, form other bases for **NSym** satisfying the following relations:

$$\sum_{k=0}^{n-1} H_k P_{n-k} = nH_n, \quad \text{and} \quad \sum_{k=0}^{n} (-1)^{n-k} H_k S_{n-k} = \sum_{k=0}^{n} (-1)^{n-k} S_k H_{n-k} = 0.$$

In Section 5, by means of the generating functions with two parameters defined in [11, Section 3], we introduce families of noncommutative elementary and power sum functions in superspace, which form new bases $\{S_\alpha\}$ and $\{P_\alpha\}$ of **sNSym**. Here, we study these bases and extend results from the classic theory. Noncommutative elementary functions in superspace consist of two subfamilies, the classical functions S_n and a new kind of functions $\tilde{S}_n$. We show relations of these functions with the generators of **sNSym** (Proposition 16) and that the antipode of $\tilde{H}_n$ is given by $S(\tilde{H}_n) = (-1)^{n+1}\tilde{S}_n$ (Proposition 17). Also, as $\tilde{S}_n$ is a linear combination of products $\Psi_i S_j$ (Proposition 18), the coproduct of $\tilde{S}_n$ can be analogously written as the one of $\tilde{H}_n$, and in consequence $S(\tilde{S}_n) = (-1)^{n+1}\tilde{H}_n$ (Proposition 19). Similarly, noncommutative power sum functions in superspace consist of two subfamilies, the functions P_n mentioned above and a new kind of functions $\tilde{P}_n$. We show that each $\tilde{P}_n$ can be written as a sum of Lie superbrackets $[P_i, \Psi_j]$ (Proposition 22) and, as a consequence, we get that $\tilde{P}_n$ is a primitive element of **sNSym**. Moreover, we conclude that the Lie superalgebras generated by P_i, Ψ_i and $P_i, \tilde{P}_i$ coincide (Corollary 25).

The classical noncommutative ribbon Schur functions were introduced in [20] via quasi determinants. In Section 6, we introduce these functions in superspace. As there are two kinds of skew Schur functions in superspace, there are two kinds of *fundamental quasisymmetric functions in superspace* [16, Subsection 5.5]. Here we give two families of *noncommutative ribbon Schur functions in superspace*, dual to the fundamental ones mentioned above, by means of two partial orders on dotted compositions introduced in [16]:

$$R_\alpha = H_\alpha - \sum_{\alpha \prec \beta} R_\beta, \qquad \hat{R}_\alpha = H_\alpha - \sum_{\alpha \prec_r \beta} \hat{R}_\beta.$$

We provide explicit formulas for the product of these functions (Theorem 26 and Subsection 6.1), which can be described combinatorially via both a new kind of ribbon diagrams and planar rooted trees:

$$R_\alpha R_\beta = \begin{cases} R_{\alpha\beta} & \text{if } \operatorname{rg}(\alpha), \beta_1 \in \dot{\mathbb{N}}_0, \\ R_{\alpha\beta} + R_{\alpha\odot\beta} & \text{otherwise,} \end{cases}$$

$$\hat{R}_\alpha \hat{R}_\beta = \begin{cases} \hat{R}_{\alpha\beta} + \hat{R}_{\alpha\odot\beta} & \text{if } \operatorname{rg}(\alpha), \beta_1 \in \mathbb{N}, \\ \hat{R}_{\alpha\beta} & \text{otherwise.} \end{cases}$$

Also, we show their relation with other bases of **sNSym** (Proposition 30, Proposition 31, Proposition 32 and Proposition 33).

In Section 7, we conclude the paper by showing the interaction of **sNSym** with other Hopf algebras. First, we extend the identification of **NSym** as a substructure of the noncommutative version of the Connes–Kreimer Hopf algebra [17, 25, 18, 24], obtaining a Hopf algebra generated by planar rooted trees. Thus, the description of the coproduct of **sNSym** can be given in terms of the admissible cuts of these trees (Proposition 34). Secondly, due to the dual relation between **sNSym** and **sQSym**, we show that, for every dotted composition α, M_α can be written in terms of the fundamental quasisymmetric functions in superspace. This implies that the set of these functions is a basis of **sQSym**. Furthermore, we provide formulas for the coproduct of these functions, which can be described by means of a new type of diagrams that extend the classical ribbon diagrams (Theorem 36). Finally, in Subsection 7.3, we study the morphism π : **sNSym** $\to$ **sSym**, introduced in [16], obtaining new relations for symmetric functions in superspace and new bases for **sSym** by means the projection of noncommutative ribbon Schur functions in superspace on **sSym** (Theorem 44): $r_\alpha := \pi(R_\alpha)$ and $\hat{r}_\alpha := \pi(\hat{R}_\alpha)$.

In the present paper, every linear algebraic structure will be taken over the field of rational numbers $\mathbb{Q}$. The word *algebra* will mean associative algebra with unit and the word *coalgebra* will mean coassociative coalgebra with counit.

For every positive integer n, we will denote by $[n]$ the set $\{1, \ldots, n\}$, and set $[n]_0 = \{0\} \cup [n]$. As usual, the *symmetric group* on $[n]$ will be denoted by $\mathfrak{S}_n$.

2 Hopf algebras

Here, we recall some notions about bialgebras and Hopf algebras.

A *bialgebra* is a vector space $\mathcal{H}$ endowed with an associative product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ together with a unit $u : \mathbb{Q} \to \mathcal{H}$, and a coassociative coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ together with a counit $\epsilon : \mathcal{H} \to \mathbb{Q}$, which are compatible, that is, the coproduct and the counit are algebra morphisms. The *reduced coproduct* of $\mathcal{H}$ is defined by $\bar{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$ for all $x \in \mathcal{H}$. We say that $\mathcal{H}$ is *graded* if there are subspaces $\mathcal{H}_0, \mathcal{H}_1, \ldots$ of $\mathcal{H}$, called *homogeneous components*, satisfying the conditions

$$\mathcal{H} = \bigoplus_{n \geqslant 0} \mathcal{H}_n, \quad \mathcal{H}_i \mathcal{H}_j \subseteq \mathcal{H}_{i+j},\ i, j \geqslant 0, \quad \Delta(\mathcal{H}_n) \subseteq \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j.$$

Moreover, $\mathcal{H}$ is called *connected* whenever $\mathcal{H}_0 \simeq \mathbb{Q}$.

A bialgebra $\mathcal{H}$, as above, is said to be a *Hopf algebra* if there is a linear map $S : \mathcal{H} \to \mathcal{H}$, called *antipode*, satisfying the relations

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Note that, as $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ must be an algebra morphism, the bialgebra structure of $\mathcal{H}$ depends on the product of $\mathcal{H} \otimes \mathcal{H}$. In particular, if $\mathcal{H}$ is graded, this product may be defined as $(m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$, where $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the *twist map* defined by $\tau(x \otimes y) = (-1)^{ab}(y \otimes x)$ with $x \in \mathcal{H}_a$ and $y \in \mathcal{H}_b$.

An element x of a Hopf algebra $\mathcal{H}$ is called *primitive*, if $\Delta(x) = 1 \otimes x + x \otimes 1$ or equivalently $\bar{\Delta}(x) = 0$.

It is known that if $\mathcal{H}$ is a connected graded bialgebra, then it is a Hopf algebra with antipode defined recursively by $S(1) = 1$ and for x in some homogeneous component,

$$S(x) = -\left(x + \sum S(x_{(1)}) x_{(2)}\right), \quad \text{where} \quad \bar{\Delta}(x) = x_{(1)} \otimes x_{(2)} \quad \text{(Sweedler's notation)}. \tag{1}$$

See for instance [23, Proposition 1.4.14].

Now, we describe the dual notion of a connected graded Hopf algebra $\mathcal{H} = \bigoplus_{n \geqslant 0} \mathcal{H}_n$ with finite dimensional homogeneous components.

First, observe that $(\mathcal{H}\otimes\mathcal{H})^* \simeq \mathcal{H}^*\otimes\mathcal{H}^*$, and that $\mathcal{H}\otimes\mathcal{H}$ is also graded as a vector space. More specifically,

$$\mathcal{H}\otimes\mathcal{H} = \bigoplus_{n\geqslant 0}(\mathcal{H}\otimes\mathcal{H})_n, \quad \text{where} \quad (\mathcal{H}\otimes\mathcal{H})_n = \bigoplus_{k=0}^{n}\mathcal{H}_k\otimes\mathcal{H}_{n-k}.$$

This, together with the coproduct Δ of $\mathcal{H}$ induce a product m_d on

$$\mathcal{H}^* = \bigoplus_{n\geqslant 0}\mathcal{H}_n^*$$

given by the composition

$$m_d : \mathcal{H}^*\otimes\mathcal{H}^* \xrightarrow{\sim} (\mathcal{H}\otimes\mathcal{H})^* \xrightarrow{\Delta^*} \mathcal{H}^*,$$

where $\Delta^*(f)(x) := f\circ\Delta(x)$ with $f\in(\mathcal{H}\otimes\mathcal{H})^*$, $x\in\mathcal{H}$.

The product m of $\mathcal{H}$ induces a coproduct Δ_d of $\mathcal{H}^*$ defined as follows

$$\Delta_d : \mathcal{H}^* \xrightarrow{m^*} (\mathcal{H}\otimes\mathcal{H})^* \xrightarrow{\sim} \mathcal{H}^*\otimes\mathcal{H}^*,$$

where $m^*(f)(x\otimes y) := f\circ m(x\otimes y)$ with $x\otimes y\in\mathcal{H}\otimes\mathcal{H}$, $f\in\mathcal{H}^*$.

The structure $(\mathcal{H}^*, m_d, \Delta_d)$ is called the *dual Hopf algebra* of $\mathcal{H}$.

For more details about Hopf algebras, see, for example [23, 18].

We conclude this section with some results about Hopf superalgebras (Theorem 1).

Recall that a vector space V is called $\mathbb{Z}_2$*–graded* if it can be written as a direct sum $V = V_0\oplus V_1$. An element $v\in V$ is called *homogeneous* if it belongs to $V_0\cup V_1$, and its *degree* $\bar{v}$ is the index $i\in\mathbb{Z}_2$ such that $v\in V_i$. A $\mathbb{Z}_2$–graded algebra is called a *superalgebra*, and a $\mathbb{Z}_2$–graded coalgebra is called a *supercoalgebra*.

A *Hopf superalgebra* H is a Hopf algebra in the category of $\mathbb{Z}_2$–graded vector spaces. The compatibility between the product and coproduct is respect to the structure of superalgebra of $H\otimes H$ given by

$$(a\otimes b)(c\otimes d) = (-1)^{\bar{b}\bar{c}}ac\otimes bd.$$

For a $\mathbb{Z}_2$–graded vector space $V = V_0\otimes V_1$, the free algebra $T(V) = \bigoplus_{n\geqslant 0}V^{\otimes n}$ is a Hopf superalgebra with $\mathbb{Z}_2$–grading induced by V and coproduct defined by $\Delta(v) = v\otimes 1 + 1\otimes v$ for all $v\in V$.

A *Lie superalgebra* is a $\mathbb{Z}_2$–graded vector space $L = L_0\oplus L_1$ together with a bilinear map $[,] : L\times L\to L$, called *Lie superbracket*, satisfying the following conditions:

1. $[L_i, L_j]\subseteq L_{i+j}$ for $i,j\in\mathbb{Z}_2$.
2. $[u,v] = -(-1)^{\bar{u}\bar{v}}[v,u]$ for all $u,v\in L$.
3. $(-1)^{\bar{u}\bar{w}}[u,[v,w]] + (-1)^{\bar{u}\bar{v}}[v,[w,u]] + (-1)^{\bar{v}\bar{w}}[w,[u,v]] = 0$ for all $u,v,w\in L$.

Observe that if A is a superalgebra, then A is a Lie superalgebra with the Lie superbracket

$$[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba.$$

Free Lie superalgebras and enveloping algebras of a Lie superalgebra are defined similarly as for classical Lie algebras, by considering the $\mathbb{Z}_2$–grading. See [2, 31] for more details.

Theorem 1 ([2, Theorem 2.10], [31, Theorem 6.2.1])**.** *Let V be a $\mathbb{Z}_2$–graded vector space, let X be a basis of homogeneous elements of V, and let $[X]$ be the Lie subsuperalgebra of $T(V)$ generated by X. Then, $[X]$ is the free Lie superalgebra generated by X. Furthermore, $[X]$ coincides with the primitive part of the Hopf superalgebra $T(V)$.*

3 Dotted compositions

In what follows, $\mathbb{N} = \{1, 2, \ldots\}$ and $\dot{\mathbb{N}}_0 = \{\dot{0}, \dot{1}, \ldots\}$.

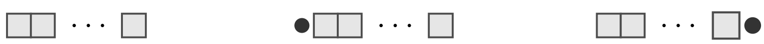
A *dotted composition* is a tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ whose *components*, denoted by $\alpha_1, \ldots, \alpha_k$, belong to the set $\mathbb{N} \cup \dot{\mathbb{N}}_0$. A component of α is called *dotted* if it belongs to $\dot{\mathbb{N}}_0$. We call k the *length* of α and denoted it by $\ell(\alpha)$. The *fermionic degree* of α is the number $\mathrm{df}(\alpha)$ of dotted components of it. The *degree* of α is the number $|\alpha|$ obtained by adding its fermionic degree with the numerical values of its components.

For instance, below the six dotted compositions of degree two:

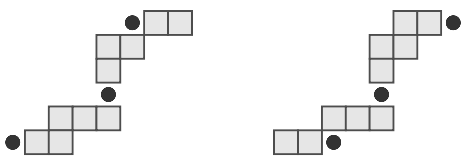
$$(1, 1) \qquad (\dot{0}, 1) \qquad (1, \dot{0}) \qquad (\dot{0}, \dot{0}) \qquad (2) \qquad (\dot{1})$$

For a length k dotted composition α, we will denote by $\mathrm{rg}(\alpha)$ the *rightmost component* of α, that is, $\mathrm{rg}(\alpha) = \alpha_k$. Note that α_1 is its *leftmost component.* Also, if $\alpha_1 = \cdots = \alpha_k$, we denote α by (α_1^k) instead.

Graphically, dotted compositions are represented by *ribbon diagrams* that extend the classical ones, that is, $m \in \mathbb{N} \cup \dot{\mathbb{N}}$ is represented by m boxes, and if m is dotted, a bullet is added either on the left or on the right.



The diagram with all its dots on the left is called the *left ribbon diagram* of the dotted composition. Similarly, we define the *right ribbon diagram.* For instance, below the left and right ribbon diagrams of $(\dot{2}, 3, \dot{0}, 1, 2, \dot{2})$.



Proposition 2. *For every positive integer n, the number of dotted compositions of degree n is $2 \cdot 3^{n-1}$. See [26, A025192].*

Proof. Observe that every dotted composition of degree n can be obtained from a usual composition $(b_1, \ldots, b_k)$ of the same degree by replacing some of its components b_i with $\dot{a}_i \in \dot{\mathbb{N}}$, where $a_i = b_i - 1$. In particular, $\dot{0}$ is gotten when 1 is replaced with it. Thus, there are 2^k dotted compositions of degree n that can be obtained in this manner from a usual composition as above. For each $k \in [n]$, there are $\binom{n-1}{k-1}$ usual compositions of degree n and length k [26, A007318]. Hence, there are $\binom{n-1}{k-1}2^k$ dotted compositions of degree n and length k. Therefore, the number of dotted compositions of degree n is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} 2^k = \sum_{k=0}^{n} \binom{n-1}{k} 2^{k+1} = 2 \sum_{k=0}^{n} \binom{n-1}{k} 2^k = 2(1+2)^{n-1} = 2 \cdot 3^{n-1}. \qquad \square$$

Given two dotted compositions α and β, we denote by $\alpha\beta$ the dotted composition obtained by concatenating α with β. If $\beta = (x)$ for some $x \in \mathbb{N} \cup \dot{\mathbb{N}}$, we denote $\alpha\beta$ by αx instead. A *convex partition* of a dotted composition α is a tuple of dotted compositions $(\beta_1, \ldots, \beta_k)$ such that $\alpha = \beta_1 \cdots \beta_k$.

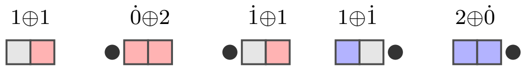
In order to describe one of the partial orders on dotted compositions introduced in [16], we first extend the sum operation of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ to $\mathbb{N}_0 \cup \dot{\mathbb{N}}_0$ as follows:

$$m \oplus n = r, \qquad m \oplus \dot{n} = \dot{r}, \qquad \dot{m} \oplus n = \dot{r}, \qquad \dot{m} \oplus \dot{n} = 0, \quad \text{where} \quad r = m + n.$$

For instance,

$$1 \oplus 1 = 2 \qquad \dot{0} \oplus 1 = \dot{1} \qquad 1 \oplus \dot{0} = \dot{1} \qquad \dot{0} \oplus \dot{0} = 0$$

Note that the length two compositions of $m \in \mathbb{N} \cup \mathbb{N}_0$, with respect to $\oplus$, are given by cutting internally its left and right ribbon diagrams. For instance, for 2 and $\dot{2}$, we have:



Remark 3. Note that $\oplus$ is not associative in general, indeed $(1 \oplus \dot{1}) \oplus \dot{1} = \dot{2} \oplus \dot{1} = 0$ and on the other hand $1 \oplus (\dot{1} \oplus \dot{1}) = 1 \oplus 0 = 1$. However, if at most one of x, y, z belongs to $\dot{\mathbb{N}}$, then $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. This property will be important to describe the partial order on dotted compositions.

As defined in [16, Subsection 5.3], for α, β dotted compositions of degree n, we say that β *covers* α, if β is obtained by $\oplus$–summing two consecutive components of α that are not both dotted. The closure of this relation is the partial order simply denoted by $\preceq$, which gives a structure of poset to the collection of all dotted compositions of degree n.

Note that two dotted compositions that are comparable with respect to $\preceq$ have the same degree and fermionic degree. Also, a dotted composition is maximal respect to $\preceq$ if it has a unique component or its components are all dotted, that is, the length and fermionic degree coincide.

Proposition 4. *For every positive integer n, the number of dotted compositions $\preceq (\dot{n})$ is $(n+2)2^{n-1}$.*

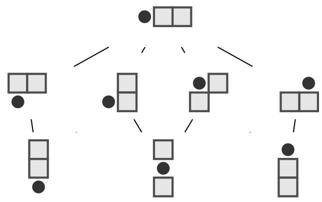
Proof. Let α be a dotted composition satisfying $\alpha \preceq (\dot{n})$. Denote by $\bar{\alpha}$ the tuple obtained by removing the zeros and all dots from the components of α. Note that α has a unique component in $\dot{\mathbb{N}}$ and that $\bar{\alpha}$ is a composition of n. For $k \in [n]$, we will count the number of dotted compositions $\alpha \preceq (\dot{n})$ such that $\ell(\bar{\alpha}) = k$.

Recall that n has $\binom{n-1}{k-1}$ compositions of length k. Then, there are $(k+1)\binom{n-1}{k-1}$ dotted compositions $\alpha \preceq (\dot{n})$ such that $\dot{0}$ is a component of α and $\ell(\bar{\alpha}) = k$. On the other hand, there are $k\binom{n-1}{k-1}$ dotted compositions $\alpha \preceq (\dot{n})$ in which the unique component belonging to $\dot{\mathbb{N}}$ is not $\dot{0}$. Hence, there are $(2k+1)\binom{n-1}{k-1}$ dotted compositions $\alpha \preceq (\dot{n})$ satisfying $\ell(\bar{\alpha}) = k$. Therefore, the number of dotted compositions $\preceq (\dot{n})$ is

$$\sum_{k=1}^{n}(2k+1)\binom{n-1}{k-1} = (n+2)2^{n-1}.$$

□

For instance, there are eight dotted compositions $\preceq (\dot{2})$.



(2)

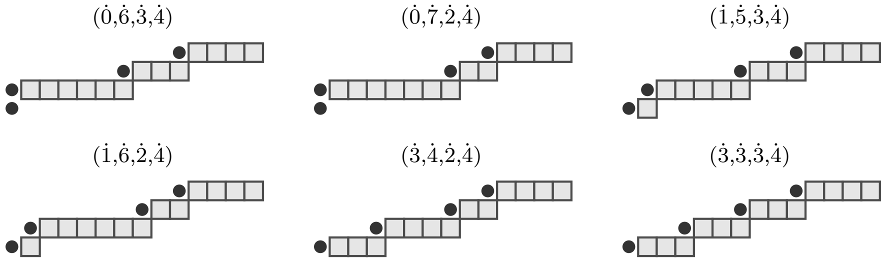
For a dotted composition α, we denote by $\alpha^{\uparrow}$ the *upper closure* of α with respect to $\preceq$, that is, $\alpha^{\uparrow}$ is the upset $\{\beta \mid \alpha \preceq \beta\}$. Note that $\alpha^{\uparrow} = \{\alpha\}$ if α is maximal with respect to $\preceq$. Moreover, if α is a usual composition of degree n, then $\alpha^{\uparrow}$ has a maximum given by (n).

If $\mathrm{df}(\alpha) \geqslant 1$, we characterize the maximal elements in $\alpha^{\uparrow}$ by means of the following proposition.

Proposition 5. *Let α be a dotted composition with $s := \mathrm{df}(\alpha) \geqslant 1$, and let m be a maximal element in $\alpha^{\uparrow}$. Then, each component of m is obtained by $\oplus$–summing the components of a dotted composition in some convex partition $(\beta_1, \dots, \beta_s)$ of α whose elements have exactly one dotted component. More specifically, $m = (\hat{\beta}_1, \dots, \hat{\beta}_s)$, where $\hat{\beta}_i$ is the $\oplus$–sum of all components of β_i.*

Proof. It follows directly from the definition of the partial order $\preceq$. □

For instance, for $\alpha = (\dot{0}, 1, 2, \dot{3}, 1, \dot{2}, \dot{4})$, the upset $\alpha^{\uparrow}$ contains the following six maximal elements:



4 The Hopf structure of sNSym

In this section, we describe the Hopf structure of noncommutative symmetric functions in superspace **sNSym**, introduced in [16].

Recall that, as in the classical case, the set of monomial quasisymmetric functions in superspace M_α, with α a dotted composition, form a basis of the Hopf algebra **sQSym**.

The *Hopf algebra of noncommutative symmetric functions in superspace* **sNSym** is the dual structure of the Hopf algebra of quasisymmetric functions in superspace, given by

$$\langle H_\alpha, M_\beta \rangle = \delta_{\alpha\beta}, \quad \text{where} \quad \alpha, \beta \quad \text{are dotted compositions.}$$

As it was shown in [16], the algebra structure of **sNSym** is freely generated by the set $\{H_r \mid r \in \mathbb{N} \cup \dot{\mathbb{N}}\}$. The generators H_r will be called *noncommutative complete homogeneus symmetric function in superspace.* For a length k dotted composition α, we write $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_k}$. The set $\{H_\alpha \mid \alpha \text{ is a dotted composition}\}$ is a basis of **sNSym**.

As with **sQSym**, the algebra **sNSym** admits a grading by the fermionic degree. With respect to this grading, **sNSym** $\otimes$ **sNSym** can be given with the following product:

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = (-1)^{\mathrm{df}(g_1)\,\mathrm{df}(f_2)}\, f_1\, f_2 \otimes g_1 g_2, \quad \text{where} \quad f_1, g_1, f_2, g_2 \in \mathbf{sNSym}. \quad (3)$$

Since **sNSym** is a free algebra, it is enough to define the coproduct Δ on its generators and so extend it as an algebra morphism, with respect to the product in (3). Thus, we have:

$$\Delta(H_r) = \sum_{p \oplus q = r} H_p \otimes H_q, \quad \text{where} \quad p, q, r \in \mathbb{N}_0 \cup \dot{\mathbb{N}}_0 \quad \text{and} \quad H_0 = 1.$$

As in [16], for every nonnegative integer n, we will write $H_{\dot{n}}$ by $\tilde{H}_n$ instead. Note that $\Delta(1) = 1 \otimes 1$ and, for every $n \in \mathbb{N}$, we have the following:

$$\Delta(H_n) = \sum_{k=0}^{n} H_k \otimes H_{n-k}, \qquad \Delta(\tilde{H}_n) = \sum_{k=0}^{n} (\tilde{H}_k \otimes H_{n-k} + H_{n-k} \otimes \tilde{H}_k).$$

There is another grading of **sNSym** given by the degree of dotted compositions. Thus, if α is a dotted composition, then H_α is an homogeneous element of degree $|\alpha|$. For every $n \in \mathbb{N}$, we denote by $\mathbf{sNSym}_n$ the subspace spanned by the elements H_α such that α is a dotted composition of degree n. By Proposition 2, $\dim(\mathbf{sNSym}_n) = 2 \cdot 3^{n-1}$. Fixing $\mathbf{sNSym}_0 = K$, we have that

$$\mathbf{sNSym} = \bigoplus_{n \geqslant 0} \mathbf{sNSym}_n,$$

is a connected graded bialgebra, and then a Hopf algebra. Note that **sNSym** is cocommutative up a sign and that **NSym** is a Hopf subalgebra of it.

Remark 6. Note that if α is a dotted composition, the coproduct of H_α depends both on the fermionic degree of α and on the action of Δ on each H_{α_i}. In particular, if $\mathrm{df}(\alpha) \in \{0, 1\}$, the summands of $\Delta(H_\alpha)$ are all positive. Otherwise, some of these summands can be negative. For instance, if $\alpha = (\dot{0}, 1, \dot{0})$, we have

$$\begin{aligned} \Delta(H_{\substack{\bullet\\ \square\\ \bullet}}) &= H_{\substack{\bullet\\ \square\\ \bullet}} \otimes 1 + H_{\substack{\square\\ \bullet}} \otimes H_{\bullet} + H_{\substack{\bullet\\ \bullet}} \otimes H_{\square} + H_{\bullet} \otimes H_{\substack{\bullet\\ \square}} \\ &\quad - H_{\substack{\bullet\\ \square}} \otimes H_{\bullet} + H_{\square} \otimes H_{\substack{\bullet\\ \bullet}} - H_{\bullet} \otimes H_{\substack{\square\\ \bullet}} + 1 \otimes H_{\substack{\bullet\\ \square\\ \bullet}}. \end{aligned}$$

In Subsection 7.1, we will give formulas to compute the coproduct of H_α in terms of trees.

4.1 Polynomial Hopf subalgebras

Here, we present two simple Hopf subalgebras of **sNSym**, which can be realized as Hopf algebras of polynomials in one variable.

The first one is the well known Hopf subalgebra of **sNSym** generated by H_1, which is isomorphic to the classical Hopf structure of polynomials in one variable X, by identifying H_1 with this variable. It is easy to check the following

$$\Delta(H_{(1^n)}) = \sum_{k=0}^{n} \binom{n}{k} H_{(1^{n-k})} \otimes H_{(1^k)}, \quad \text{where} \quad H_{(1^0)} = 1.$$

The second substructure is the Hopf subalgebra of **sNSym** generated by $\tilde{H}_0$. To describe the coproduct of $H_{(\dot{0}^n)} = \tilde{H}_0 \cdots \tilde{H}_0$, n times, for $0 \leqslant k \leqslant n$ we define the following number

$$\tilde{C}(n,k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{if } n \text{ is even and } k \text{ is odd}, \\ \tilde{C}(n-1,k-1) + \tilde{C}(n-1,k) & \text{otherwise.} \end{cases}$$

Note that if n is odd and k is even, then

$$\tilde{C}(n,k) = \tilde{C}(n,k+1) = \tilde{C}(n-1,k). \tag{4}$$

Diagrammatically,

$$
\begin{array}{lcc}
n=0 & 1 & \\
n=1 & 1 \quad 1 & \\
n=2 & 1 \quad 0 \quad 1 & 1 \\
n=3 & 1 \quad 1 \quad 1 \quad 1 & 1 \quad 1 \\
n=4 & 1 \quad 0 \quad 2 \quad 0 \quad 1 & 1 \quad 2 \quad 1 \\
n=5 & 1 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 & 1 \quad 3 \quad 3 \quad 1 \\
n=6 & 1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 1 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
n=7 & 1 \quad 1 \quad 3 \quad 3 \quad 3 \quad 3 \quad 1 \quad 1 & \\
n=8 & 1 \quad 0 \quad 4 \quad 0 \quad 6 \quad 0 \quad 4 \quad 0 \quad 1 &
\end{array}
$$

Observe that the classic Pascal triangle appears whenever n and k are both even. More specifically, we have the following result.

Proposition 7. *We have* $\tilde{C}(2n, 2k) = \binom{n}{k}$ *for all* $0 \leqslant k \leqslant n$. *In consequence, we obtain*

$$\tilde{C}(2n-1, 2k) = \binom{n-1}{k} \quad \textit{and} \quad \tilde{C}(2n-1, 2k-1) = \binom{n-1}{k-1}.$$

Proof. We proceed by induction on n. If $n = 0$, it is clear because $k = 0$ as well. If $n > 0$, by applying (4) and the induction hypothesis, we have

$$
\begin{aligned}
\tilde{C}(2n, 2k) &= \tilde{C}(2n-1, 2k-1) + \tilde{C}(2n-1, 2k) \\
&= \tilde{C}(2n-2, 2k-2) + \tilde{C}(2n-2, 2k) \\
&= \tilde{C}(2(n-1), 2(k-1)) + \tilde{C}(2(n-1), 2k) \\
&= \binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \binom{n}{k}.
\end{aligned}
$$

□

Proposition 8. *For* $0 \leqslant k \leqslant n$, *we have* $\Delta(H_{(\dot{0}^n)}) = \sum_{k=0}^{n} \tilde{C}(n, k)\, H_{(\dot{0}^{n-k})} \otimes H_{(\dot{0}^k)}$, *where* $H_{(\dot{0}^0)} = 1$.

Proof. We proceed by induction on n. The result is obvious if $n \in \{1, 2\}$. If $n \geqslant 3$, assume that claim is true for smaller values. By the induction hypothesis, we have

$$\begin{aligned}
\Delta(H_{(\dot{0}^n)}) &= \Delta(H_{(\dot{0}^{n-1})})\Delta(H_{\dot{0}}),\\
&= \left(H_{(\dot{0}^{n-1})}\otimes 1+\sum_{k=1}^{n-1}\tilde{C}(n-1,k)\,H_{(\dot{0}^{n-1-k})}\otimes H_{(\dot{0}^k)}\right)(H_{\dot{0}}\otimes 1+1\otimes H_{\dot{0}})\\
&= H_{(\dot{0}^n)}\otimes 1+H_{(\dot{0}^{n-1})}\otimes H_{\dot{0}}+\sum_{k=1}^{n-1}(-1)^k\tilde{C}(n-1,k)\,H_{(\dot{0}^{n-k})}\otimes H_{(\dot{0}^k)}\\
&\quad+\sum_{k=1}^{n-1}\tilde{C}(n-1,k)\,H_{(\dot{0}^{n-(k+1)})}\otimes H_{(\dot{0}^{k+1})}\\
&= H_{(\dot{0}^n)}\otimes 1+1\otimes H_{(\dot{0}^n)}+(1-\tilde{C}(n-1,1))H_{(\dot{0}^{n-1})}\otimes H_{\dot{0}}\\
&\quad+\sum_{k=2}^{n-1}\left[\tilde{C}(n-1,k-1)+(-1)^k\tilde{C}(n-1,k)\right]H_{(\dot{0}^{n-k})}\otimes H_{(\dot{0}^k)}.
\end{aligned}$$

If n is even, we have $\tilde{C}(n-1,1)=\tilde{C}(n,0)=1$, and so $1-\tilde{C}(n-1,1)=0$. For k odd, we have $\tilde{C}(n-1,k-1)=\tilde{C}(n-1,k)$, then $\tilde{C}(n-1,k-1)+(-1)^k\tilde{C}(n-1,k)=0$, because $(-1)^k=-1$. For k even, we have $\tilde{C}(n-1,k-1)+(-1)^k\tilde{C}(n-1,k)=\tilde{C}(n-1,k-1)+\tilde{C}(n-1,k)=\tilde{C}(n,k)$.

Now, assume that n is odd. If k is even, by using (4), we have $\tilde{C}(n-1,k-1)+(-1)^k\tilde{C}(n-1,k)=\tilde{C}(n-1,k)=\tilde{C}(n,k)$. Finally, if k is odd, we have $\tilde{C}(n-1,k-1)+(-1)^k\tilde{C}(n-1,k)=\tilde{C}(n-1,k-1)=\tilde{C}(n,k)$. □

By using Proposition 8 and Proposition 7, we obtain

- $\Delta(H_{(\dot{0}^{2n})})=\sum_{k=0}^{n}\binom{n}{k}H_{(\dot{0}^{2n-2k})}\otimes H_{(\dot{0}^{2k})}$.
- $\Delta(H_{(\dot{0}^{2n-1})})=\sum_{k=0}^{n-1}\binom{n-1}{k}H_{(\dot{0}^{2n-1-2k})}\otimes H_{(\dot{0}^{2k})}+\sum_{k=1}^{n}\binom{n-1}{k-1}H_{(\dot{0}^{2n-2k})}\otimes H_{(\dot{0}^{2k-1})}$.

By identifying $\tilde{H}_0$ with the variable X, we obtain a new structure on the set $\mathcal{P}$ of polynomials in one variable. In particular, the fermionic degree inherits an algebraic structure on $\mathcal{P}\otimes\mathcal{P}$ given by

$$(X^{m_1}\otimes X^{m_2})\cdot(X^{n_1}\otimes X^{n_2})=(-1)^{m_2n_1}X^{m_1+n_1}\otimes X^{m_2+n_2}.$$

Note that the algebraic structures presented above differ in their coalgebraic structure. For instance, the coproducts of X^4, with respect to the identifications above, are given respectively by

$$X^4\otimes 1+4X^3\otimes X+6X^2\otimes X^2+4X\otimes X^3+1\otimes X^4 \quad\text{and}\quad X^4\otimes 1+2X^2\otimes X^2+1\otimes X^4.$$

4.2 Antipode

Recall that the antipode of a connected graded bialgebra $\mathcal{H}$ can be defined as in (1).

For a dotted composition α, denote by $\operatorname{rev}(\alpha)$ the dotted composition obtained by reversing its components.

It is well known that [23, Remark 5.3.4], for a usual composition α, we have

$$S(H_\alpha) = \sum_{\beta \preceq \operatorname{rev}(\alpha)} (-1)^{\ell(\beta)} H_\beta. \tag{5}$$

A similar result can be generalized for dotted compositions up a sign that depends on the fermionic degree. Previously, we determine the antipode of generators indexed by dotted compositions of length one.

Proposition 9. *For every integer $n \geqslant 0$, we have $S(\tilde{H}_n) = \sum_{\alpha \preceq (\dot{n})} (-1)^{\ell(\alpha)} H_\alpha$.*

Proof. We proceed by induction on n. Since $\bar{\Delta}(\tilde{H}_0) = 0$, $S(\tilde{H}_0) = -\tilde{H}_0$. Consider $n \geqslant 1$ and assume the claim is true for all $k < n$. By applying (1), as $\bar{\Delta}(\tilde{H}_n) = \sum_{k=0}^{n-1}(\tilde{H}_k \otimes H_{n-k} + H_{n-k} \otimes \tilde{H}_k)$, we obtain the following formula

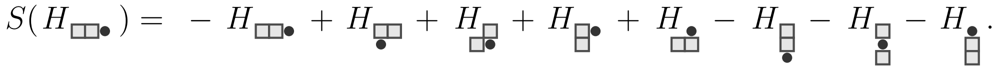
$$S(\tilde{H}_n) = -\left(\tilde{H}_n + \sum_{k=0}^{n-1}(S(\tilde{H}_k)H_{n-k} + S(H_{n-k})\tilde{H}_k)\right).$$

By rewriting $\displaystyle\sum_{k=0}^{n-1} S(\tilde{H}_k)H_{n-k}$ as $\displaystyle\sum_{k=1}^{n} S(\tilde{H}_{n-k})H_k$ and applying the induction hypothesis, we obtain

$$S(\tilde{H}_n) = -\tilde{H}_n + \sum_{k=1}^{n} \sum_{\beta \preceq (\dot{m})} (-1)^{\ell(\beta)+1} H_\beta H_k + \sum_{k=0}^{n-1} \sum_{\beta \preceq (m)} (-1)^{\ell(\beta)+1} H_\beta \tilde{H}_k,$$

where $m = n - k$. On the other hand, if $\alpha \preceq (\dot{n})$, then either $H_\alpha = H_\beta \tilde{H}_k$ for some $k \in [n]$ and $\beta \preceq (m)$, or $H_\alpha = H_\beta H_k$ for some $k \in [n]_0$ and $\beta \preceq (\dot{m})$. Hence, the result follows. $\square$

For instance, from the lattice in (2), we obtain



Proposition 10. *For a dotted composition α, we have*

$$S(H_\alpha) = (-1)^{\binom{\operatorname{df}(\alpha)}{2}} \sum_{\beta \preceq \operatorname{rev}(\alpha)} (-1)^{\ell(\beta)} H_\beta.$$

Proof. Let α be a dotted composition with $k = \ell(\alpha)$, and, for each $i \in [k]$, let a_i be the fermionic degree of (α_i). Note that $a_i = 1$ if $\alpha_i \in \dot{\mathbb{N}}$ and $a_i = 0$ otherwise. If $k = 1$, the result follows from (5) or Proposition 9. Now, assume that $k \geqslant 2$. Since the antipode is a signed antihomomorphism, Proposition 9 implies that

$$S(H_\alpha) = (-1)^a S(H_{\alpha_k}) \cdots S(H_{\alpha_1}) = (-1)^a \sum_{\beta \preceq \mathrm{rev}(\alpha)} (-1)^{\ell(\beta)} H_\beta,$$

where

$$a = \sum_{i=1}^{k-1} a_i (a_{i+1} + \cdots + a_k).$$

Now, we will show that $a = \binom{\mathrm{df}(\alpha)}{2}$. Let $\alpha_{i_1}, \ldots, \alpha_{i_s}$, with $i_1 < \cdots < i_s$, be the dotted components of α, that is, $s = \mathrm{df}(\alpha)$. We have $a_{i_1} = \cdots = a_{i_s} = 1$ and $a_i = 0$ otherwise. This implies that

$$\begin{aligned} a &= \sum_{j=1}^{s-1} a_{i_j} (a_{i_{j+1}} + \cdots + a_{i_s}) \\ &= \sum_{j=1}^{s-1} (s-j) = s(s-1) - \frac{(s-1)s}{2} = \frac{(s-1)s}{2} = \binom{s}{2} = \binom{\mathrm{df}(\alpha)}{2}. \end{aligned}$$ □

Note that Proposition 10 is dual to [16, Proposition 5.10] for the antipode of M_α in **sQSym**.

4.3 Primitive elements

In this section we study the primitive part of **sNSym**. Recall that the classical noncommutative power sum functions P_n are defined recursively by $P_1 = H_1$ and

$$\sum_{k=0}^{n-1} H_k P_{n-k} = n H_n, \quad n \geqslant 1. \tag{6}$$

It is known that P_n is a primitive element of the Hopf algebra **NSym**, and that the primitive part of **NSym** is the free Lie algebra generated by these functions [20, Subsection 3.1]. Note that, as **NSym** is a Hopf subalgebra of **sNSym**, then P_n is a primitive element of **sNSym** as well.

Here, we determined another primitive elements of **sNSym**, which are defined recursively by $\Psi_0 = \tilde{H}_0$ and

$$\Psi_n = \tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k} \Psi_k, \quad n \geqslant 1. \tag{7}$$

Proposition 11. *For every $n \geqslant 0$, Ψ_n is a primitive of* **sNSym**.

Proof. By definition, Ψ_0 is a primitive, so we proceed by induction on n. Assume the claim is true for all $k < n$, that is $\Delta(\Psi_k) = \Psi_k \otimes 1 + 1 \otimes \Psi_k$. Since $\Delta(H_{n-k}) = \sum_{k=0}^{n-k} H_i \otimes H_{n-k-1}$, we obtain

$$\begin{aligned}\Delta(H_{n-k})\Delta(\Psi_k) &= \left(\sum_{i=0}^{n-k} H_i \otimes H_{n-k-i}\right)(\Psi_k \otimes 1 + 1 \otimes \Psi_k) \\ &= \sum_{i=0}^{n-k} H_i\Psi_k \otimes H_{n-k} + \sum_{i=0}^{n-k} H_i \otimes H_{n-k}\Psi_k.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta(\Psi_n) &= \Delta(\tilde{H}_n) - \sum_{k=0}^{n-1} \Delta(H_{n-k})\Delta(\Psi_k) \\ &= \sum_{k=0}^{n} (\tilde{H}_k \otimes H_{n-k} + H_{n-k} \otimes \tilde{H}_k) \\ &\quad - \sum_{k=0}^{n-1}\sum_{i=0}^{n-k} (H_i\Psi_k \otimes H_{n-k-i} + H_{n-k-i} \otimes H_i\Psi_k).\end{aligned}$$

Rewriting both sums, we get

$$\begin{aligned}\Delta(\Psi_n) &= \sum_{i=1}^{n} \left(\tilde{H}_{n-i} \otimes H_i + H_i \otimes \tilde{H}_{n-i}\right) + \tilde{H}_n \otimes 1 + 1 \otimes \tilde{H}_n \\ &\quad - \left(\sum_{k=0}^{n-1} H_{n-k}\Psi_k\right) \otimes 1 - 1 \otimes \left(\sum_{k=0}^{n-1} H_{n-k}\Psi_k\right) \\ &\quad - \sum_{i=1}^{n} \left[\left(\sum_{k=0}^{n-i} H_{n-i-k}\Psi_k\right) \otimes H_i + H_i \otimes \left(\sum_{k=0}^{n-i} H_{n-i-k}\Psi_k\right)\right] \\ &= \sum_{i=1}^{n} \left[\left(\tilde{H}_{n-i} - \sum_{k=0}^{n-i} H_{n-i-k}\Psi_k\right) \otimes H_i + H_i \otimes \left(\tilde{H}_{n-i} - \sum_{k=0}^{n-i} H_{n-i-k}\Psi_k\right)\right] \\ &\quad + \left(\tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k}\Psi_k\right) \otimes 1 + 1 \otimes \left(\tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k}\Psi_k\right).\end{aligned}$$

Due to (7), we have

$$\tilde{H}_{n-i} - \sum_{k=0}^{n-i} H_{n-i-k}\Psi_k = 0 \qquad \text{and} \qquad \tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k}\Psi_k = \Psi_n.$$

Therefore $\Delta(\Psi_n) = \Psi_n \otimes 1 + 1 \otimes \Psi_n$. □

Recall that P_n can be written as, see for instance [23, Equation (5.3.13)],

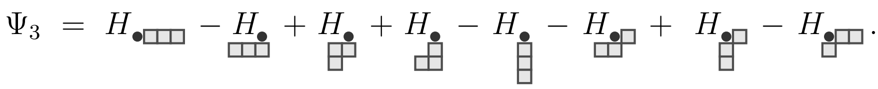
$$P_n = \sum_{\alpha \preceq (n)} (-1)^{\ell(\alpha)-1} \operatorname{rg}(\alpha) H_\alpha. \tag{8}$$

Now, by an inductive argument, we have the following proposition for Ψ_n.

Proposition 12. *For every integer $n \geqslant 1$, Ψ_n is the sum of 2^n terms given as*

$$\Psi_n = \tilde{H}_n + \sum_{k=0}^{n-1} \sum_{\alpha \preceq (n-k)} (-1)^{\ell(\alpha)} H_\alpha \tilde{H}_k.$$

For instance, Ψ_3 is given by the following sum of eight terms.



Recall that if α is a usual composition of length k, then $P_\alpha = P_{\alpha_1} \cdots P_{\alpha_k}$. It is known that, see for instance [23, Equation (5.3.14)],

$$H_n = \sum_{\alpha \preceq (n)} \frac{1}{\nu_\alpha} P_\alpha, \quad \text{where} \quad \nu_\alpha = \alpha_1(\alpha_1 + \alpha_2) \cdots (\alpha_1 + \cdots + \alpha_k). \tag{9}$$

This together with (7) imply the following

$$\tilde{H}_n = \Psi_n + \sum_{k=0}^{n-1} \left(\sum_{\alpha \preceq (n-k)} \frac{1}{\nu_\alpha} P_\alpha \right) \Psi_k.$$

Remark 13. Since **sNSym** is free on the H_n's and $\tilde{H}_n$'s, and the results above imply that these generators can be written in terms of the P_n's and Ψ_n's, we get that **sNSym** is free on $\Psi := \{P_n \mid n \in \mathbb{N}\} \cup \{\Psi_n \mid n \in \mathbb{N}_0\}$. Note that **sNSym** can be regarded as a Hopf superalgebra by considering the $\mathbb{Z}_2$–grading induced by the fermionic degree. Thus, **sNSym** is isomorphic, as a Hopf superalgebra, to the free algebra $T(\Psi)$.

To describe the primitive part of **sNSym**, we consider the bilinear map $[,] :$ **sNSym** $\times$ **sNSym** $\to$ **sNSym** defined by $[H_\alpha, H_\beta] = H_\alpha H_\beta - (-1)^{\mathrm{df}(\alpha)\,\mathrm{df}(\beta)} H_\beta H_\alpha$ for all dotted compositions α, β. With this operation, **sNSym** is a Lie superalgebra. The following result is immediate.

Proposition 14. *If x, y are primitive elements of* **sNSym**, *then $[x, y]$ is primitive as well.*

As a consequence of Remark 13 and Theorem 1, we get the following result.

Proposition 15. *The primitive part of* **sNSym** *is the free Lie superalgebra generated by $\{P_n \mid n \in \mathbb{N}\} \cup \{\Psi_n \mid n \in \mathbb{N}_0\}$.*

5 Noncommutative elementary and power sum functions in superspace

In this section, we introduce noncommutative analogues of elementary (Subsection 5.1) and power sum (Subsection 5.2) functions in superspace by means of generating functions. More precisely, we will use two generating functions given in [11, Section 3], adapted to the noncommutative context. This approach follows the one given in [20, Subsection 3.1]. In each subsection we get that these families of functions form bases of **sNSym**.

As in [11, Section 3], we consider t and τ be two indeterminate parameters with $\tau^2 = 0$. In what follows, we write the generating function [11, Equation (3.16)] of the complete homogeneous functions H_n and $\tilde{H}_n$, by

$$\lambda(t,\tau) := \sum_{n\geqslant 0} t^n (H_n + \tau \tilde{H}_n),$$

5.1 Elementary functions

The *elementary functions in superspace* are defined by means of the following generating function [11, Equation (3.4)]

$$\sigma(t,\tau) := \sum_{n\geqslant 0} t^n (S_n + \tau \tilde{S}_n) \quad \text{satisfying} \quad \sigma(t,\tau)\lambda(-t,-\tau) = 1. \tag{10}$$

Observe that (10) coincides with the generating function in [20, Equation (22)], that defines the classical noncommutative elementary functions, whenever $\tau = 0$.

The following proposition is a noncommutative analogue of [11, Lemma 22].

Proposition 16. *For $n \geqslant 1$, we obtain the following recursive formulas:*

$$\sum_{k=0}^{n} (-1)^{n-k} H_k S_{n-k} = \sum_{k=0}^{n} (-1)^{n-k} S_k H_{n-k} = 0 \quad \textit{and}$$

$$\sum_{k=0}^{n} (-1)^{n-k} (S_k \tilde{H}_{n-k} - \tilde{S}_k H_{n-k}) = 0.$$

Note that S_n is the classical noncommutative elementary function and

$$\tilde{S}_n = S_n \tilde{H}_0 + \sum_{k=0}^{n-1} (-1)^{n-k} (S_k \tilde{H}_{n-k} - \tilde{S}_k H_{n-k}).$$

Proposition 17. *For every integer $n \geqslant 0$, we have $S(\tilde{H}_n) = (-1)^{n+1} \tilde{S}_n$. As a consequence,*

$$\tilde{S}_n = (-1)^{n+1} \sum_{\alpha \preceq (\dot{n})} (-1)^{\ell(\alpha)} H_\alpha.$$

Proof. We proceed by induction on n. Note that $S(\tilde{H}_0) = -\tilde{S}_0$ because $\tilde{S}_0 = \tilde{H}_0$. If $n \geqslant 1$, assume the claim is true for all $k < n$. By rewriting the coproduct of $\tilde{H}_n$ as

$$\Delta(\tilde{H}_n) = \sum_{k=0}^{n} H_k \otimes \tilde{H}_{n-k} + \sum_{k=0}^{n} \tilde{H}_k \otimes H_{n-k},$$

we obtain

$$S(\tilde{H}_n) = -\left(\tilde{H}_n + \sum_{k=1}^{n} S(H_k)\tilde{H}_{n-k} + \sum_{k=0}^{n-1} S(\tilde{H}_k)H_{n-k}\right).$$

Proposition 16, the fact that $S(H_k) = (-1)^k S_k$ and the induction hypothesis imply that

$$\begin{aligned} S(\tilde{H}_n) &= -\left(\tilde{H}_n + \sum_{k=1}^{n}(-1)^k S_k \tilde{H}_{n-k} + \sum_{k=0}^{n-1}(-1)^{k+1}\tilde{S}_k H_{n-k}\right) \\ &= -\left((-1)^n S_n \tilde{H}_0 + \sum_{k=0}^{n-1}(-1)^k \left(S_k \tilde{H}_{n-k} - \tilde{S}_k H_{n-k}\right)\right) \\ &= (-1)^{n+1}\left(S_n \tilde{H}_0 + \sum_{k=0}^{n-1}(-1)^{n-k}\left(S_k \tilde{H}_{n-k} - \tilde{S}_k H_{n-k}\right)\right) \\ &= (-1)^{n+1}\tilde{S}_n. \end{aligned}$$

Finally, Proposition 9 implies that $\tilde{S}_n = (-1)^{n+1} \sum_{\alpha \preceq (\dot{n})} (-1)^{\ell(\alpha)} H_\alpha$. □

Proposition 18. *For every $n \geqslant 0$, $\tilde{S}_n = \sum_{k=0}^{n}(-1)^{n-k}\Psi_{n-k}S_k$.*

Proof. Proposition 17 together with (7) and Proposition 11 imply the following

$$(-1)^{n+1}\tilde{S}_n = S(\tilde{H}_n) = \sum_{k=0}^{n} S(H_{n-k})S(\Psi_k) = \sum_{k=0}^{n} S(H_{n-k})(-\Psi_k).$$

Then, as $S(H_{n-k}) = (-1)^{n-k}S_{n-k}$, we get

$$(-1)^{n+1}\tilde{S}_n = \sum_{k=0}^{n}(-1)^{n-k}S_{n-k}(-\Psi_k) = (-1)^{n+1}\sum_{k=0}^{n}(-1)^{-k}S_{n-k}\Psi_k.$$

Therefore $\tilde{S}_n = \sum_{k=0}^{n}(-1)^k \Psi_k S_{n-k} = \sum_{k=0}^{n}(-1)^{n-k}\Psi_{n-k}S_k$. □

Proposition 19. *For every $n \geqslant 0$, we have*

$$\Delta(\tilde{S}_n) = \sum_{k=0}^{n}\left(\tilde{S}_k \otimes S_{n-k} + S_{n-k} \otimes \tilde{S}_k\right).$$

In consequence, $S(\tilde{S}_n) = (-1)^{n+1}\tilde{H}_n$.

Proof. Proposition 11 implies that $\Delta(\Psi_q) = \Psi_q \otimes 1 + 1 \otimes \Psi_q$. Also, it is known that

$$\Delta(S_k) = \sum_{i=0}^{k} S_i \otimes S_{k-i} = \sum_{i=0}^{k} S_{k-i} \otimes S_i.$$

Thus, by using Proposition 18, we have

$$\begin{aligned}
\Delta(\tilde{S}_n) &= \sum_{k=0}^{n} (-1)^{n-k} \Delta(\Psi_{n-k}) \Delta(S_k) \\
&= \sum_{k=0}^{n} (-1)^{n-k} \left[(\Psi_{n-k} \otimes 1) \left(\sum_{i=0}^{k} S_i \otimes S_{k-i} \right) + (1 \otimes \Psi_{n-k}) \left(\sum_{i=0}^{k} S_{k-i} \otimes S_i \right) \right] \\
&= \sum_{k=0}^{n} \sum_{i=0}^{k} \left[(-1)^{n-k} \Psi_{n-k} S_i \otimes S_{k-i} + S_{k-i} \otimes (-1)^{n-k} \Psi_{n-k} S_i \right] \\
&= \sum_{i+j+k=n} \left[(-1)^i \Psi_i S_j \otimes S_k + S_k \otimes (-1)^i \Psi_i S_j \right] \\
&= \sum_{k=0}^{n} \left[\left(\sum_{i=0}^{k} (-1)^{k-i} \Psi_{k-i} S_i \right) \otimes S_{n-k} + S_{n-k} \otimes \left(\sum_{i=0}^{k} (-1)^{k-i} \Psi_{k-i} S_i \right) \right] \\
&= \sum_{k=0}^{n} \left(\tilde{S}_k \otimes S_{n-k} + S_{n-k} \otimes \tilde{S}_k \right).
\end{aligned}$$

□

For a dotted composition $\alpha = (\alpha_1, \dots, \alpha_k)$, we set $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$. By Proposition 16 and an inductive argument, we get the following formulas:

$$H_n = \sum_{\alpha \preceq (n)} (-1)^{\ell(\alpha)} S_\alpha \quad \text{and} \quad \tilde{H}_n = \sum_{\alpha \preceq (\dot{n})} (-1)^{\ell(\alpha)} S_\alpha.$$

This implies that $\{S_\alpha \mid \alpha \text{ is a dotted composition}\}$ is a basis of **sNSym**.

5.2 Power sum functions

The *power sum functions in superspace*, formed by the classical P_n and its analogue in superspace $\tilde{P}_n$, are defined by means of the following generating function [11, Equation (3.39)]

$$\Pi(t,\tau) := \sum_{n \geqslant 0} t^n (P_n + \tau(n+1)\tilde{P}_n) \quad \text{satisfying} \quad \lambda(t,\tau)\Pi(t,\tau) = (t\partial_t + \tau\partial_\tau)\lambda(t,\tau). \quad (11)$$

Observe that (11) coincides with the generating function in [20, Equation (23) and Equation (24)], that defines the classical noncommutative power sum functions given in (6), whenever $\tau = 0$.

Due to this relation, we get the following noncommutative analogue of [11, Lemma 26].

Proposition 20. *For every $n \geqslant 0$, we have*

$$\sum_{k=0}^{n}\left(\tilde{H}_{n-k}P_k+(k+1)H_{n-k}\tilde{P}_k\right)=(n+1)\tilde{H}_n, \quad \textit{where} \quad P_0=0. \tag{12}$$

Proof. We have $(t\partial_t+\tau\partial_\tau)\lambda(t,\tau)=\sum_{n\geqslant 0}t^n\left[nH_n+\tau(n+1)\tilde{H}_n\right]$. On the other hand,

$$\begin{aligned}\lambda(t,\tau)\Pi(t,\tau) &= \sum_{n\geqslant 0}\left[\sum_{k=0}^{n}t^{n-k}(H_{n-k}+\tau\tilde{H}_{n-k})t^k(P_k+\tau(k+1)\tilde{P}_k)\right]\\ &= \sum_{n\geqslant 0}t^n\left[\sum_{k=0}^{n}H_{n-k}P_k+\tau\sum_{k=0}^{n}((k+1)H_{n-k}\tilde{P}_k+\tilde{H}_{n-k}P_k)\right].\end{aligned}$$

Then, due to (11), we get (6) and (12). □

Proposition 21. *For every $n \geqslant 0$, we have*

$$(n+1)\tilde{P}_n=\sum_{\alpha\preceq(\dot{n})}(-1)^{\ell(\alpha)-1}|\operatorname{rg}(\alpha)|H_\alpha,$$

where $|\operatorname{rg}(\alpha)|$ *denotes the degree of dotted composition* $(\operatorname{rg}(\alpha))$.

Proof. We proceed by induction on n. Proposition 20 implies that $\tilde{P}_0=\tilde{H}_0$. Let $n\geqslant 1$ and assume the claim is true for $0\leqslant k<n$. By Proposition 20, we have:

$$(n+1)\tilde{P}_n=(n+1)\tilde{H}_n-\sum_{k=1}^{n}\tilde{H}_{n-k}P_k-\sum_{k=0}^{n-1}H_{n-k}(k+1)\tilde{P}_k.$$

By replacing P_k as in (8) and applying the induction hypothesis to $(k+1)\tilde{P}_k$, we obtain the following

$$\begin{aligned}(n+1)\tilde{P}_n &= (n+1)\tilde{H}_n-\sum_{k=1}^{n}\tilde{H}_{n-k}\left(\sum_{\alpha\preceq(k)}(-1)^{\ell(\alpha)-1}\operatorname{rg}(\alpha)H_\alpha\right)\\ &\quad -\sum_{k=0}^{n-1}H_{n-k}\left(\sum_{\alpha\preceq(\dot{k})}(-1)^{\ell(\alpha)-1}|\operatorname{rg}(\alpha)|H_\alpha\right)\\ &= (n+1)\tilde{H}_n-\sum_{k=1}^{n}\sum_{\alpha\preceq(k)}(-1)^{\ell(\alpha)-1}\operatorname{rg}(\alpha)H_{(\dot{n-k})\alpha}\\ &\quad -\sum_{k=0}^{n-1}\sum_{\alpha\preceq(\dot{k})}(-1)^{\ell(\alpha)-1}|\operatorname{rg}(\alpha)|H_{(n-k)\alpha}\\ &= \sum_{k=1}^{n}\sum_{\alpha\preceq(k)}(-1)^{\ell(\alpha)}\operatorname{rg}(\alpha)H_{(\dot{n-k})\alpha}\\ &\quad +\sum_{k=0}^{n-1}\sum_{\alpha\preceq(\dot{k})}(-1)^{\ell(\alpha)}|\operatorname{rg}(\alpha)|H_{(n-k)\alpha}+(n+1)\tilde{H}_n.\end{aligned}$$

Since $\{\alpha \mid \alpha \prec (\dot{n})\} = \{(n \dot{-} k)\alpha \mid k \in [n] \text{ and } \alpha \preceq (k)\} \cup \{(n-k)\alpha \mid k \in [n-1]_0 \text{ and } \alpha \preceq (\dot{k})\}$ and $\ell((n \dot{-} k)\alpha) = \ell((n-k)\alpha) = \ell(\alpha)+1$, the result follows. □

Proposition 22. $\tilde{P}_0 = \Psi_0$, *and for every* $n \geqslant 1$, *we have*

$$(n+1)\tilde{P}_n = (n+1)\Psi_n + \sum_{k=0}^{n-1}[P_{n-k}, \Psi_k]. \tag{13}$$

Proof. As $\tilde{H}_1 = H_1\Psi_0 + \Psi_1$, Proposition 20 implies that $\tilde{P}_0 = \tilde{H}_0 = \Psi_0$ and $2\tilde{P}_1 = 2\Psi_1 + [P_1, \Psi_0]$. Now, we proceed by induction by assuming the claim is true for all $1 < k < n$. By Proposition 20, we have

$$(n+1)\tilde{P}_n = (n+1)\tilde{H}_n - H_n\Psi_0 - \sum_{k=1}^{n-1} H_{n-k}(k+1)\tilde{P}_k - \sum_{k=1}^{n} \tilde{H}_{n-k}P_k.$$

The induction hypothesis implies that, for each $1 \leqslant k < n$, we have

$$(k+1)\tilde{P}_k = (k+1)\Psi_k + \sum_{i=0}^{k-1}[P_{k-i}, \Psi_i].$$

Due to (7), we have $(n+1)\tilde{H}_n = (n+1)\sum_{k=0}^{n} H_{n-k}\Psi_k$. Then, applying the induction hypothesis, we have

$$(n+1)\tilde{P}_n = (n+1)\Psi_n + \sum_{k=0}^{n-1}(n-k)H_{n-k}\Psi_k - \sum_{k=1}^{n-1}\left(H_{n-k}\sum_{i=0}^{k-1}[P_{k-i}, \Psi_i]\right) - \sum_{k=1}^{n}\tilde{H}_{n-k}P_k.$$

Now, by (6), we have $(n-k)H_{n-k} = \sum_{j=1}^{n-k} H_{n-k-j}P_j$, which implies

$$\begin{aligned}\sum_{k=0}^{n-1}(n-k)H_{n-k}\Psi_k &= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{n-k} H_{n-k-j}P_j\right)\Psi_k \\ &= \sum_{k=0}^{n-1} P_{n-k}\Psi_k + \sum_{k=1}^{n-1} H_k(P_{n-k}\Psi_0 + \cdots + P_1\Psi_{n-k-1}).\end{aligned}$$

On the other hand, because of (7), we have

$$\sum_{k=1}^{n}\tilde{H}_{n-k}P_k = \sum_{k=1}^{n}\left(\sum_{j=0}^{n-k} H_{n-k-j}\Psi_j\right)P_k = \sum_{k=0}^{n-1}\Psi_k P_{n-k} + \sum_{k=1}^{n-1} H_k(\Psi_0 P_{n-k} + \cdots + \Psi_{n-k-1}P_1).$$

Then,

$$\sum_{k=0}^{n-1}(n-k)H_{n-k}\Psi_k - \sum_{k=1}^{n}\tilde{H}_{n-k}P_k = \sum_{k=0}^{n-1}[P_{n-k}, \Psi_k] + \sum_{k=1}^{n-1}\left(\sum_{i=0}^{n-k-1}[P_{n-k-i}, \Psi_i]\right).$$

The results above implies that

$$(n+1)\tilde{P}_n = (n+1)\Psi_n + \sum_{k=0}^{n-1}[P_{n-k},\Psi_k] + \sum_{k=1}^{n-1} H_k \left(\sum_{i=0}^{n-k-1}[P_{n-k-i},\Psi_i]\right)$$
$$-\sum_{k=1}^{n-1}\left(H_{n-k}\sum_{i=0}^{k-1}[P_{k-i},\Psi_i]\right)$$
$$= (n+1)\Psi_n + \sum_{k=0}^{n-1}[P_{n-k},\Psi_k]. \qquad \square$$

Since P_{n-k} and Ψ_k are primitive elements (Subsection 4.3), Proposition 14 implies that $[P_{n-k},\Psi_k]$ is also a primitive element. As an immediate consequence we have the following result.

Corollary 23. *$\tilde{P}_n$ is a primitive element of* **sNSym** *for all $n \geqslant 0$.*

In the following proposition, we write Ψ_n in terms of the P_i's and $\tilde{P}_i$'s. To simplify the notation, for $g_1,\dots,g_n$ in a Lie algebra, we set

$$[g_1,\dots,g_n] = \begin{cases} g_1 & \text{if } n=1, \\ [g_1,[g_2,\dots,g_n]] & \text{if } n>1 \end{cases}$$

Proposition 24. *For every $n \geqslant 0$, we have*

$$\Psi_n = \sum_{\alpha=(\alpha_1,\dots,\alpha_k)} \left(\frac{(-1)^{\ell(\alpha)+1}}{t_\alpha}[P_{\alpha_1},\dots,P_{\alpha_{k-1}},\tilde{P}_{\alpha_k}] + \frac{(-1)^{\ell(\alpha)}}{(\alpha_k+1)t_\alpha}[P_{\alpha_1},\dots,P_{\alpha_k},\tilde{P}_0] \right),$$

where the sum is over all usual compositions α of n, and $t_\alpha = \prod_{i=1}^{k-1}(\alpha_i+\cdots+\alpha_k+1)$.

Proof. The proof follows by applying Proposition 22 and an inductive argument on n. $\square$

As a consequence, we obtain the following result.

Corollary 25. *The Lie superalgebras generated by P_i, Ψ_i and $P_i, \tilde{P}_i$ respectively, coincide.*

For a dotted composition $\alpha = (\alpha_1,\dots,\alpha_k)$, we set $P_\alpha = P_{\alpha_1}\cdots P_{\alpha_k}$. Due to (9), Proposition 20 and an inductive argument, we get the following formula:

$$\tilde{H}_n = \sum_{\alpha \preceq (\dot{n})} q_\alpha P_\alpha,$$

where $\alpha = (\alpha_1,\dots,\alpha_{i-1},\dot{\alpha}_i,\alpha_{i+1},\dots,\alpha_k)$, and

$$q_\alpha = \frac{\alpha_i+1}{\alpha_1(\alpha_1+\alpha_2)\cdots(\alpha_1+\cdots+\alpha_{i-1})(\alpha_1+\cdots+\alpha_i+1)\cdots(\alpha_1+\cdots+\alpha_k+1)}.$$

This implies that $\{P_\alpha \mid \alpha \text{ is a dotted composition}\}$ is a basis of **sNSym**.

6 Ribbon Schur functions in superspace

Classical noncommutative ribbon Schur functions were introduced in [20]. These functions form a dual basis of the so–called *fundamental quasisymmetric functions* [21]. In superspace, there are two kinds of fundamental quasisymmetric functions [16, Subsection 5.5], which depend on two different partial orders on dotted compositions.

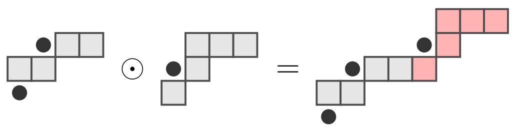
In this section, we extend noncommutative ribbon Schur functions to superspace. We introduce a first kind of these functions via the partial order defined in Section 3. Then, we introduce a second kind of these functions via a restriction of this order (Subsection 6.1). These functions form new bases of the Hopf algebra **sNSym**, which will be shown to be dual to the fundamental quasisymmetric functions in superspace (Subsection 7.2). We give explicit formulas for the product of noncommutative ribbon Schur functions in superspace and write other bases of **sNSym** in terms of these functions.

Let α be a dotted composition. The *noncommutative ribbon Schur function in superspace* R_α is defined inductively as follows

$$R_\alpha = H_\alpha - \sum_{\alpha \prec \beta} R_\beta. \tag{14}$$

Notice that $R_\alpha = H_\alpha$ whenever α is maximal. Furthermore, if α is a usual composition, R_α is a classical noncommutative ribbon Schur function.

To characterize the product of ribbon functions in superspace, we introduce a new (partial) operation $\odot$ on dotted compositions. Given $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_s)$ be two dotted compositions such that α_r and β_1 are not dotted at once, we define $\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r \oplus \beta_1, \beta_2, \ldots, \beta_s)$. For instance,



In particular, due to Remark 3, we obtain the following identities relating $\odot$ and the usual concatenation:

$$(\alpha \odot \beta) \odot x = \alpha \odot (\beta \odot x), \quad (\alpha\beta) \odot x = \alpha(\beta \odot x), \quad (\alpha \odot \beta)x = \alpha \odot (\beta x), \qquad x \in \mathbb{N} \cup \dot{\mathbb{N}}.$$

The following is the main result of this section.

Theorem 26. *Let α, β be two dotted compositions. Then*

$$R_\alpha R_\beta = \begin{cases} R_{\alpha\beta} & \text{if } \operatorname{rg}(\alpha), \beta_1 \in \dot{\mathbb{N}}_0, \\ R_{\alpha\beta} + R_{\alpha\odot\beta} & \text{otherwise.} \end{cases}$$

To prove Theorem 26, first we need to show Lemma 27, Lemma 28 and Lemma 29.

Lemma 27. *Let α be a dotted composition, and let $x \in \mathbb{N} \cup \dot{\mathbb{N}}_0$.*

1. *If $x \in \mathbb{N}$, then* $(\alpha x)^{\uparrow} = \{\beta x \mid \alpha \preceq \beta\} \cup \{\beta \odot x \mid \alpha \preceq \beta\}$.

2. *If $x \in \dot{\mathbb{N}}$, then* $(\alpha x)^{\uparrow} = \{\beta x \mid \alpha \preceq \beta\}$ *whenever* $\mathrm{rg}(\alpha)$ *is dotted. Otherwise, we have* $(\alpha x)^{\uparrow} = \{\beta x \mid \alpha \preceq \beta\} \cup \{\beta \odot x \mid \alpha \preceq \beta, \mathrm{rg}(\beta) \in \mathbb{N}\}$.

Proof. By definition, $(\alpha x)^{\uparrow} = \{\gamma \mid \alpha x \preceq \gamma\}$. Note that, in all cases, the set on the right is contained in $(\alpha x)^{\uparrow}$. Conversely, for $\gamma \in (\alpha x)^{\uparrow}$, we distinguish three cases.

Let $x \in \mathbb{N}$. If $\mathrm{rg}(\gamma) = x$, the other components of γ are obtained as $\oplus$–sums of consecutive components of α. So, $\gamma = \beta x$ for some $\beta \succeq \alpha$. Now, if $\mathrm{rg}(\gamma) \neq x$, there is i such that $\mathrm{rg}(\gamma) = \alpha[i] \oplus x$, where $\alpha[i]$ is the $\oplus$–sum of the last $\ell(\alpha) - i - 1$ components of α. Note that at most one of the components that define $\alpha[i]$ is dotted. Consider γ' such that $\gamma = \gamma' \,\mathrm{rg}(\gamma)$ and define the dotted composition $\beta = \gamma'\alpha[i]$. Since γ' is obtained by $\oplus$–summing consecutive components of the dotted composition formed by the the first i components of α, we obtain that $\beta \succeq \alpha$ and $\gamma = \beta \odot x$.

If $x, \mathrm{rg}(\alpha) \in \dot{\mathbb{N}}$, they are not $\oplus$–summed when consider elements in $(\alpha x)^{\uparrow}$. Then $\gamma = \beta x$ for some dotted composition $\beta \succeq \alpha$.

Finally, consider $x \in \dot{\mathbb{N}}$ and $\mathrm{rg}(\alpha) \in \mathbb{N}$. As in the first case, if $\mathrm{rg}(\gamma) = x$, there is $\beta \succeq \alpha$ such that $\gamma = \beta x$. Now, if $\mathrm{rg}(\gamma) \neq x$, as above, $\gamma = \beta \odot x$, where $\beta = \gamma'\alpha[i]$ with $\alpha[i] \in \mathbb{N}$ because x is already dotted. ☐

Notice that, from the previous lemma, for $x, y \in \mathbb{N} \cup \dot{\mathbb{N}}$, we have

$$(x, y)^{\uparrow} = \begin{cases} \{(x, y), (x \oplus y)\} & \text{if } \{x, y\} \not\subset \dot{\mathbb{N}}_0, \\ \{(x, y)\} & \text{otherwise.} \end{cases} \tag{15}$$

Lemma 28. *Let α be a dotted composition, and let $x \in \mathbb{N} \cup \dot{\mathbb{N}}$. Then*

$$R_\alpha H_x = \begin{cases} R_{\alpha x} & \text{if } \mathrm{rg}(\alpha), x \in \dot{\mathbb{N}}_0, \\ R_{\alpha x} + R_{\alpha \odot x} & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on the length of α. If $\ell(\alpha) = 1$, we have $R_\alpha = H_\alpha$. So, the result follows because of (14) and (15). Now, if $\ell(\alpha) > 1$, assume the result is true for all dotted compositions of smaller length. Recall that if $\alpha \prec \beta$, then $\ell(\beta) < \ell(\alpha)$, and that, by definition, we have

$$R_\alpha = H_\alpha - \sum_{\alpha \prec \beta} R_\beta, \quad \text{then} \quad R_\alpha H_x = H_{\alpha x} - \sum_{\alpha \prec \beta} R_\beta H_x = \sum_{\alpha x \preceq \gamma} R_\gamma - \sum_{\alpha \prec \beta} R_\beta H_x.$$

We distinguish two cases. If $x \in \mathbb{N}$, Lemma 27 implies that $(\alpha x)^{\uparrow} = \{\beta x \mid \alpha \preceq \beta\} \cup \{\beta \odot x \mid \alpha \preceq \beta\}$. This together with the induction hypothesis implies that

$$R_\alpha H_x = \sum_{\alpha \preceq \beta} (R_{\beta x} + R_{\beta \odot x}) - \sum_{\alpha \prec \beta} (R_{\beta x} + R_{\beta \odot x}) = R_{\alpha x} + R_{\alpha \odot x}.$$

Let $x \in \dot{\mathbb{N}}_0$. If $\mathrm{rg}(\alpha) \in \dot{\mathbb{N}}_0$, then $\mathrm{rg}(\beta) \in \dot{\mathbb{N}}_0$ for all $\beta \succeq \alpha$, and, by Lemma 27, we have $(\alpha x)^{\uparrow} = \{\beta x \mid \alpha \preceq \beta\}$. By the induction hypothesis, we obtain

$$R_\alpha H_x = \sum_{\alpha \preceq \beta} R_{\beta x} - \sum_{\alpha \prec \beta} R_{\beta x} = R_{\alpha x}.$$

Now, if $\mathrm{rg}(\alpha) \in \mathbb{N}$, since $\{\beta \succeq \alpha \mid \mathrm{rg}(\beta) \in \mathbb{N}\}$ is nonempty, Lemma 27 implies that

$$H_{\alpha x} = \sum_{\alpha \preceq \beta} R_{\beta x} + \sum_{\alpha \preceq \beta,\, \mathrm{rg}(\beta) \in \mathbb{N}} R_{\beta \odot x}.$$

On the other hand, by applying the induction hypothesis for $\beta \succ \alpha$, we have $R_\beta H_x = R_{\beta x}$ if $\mathrm{rg}(\beta) \in \dot{\mathbb{N}}_0$, and $R_\beta H_x = R_{\beta x} + R_{\beta \odot x}$ if $\mathrm{rg}(\beta) \in \mathbb{N}$. Hence,

$$R_\alpha H_x = \sum_{\alpha \preceq \beta} R_{\beta x} + \sum_{\alpha \preceq \beta,\, \mathrm{rg}(\beta) \in \mathbb{N}} R_{\beta \odot x} - \sum_{\alpha \prec \beta} R_{\beta x} - \sum_{\alpha \prec \beta,\, \mathrm{rg}(\beta) \in \mathbb{N}} R_{\beta \odot x} = R_{\alpha x} + R_{\alpha \odot x}.$$

This concludes the proof. □

Lemma 29. *Let α be a dotted composition, and let $\beta = (x, y)$ for some $x, y \in \mathbb{N} \cup \dot{\mathbb{N}}_0$. Then*

$$R_\alpha R_\beta = \begin{cases} R_{\alpha\beta} & \text{if } \mathrm{rg}(\alpha), x \in \mathbb{N}_0, \\ R_{\alpha\beta} + R_{\alpha \odot \beta} & \text{otherwise.} \end{cases}$$

Proof. Notice that, by Lemma 28, we have

$$R_\beta = \begin{cases} R_x R_y & \text{if } x, y \in \dot{\mathbb{N}}_0, \\ R_x R_y - R_{x \odot y} & \text{otherwise.} \end{cases}$$

First, we assume that $\mathrm{rg}(\alpha), x \in \dot{\mathbb{N}}_0$. If $y \in \dot{\mathbb{N}}_0$, then

$$R_\alpha R_\beta = R_\alpha (R_x R_y) = (R_\alpha R_x) R_y = R_{\alpha x} R_y = R_{\alpha\beta}.$$

On the other hand, if $y \in \mathbb{N}$, then

$$\begin{aligned} R_\alpha R_\beta &= R_\alpha (R_x R_y - R_{x \odot y}) \\ &= (R_\alpha R_x) R_y - R_\alpha R_{x \odot y} \\ &= R_{\alpha x} R_y - R_{\alpha (x \odot y)} \\ &= R_{\alpha x y} + R_{(\alpha x) \odot y} - R_{\alpha (x \odot y)} \\ &= R_{\alpha\beta}, \end{aligned}$$

because $x \odot y \in \dot{\mathbb{N}}_0$ and $(\alpha x) \odot y = \alpha(x \odot y)$.

Now, we assume that $\{\mathrm{rg}(\alpha), x\} \not\subset \dot{\mathbb{N}}_0$. We will distinguish several cases.

In general, if $x \in \mathbb{N}$, we have

$$\begin{aligned} R_\alpha R_\beta &= R_\alpha (R_x R_y - R_{x \odot y}) \\ &= (R_\alpha R_x) R_y - R_\alpha R_{x \odot y} \\ &= (R_{\alpha x} + R_{\alpha \odot x}) R_y - R_\alpha R_{x \odot y} \\ &= R_{\alpha x} R_y + R_{\alpha \odot x} R_y - R_\alpha R_{x \odot y} \\ &= R_{\alpha\beta} + R_{(\alpha x) \odot y} + R_{\alpha \odot x} R_y - R_\alpha R_{x \odot y}. \end{aligned}$$

This equation depends on the set the elements $\mathrm{rg}(\alpha)$ and y belong to. If $\mathrm{rg}(\alpha) \in \mathbb{N}$, then $\mathrm{rg}(\alpha \odot x) \in \mathbb{N}$, hence

$$R_\alpha R_\beta = R_{\alpha\beta} + R_{(\alpha x) \odot y} + R_{(\alpha \odot x) y} + R_{(\alpha \odot x) \odot y} - R_{\alpha (x \odot y)} - R_{\alpha \odot (x \odot y)} = R_{\alpha\beta} + R_{\alpha \odot \beta}$$

because $(\alpha x) \odot y = \alpha(x \odot y)$ and $\alpha \odot (x \odot y) = (\alpha \odot x) \odot y$. Now, consider $\mathrm{rg}(\alpha) \in \dot{\mathbb{N}}_0$. Notice that $\mathrm{rg}(\alpha \odot x)$ belongs to $\dot{\mathbb{N}}_0$ as well. If $y \in \mathbb{N}$, then

$$R_\alpha R_\beta = R_{\alpha\beta} + R_{(\alpha x)\odot y} + R_{(\alpha\odot x)y} + R_{(\alpha\odot x)\odot y} - R_{\alpha(x\odot y)} - R_{\alpha\odot(x\odot y)} = R_{\alpha\beta} + R_{\alpha\odot\beta}.$$

On the other hand, if $y \in \dot{\mathbb{N}}_0$, then

$$R_\alpha R_\beta = R_{\alpha\beta} + R_{(\alpha x)\odot y} + R_{(\alpha\odot x)y} - R_{\alpha(x\odot y)} = R_{\alpha\beta} + R_{\alpha\odot\beta}.$$

Finally, consider $x \in \dot{\mathbb{N}}_0$. As $\{\mathrm{rg}(\alpha), x\} \not\subset \dot{\mathbb{N}}_0$, then $\mathrm{rg}(\alpha) \in \mathbb{N}$. If $y \in \dot{\mathbb{N}}_0$, then

$$R_\alpha R_\beta = (R_\alpha R_x)R_y = (R_{\alpha x} + R_{\alpha\odot x})R_y = R_{\alpha\beta} + R_{\alpha\odot\beta},$$

because $(\alpha \odot x)y = \alpha \odot (xy)$. If $y \in \mathbb{N}$, then

$$\begin{aligned} R_\alpha R_\beta &= (R_\alpha R_x)R_y - R_\alpha R_{x\odot y} \\ &= (R_{\alpha x} + R_{\alpha\odot x})R_y - R_{\alpha(x\odot y)} - R_{\alpha\odot(x\odot y)} \\ &= R_{\alpha\beta} + R_{(\alpha x)\odot y} + R_{(\alpha\odot x)y} + R_{(\alpha\odot x)\odot y} - R_{\alpha(x\odot y)} - R_{\alpha\odot(x\odot y)} \\ &= R_{\alpha\beta} + R_{\alpha\odot\beta}. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 26. Due to Lemma 28 and Lemma 29, the result is true when $\ell(\beta) \leqslant 2$. So, we will proceed by induction on the length of β. Assume that $\ell(\beta) \geqslant 3$, that is, $\beta = \gamma x$ for some dotted composition γ satisfying $\ell(\gamma) = \ell(\beta) - 1$, where $x = \mathrm{rg}(\beta)$. In particular, we have $\beta_1 = \gamma_1$ and $\mathrm{rg}(\gamma) \neq \beta_1$. Moreover, by Lemma 28, we obtain

$$R_\beta = \begin{cases} R_\gamma R_x & \text{if } \mathrm{rg}(\gamma), x \in \dot{\mathbb{N}}_0, \\ R_\gamma R_x - R_{\gamma\odot x} & \text{otherwise.} \end{cases}$$

We will distinguish some cases.

First, assume that $\mathrm{rg}(\alpha), \beta_1 \in \dot{\mathbb{N}}_0$. If $\mathrm{rg}(\gamma), x \in \dot{\mathbb{N}}_0$, by the induction hypothesis, we obtain

$$R_\alpha R_\beta = R_\alpha(R_\gamma R_x) = (R_\alpha R_\gamma)R_x = R_{\alpha\gamma} R_x = R_{\alpha\beta}.$$

On the other hand, if $\{\mathrm{rg}(\gamma), x\} \not\subset \dot{\mathbb{N}}_0$, the induction hypothesis implies that

$$\begin{aligned} R_\alpha R_\beta &= R_\alpha(R_\gamma R_x - R_{\gamma\odot x}) \\ &= (R_\alpha R_\gamma)R_x - R_\alpha R_{\gamma\odot x} \\ &= R_{\alpha\gamma} R_x - R_{\alpha(\gamma\odot x)} \\ &= R_{\alpha\beta} + R_{(\alpha\gamma)\odot x} - R_{\alpha(\gamma\odot x)} \\ &= R_{\alpha\beta} \end{aligned}$$

because $(\gamma \odot x)_1 \in \dot{\mathbb{N}}$ and $(\alpha\gamma) \odot x = \alpha(\gamma \odot x)$.

Now, assume that $\{\mathrm{rg}(\alpha), \beta_1\} \not\subset \dot{\mathbb{N}}_0$. If $\mathrm{rg}(\gamma), x \in \dot{\mathbb{N}}_0$, by the induction hypothesis, we obtain

$$\begin{aligned} R_\alpha R_\beta &= (R_\alpha R_\gamma)R_x \\ &= (R_{\alpha\gamma} + R_{\alpha\odot\gamma})R_x \\ &= R_{\alpha\gamma}R_x + R_{(\alpha\odot\gamma)}R_x \\ &= R_{\alpha\gamma x} + R_{(\alpha\odot\gamma)x} \\ &= R_{\alpha\beta} + R_{\alpha\odot\beta} \end{aligned}$$

because $\mathrm{rg}(\alpha\gamma) = \mathrm{rg}(\gamma), \mathrm{rg}(\alpha \odot \gamma) \in \dot{\mathbb{N}}_0$ and $(\alpha \odot \gamma)x = \alpha \odot (\gamma x) = \alpha \odot \beta$. On the other hand, if $\{\mathrm{rg}(\gamma), x\} \not\subset \dot{\mathbb{N}}_0$, then

$$R_\alpha R_\beta = R_\alpha(R_\gamma R_x - R_{\gamma\odot x}) = (R_\alpha R_\gamma)R_x - R_\alpha R_{\gamma\odot x}.$$

Since $(\gamma \odot x)_1 = \beta_1 \neq x$ and $\mathrm{rg}(\alpha \odot \gamma) = \mathrm{rg}(\gamma)$, the induction hypothesis implies that

$$\begin{aligned} R_\alpha R_\beta &= (R_{\alpha\gamma} + R_{\alpha\odot\gamma})R_x - R_{\alpha(\gamma\odot x)} - R_{\alpha\odot(\gamma\odot x)} \\ &= R_{\alpha\gamma}R_x + R_{\alpha\odot\gamma}R_x - R_{\alpha(\gamma\odot x)} - R_{\alpha\odot(\gamma\odot x)} \\ &= R_{\alpha\beta} + R_{(\alpha\gamma)\odot x} + R_{(\alpha\odot\gamma)x} + R_{(\alpha\odot\gamma)\odot x} - R_{\alpha(\gamma\odot x)} - R_{\alpha\odot(\gamma\odot x)} \\ &= R_{\alpha\beta} + R_{\alpha\odot\beta} \end{aligned}$$

because $\alpha \odot (\gamma \odot x) = (\alpha \odot \gamma)x = \alpha \odot (\gamma x) = \alpha \odot \beta$ and $\alpha(\gamma \odot x) = (\alpha\gamma) \odot x$. This concludes the proof. □

Now, by using the previous results, we write noncommutative ribbon Schur function in superspace in terms of the noncommutative homogeneous functions in superspace.

Proposition 30. *Let α be a dotted composition. Then*

$$R_\alpha = \sum_{\alpha\preceq\beta} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta.$$

Proof. We proceed by induction on the length of α. If $\ell(\alpha) = 1$ the result is obvious. If $\ell(\alpha) > 1$, assume that the result is true for dotted compositions of smaller length. Let α' be such that $\alpha = \alpha' x$ with $x = \mathrm{rg}(\alpha)$. We distinguish two cases.

If $\mathrm{rg}(\alpha'), x \in \dot{\mathbb{N}}_0$, Lemma 28 implies that $R_\alpha = R_{\alpha'}H_x$. So, by the induction hypothesis

$$R_\alpha = \sum_{\alpha'\preceq\gamma} (-1)^{\ell(\alpha')-\ell(\gamma)} H_\gamma H_x = \sum_{\alpha'\preceq\gamma} (-1)^{\ell(\alpha')+1-\ell(\gamma x)} H_{\gamma x}.$$

Since $\mathrm{rg}(\alpha'), x \in \dot{\mathbb{N}}_0$, then $\alpha^\uparrow = \{\beta \mid \alpha \preceq \beta\} = \{\gamma x \mid \alpha' \preceq \gamma\} = (\alpha' x)^\uparrow$, hence

$$R_\alpha = \sum_{\alpha\preceq\beta} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta.$$

Now, if $\{\mathrm{rg}(\alpha'), x\} \not\subset \dot{\mathbb{N}}_0$, Lemma 28 implies that $R_\alpha = R_{\alpha'} H_x - R_{\alpha' \odot x}$. So, by the induction hypothesis, we have

$$\begin{aligned} R_\alpha &= \sum_{\alpha' \preceq \gamma} (-1)^{\ell(\alpha') - \ell(\gamma)} H_\gamma H_x - \sum_{\alpha' \odot x \preceq \gamma} (-1)^{\ell(\alpha' \odot x) - \ell(\gamma)} H_\gamma \\ &= \sum_{\alpha' \preceq \gamma} (-1)^{\ell(\alpha') + 1 - \ell(\gamma x)} H_{\gamma x} + \sum_{\alpha' \odot x \preceq \gamma} (-1)^{\ell(\alpha' \odot x) + 1 - \ell(\gamma)} H_\gamma \\ &= \sum_{\alpha' \preceq \gamma} (-1)^{\ell(\alpha) - \ell(\gamma x)} H_{\gamma x} + \sum_{\alpha' \odot x \preceq \gamma} (-1)^{\ell(\alpha) - \ell(\gamma)} H_\gamma \\ &= \sum_{\alpha \preceq \beta} (-1)^{\ell(\alpha) - \ell(\beta)} H_\beta. \end{aligned}$$

□

In what follows of this section, we give explicit formulas to write other noncommutative functions in superspace, defined in previous sections, in terms of the noncommutative ribbon Schur functions in superspace.

Proposition 31. *$\Psi_0 = R_{\dot{0}}$, and for every $n \geqslant 1$, we have $\Psi_n = (-1)^n R_{(1^n, \dot{0})}$.*

Proof. We proceed by induction on n. Note that $\Psi_0 = \tilde{H}_0 = R_{\dot{0}}$, and that

$$\Psi_1 = \tilde{H}_1 - H_1 \Psi_0 = R_{\dot{1}} - R_1 R_{\dot{0}} = R_{\dot{1}} - R_{(1,\dot{0})} - R_{\dot{1}} = -R_{(1,\dot{0})}.$$

Assume the result is true for all $k < n$. So, by definition and the induction hypothesis, we obtain the following

$$\Psi_n = \tilde{H}_n - \sum_{k=0}^{n-1} H_{n-k} \Psi_k = R_{\dot{n}} - R_n R_{\dot{0}} + R_{n-1} R_{(1,\dot{0})} - \sum_{k=2}^{n-1} R_{n-k} (-1)^k R_{(1^k,\dot{0})}.$$

Now, Theorem 26 implies that $R_{n-k} R_{(1^k,\dot{0})} = R_{(n-k,1^k,\dot{0})} + R_{(n-(k-1),1^{k-1},\dot{0})}$, hence

$$\begin{aligned} \Psi_n &= R_{(n-1,1,\dot{0})} - \sum_{k=2}^{n-1} (-1)^k R_{(n-k,1^k,\dot{0})} - \sum_{k=2}^{n-1} (-1)^k R_{(n-(k-1),1^{k-1},\dot{0})} \\ &= R_{(n-1,1,\dot{0})} - \sum_{k=2}^{n-1} (-1)^k R_{(n-k,1^k,\dot{0})} + \sum_{k=2}^{n-1} (-1)^{k-1} R_{(n-(k-1),1^{k-1},\dot{0})} \\ &= -\sum_{k=1}^{n-2} (-1)^k R_{(n-k,1^k,\dot{0})} + (-1)^n R_{(1^n,\dot{0})} + \sum_{k=1}^{n-2} (-1)^k R_{(n-k,1^k,\dot{0})}. \end{aligned}$$

Therefore $\Psi_n = (-1)^n R_{(1^n,\dot{0})}$. □

Note that 1^n is the minimal element in the lattice of compositions of n with the usual order. This implies that

$$\begin{aligned} R_{(1^n)} &= \sum_{1^n \preceq \beta} (-1)^{n-\ell(\beta)} H_\beta \\ &= (-1)^n \sum_{1^n \preceq \beta} (-1)^{\ell(\beta)} H_\beta \\ &= (-1)^n \sum_{\beta \preceq (n)} (-1)^{\ell(\beta)} H_\beta \\ &= (-1)^n S(H_n) \\ &= S_n. \end{aligned} \tag{16}$$

In the following proposition, we write $\tilde{S}_n$ in terms of noncommutative ribbon Schur functions in superspace.

Proposition 32. *We have* $\tilde{S}_n = \sum_{k=0}^{n} R_{(1^k,\dot{0},1^{n-k})} + \sum_{k=1}^{n} R_{(1^{k-1},\dot{1},1^{n-k})}$, *for all* $n \geqslant 1$.

Proof. By Proposition 18, Proposition 31 and (16), we have

$$\tilde{S}_n = (-1)^n \Psi_n + \sum_{k=0}^{n-1} (-1)^k \Psi_k S_{n-k} = R_{(1^n,\dot{0})} + \sum_{k=0}^{n-1} R_{(1^k,\dot{0})} R_{(1^{n-k})}.$$

Now, by using Theorem 26, we obtain the result

$$\begin{aligned} \tilde{S}_n &= R_{(1^n,\dot{0})} + \sum_{k=0}^{n-1} \left[R_{(1^k,\dot{0},1^{n-k})} + R_{(1^k,\dot{1},1^{n-k-1})} \right] \\ &= \sum_{k=0}^{n} R_{(1^k,\dot{0},1^{n-k})} + \sum_{k=1}^{n} R_{(1^{k-1},\dot{1},1^{n-k})}. \end{aligned}$$ □

Recall, for the classical case [20, Corollary 3.14], we have

$$P_n = \sum_{k=0}^{n-1} (-1)^k R_{(1^k,n-k)}, \qquad n \geqslant 1. \tag{17}$$

To finish, we give a explicit formula of $\tilde{P}_n$ in terms of noncommutative ribbon Schur functions in superspace.

Proposition 33. *For every* $n \geqslant 1$, *we have*

$$\begin{aligned} (n+1)\tilde{P}_n &= -R_{(\dot{0},n)} + (-1)^n R_{(1^n,\dot{0})} \\ &\quad + \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{i=1}^{k} R_{(1^{i-1},\dot{1},1^{k-i},n-k)} + \sum_{i=0}^{k} R_{(1^i,\dot{0},1^{k-i},n-k)} \right). \end{aligned}$$

Proof. The result follows by applying inductively Proposition 20, (17) and Theorem 26. □

6.1 Ribbon Schur functions of the second kind

As it was shown in [16, Subsection 5.3], there is another partial order on dotted compositions, obtained by restricting $\preceq$. Specifically, given dotted compositions α and β, we say that β *r–covers* α, if β is obtained by summing two consecutive components of α that are not dotted. The partial order obtained by the closure of this relation is denoted by $\preceq_r$ and gives another poset structure to the collection of dotted compositions. Note that $\alpha \preceq_r \beta$ implies $\alpha \preceq \beta$. Also, with this order, a dotted composition whose components are all dotted is comparable only with itself.

For a dotted composition α, we denote by $\alpha^{\uparrow_r}$ the *upper closure* of α with respect to $\preceq_r$, that is, $\alpha^{\uparrow_r}$ is the upset $\{\beta \mid \alpha \preceq_r \beta\}$. Note that $\alpha^{\uparrow_r}$ has a unique maximal element obtained by summing all the undotted consecutive components of α. For instance, if $\alpha = (\dot{0}, 1, 2, \dot{1}, 3, 5, 1, \dot{2})$, the maximal of $\alpha^{\uparrow_r}$ is $(\dot{0}, 3, \dot{1}, 9, \dot{2})$.

By using this partial order, we define, inductively, a second kind of ribbon Schur functions

$$\hat{R}_\alpha = H_\alpha - \sum_{\alpha \prec_r \beta} \hat{R}_\beta.$$

For instance, $\hat{R}_{(\dot{0},1,1,1,\dot{0})} = H_{(\dot{0},1,1,1,\dot{0})} - H_{(\dot{0},1,1,1,\dot{0})} - H_{(\dot{0},2,1,\dot{0})} + H_{(\dot{0},3,\dot{0})}$.

Note that $R_\alpha = \hat{R}_\alpha$ whenever α is a usual composition. Furthermore, if α has no consecutive undotted components, then $\hat{R}_\alpha = H_\alpha$. For instance, $\hat{R}_{(\dot{1},3,\dot{2},5)} = H_{(\dot{1},3,\dot{2},5)}$.

By adapting Theorem 26, for dotted compositions α, β, we obtain the following result:

$$\hat{R}_\alpha \hat{R}_\beta = \begin{cases} \hat{R}_{\alpha\beta} + \hat{R}_{\alpha\odot\beta} & \text{if } \mathrm{rg}(\alpha), \beta_1 \in \mathbb{N}, \\ \hat{R}_{\alpha\beta} & \text{otherwise.} \end{cases} \tag{18}$$

For instance, $\hat{R}_{(2,1)} \cdot \hat{R}_{(1,1,\dot{3})} = \hat{R}_{(2,1,1,1,\dot{3})} + \hat{R}_{(2,2,1,\dot{3})}$ and $\hat{R}_{(1,1,\dot{3})} \cdot \hat{R}_{(2,1)} = \hat{R}_{(1,1,\dot{3},2,1)}$.

As a consequence of (18), we have:

$$\hat{R}_\alpha = \sum_{\alpha \preceq_r \beta} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta.$$

The following follows directly by induction:

$$\Psi_0 = \hat{R}_0, \quad \text{and} \quad \Psi_n = \sum_{k=0}^{n} (-1)^{n-k} \hat{R}_{(1^{n-k},\, \dot{k})} \quad \text{for all} \quad n \geqslant 1,$$

$$\tilde{S}_n = \sum_{m=0}^{n} (-1)^m \hat{R}_{(1^{n-m},\, \dot{m})} + \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k \hat{R}_{(1^{m-k},\, \dot{k}, 1^{n-m})} \quad \text{for all} \quad n \geqslant 1.$$

7 Related structures

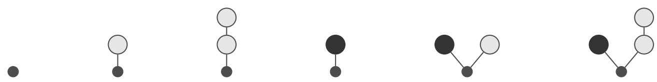
7.1 sNSym as a Hopf algebra of trees

In this subsection, we give a realization of **sNSym** as a Hopf algebra of planar rooted trees.

Hopf algebras generated by trees have been widely studied and there is an extensive literature on it. See for example [25, 18, 1]. The *Connes–Kreimer Hopf algebra*, generated by rooted trees, was introduced in [6] and its noncommutative version H_{PR} for planar rooted trees was given simultaneously in [17, 25]. The coalgebra structure of these Hopf algebras can be described in terms of cuts of trees.

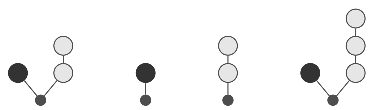
It was shown in [18, 24] that **NSym** can be realized as a Hopf subalgebra of H_{PR}. Here, we extend this description for **sNSym**, by identifying its generators with certain type of planar rooted trees. Cf. [14].

For a planar rooted tree t, we define the *degree* of it, denoted by $\deg(t)$, as the number of its non-root nodes. Recall that a *ladder tree* is a planar rooted tree with only one branch. For $n \geqslant 0$, we will denote by t_n the unique ladder tree of degree n, and we will identify H_n with this tree. The generator $\tilde{H}_n$ is identified with the planar rooted tree $t_{\dot{n}}$ of degree $n+1$ obtained by gluing the roots of a coloured ladder tree of degree one with t_n. The coloured node of $t_{\dot{n}}$ represents the fermionic degree of $\tilde{H}_n$. For instance, for $n \leqslant 2$, we have:



$1 \qquad H_1 \qquad H_2 \qquad \tilde{H}_0 \qquad \tilde{H}_1 \qquad \tilde{H}_2$

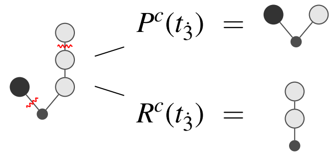
Given a dotted composition α with $k = \ell(\alpha)$, we identify $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_k}$ with the forest $t_\alpha = t_{\alpha_1} \cdots t_{\alpha_k}$, and 1 is identified with t_0. For instance, for $\alpha = (\dot{2}, \dot{0}, 2, \dot{3})$, we have:

$t_\alpha \quad =$ 

Hence, the product is obtained by concatenating forests with the assumption that $t_\alpha t_0 = t_0 t_\alpha = t_\alpha$ for all dotted composition α.

To the describe the coproduct, we consider *admissible cuts* on trees. A *cut* on a planar rooted tree t is any subset of edges of it. A cut is called *admissible* if each branch of the tree contains at most one edge of it. The set of all admissible cuts of a tree t is denoted by $\mathrm{adm}(t)$. Note that the empty cut of t is admissible.

Given a planar rooted tree t and an admissible cut c of it, we call *components* of t respect to c, the subtrees of t obtained by removing the edges of c from t. We denote by $R^c(t)$ the component containing the root of t. On the other hand, by adapting the classical case, we denote by $P^c(t)$ the planar tree obtained by removing the non-root nodes of $R^c(t)$ from t and then contracting its edges. For instance,



It is easy to see that via the identification $H_m \mapsto t_m$, with $m \in \mathbb{N} \cup \dot{\mathbb{N}}_0$, the coproduct of **sNSym** can be described by means of the following formula

$$\Delta(t_m) = \sum_{c \in \mathrm{adm}(t)} P^c(t) \otimes R^c(t). \tag{19}$$

The notions described above can be extended to the forests $t_\alpha = t_{\alpha_1} \cdots t_{\alpha_k}$, with α a dotted composition of length k. Indeed, an *admissible cut* c of t_α is a tuple $c = (c_1, \ldots, c_k)$, where each c_i is an admissible cut of t_{α_i}, possibly empty. Thus, $P^c(t_\alpha)$ and $R^c(t_\alpha)$ are given by the following forests:

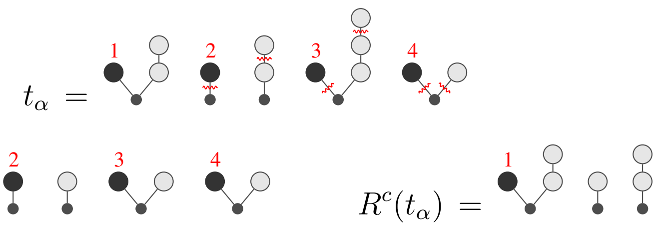
$$P^c(t_\alpha) = P^{c_1}(t_{\alpha_1}) \cdots P^{c_k}(t_{\alpha_k}), \qquad R^c(t_\alpha) = R^{c_1}(t_{\alpha_1}) \cdots R^{c_k}(t_{\alpha_k}).$$

We will see in Proposition 34 that the summands of $\Delta(t_\alpha)$ can be described by $P^c(t_\alpha) \otimes R^c(t_\alpha)$ up a sign induced by the fermionic components of α.

Let α be a dotted composition with $j = \mathrm{df}(\alpha)$, and let c be a cut of t_α. If $j \geqslant 1$, we denote by σ_c the unique permutation in $\mathfrak{S}_j$ given by the reordering of the coloured nodes of t_α in $P^c(t_\alpha) \otimes R^c(t_\alpha)$. We define the sign of c as follows:

$$\mathrm{sgn}(c) = \begin{cases} 1 & \text{if } j = 0, \\ \mathrm{sgn}(\sigma_c) & \text{otherwise.} \end{cases}$$

For instance, if $\alpha = (\dot{2}, \dot{0}, 2, \dot{3}, \dot{1})$, we have:



where $\sigma_c = (2, 3, 4, 1)$ and $\mathrm{sgn}(c) = \mathrm{sgn}(\sigma_c) = (-1)^3 = -1$.

Proposition 34. *For a dotted composition α, we have*

$$\Delta(t_\alpha) = \sum_{c \in \mathrm{adm}(t_\alpha)} \mathrm{sgn}(c) P^c(t_\alpha) \otimes R^c(t_\alpha).$$

Proof. Let $j = \mathrm{df}(\alpha)$. We proceed by induction on $k := \ell(\alpha)$. If $k = 1$, the result follows from (19). If $k > 1$, assume the claim is true for smaller values. We will show that the summands on both sides of the equation coincide. Notice that $\Delta(t_\alpha) = \Delta(t_{\alpha'})\Delta(t_{\alpha_k})$, where $\alpha' = (\alpha_1, \ldots, \alpha_{k-1})$. Now, by the induction hypothesis, the summands of $\Delta(t_{\alpha'})$ can be written as $\mathrm{sgn}(c')P^{c'}(t_{\alpha'}) \otimes R^{c'}(t_{\alpha'})$, where $c' = (c_1, \ldots, c_{k-1})$ is an admissible cut of $t_{\alpha'}$. The summands of $\Delta(t_{\alpha_k})$ can be written as $P^{c_k}(t_{\alpha_k}) \otimes R^{c_k}(t_{\alpha_k})$, where c_k is an admissible cut of t_{α_k}. Hence, a summand of $\Delta(t_\alpha)$ has the following form:

$$\begin{aligned} &[\mathrm{sgn}(c')P^{c'}(t_{\alpha'}) \otimes R^{c'}(t_{\alpha'})] \cdot [P^{c_k}(t_{\alpha_k}) \otimes R^{c_k}(t_{\alpha_k})] \\ &\qquad = \mathrm{sgn}(c')(-1)^{ab} P^{c'}(t_{\alpha'})P^{c_k}(t_{\alpha_k}) \otimes R^{c'}(t_{\alpha'})R^{c_k}(t_{\alpha_k}), \end{aligned}$$

where $a = \mathrm{df}(R^{c'}(t_{\alpha'}))$ and $b = \mathrm{df}(P^{c_k}(t_{\alpha_k}))$. Observe that $c = (c_1, \ldots, c_{k-1}, c_k)$ is an admissible cut of t_α, satisfying

$$P^c(t_\alpha) \otimes R^c(t_\alpha) = P^{c'}(t_{\alpha'})P^{c_k}(t_{\alpha_k}) \otimes R^{c'}(t_{\alpha'})R^{c_k}(t_{\alpha_k}). \tag{20}$$

Now, we will show that $\mathrm{sgn}(c) = \mathrm{sgn}(c')(-1)^{ab}$. If $j = 0$ it is obvious. For $j \geqslant 1$, we distinguish three cases.

If $\alpha_k \in \mathbb{N}$, then $\sigma_c = \sigma_{c'}$ and $b = 0$.

If $\alpha_k \in \dot{\mathbb{N}}_0$ and $b = 0$, then $\mathrm{sgn}(c')(-1)^{ab} = \mathrm{sgn}(\sigma_{c'})$. Since the coloured node of $\Delta(t_{\alpha_k})$ belongs to $R^{c_k}(t_{\alpha_k})$, then $\sigma_c = (\sigma_{c'}(1), \ldots, \sigma_{c'}(j-1), j)$. Thus, the number of inversions of $\sigma_{c'}$ and σ_c coincide, and so $\mathrm{sgn}(\sigma_{c'}) = \mathrm{sgn}(c)$.

If $\alpha_k \in \dot{\mathbb{N}}_0$ and $b = 1$, then $\mathrm{sgn}(c')(-1)^{ab} = \mathrm{sgn}(\sigma_{c'})(-1)^a$. Since $\sigma_{c'}$ is determined by the order of coloured nodes of $P^{c'}(t_{\alpha'}) \otimes R^{c'}(t_{\alpha'})$, there is $i \in [j-1]$ such that the nodes positioned in $\sigma_{c'}(1), \ldots, \sigma_{c'}(i)$ belong to $P^{c'}(t_{\alpha'})$ and the nodes positioned in $\sigma_{c'}(i+1), \ldots, \sigma_{c'}(j-1)$ belong to $R^{c'}(t_{\alpha'})$. This together with (20) imply that $a = j-1-i$ and $\sigma_c = (\sigma_{c'}(1), \ldots, \sigma_{c'}(i), j, \sigma_{c'}(i+1), \ldots, \sigma_{c'}(j-1))$. Thus, the number of inversions of σ_c is the one of $\sigma_{c'}$ plus $j-1-i$. So, $\mathrm{sgn}(\sigma_c) = \mathrm{sgn}(\sigma_{c'})(-1)^{j-1-i} = \mathrm{sgn}(\sigma_{c'})(-1)^a$.

Similarly, we prove that for each admissible cut c of t_α, $\mathrm{sgn}(c)P^c(t_\alpha) \otimes R^c(t_\alpha)$ can be represented as a summand of $\Delta(t_\alpha)$. This concludes the proof. □

7.2 Fundamental quasisymmetric functions in superspace

The dual structure of **sNSym** is the Hopf algebra of quasisymmetric functions in superspace **sQSym**. This relation is determined by a pairing $\langle \cdot, \cdot \rangle : \mathbf{sQSym} \otimes \mathbf{sNSym} \to \mathbb{Q}$, which satisfies $\langle M_\alpha, H_\beta \rangle = \delta_{\alpha\beta}$.

Now we introduce the set of *fundamental quasisymmetric functions in superspace* $\{L_\alpha\}$ as the basis of **sQSym** obtained by dualizing the noncommutative ribbon Schur functions in superspace $\{R_\beta\}$ defined in Section 6.

Definition 35. The *fundamental quasisymmetric functions in superspace* $\{L_\alpha\}$ are defined by

$$\langle L_\alpha, R_\beta \rangle = \delta_{\alpha\beta}.$$

In the following theorem, we show that this basis coincides with the set of fundamental quasisymmetric functions introduced in [16], with respect to the partial order $\preceq$ defined in Section 3. Moreover, we write M_α in terms of L_β and provide a formula for the coproduct of fundamental quasisymmetric functions in superspace.

Theorem 36. *Let α be a dotted composition. Then*

$$L_\alpha = \sum_{\beta \preceq \alpha} M_\beta, \qquad M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} L_\beta, \qquad \Delta(L_\alpha) = \sum_{\beta\gamma=\alpha \text{ or } \beta\odot\gamma=\beta} L_\beta \otimes L_\gamma.$$

Proof. As $\{M_\gamma\}$ forms a basis of **sQSym**, we can write L_α as a linear combination $L_\alpha = \sum_\gamma c_\gamma M_\gamma$.

For a dotted composition β, we have $\langle L_\alpha, H_\beta\rangle = \sum_\gamma c_\gamma \langle M_\gamma, H_\beta\rangle$.

Since $\langle M_\gamma, H_\beta\rangle = 0$ if $\gamma \neq \beta$ and $\langle M_\gamma, H_\beta\rangle = 0$ if $\gamma = \beta$, then $c_\beta = \langle L_\alpha, H_\beta\rangle$. This implies that

$$L_\alpha = \sum_\beta \langle L_\alpha, H_\beta\rangle M_\beta.$$

Now, as $H_\beta = \sum_{\beta \preceq \gamma} R_\gamma$, then

$$\langle L_\alpha, H_\beta\rangle = \sum_{\beta\preceq\gamma} \langle L_\alpha, R_\gamma\rangle = \begin{cases} 0 & \text{if } \alpha \prec \beta, \\ 1 & \text{if } \alpha \succeq \beta. \end{cases}$$

Thus, $L_\alpha = \sum_{\beta\preceq\alpha} M_\beta$.

Similarly, as $\{L_\alpha\}$ is dual to the basis $\{R_\alpha\}$, then it is also a basis of **sQSym**. Thus,

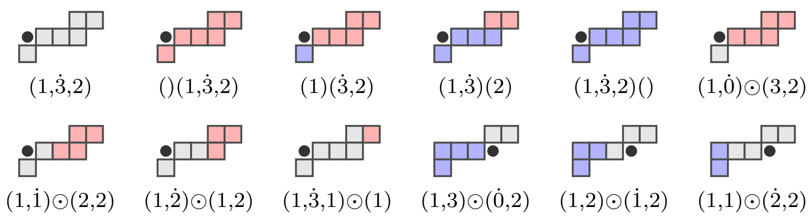
$$\begin{aligned} M_\alpha &= \textstyle\sum_\beta \langle M_\alpha, R_\beta\rangle L_\beta \\ &= \sum_\beta \langle M_\alpha, \sum_{\beta\preceq\gamma} (-1)^{\ell(\beta)-\ell(\gamma)} H_\gamma\rangle L_\beta \\ &= \sum_\beta \left[\sum_{\beta\preceq\gamma} \langle M_\alpha, (-1)^{\ell(\beta)-\ell(\gamma)} H_\gamma\rangle\right] L_\beta. \end{aligned}$$

Now, $\sum_{\beta\preceq\gamma} \langle M_\alpha, (-1)^{\ell(\beta)-\ell(\gamma)} H_\gamma\rangle$ is 0 if $\alpha \prec \beta$ and it is $(-1)^{\ell(\beta)-\ell(\alpha)}$ otherwise. Hence,

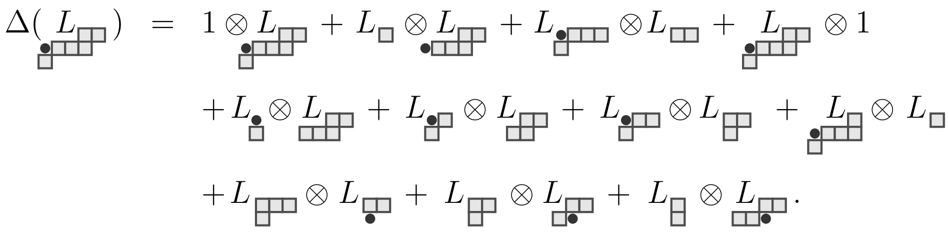
$$M_\alpha = \sum_{\beta\preceq\alpha} (-1)^{\ell(\beta)-\ell(\alpha)} L_\beta.$$

The last assertion follows from Theorem 26 and the duality. □

For a dotted composition α, the coproduct of L_α can be obtained by considering all possible horizontal ($\alpha = \beta\gamma$) and vertical ($\alpha = \beta \odot \gamma$) splittings of the ribbon diagram of α. For instance, for $\alpha = (1, \dot{3}, 2)$, we have



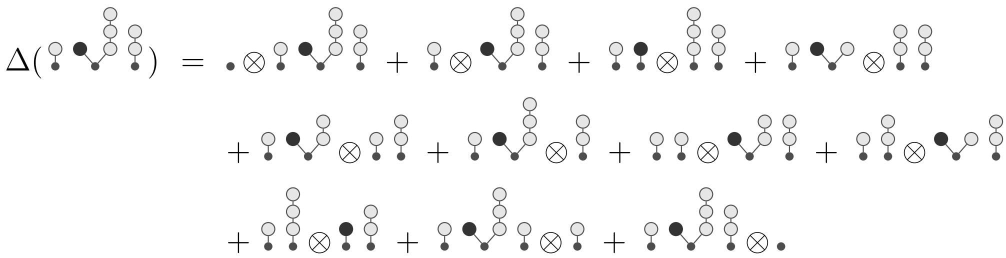
Note that, to obtain all possible vertically splittings of α we need to consider both the left and the right diagram of it. This description extends the one for classical ribbon diagrams, see [23, Proposition 5.2.15]. Thus,



We also can describe the coproduct of L_α, with α a length k dotted composition, by identifying L_α with the forest t_α. Thus,

$$\Delta(t_\alpha) = 1 \otimes t_\alpha + \sum_{i=1}^{k} \sum_{c \neq \emptyset} t_{(\alpha_1,\dots,\alpha_{i-1})} P^c(t_{\alpha_i}) \otimes R^c(t_{\alpha_i}) t_{(\alpha_{i+1},\dots,\alpha_k)}.$$

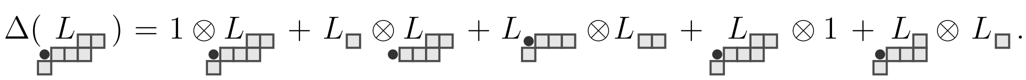
For instance, for $\alpha = (1, \dot{3}, 2)$, we have



Remark 37. By using the ribbon Schur functions of the second kind, defined in Subsection 6.1, we obtain a second kind of fundamental quasisymmetric functions in superspace $\hat{L}_\alpha$, which coincide with the ones introduced in [16] with respect to the partial order $\preceq_r$. As in Theorem 36, we obtain analogous results:

$$\hat{L}_\alpha = \sum_{\beta \preceq_r \alpha} M_\beta, \qquad M_\alpha = \sum_{\beta \preceq_r \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \hat{L}_\beta, \qquad \Delta(\hat{L}_\alpha) = \sum_{\beta\gamma=\alpha} L_\beta \otimes L_\gamma + \sum_{\substack{\beta \odot \gamma = \gamma, \\ \mathrm{rg}(\beta), \gamma_1 \in \mathbb{N}}} L_\beta \otimes L_\gamma.$$

The coproduct above can be described by splitting ribbon diagrams, where the first sum represents the horizontal splitting and the second sum represents vertical splitting on undotted components. For instance,



7.3 Symmetric functions in superspace

In this subsection, we present some results on symmetric functions in superspace, which are obtained by the projection π from **sNSym** to **sSym** defined in [16]. Additionally, we obtain a new basis of **sSym** formed by a new class of functions that we will call *ribbon Schur functions in superspace*, which extends the classical ribbon Schur functions in **Sym**.

Recall that $\pi : \mathbf{sNSym} \to \mathbf{sSym}$ is determined by $\pi(H_n) := h_n$ and $\pi(\tilde{H}_n) := \tilde{h}_n$. In particular, this morphism extends the classical morphism from **NSym** to **Sym**.

The following proposition characterizes the action of π on the families of noncommutative functions in superspace.

Proposition 38. *For every $n \geqslant 0$, we have $\pi(\tilde{S}_n) = \tilde{e}_n$ and $\pi(\tilde{P}_n) = \pi(\Psi_n) = \tilde{p}_n$.*

Proof. First, as π is a Hopf algebra morphism, we have $\pi \circ S = S \circ \pi$, where S denotes the antipodes of **sNSym** and **sSym** respectively. Thus, we obtain $\pi(S(\tilde{H}_n)) = \pi((-1)^{n+1}\tilde{S}_n) = (-1)^{n+1}\pi(\tilde{S}_n)$, and $S(\pi(\tilde{H}_n)) = S(\tilde{h}_n) = (-1)^{n+1}\tilde{e}_n$ [16, Corollary 4.5]. So, $(-1)^{n+1}\pi(\tilde{S}_n) = (-1)^{n+1}\tilde{e}_n$. Hence, $\pi(\tilde{S}_n) = \tilde{e}_n$.

For the second part, we proceed by induction on n. Since $\tilde{p}_0 = \tilde{h}_0$ and $\tilde{P}_0 = \tilde{H}_0$, then $\pi(\tilde{P}_0) = \tilde{p}_0$. Now, let $n \geqslant 1$, and assume the result is true for $0 \leqslant k < n$. By [11, Lemma 26], we have

$$(n+1)\tilde{p}_n = (n+1)\tilde{h}_n - p_n\tilde{h}_0 - \sum_{k=0}^{n-1}\left(p_k\tilde{h}_{n-k} + (k+1)\tilde{p}_k h_{n-k}\right).$$

By applying π on both sides of (12), the induction hypothesis implies that $\pi(\tilde{P}_n) = \tilde{p}_n$. Similarly, by applying π on (13), we obtain $\pi(\Psi_n) = \pi(\tilde{P}_n)$, because the Lie brackets become zero due to the commutativity of the product in **sSym** whenever one of the elements has null fermionic degree. □

Proposition 39. *For $n \geqslant 1$, we have $\tilde{p}_n = \tilde{h}_n - \sum_{k=0}^{n-1} h_{n-k}\tilde{p}_k$ and $\tilde{e}_n = \sum_{k=0}^{n}(-1)^{n-k}\tilde{p}_{n-k}e_k$.*

Proof. It is a consequence of Proposition 38, (7) and Proposition 17. □

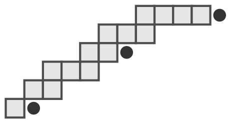
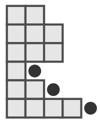
Now, we introduce the *ribbon Schur functions in superspace*. In the classical case, ribbon Schur functions can be regarded as a special case of *skew Schur functions* $s_{\lambda/\mu}$, which are indexed by the so-called *skew partitions* λ/μ, where λ, μ are partitions such that the Young diagram of μ is contained in the one of λ. The Young diagram of λ/μ is obtained by removing the boxes of the diagram of μ from the one of λ. This diagram is called a *ribbon diagram* if it is connected and contains no 2×2 block. In this case, we can identity λ/μ with a composition α. Thus, the *ribbon Schur function* r_α is defined as $s_{\lambda/\mu}$. See [32, Section 7.15] for details.

In superspace, two kinds of Schur functions were defined by means of a specialization of the parameters of the so-called *Macdonald polynomials in superspace*. On the other hand, two kinds of *skew Schur functions in superspace* were defined in relation with a generalization of the Littlewood–Richardson coefficients, see [27] for details.

Here, for a dotted composition α, we define the *ribbon Schur functions in superspace* r_α by projecting the noncommutative ribbon Schur functions in superspace R_α on **sSym**, that is, $r_\alpha = \pi(R_\alpha)$.

In what follows of this subsection we will identify a superpartition with the unique dotted composition obtained by dotting its fermionic part.

Recall that if α is a usual composition, $\tilde{\alpha}$ denotes the partition obtained by sorting its component in nonincreasing order. Similarly, for a dotted composition α, we will denote by $\tilde{\alpha}$ the tuple obtained by sorting in nonincreasing order, both the dotted components and the undotted components. Note that, since the fermionic part of a superpartition must be strictly decreasing, $\tilde{\alpha}$ is a superpartition only if the dotted components of α are all different. For instance, if $\alpha = (\dot{1}, 2, 3, \dot{2}, 3, 1, \dot{4})$, then $\tilde{\alpha} = (\dot{4}, \dot{2}, \dot{1}, 3, 3, 2, 1)$, that is

$\alpha =$  $\tilde{\alpha} =$ 

For a dotted composition α with $k = \ell(\alpha)$, we define $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}$. Note that, as $h_m^2 = 0$ for all $m \in \dot{\mathbb{N}}_0$, then $h_\alpha = 0$ whenever α has repeated dotted components. Further, if $\tilde{\alpha}$ is a superpartition, we have $h_\alpha = (-1)^{\sigma(\alpha)} h_{\tilde{\alpha}}$, where $\sigma(\alpha)$ is the number of inversions of the permutation of the dotted components of α, obtained when computing $\tilde{\alpha}$. For instance, $h_{(\dot{1},2,3,\dot{2},3,1,\dot{4})} = -h_{(\dot{4},\dot{2},\dot{1},3,3,2,1)}$.

Below, we obtain an expansion of a ribbon Schur function in superspace in terms of complete homogeneous functions in superspace, which generalizes the well known formula for classical ribbon Schur functions [29].

Proposition 40. *For a dotted composition α, we have*

$$r_\alpha = \sum_{\alpha \preceq \beta} (-1)^{\ell(\alpha)-\ell(\beta)+\sigma(\beta)} h_{\tilde{\beta}}.$$

Conversely, for a superpartition Λ, we have $h_\Lambda = \sum_{\Lambda \preceq \beta} r_\beta$.

Proof. This is a consequence of Proposition 30, the definition of r_α and the fact that $\pi(H_\beta) = 0$ whenever β has repeated dotted components. □

For instance, for $\alpha = (\dot{0}, 1, \dot{2}, 1)$, we have

$$r_\alpha = -h_{(\dot{2},\dot{0},1,1)} + 2h_{(\dot{3},\dot{0},1)} + h_{(\dot{2},\dot{1},1)} - h_{(\dot{3},\dot{1})} - h_{(\dot{4},\dot{0})}.$$

It follows from Theorem 26 that, for dotted compositions α, β, the product of r_α with r_β is given as follows:

$$r_\alpha r_\beta = \begin{cases} r_{\alpha\beta} & \text{if } \operatorname{rg}(\alpha), \beta_1 \in \dot{\mathbb{N}}_0, \\ r_{\alpha\beta} + r_{\alpha\odot\beta} & \text{otherwise.} \end{cases} \tag{21}$$

Proposition 41. *For every $n \geqslant 0$, we have* $r_{(1^n,\dot{0})} = \sum_{k=0}^{n} (-1)^k r_{(1^{n-k})} r_{(\dot{k})}$.

Proof. We proceed by induction on n. The result is obvious if $n = 0$. Now, assume the claim is true for values less than or equal to n. Thus,

$$r_{(1^n,\dot{0})} = \sum_{k=0}^{n} (-1)^k r_{(1^{n-k})} r_{(\dot{k})}.$$

By multiplying by $r_{(1)}$ on both sides of the equation above, with respect to the product in (21), and by applying the induction hypothesis, we obtain

$$\begin{aligned} r_{(1^{n+1},\dot{0})} + r_{(2,1^{n-1},\dot{0})} &= \sum_{k=0}^{n-1} (-1)^k r_{(1^{n+1-k})} r_{(\dot{k})} + \sum_{k=0}^{n-1} (-1)^k r_{(2,1^{n-1-k})} r_{(\dot{k})} + (-1)^n r_{(1)} r_{(\dot{n})} \\ &= \sum_{k=0}^{n-1} (-1^k) r_{(1^{n+1-k})} r_{(\dot{k})} + \sum_{k=0}^{n-1} (-1)^k (r_{(2,1^{n-1-k},\dot{k})} + r_{(2,1^{n-k-2},(\dot{k+1}))}) \\ &\quad + (-1)^{n-1} r_{(\dot{n+1})} + (-1)^n r_{(1)} r_{(\dot{n})} \\ &= \sum_{k=0}^{n-1} (-1^k) r_{(1^{n+1-k})} r_{(\dot{k})} + r_{(2,1^{n-1},\dot{0})} + (-1)^{n-1} r_{(\dot{n+1})} \\ &\quad + (-1)^n r_{(1)} r_{(\dot{n})}. \end{aligned}$$

Therefore, $r_{(1^{n+1},\dot{0})} = \displaystyle\sum_{k=0}^{n+1} (-1)^k r_{(1^{n+1-k})} r_{(\dot{k})}$. ☐

Proposition 42. *We have:*

$$\tilde{p}_n = (-1)^n r_{(1^n,\dot{0})}, \qquad r_{(1^n,\dot{0})} = \sum_{k=0}^{n} (-1)^k e_{n-k} \tilde{h}_k, \qquad \tilde{p}_n = \sum_{k=0}^{n} (-1)^{n-k} e_{n-k} \tilde{h}_k.$$

Proof. The first equality follows directly by applying π in Proposition 31. The second assertion is a consequence of Proposition 41 and the fact that, in the classical case, $r_{(1^i)} = e_i$, where e_i is the classical elementary symmetric function and $r_{(\dot{k})} = \tilde{h}_k$. The last result follows directly from the first two equalities. ☐

Using the bijection between ribbon diagrams and compositions discussed in Section 3, the set of classical ribbon Schur functions indexed by partitions is a basis of **Sym** [3]. We conclude this section by extending this result to superspace.

For dotted compositions α, β with $a = \mathrm{df}(\alpha)$ and $b = \mathrm{df}(\beta)$, we set $r_\alpha * r_\beta = 0$ if $\mathrm{rg}(\alpha)$ and β_1 are dotted, and $r_\alpha * r_\beta = r_{\alpha \odot \beta}$ otherwise. Thus, as $r_\alpha r_\beta = (-1)^{ab} r_\beta r_\alpha$, the product (21) implies the following:

$$r_{\alpha\beta} = (-1)^{ab} r_\beta r_\alpha - r_\alpha * r_\beta. \tag{22}$$

Lemma 43. *Let α be a dotted composition such that $\tilde{\alpha}$ is a superpartition and* $\mathrm{df}(\alpha) \geqslant 1$. *Then*

$$r_\alpha = (-1)^{\sigma(\alpha)} r_{\tilde{\alpha}} + \sum_\mu c_\mu r_\mu \quad \textit{where} \quad \ell(\mu) < \ell(\alpha) \quad \textit{and} \quad c_\mu \in \mathbb{Z}.$$

Proof. We proceed by induction on $k := \ell(\alpha)$. If $k = 1$ the result is obvious. If $k > 1$, assume the claim is true for smaller values. Let α_i be the maximal dotted component of α. Consider $\beta = (\alpha_1, \ldots, \alpha_{i-1})$ with $b = \mathrm{df}(\beta)$ and $\gamma = (\alpha_i, \ldots, \alpha_k)$ with $c = \mathrm{df}(\gamma)$. Then, by (21) and (22), we have

$$\begin{aligned} r_\alpha &= (-1)^{bc} r_\gamma r_\beta - r_\beta * r_\gamma \\ &= (-1)^{bc} r_{(\alpha_i,\ldots,\alpha_k,\alpha_1,\ldots,\alpha_{i-1})} + (-1)^{bc} r_\gamma * r_\beta - r_\beta * r_\gamma \\ &= (-1)^{bc} r_{\alpha_i} r_{(\alpha_{i+1},\ldots,\alpha_k,\alpha_1,\ldots,\alpha_{i-1})} + (-1)^{bc+1} r_{\alpha_i} * r_{(\alpha_{i+1},\ldots,\alpha_k,\alpha_1,\ldots,\alpha_{i-1})} \\ &\quad + (-1)^{bc} r_\gamma * r_\beta - r_\beta * r_\gamma. \end{aligned}$$

Now, by applying the induction hypothesis on $\omega := (\alpha_{i+1}, \ldots, \alpha_k, \alpha_1, \ldots, \alpha_{i-1})$, we obtain

$$r_\alpha = (-1)^{bc+\sigma(\omega)} r_{\alpha_i \tilde{\omega}} + \sum_\mu c_\mu r_\mu = (-1)^{\sigma(\alpha)} r_{\tilde{\alpha}} + \sum_\mu c_\mu r_\mu,$$

where $\ell(\mu) < \ell(\alpha)$. □

Theorem 44. *The set* $\{r_\Lambda \mid \Lambda$ *is a superpartition*$\}$ *is a basis of* **sSym**.

Proof. Let α be a dotted composition. It is enough to show that r_α is a linear combination of elements in $\{r_\Lambda\}$. If $\mathrm{df}(\alpha) = 0$, the result is true due to the classical case. For $\mathrm{df}(\alpha) \geqslant 1$, we proceed by induction on $k := \ell(\alpha)$. If $k = 1$, the result is obvious. If $k > 1$, we assume the claim is true for dotted compositions of smaller length. We will distinguish two cases.

If α has repeated dotted components, then $h_\alpha = 0$. So, by Proposition 40, we have

$$r_\alpha = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha)-\ell(\beta)+\sigma(\beta)} h_{\tilde{\beta}} = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha)-\ell(\beta)+\sigma(\beta)} \sum_{\alpha_{\tilde{\beta}} \preceq \gamma} r_\gamma.$$

Since $\ell(\gamma) \leqslant \ell(\beta) < \ell(\alpha)$ for each γ above, the result is obtained by applying the induction hypothesis to r_γ.

Now, assume that $\tilde{\alpha}$ is a superpartition. Lemma 43 implies that

$$r_\alpha = (-1)^{\sigma(\alpha)} r_{\tilde{\alpha}} + \sum_\mu c_\mu r_\mu \quad \text{where} \quad \ell(\mu) < \ell(\alpha) \quad \text{and} \quad c_\mu \in \mathbb{Z}.$$

We conclude the proof by applying the induction hypothesis on each r_μ. □

Remark 45. Analogously, we can extend the results above to a second kind of ribbon Schur function in superspace, obtained by projecting $\hat{R}_\alpha$, that is, $\hat{r}_\alpha = \pi(\hat{R}_\alpha)$. In particular, we obtain the following formulas.

$$\hat{r}_\alpha \hat{r}_\beta = \begin{cases} \hat{r}_{\alpha\beta} + \hat{r}_{\alpha\odot\beta} & \text{if } \mathrm{rg}(\alpha), \beta \in \mathbb{N}, \\ \hat{r}_{\alpha\beta} & \text{otherwise.} \end{cases} \qquad \tilde{p}_n = \sum_{k=0}^{n} (-1)^{n-k} \hat{r}_{(1^{n-k}, \dot{k})}.$$

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