Forbidden subgraphs restricting vertices of degree two in a spanning tree

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Abstract

For a tree T, let $V_2(T)$ denote the set of vertices of T having degree 2. Let G be a connected graph. A spanning tree T of G with $V_2(T) = \emptyset$ is called a *homeomorphically irreducible spanning tree* (or a *HIST*) of G.

We focus on two relaxations of HISTs as follows:
(1) A spanning tree T of G such that the maximum order of components of the subgraph of T induced by $V_2(T)$ is bounded.
(2) A spanning tree T of G such that $|V_2(T)|$ is bounded.
A spanning tree satisfying (1) was recently introduced by Lyngsie and Merker, and a spanning tree satisfying (2) is known as a tool for constructing a HIST. In this paper, we define a *star-path system*, which is a useful concept for finding a spanning tree satisfying (1) or (2) (or both). To demonstrate how the concept works, we characterize forbidden subgraph conditions forcing connected graphs to have such spanning trees.

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1 Introduction

In this paper, all graphs are finite, simple, and undirected. Let G be a graph. Let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $u \in V(G)$, let $N_G(u)$ and $d_G(u)$ denote the *neighborhood* and the *degree* of u, respectively; thus $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. For an integer $i \geqslant 0$, let $V_i(G) = \{u \in V(G) : d_G(u) = i\}$. For a subset U of $V(G)$, let $G[U]$ denote the subgraph of G induced by U. Let $\mathcal{C}(G)$ be the family of components of G. For a graph H, G is said to be *H-free* if G contains no induced copy of H. For a family $\mathcal{H}$ of graphs, G is said to be *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. In this context, the members of $\mathcal{H}$ are called

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forbidden subgraphs. For two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of graphs, we write $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ such that H_2 contains a copy of H_1 as an induced subgraph. Note that if $\mathcal{H}_1 \leqslant \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

A spanning tree of a graph G without vertices of degree 2 is called a *homeomorphically irreducible spanning tree* (or a *HIST*) of G, i.e., a spanning tree T of G is a HIST if and only if $V_2(T) = \emptyset$. A structure of HISTs is sometimes used as an essential tool to construct graph classes; for example, in an explicit class of edge-minimal 3-connected plane graphs given by Halin [11], HISTs play a key role. Motivated from such importance, the existence of a HIST has been widely studied (for example, see [1,3,4,10,12,13,15,16]). In [8], the authors have characterized the forbidden subgraph conditions for the existence of a HIST. Further relationships between forbidden subgraphs and homeomorphically irreducible trees are studied in [5,9].

Let G be a connected graph. For a spanning tree T of G, let $\xi(G,T)$ be the maximum order of components of the subgraph of T induced by $V_2(T)$, i.e., $\xi(G,T) = \max\{|V(P)| : P \in \mathcal{C}(T[V_2(T)])\}$. Let $\xi(G) = \min\{\xi(G,T) : T \text{ is a spanning tree of } G\}$. Note that G has a HIST if and only if $\xi(G) = 0$. Recently, Lyngsie and Merker [14] weakened the concept of HISTs by focusing on the value $\xi(G)$, and they proved the following two results:

(1) There exists an integer d such that every connected graph G with minimum degree at least d satisfies $\xi(G) \leqslant 1$.
(2) Every connected graph G with minimum degree at least 3 satisfies $\xi(G) \leqslant 2$.

Furthermore, other HIST-like structures also received a lot of attention. In the research of HISTs, we often find a large subtree T of a target graph G such that $|V_2(T)|$ is small, and after that, construct a HIST by properly joining each vertex in $V_2(T)$ with a vertex in $V(G) \setminus V(T)$. Indeed, such a strategy was adopted in many papers (for example, [10,13,15]).

Motivated by the above facts, we study the existence of a spanning tree T with some restriction on the vertices in $V_2(T)$, and in this paper, we propose a new concept (a star-path system defined in Section 3) for such problems. To demonstrate how the concept works, we focus on the existence of spanning trees T such that $\xi(G,T)$ or $|V_2(T)|$ (or both) is bounded by a constant (compared to the order of G), and characterize the forbidden subgraph conditions forcing connected graphs to have such spanning trees by using a star-path system. Specifically, we judge whether or not a family $\mathcal{H}$ of connected graphs satisfies one of the following conditions (but we do not use a star-path system for (F1) because we find a more concise proof).

(F1) There is no long sequence of vertices of degree 2:
There exists a constant $c(\mathcal{H})$ such that every connected $\mathcal{H}$-free graph G satisfies $\xi(G) \leqslant c(\mathcal{H})$.

(F2) The number of vertices of degree 2 is small:
There exists a constant $c(\mathcal{H})$ such that every connected $\mathcal{H}$-free graph has a spanning tree T with $|V_2(T)| \leqslant c(\mathcal{H})$.

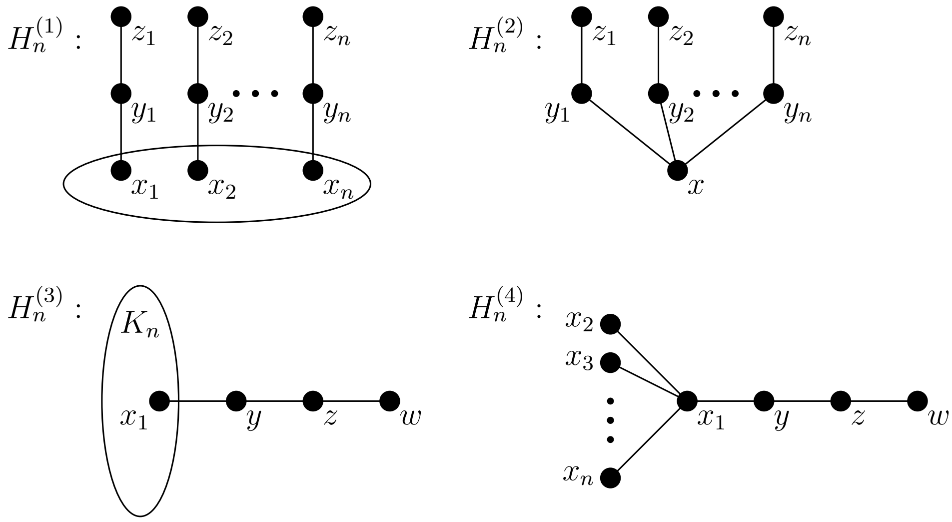

Figure 1: The graphs $H_n^{(i)}$ $(1 \leqslant i \leqslant 4)$.

(F3) No vertices of degree 2 are adjacent:
There exists a constant $n(\mathcal{H})$ such that every connected $\mathcal{H}$-free graph G of order at least $n(\mathcal{H})$ satisfies $\xi(G) \leqslant 1$ (i.e., $V_2(T)$ is an independent set of T).

(F4) Both (F2) and (F3) are satisfied:
There exist two constants $c(\mathcal{H})$ and $n(\mathcal{H})$ such that every connected $\mathcal{H}$-free graph G of order at least $n(\mathcal{H})$ has a spanning tree T with $|V_2(T)| \leqslant c(\mathcal{H})$ and $\xi(G, T) \leqslant 1$.

Let $n \geqslant 1$ be an integer. We let P_n denote the *path* of order n. Furthermore, we construct four graphs as follows (see Figure 1):

- Let $H_n^{(1)}$ be the graph consisting of $3n$ vertices x_i, y_i, z_i $(i \in \{1, 2, \dots, n\})$ such that $E(H_n^{(1)}) = \{x_i x_j : 1 \leqslant i < j \leqslant n\} \cup \{x_i y_i, y_i z_i : 1 \leqslant i \leqslant n\}$.
- Let $H_n^{(2)}$ be the graph obtained from $H_n^{(1)}$ by contracting n vertices $x_1, x_2, \dots, x_n$ to a vertex x.
- Let $H_n^{(3)}$ be the graph consisting of $n+3$ vertices x_i $(i \in \{1, 2, \dots, n\}), y, z, w$ such that $E(H_n^{(3)}) = \{x_i x_j : 1 \leqslant i < j \leqslant n\} \cup \{x_1 y, yz, zw\}$.
- Let $H_n^{(4)} = H_n^{(3)} - \{x_i x_j : 2 \leqslant i < j \leqslant n\}$.

Our main results are the following.

Theorem 1. *A family $\mathcal{H}$ of connected graphs satisfies (F1) if and only if $\mathcal{H} \leqslant \{P_n\}$ for an integer $n \geqslant 2$.*

Theorem 2. *A family $\mathcal{H}$ of connected graphs satisfies (F2) if and only if $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(2)}\}$ for an integer $n \geqslant 2$.*

Theorem 3. *A family $\mathcal{H}$ of connected graphs satisfies (F3) if and only if $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(3)}, H_n^{(4)}\}$ for an integer $n \geqslant 2$.*

Theorem 4. *A family $\mathcal{H}$ of connected graphs satisfies (F4) if and only if $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(2)}, H_n^{(3)}, H_n^{(4)}\}$ for an integer $n \geqslant 2$.*

In Section 2, we prove Theorem 1. In Section 3, we define a star-path system, which is the key concept of this paper, and give its fundamental properties. We prove Theorem 2 in Section 4. In Section 5, we simultaneously prove Theorems 3 and 4.

1.1 Further notations and preliminaries

In this subsection, we specify additional notation and introduce useful preliminaries. For terms and symbols not defined in this paper, we refer the reader to [6].

Let G be a graph. For an integer $i \geqslant 0$, let $V_{\geqslant i}(G) = \{u \in V(G) : d_G(u) \geqslant i\}$ and $V_{\neq i}(G) = \{u \in V(G) : d_G(u) \neq i\}$. For a subset U of $V(G)$, let $N_G(U) = (\bigcup_{u \in U} N_G(u)) \setminus U$. For two subsets U_1 and U_2 of $V(G)$, U_1 *dominates* U_2 in G if $U_2 \subseteq N_G(U_1) \cup U_1$. A subset U of $V(G)$ is a *connected dominating set* of G if U dominates $V(G)$ in G and $G[U]$ is connected. For two disjoint subsets U_1 and U_2 of $V(G)$, let $E_G(U_1, U_2) = \{u_1u_2 \in E(G) : u_1 \in U_1,\ u_2 \in U_2\}$. For $u, v \in V(G)$, the *distance* between u and v, denoted by $\mathrm{dist}_G(u, v)$, is the minimum length of a path of G connecting u and v. The value $\mathrm{diam}(G) := \max\{\mathrm{dist}_G(u, v) : u, v \in V(G)\}$ is called the *diameter* of G. For a subgraph H of G and a subset F of $E(G)$, let $H + F$ be the subgraph of G with $V(H + F) = V(H) \cup \{u, v : uv \in F\}$ and $E(H + F) = E(H) \cup F$. Let $\alpha(G)$ denote the *independence number* of G, i.e., the maximum cardinality of an independent set of G. Let $q(G) = |\mathcal{C}(G)|$, and let $q_2(G)$ denote the number of components of G consisting of two vertices. For two positive integers n_1 and n_2, the *Ramsey number* $R(n_1, n_2)$ is the minimum positive integer R such that any graph of order at least R contains a clique of cardinality n_1 or an independent set of cardinality n_2.

The following lemma, that will be used when we prove the "only if" parts of Theorems 2–4, clearly holds.

Lemma 5. *Let G be a connected graph. If a cut-vertex u of G satisfies $d_G(u) = 2$, then $d_T(u) = 2$ for every spanning tree T of G.*

In the process of characterizing P_n-free graphs, Camby and Schaudt [2] proved the following lemma.

Lemma 6 (Camby and Schaudt [2])**.** *Let $n \geqslant 4$ be an integer, and let G be a connected P_n-free graph. Then there exists a connected dominating set X of G such that either $G[X]$ is P_{n-2}-free or $G[X] \simeq P_{n-2}$.*

2 Proof of Theorem 1

The following proposition gives the "only if" part of Theorem 1.

Proposition 7. *If a family $\mathcal{H}$ of connected graphs satisfies (F1), then $\mathcal{H} \leqslant \{P_n\}$ for an integer $n \geqslant 2$.*

Proof. Let $c = c(\mathcal{H})$ be a constant such that every connected $\mathcal{H}$-free graph G satisfies $\xi(G) \leqslant c$. Since $\xi(P_{c+3}) = c+1$, it follows from the definition of c that P_{c+3} is not $\mathcal{H}$-free. This implies that $\mathcal{H} \leqslant \{P_{c+3}\}$, as desired. □

On the other hand, the following theorem implies that the "if" part of Theorem 1 holds.

Theorem 8. *Let $n \geqslant 2$ be an integer, and let G be a connected P_n-free graph. Then there exists a spanning tree T of G with $\xi(G,T) \leqslant n-2$.*

Proof. We proceed by induction on n. If $n = 2$, then $G \simeq K_1$; if $n = 3$, then G is a complete graph. In either case, the desired conclusion clearly holds. Thus we may assume that $n \geqslant 4$. By Lemma 6, there exists a connected dominating set X of G such that either $G[X]$ is P_{n-2}-free or $G[X] \simeq P_{n-2}$. If $G[X]$ is P_{n-2}-free, then by the induction hypothesis, there exists a spanning tree T_0 of $G[X]$ with $\xi(G[X], T_0) \leqslant n-4$; if $G[X] \simeq P_{n-2}$, then $T_0 := G[X]$ is a tree with $\xi(G[X], T_0) = |X| - 2 = n-4$. In either case, there exists a spanning tree T_0 of $G[X]$ with $\xi(G[X], T_0) \leqslant n-4$.

For each $u \in V(G) \setminus X$, since X dominates $V(G) \setminus X$ in G, there exists a vertex $v_u \in X$ with $uv_u \in E(G)$. Let $T = T_0 + \{uv_u : u \in V(G) \setminus X\}$. Since T is a spanning tree of G, it suffices to show that for an element P of $\mathcal{C}(T[V_2(T)])$, $|V(P)| \leqslant n-2$. Note that P is a path. Write $P = v_1v_2 \cdots v_l$. We may assume that $l \geqslant 3$. Since every vertex in $V(G) \setminus X$ is a leaf of T, $V(P) \subseteq X$. In particular, P is a subgraph of T_0. For an integer i with $2 \leqslant i \leqslant l-1$, since $T_0 = T[X]$ and $N_T(v_i) = \{v_{i-1}, v_{i+1}\}$, we have $N_{T_0}(v_i) = \{v_{i-1}, v_{i+1}\}$. This implies that $v_2, v_3, \ldots, v_{l-1}$ belong to a common element of $\mathcal{C}(T_0[V_2(T_0)])$. Consequently, $|V(P)| - 2 = l - 2 \leqslant \xi(G[X], T_0) \leqslant n-4$, as desired. □

3 SP-systems

A tree T is a *star* if T has a vertex of degree $|V(T)|-1$ (where a connected graph of order at most two is regarded as a star). For a star T, a vertex $u \in V(T)$ is a *center* of T if $d_T(u) = |V(T)| - 1$. Note that a star T has two centers if and only if $|V(T)| = 2$. For a path P, a vertex $u \in V(P)$ is an *endvertex* of P if $d_P(u) \leqslant 1$.

In this section, we introduce star-path systems (or SP-systems for short), that play a key role in our argument. We start with an overview of SP-systems: Let G be a graph, and fix two sets $X \subseteq V(G)$ and $X' \subseteq N_G(X)$. (When we use the star-path systems in practice, we fix a vertex x and let X and X' be the set of vertices at distance $i-1$ and i from x, respectively.) Let F be a spanning subgraph of $G[X']$ whose components are stars, and take a subgraph L of $G[X]$ so that the components of L are paths and $V(L)$ dominates the centers of components of F. In fact, we impose more detailed conditions (S1)–(S9), and prove that we can always find a structure satisfying the conditions (see Proposition 9). We consider a subgraph of G obtained from F and L by adding some

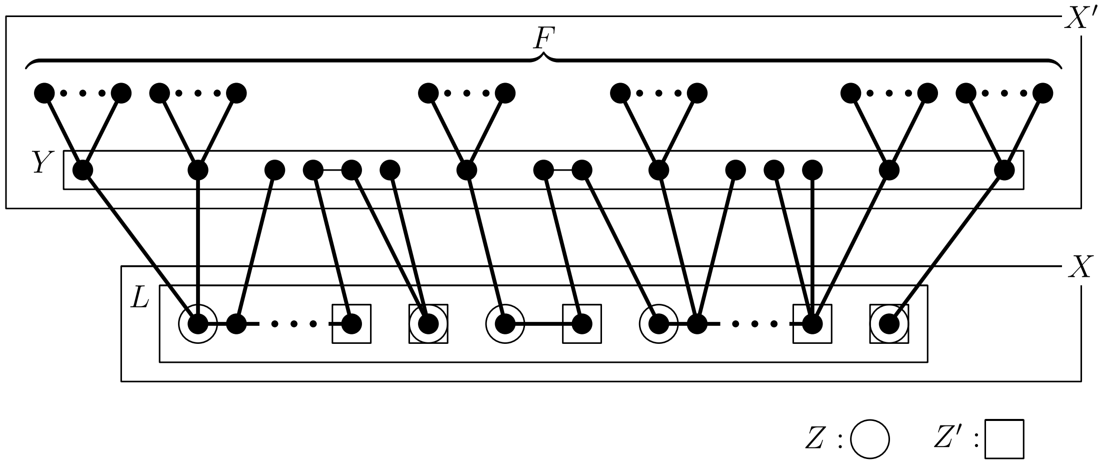

Figure 2: An SP-system $(F, Y; L, Z, Z')$ of (G, X, X'), where the bold lines form an $(F, Y; L, Z, Z')$-typical subgraph.

edges between $V(F)$ and $V(L)$ (and such a subgraph is said to be typical). We prove that a typical subgraph is a forest and every vertex of degree 2 is an endvertex of a component of L (see Lemma 11). In our proof of Theorems 2–4, we will construct a spanning tree with restricted vertices of degree 2 by connecting some typical subgraphs.

Now we define the structure strictly. Let G be a graph, and let $X \subseteq V(G)$ and $X' \subseteq N_G(X)$. We consider a 5-tuple $(F, Y; L, Z, Z')$ satisfying the following nine conditions (see Figure 2):

(S1) F is a spanning subgraph of $G[X']$ such that every element of $\mathcal{C}(F)$ is a star;
(S2) if u and u' are centers of distinct elements of $\mathcal{C}(F)$, then $uu' \notin E(G)$;
(S3) $Y = \{u : u \text{ is a center of an element of } \mathcal{C}(F)\}$;
(S4) L is a subgraph of $G[X]$ such that every element of $\mathcal{C}(L)$ is a path;
(S5) $Y \subseteq N_G(V(L))$;
(S6) $\{u \in Y : N_G(u) \cap V(L) = \{v\}\} \neq \emptyset$ for every vertex $v \in V(L)$;
(S7) $Z, Z' \subseteq \{v : v \text{ is an endvertex of an element of } \mathcal{C}(L)\}$;
(S8) both Z and Z' are independent sets of G; and
(S9) for an element P of $\mathcal{C}(L)$, $|Z \cap V(P)| = |Z' \cap V(P)| = 1$, and if $|V(P)| \geqslant 2$, then $Z \cap Z' \cap V(P) = \emptyset$.

A 5-tuple $(F, Y; L, Z, Z')$ satisfying (S1)–(S9) is called a *star-path system* (or an *SP-system*) of (G, X, X'). Note that if $X' = \emptyset$, then $(\emptyset, \emptyset; \emptyset, \emptyset, \emptyset)$ is the unique SP-system of (G, X, X').

Proposition 9. *Let G be a graph, and let $X \subseteq V(G)$ and $X' \subseteq N_G(X)$. Then there exists an SP-system of (G, X, X').*

Proof. We may assume that $X' \neq \emptyset$. We first prove that there exists a pair (F, Y') of a graph F and a set Y' of centers of elements of F satisfying (S1) and

(S2') Y' is an independent set of G and $|Y' \cap V(S)| = 1$ for every $S \in \mathcal{C}(F)$.

Take a maximal independent set Y' of $G[X']$. Then for each $u \in X' \setminus Y'$, it follows from the maximality of Y' that there exists a vertex $w_u \in N_G(u) \cap Y'$. Let F be a spanning subgraph of $G[X']$ with $E(F) = \{uw_u : u \in X' \setminus Y'\}$. Then every element of $\mathcal{C}(F)$ is a star and Y' is a set of centers of elements of F such that $|Y' \cap V(S)| = 1$ for every $S \in \mathcal{C}(F)$. Thus (F, Y') satisfying (S1) and (S2').

Choose a pair (F, Y') satisfying (S1) and (S2') so that $q_2(F)$ is as small as possible. We prove the following claim which implies that F satisfies (S2).

Claim 10. *Write $\mathcal{C}(F) = \{S_1, S_2, \ldots, S_k\}$ and $V(S_i) \cap Y' = \{y_i\}$ for each integer i with $1 \leqslant i \leqslant k$.*

(i) *For an integer i with $1 \leqslant i \leqslant k$, if $|V(S_i)| = 2$, then $E_G(V(S_i), Y' \setminus \{y_i\}) = \emptyset$.*

(ii) *For integers i and j with $1 \leqslant i < j \leqslant k$, if $|V(S_i)| = |V(S_j)| = 2$, then $E_G(V(S_i), V(S_j)) = \emptyset$.*

Proof. (i) Fix an integer i with $1 \leqslant i \leqslant k$, and suppose that $|V(S_i)| = 2$. Write $V(S_i) \setminus \{y_i\} = \{x_i\}$.

Let j be an integer with $1 \leqslant j \leqslant k$ and $j \neq i$. It suffices to show that $x_iy_j \notin E(G)$. By way of contradiction, suppose that $x_iy_j \in E(G)$. For the moment, we assume that $|V(S_j)| \geqslant 2$. Let $S'_i = S_i - \{x_i\}$, $S'_j = S_j + y_jx_i$ and $F' = (F - (V(S_i) \cup V(S_j))) \cup S'_i \cup S'_j$. Then $V(S'_i) \cap Y' = V(S'_i) = \{y_i\}$, $V(S'_j) \cap Y' = \{y_j\}$ and y_j is a center of S'_j. In particular, (F', Y') satisfies (S1) and (S2') with $q_2(F') < q_2(F)$, which contradicts the minimality of $q_2(F)$. Thus $|V(S_j)| = 1$, i.e., $V(S_j) = \{y_j\}$, and we have proved that the following holds:

$$N_G(x_i) \cap \{y_l : 1 \leqslant l \leqslant k,\ l \neq i,\ |V(S_l)| \geqslant 2\} = \emptyset. \tag{1}$$

Let $I = \{l : 1 \leqslant l \leqslant k,\ l \neq i,\ x_iy_l \in E(G)\}$. Note that $I \neq \emptyset$. By (1), if $l \in I$, then $|V(S_l)| = 1$. Let $S''_i = S_i + \{x_iy_l : l \in I\}$. Since $I \neq \emptyset$, S''_i is a star of order at least three, and x_i is the unique center of S''_i. Let $F'' = (F - (\bigcup_{l \in \{i\} \cup I} V(S_l))) \cup S''_i$ and $Y'' = (Y' \setminus (\{y_i\} \cup \{y_l : l \in I\})) \cup \{x_i\}$. Then by (1) and the definition of I, Y'' is an independent set of G and $|Y'' \cap V(S)| = 1$ for every $S \in \mathcal{C}(F'')$. Hence (F'', Y') satisfying (S1) and (S2') with $q_2(F'') < q_2(F)$, which contradicts the minimality of $q_2(F)$.

(ii) Fix integers i and j with $1 \leqslant i < j \leqslant k$, and suppose that $|V(S_i)| = |V(S_j)| = 2$. Write $V(S_i) \setminus \{y_i\} = \{x_i\}$ and $V(S_j) \setminus \{y_j\} = \{x_j\}$. By (i), it suffices to show that $x_ix_j \notin E(G)$. By way of contradiction, suppose that $x_ix_j \in E(G)$. Let $S^*_i = S_i + \{x_ix_j\}$, $S^*_j = S_j - \{x_j\}$, $F^* = (F - (V(S_i) \cup V(S_j))) \cup S^*_i \cup S^*_j$ and $Y^* = (Y' \setminus \{y_i\}) \cup \{x_i\}$. Then by (i), $N_G(x_i) \cap (Y^* \setminus \{x_i\}) = N_G(x_i) \cap (Y' \setminus \{y_i\}) = \emptyset$. Hence (F^*, Y^*) satisfying (S1) and (S2') with $q_2(F^*) < q_2(F)$, which contradicts the minimality of $q_2(F)$.

□

Let

$$Y = Y' \cup \left(\bigcup_{\substack{S \in \mathcal{C}(F) \\ |V(S)|=2}} V(S) \right).$$

Then (F, Y) satisfies (S3).

Recall that $Y \subseteq X' \subseteq N_G(X)$. Take a set $A \subseteq X$ dominating Y in G so that $|A|$ is as small as possible. Then by the minimality of $|A|$, we have

$$\{u \in Y : N_G(u) \cap A = \{v\}\} \neq \emptyset \text{ for every vertex } v \in A. \tag{2}$$

Note that for the graph L_0 with $V(L_0) = A$ and $E(L_0) = \emptyset$, $\mathcal{C}(L_0)$ consists of paths of order one. Take a spanning subgraph L of $G[A]$ such that every element of $\mathcal{C}(L)$ is a path so that $q(L)$ is as small as possible. Considering (2), we can easily verify that $(F, Y; L)$ satisfies (S4)–(S6).

Write $\mathcal{C}(L) = \{Q_1, Q_2, \dots, Q_m\}$, and for each integer i with $1 \leqslant i \leqslant m$, write $Q_i = z_{i,1}z_{i,2} \cdots z_{i,\eta_i}$. Let $Z = \{z_{i,1} : 1 \leqslant i \leqslant m\}$ and $Z' = \{z_{i,\eta_i} : 1 \leqslant i \leqslant m\}$. Suppose that Z is not an independent set of G. Then $z_{i,1}z_{j,1} \in E(G)$ for some integers i and j with $1 \leqslant i < j \leqslant m$, and hence the path $Q' := z_{i,\eta_i}z_{i,\eta_i-1} \cdots z_{i,1}z_{j,1}z_{j,2} \cdots z_{j,\eta_j}$ is a subgraph of $G[A]$. Consequently, $L' := (L - (V(Q_i) \cup V(Q_j))) \cup Q'$ is a spanning subgraph of $G[A]$ such that every element of $\mathcal{C}(L')$ is a path and $q(L') < q(L)$, which contradicts the minimality of $q(L)$. Thus Z is an independent set of G. By symmetry, Z' is also an independent set of G. Therefore, $(F, Y; L, Z, Z')$ satisfies (S7)–(S9). $\square$

We remark that the proof of Proposition 9 is constructive. For an SP-system $(F, Y; L, Z, Z')$ of (G, X, X'), we construct a subgraph of G as follows: For each $u \in Y$, it follows from (S5) that there exists a vertex $v_u \in V(L)$ with $uv_u \in E(G)$. Let

$$H = \left(\left(F - \bigcup_{\substack{S \in \mathcal{C}(F) \\ |V(S)|=2}} E(S) \right) \cup L \right) + \{uv_u : u \in Y\}.$$

Such a subgraph H of G is said to be $(F, Y; L, Z, Z')$-*typical* (again see Figure 2).

Lemma 11. *Let G be a graph, and let $X \subseteq V(G)$ and $X' \subseteq N_G(X)$. Let $(F, Y; L, Z, Z')$ be an SP-system of (G, X, X'), and let H be an $(F, Y; L, Z, Z')$-typical subgraph of G. Then the following hold:*

(i) *The graph H is a subforest of G with $V(H) = V(L) \cup X'$.*

(ii) *For each component C of H, $|V(C) \cap Z| = 1$. In particular, $q(H) = |Z| = q(L)$.*

(iii) *We have $V_2(H) \subseteq Z \cup Z'$.*

(iv) *For $P \in \mathcal{C}(L)$, if $|V(P)| = 1$, then $V(P) \cap Z = V(P) \cap Z' \subseteq V_{\geqslant 1}(H)$; if $|V(P)| \geqslant 2$, then $V(P) \cap (Z \cup Z') \subseteq V_{\geqslant 2}(H)$.*

Proof. By the definition of H, (i) and (ii) clearly hold. For each $u \in Y$, let v_u be the vertex as in the definition of H.

We first show that

$$X' \cap V_2(H) = \emptyset. \tag{3}$$

Recall that $X' = \bigcup_{S \in \mathcal{C}(F)} V(S)$. Thus it suffices to show that $V(S) \subseteq V_{\neq 2}(H)$ for every $S \in \mathcal{C}(F)$. If $|V(S)| \geqslant 3$, then

- for the unique vertex u in $V(S) \cap Y$, $N_H(u) = (V(S) \setminus \{u\}) \cup \{v_u\}$, and so $d_H(u) = (|V(S)| - 1) + 1 \geqslant 2 + 1$, and
- for every vertex $u' \in V(S) \setminus Y$, the neighborhood of u' in H consists of the unique vertex in $V(S) \cap Y$, and so $d_H(u') = 1$.

Thus we may assume that $1 \leqslant |V(S)| \leqslant 2$, and let $u \in V(S)$. Since $V(S) \subseteq Y$, the vertex v_u has been defined. By the definition of H, we have $N_H(u) = \{v_u\}$, and in particular, $d_H(u) = 1$. Since $u \in V(S)$ is arbitrary, we have $V(S) \subseteq V_1(H)$. Consequently, (3) holds.

Now we prove (iii) and (iv). Note that

$$V(H) \setminus X' = \left(\bigcup_{P \in \mathcal{C}(L)} V(P) \setminus (Z \cup Z') \right) \cup \left(\bigcup_{P \in \mathcal{C}(L)} V(P) \cap (Z \cup Z') \right).$$

Furthermore, for $P \in \mathcal{C}(L)$, if $|V(P)| = 1$, then $V(P) = V(P) \cap Z = V(P) \cap Z'$. Hence by (3), it suffices to show that for $P \in \mathcal{C}(L)$,

$$V(P) \subseteq V_{\geqslant 1}(H), \text{ and} \tag{4}$$

$$\text{if } |V(P)| \geqslant 2, \text{ then } V(P) \subseteq V_{\geqslant 2}(H) \text{ and } V(P) \cap V_2(H) \subseteq Z \cup Z'. \tag{5}$$

By (S6),

$$\{u \in Y : v_u = v\} \neq \emptyset \text{ for every vertex } v \in V(P), \tag{6}$$

which implies (4). Assume that $|V(P)| \geqslant 2$, and let $v \in V(P)$. Since $N_P(v) \subseteq N_H(v)$, it follows from (6) that $d_H(v) \geqslant d_P(v) + 1$. Hence if $v \notin Z \cup Z'$, then $d_H(v) \geqslant d_P(v) + 1 = 2 + 1$; if $v \in Z \cup Z'$, then $d_H(v) \geqslant d_P(v) + 1 = 1 + 1$. This implies that (5) holds. □

4 Proof of Theorem 2

4.1 The "if" part of Theorem 2

Throughout this subsection, we fix an integer $n \geqslant 2$. We recursively define s_i ($i \geqslant 0$) with $s_0 = 1$ and $s_i = (n-1)R(n, s_{i-1} + 1) - 1$ for each integer $i \geqslant 1$. Then $4 \sum_{1 \leqslant i \leqslant n-3} s_i + 1$ is a constant depending only on n. We prove the following theorem, which implies that the "if" part of Theorem 2 holds.

Theorem 12. *Let $n \geqslant 2$ be an integer, and let G be a connected $\{P_n, H_n^{(1)}, H_n^{(2)}\}$-free graph. Then there exists a spanning tree T of G with $|V_2(T)| \leqslant 4\sum_{1\leqslant i\leqslant n-3} s_i + 1$.*

Proof. Let $x \in V(G)$. For each integer $i \geqslant 0$, let $X_i = \{y \in V(G) : \mathrm{dist}_G(x, y) = i\}$. Let $p = \max\{i \geqslant 0 : X_i \neq \emptyset\}$. If $p \geqslant n-1$, then a shortest path of G connecting x and X_{n-1} is an induced path of order n, which contradicts the P_n-freeness of G. Thus $p \leqslant n-2$. If $p \leqslant 1$, then the spanning subgraph T of G with $E(T) = \{xy : y \in X_1\}$ is a tree with $V_2(T) \subseteq \{x\}$. Thus we may assume that $p \geqslant 2$.

Let $W_p = X_p$. We recursively define an SP-system $(F_i, Y_i; L_{i-1}, Z_{i-1}, Z'_{i-1})$ of (G, X_{i-1}, W_i) for $i = p, p-1, \dots, 1$ and a subset W_i of X_i for $i = p-1, p-2, \dots, 0$ as follows: Let i be an integer with $1 \leqslant i \leqslant p$, and assume that W_i $(\subseteq X_i)$ has been defined. Since $W_i \subseteq X_i \subseteq N_G(X_{i-1})$, it follows from Proposition 9 that there exists an SP-system $(F_i, Y_i; L_{i-1}, Z_{i-1}, Z'_{i-1})$ of (G, X_{i-1}, W_i). Let $W_{i-1} = X_{i-1} \setminus V(L_{i-1})$. Note that if $W_i = \emptyset$, then $(F_i, Y_i; L_{i-1}, Z_{i-1}, Z'_{i-1}) = (\emptyset, \emptyset; \emptyset, \emptyset, \emptyset)$ and $W_{i-1} = X_{i-1}$.

Claim 13. *For every integer i with $0 \leqslant i \leqslant p-1$, $|Z_i| \leqslant 2s_i$.*

Proof. Fix an integer i with $0 \leqslant i \leqslant p-1$, and suppose that $|Z_i| \geqslant 2s_i + 1$. Since $|Z_0| \leqslant |X_0| = |\{x\}| = 1 = s_0$, we have $i \geqslant 1$. For each $u \in Z_i$, it follows from (S6) that there exists a vertex $y_u \in Y_{i+1}$ with $N_G(y_u) \cap Z_i = \{u\}$. Take a set $I_i \subseteq Z_i$ so that

(I1) $\{y_u : u \in I_i\}$ is an independent set of G, and
(I2) subject to (I1), $|I_i|$ is as large as possible.

By (S2) and (S3), no two edges of $G[Y_{i+1}]$ are adjacent. Hence $|I_i| \geqslant \lceil \frac{|Z_i|}{2} \rceil \geqslant \lceil \frac{2s_i+1}{2} \rceil = s_i + 1$.

For $j = i-1, i-2, \dots, 0$, we recursively define subsets U_j and I_j of X_j as follows: Assume that I_{j+1} has been defined. Since X_j dominates I_{j+1} in G, we can take a minimum set $U_j \subseteq X_j$ dominating I_{j+1} in G. Let $I_j \subseteq U_j$ be a maximum independent set of G. Note that $U_0 = I_0 = \{x\}$.

For every integer j with $0 \leqslant j \leqslant i-1$ and every vertex $u \in U_j$, it follows from the minimality of U_j that there exists a vertex $y_u \in I_{j+1}$ with $N_G(y_u) \cap U_j = \{u\}$. Since $|I_i| \geqslant s_i + 1$ and $|I_0| = |\{x\}| = 1 = s_0$, there exists an integer h with $1 \leqslant h \leqslant i$ such that $|I_h| \geqslant s_h + 1$ and $|I_{h-1}| \leqslant s_{h-1}$. Suppose that $|I_h| \geqslant (n-1)|U_{h-1}| + 1$. Then there exists a vertex $w \in U_{h-1}$ such that $|N_G(w) \cap I_h| \geqslant \lceil \frac{|I_h|}{|U_{h-1}|} \rceil \geqslant \lceil \frac{(n-1)|U_{h-1}|+1}{|U_{h-1}|} \rceil = n$. Take a set $J \subseteq N_G(w) \cap I_h$ with $|J| = n$. Recall that for each $u \in I_i \cup (\bigcup_{0\leqslant j\leqslant i-1} U_j)$, the vertex y_u has been defined. Hence $\{w\} \cup J \cup \{y_u : u \in J\}$ induces a copy of $H_n^{(2)}$ in G, which is a contradiction. Thus $(n-1)|U_{h-1}| \geqslant |I_h| \geqslant s_h + 1 = (n-1)R(n, s_{h-1}+1)$, i.e., $|U_{h-1}| \geqslant R(n, s_{h-1}+1)$. By the definitions of h and I_{h-1}, we have $\alpha(G[U_{h-1}]) = |I_{h-1}| \leqslant s_{h-1}$. Thus there exists a clique $C \subseteq U_{h-1}$ of G with $|C| = n$. Then $C \cup \{y_u, y_{y_u} : u \in C\}$ induces a copy of $H_n^{(1)}$ in G, which is a contradiction. □

Recall that $X_p = W_p$ and for an integer i with $0 \leqslant i \leqslant p-1$, X_i is the disjoint union of $V(L_i)$ and W_i. Furthermore, $W_0 \subseteq \{x\}$ and W_0 might be the empty set. Hence $V(G)$ is the disjoint union of $W_i \cup V(L_{i-1})$ $(1 \leqslant i \leqslant p)$ and W_0. For each integer i with $1 \leqslant i \leqslant p$, let H_i be an $(F_i, Y_i; L_{i-1}, Z_{i-1}, Z'_{i-1})$-typical subgraph of G. Since $V(H_i) = W_i \cup V(L_{i-1})$, $H^* := \bigcup_{1 \leqslant i \leqslant p} H_i$ is a spanning forest of $G - W_0$. For each integer i with $1 \leqslant i \leqslant p-1$ and for each $v \in Z_i$, take a vertex $z_v \in N_G(v) \cap X_{i-1}$. Let $T = H^* + (\bigcup_{1 \leqslant i \leqslant p-1} \{vz_v : v \in Z_i\})$. For each $C \in \mathcal{C}(H^*)$, it follows from Lemma 11 (ii) that $|C \cap (\bigcup_{0 \leqslant i \leqslant p-1} Z_i)| = 1$. Hence

$$\text{for an integer } i \text{ with } 2 \leqslant i \leqslant p \text{ and for a vertex } v \in V(H_i),\\ \text{there exists a path of } T \text{ connecting } v \text{ and } X_{i-2}\ (= V(L_{i-2}) \cup W_{i-2}). \tag{7}$$

Note that at least one of H_2 and H_1 is a non-empty graph. If H_2 is non-empty, then it follows from (7) that $x \in V(T)$; if H_1 is non-empty, then $\{x\} = Z_0 \subseteq V(H_1)$. In either case, we have $x \in V(T)$. This together with (7) implies that T is a spanning tree of G.

For an integer i with $1 \leqslant i \leqslant p-1$ and for an element P of $\mathcal{C}(L_i)$ with $|V(P)| \geqslant 2$, by Lemma 11 (iv) and the construction of T, $V(P) \cap Z \subseteq V_{\geqslant 3}(T)$. This together with Lemma 11 (iii) implies that

$$V_2(T) \subseteq \left(\bigcup_{0 \leqslant i \leqslant p-1} Z'_i \right) \cup \left\{ z_v : v \in \bigcup_{1 \leqslant i \leqslant p-1} Z_i \right\}.$$

Recall that $p \leqslant n-2$. It follows from Claim 13 that $|V_2(T)| \leqslant \sum_{1 \leqslant i \leqslant p-1} (|Z_i| + |Z'_i|) + |Z'_0| \leqslant 2 \sum_{1 \leqslant i \leqslant p-1} |Z_i| + 1 \leqslant 4 \sum_{1 \leqslant i \leqslant p-1} s_i + 1 \leqslant 4 \sum_{1 \leqslant i \leqslant n-3} s_i + 1$. Consequently, T is a desired spanning tree of G. □

4.2 The "only if" part of Theorem 2

In this subsection, we prove the following proposition, which gives the "only if" part of Theorem 2.

Proposition 14. *If a family $\mathcal{H}$ of connected graphs satisfies (F2), then $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(2)}\}$ for an integer $n \geqslant 2$.*

Proof. Let $c = c(\mathcal{H})$ be a constant such that every connected $\mathcal{H}$-free graph has a spanning tree T with $|V_2(T)| \leqslant c$. Since P_{c+3} is a tree and $|V_2(P_{c+3})| = c+1$, P_{c+3} is not $\mathcal{H}$-free, i.e., there exists a graph $H \in \mathcal{H}$ such that P_{c+3} contains a copy of H as an induced subgraph. Hence

$$\mathcal{H} \leqslant \{P_{c+3}\}. \tag{8}$$

For an integer $i \in \{1, 2\}$ and a spanning tree T of $H_{c+3}^{(i)}$, it follows from Lemma 5 that $\{y_j : 1 \leqslant j \leqslant c+3\} \subseteq V_2(T)$, and so $|V_2(T)| \geqslant c+3$. This implies that neither $H_{c+3}^{(1)}$ nor $H_{c+3}^{(2)}$ is $\mathcal{H}$-free, and so

$$\mathcal{H} \leqslant \{H_{c+3}^{(1)}, H_{c+3}^{(2)}\} \tag{9}$$

By (8) and (9), we obtain $\mathcal{H} \leqslant \{P_{c+3}, H_{c+3}^{(1)}, H_{c+3}^{(2)}\}$, as desired. □

5 Proof of Theorems 3 and 4

5.1 The "if" parts of Theorems 3 and 4

Throughout this subsection, we fix an integer $n \geqslant 2$, and let $R = R(n-1, n-1)$ and $R' = R(2n-1, n)$. We prove the following theorem, which implies that the "if" parts of Theorems 3 and 4 hold.

Theorem 15. *Let G be a connected $\{P_n, H_n^{(1)}, H_n^{(3)}, H_n^{(4)}\}$-free graph of order at least $(3RR' - 3R + R' + 1)^{n-2} + 2$. Then the following hold:*

(i) *There exists a spanning tree T of G such that $V_2(T)$ is an independent set of T.*

(ii) *If G is $H_n^{(2)}$-free, then there exists a spanning tree T of G such that $V_2(T)$ is an independent set of T and $|V_2(T)| \leqslant 4nR' - 2n + 1$.*

We start with two lemmas. The following lemma is well-known (see, for example, [7]).

Lemma 16. *Let G be a connected graph. Then $|V(G)| \leqslant \Delta(G)^{\mathrm{diam}(G)} + 1$.*

In the proof of Theorem 15 (ii), we will use the following lemma.

Lemma 17. *Let G be an $H_n^{(2)}$-free graph, and let $X \subseteq V(G)$ and $X' \subseteq N_G(X)$. Let $\mathcal{S} := (F, Y; L, Z, Z')$ be an SP-system of (G, X, X'), and let A be a subset of $V(G) \setminus (X \cup X')$ such that $Z \subseteq N_G(A)$ and $N_G(A) \cap X' = \emptyset$. Then $|Z'| \leqslant (2n-2)|A|$.*

Proof. By way of contradiction, suppose that $(2n-2)|A| + 1 \leqslant |Z'|$ $(= |Z|)$. Since $Z \subseteq N_G(A)$, there exists a vertex $a \in A$ such that $|N_G(a) \cap Z| \geqslant \lceil \frac{|Z|}{|A|} \rceil \geqslant \lceil \frac{(2n-2)|A|+1}{|A|} \rceil = 2n-1$. Let $\tilde{Z} = N_G(a) \cap Z$. By (S8), $\tilde{Z}$ is an independent set of G. For each $v \in \tilde{Z}$, it follows from (S6) that there exits a vertex $u_v \in Y$ with $N_G(u_v) \cap V(L) = \{v\}$. By (S2) and (S3), no two edges of $G[\{u_v : v \in \tilde{Z}\}]$ are adjacent. In particular, $\alpha(G[\{u_v : v \in \tilde{Z}\}]) \geqslant \lceil \frac{|\{u_v : v \in \tilde{Z}\}|}{2} \rceil \geqslant \lceil \frac{2n-1}{2} \rceil = n$. Take an independent set I of $G[\{u_v : v \in \tilde{Z}\}]$ with $|I| = n$. Then $I \cup \{v \in \tilde{Z} : u_v \in I\} \cup \{a\}$ induces $H_n^{(2)}$ in G, which is a contradiction. $\square$

Proof of Theorem 15. Since G is P_n-free, $\mathrm{diam}(G) \leqslant n-2$. Since $|V(G)| \geqslant (3RR' - 3R + R' + 1)^{n-2} + 2$, it follows from Lemma 16 that $\Delta(G) \geqslant 3RR' - 3R + R' + 2$. Let $x \in V(G)$ be a vertex with $d_G(x) = \Delta(G)$ $(\geqslant 3RR' - 3R + R' + 2)$. For each integer $i \geqslant 0$, let $X_i = \{y \in V(G) : \mathrm{dist}_G(x, y) = i\}$.

Claim 18. *Let $l \geqslant 3$ be an integer. Suppose that $X_l \neq \emptyset$, and let $x_0 x_1 \cdots x_l$ be a shortest path of G connecting x and X_l, where $x_0 = x$.*

(i) *We have $|(N_G(x_1) \cup N_G(x_2)) \cap X_1| \geqslant d_G(x) - R + 1$.*

(ii) *If $l \geqslant 4$, then $|N_G(x_2) \cap X_1| \geqslant d_G(x) - 2(R-1)$.*

(iii) *We have $l \leqslant 4$.*

Proof. Note that $x_i \in X_i$ for every integer i with $0 \leqslant i \leqslant l$ and $x_1 \in N_G(x_2)$. If there exists a clique $C \subseteq X_1 \setminus (N_G(x_1) \cup N_G(x_2))$ with $|C| = n-1$, then $\{x_3, x_2, x_1, x\} \cup C$ induces a copy of $H_n^{(3)}$ in G; if there exists an independent set $I \subseteq X_1 \setminus (N_G(x_1) \cup N_G(x_2))$ with $|I| = n-1$, then $\{x_3, x_2, x_1, x\} \cup I$ induces a copy of $H_n^{(4)}$ in G. In either case, we obtain a contradiction. Thus

$$|X_1 \setminus (N_G(x_1) \cup N_G(x_2))| \leqslant R - 1. \tag{10}$$

(i) By (10),

$$|(N_G(x_1) \cup N_G(x_2)) \cap X_1| = |X_1| - |X_1 \setminus (N_G(x_1) \cup N_G(x_2))| \geqslant d_G(x) - (R-1).$$

(ii) Suppose that $l \geqslant 4$. If there exists a clique $C' \subseteq (N_G(x_1) \cap X_1) \setminus N_G(x_2)$ with $|C'| = n-1$, then $\{x_4, x_3, x_2, x_1\} \cup C'$ induces a copy of $H_n^{(3)}$ in G; if there exists an independent set $I' \subseteq (N_G(x_1) \cap X_1) \setminus N_G(x_2)$ with $|I'| = n-1$, then $\{x_4, x_3, x_2, x_1\} \cup I'$ induces a copy of $H_n^{(4)}$ in G. In either case, we obtain a contradiction. Thus $|(N_G(x_1) \cap X_1) \setminus N_G(x_2)| \leqslant R-1$. This together with (10) implies that $|X_1 \setminus N_G(x_2)| = |X_1 \setminus (N_G(x_1) \cup N_G(x_2))| + |(N_G(x_1) \cap X_1) \setminus N_G(x_2)| \leqslant 2(R-1)$. Consequently, we have

$$|N_G(x_2) \cap X_1| = |X_1| - |X_1 \setminus N_G(x_2)| \geqslant d_G(x) - 2(R-1).$$

(iii) Suppose that $l \geqslant 5$. If there exists a clique $C'' \subseteq N_G(x_2) \cap X_1$ with $|C''| = n-1$, then $\{x_5, x_4, x_3, x_2\} \cup C''$ induces a copy of $H_n^{(3)}$ in G; if there exists an independent set $I'' \subseteq N_G(x_2) \cap X_1$ with $|I''| = n-1$, then $\{x_5, x_4, x_3, x_2\} \cup I''$ induces a copy of $H_n^{(4)}$ in G. In either case, we obtain a contradiction. Thus $|N_G(x_2) \cap X_1| \leqslant R-1$. On the other hand, since $R' \geqslant 2$, it follows from (ii) that

$$\begin{aligned} |N_G(x_2) \cap X_1| &\geqslant d_G(x) - 2(R-1) \\ &\geqslant (3RR' - 3R + R' + 2) - 2(R-1) \\ &\geqslant (6R - 3R + 2 + 2) - 2(R-1) \\ &= R + 6, \end{aligned}$$

which is a contradiction.

□

Claim 19. *Let $l \in \{3, 4\}$. Let $\tilde{X}_l \subseteq X_l$, and let $(F, Y; L, Z, Z')$ be an SP-system of $(G, X_{l-1}, \tilde{X}_l)$. Then there exists a set $A \subseteq X_{l-2}$ dominating Z in G such that*

(i) *$\{v \in Z : N_G(v) \cap A = \{a\}\} \neq \emptyset$ for every vertex $a \in A$,*

(ii) *$|A| \leqslant R' - 1$,*

(iii) *if $l = 3$, then $|(N_G(a) \cup N_G(b)) \cap X_1| \geqslant d_G(x) - R + 1$ for every pair of vertices $a \in A$ and $b \in N_G(a) \cap Z$, and*

(iv) *if* $l = 4$*, then* $|N_G(a) \cap X_1| \geqslant d_G(x) - 2(R-1)$ *for every vertex* $a \in A$*.*

Proof. If $\tilde{X}_l = \emptyset$, then $(F, Y; L, Z, Z') = (\emptyset, \emptyset; \emptyset, \emptyset, \emptyset)$, and hence $A = \emptyset$ is the desired set satisfying (i)–(iv). Thus we may assume that $\tilde{X}_l \neq \emptyset$.

Note that X_{l-2} dominates Z $(\subseteq X_{l-1})$ in G. Take a minimum set $A \subseteq X_{l-2}$ dominating Z in G. We prove that A is a desired set. For each $v \in Z$, it follows from (S6) that there exists a vertex $u_v \in Y$ such that $N_G(u_v) \cap V(L) = \{v\}$.

By the minimality of A, for each $a \in A$, there exists a vertex $v_a \in Z$ such that $N_G(v_a) \cap A = \{a\}$. In particular, (i) holds.

Fix two vertices $a \in A$ and $b \in N_G(a) \cap Z$, and let P be a shortest path connecting x and a in G. Then $Pabu_b$ is a shortest path of G connecting x and X_l. By Claim 18 (i), if $l = 3$, then $|(N_G(a) \cap N_G(b)) \cap X_1| \geqslant d_G(x) - R + 1$. By Claim 18 (ii),

$$\text{if } l = 4, \text{ then } |N_G(a) \cap X_1| \geqslant d_G(x) - 2(R-1). \tag{11}$$

Consequently, both (iii) and (iv) hold.

Now we prove (ii). By way of contradiction, suppose that $|A| \geqslant R'$. For the moment, we suppose that there exists a clique $C \subseteq A$ of G with $|C| = 2n-1$. Since $\{u_{v_a} : a \in C\} \subseteq Y$, it follows from (S2) and (S3) that no two edges of $G[\{u_{v_a} : a \in C\}]$ are adjacent. In particular, $\alpha(G[\{u_{v_a} : a \in C\}]) \geqslant \lceil \frac{|C|}{2} \rceil = \lceil \frac{2n-1}{2} \rceil = n$. Let $C' \subseteq C$ be a set with $|C'| = n$ such that $\{u_{v_a} : a \in C'\}$ is an independent set of G. Since $\{v_a : a \in C'\}$ $(\subseteq Z)$ is an independent set of G by (S8), $\{a, v_a, u_{v_a} : a \in C'\}$ induces a copy of $H_n^{(1)}$ in G, which is a contradiction. Since $|A| \geqslant R'$, this implies that there exists an independent set $I \subseteq A$ of G with $|I| = n$. Recall that $n \geqslant 2$ and $R' \geqslant n+1$. If $l = 3$, then x $(\in X_0)$ is adjacent to all vertices in I; if $l = 4$, then it follows from (11) that

$$\begin{aligned}
\left| \bigcap_{a \in I} (N_G(a) \cap X_1) \right| &\geqslant d_G(x) - 2(R-1)|I| \\
&\geqslant (3RR' - 3R + R' + 2) - 2(R-1)(n+1) \\
&\geqslant (3(n+1)R - 3R + (n+1) + 2) - 2(R-1)(n+1) \\
&= (n-2)R + 3n + 5 \\
&> 0.
\end{aligned}$$

In either case, there exists a vertex $c \in X_{l-3}$ adjacent to all vertices in I. Fix a vertex $a^* \in I$. Then $\{u_{v_{a^*}}, v_{a^*}, a^*, c\} \cup (I \setminus \{a^*\})$ induces a copy of $H_n^{(4)}$ in G, which is a contradiction. □

Now we derive some properties of G. We refer to Figure 3 for an illustration.

Let $\mathcal{S}_4 := (F_4, Y_4; L_3, Z_3, Z_3')$ be an SP-system of (G, X_3, X_4). Then by Claim 19, there exists a set $A_2 \subseteq X_2$ dominating Z_3 in G such that

(A1) $\{v \in Z_3 : N_G(v) \cap A_2 = \{a\}\} \neq \emptyset$ for every vertex $a \in A_2$,
(A2) $|A_2| \leqslant R' - 1$, and
(A3) $|N_G(a) \cap X_1| \geqslant d_G(x) - 2(R-1)$ for every vertex $a \in A_2$.

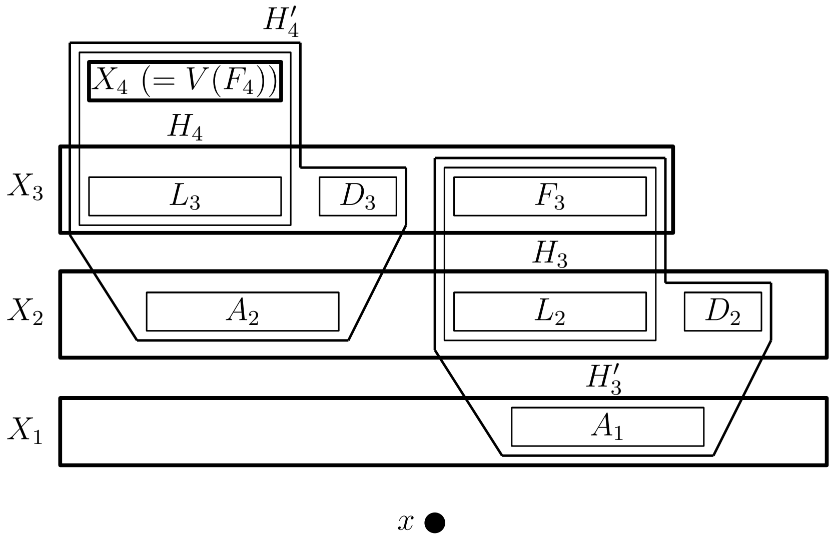

Figure 3: The structure of H'_4 and H'_3.

Note that if $p \leqslant 3$, then $L_3 = \emptyset$ and $A_2 = \emptyset$. Let $D_3 = (X_3 \setminus V(L_3)) \cap N_G(A_2)$. Since $Z_3 \cup D_3 \subseteq N_G(A_2)$, for each $v \in Z_3 \cup D_3$, we can take a vertex $a_v \in N_G(v) \cap A_2$. Let H_4 be an $\mathcal{S}_4$-typical subgraph of G, and let

$$H'_4 = H_4 + \{va_v : v \in Z_3 \cup D_3\}.$$

Let $X'_3 = X_3 \setminus (V(L_3) \cup D_3)$. Then no vertex in X'_3 is adjacent to a vertex in A_2, and hence $X_2 \setminus A_2$ dominates X'_3 in G. Let $\mathcal{S}_3 := (F_3, Y_3; L_2, Z_2, Z'_2)$ be an SP-system of $(G, X_2 \setminus A_2, X'_3)$. Then by Claim 19, there exists a set $A_1 \subseteq X_1$ dominating Z_2 in G such that

(A'1) $\{v \in Z_2 : N_G(v) \cap A_1 = \{a\}\} \neq \emptyset$ for every vertex $a \in A_1$,
(A'2) $|A_1| \leqslant R' - 1$, and
(A'3) $|(N_G(a) \cup N_G(b)) \cap X_1| \geqslant d_G(x) - R + 1$ for every pair of vertices $a \in A_1$ and $b \in N_G(a) \cap Z_2$.

Note that if $X'_3 = \emptyset$, then $L_2 = \emptyset$ and $A_1 = \emptyset$. Let $D_2 = (X_2 \setminus (A_2 \cup V(L_2))) \cap N_G(A_1)$. Since $Z_2 \cup D_2 \subseteq N_G(A_1)$, for each $v \in Z_2 \cup D_2$, we can take a vertex $a_v \in N_G(v) \cap A_1$. Let H_3 be an $\mathcal{S}_3$-typical subgraph of G, and let

$$H'_3 = H_3 + \{va_v : v \in Z_2 \cup D_2\}.$$

In the following, we list some properties of H'_4 and H'_3.

(H1) We have $V(H'_4) \cap X_4 = X_4$ and $V(H'_3) \cap X_4 = \emptyset$.
(H2) We have $V(H'_4) \cap X_3 = V(L_3) \cup D_3$ and $V(H'_3) \cap X_3 = X'_3\ (= X_3 \setminus (V(L_3) \cup D_3))$. In particular, X_3 is the disjoint union of $V(H'_4) \cap X_3$ and $V(H'_3) \cap X_3$.
(H3) We have $V(H'_4) \cap X_2 = A_2$ and $V(H'_3) \cap X_2 = V(L_2) \cup D_2\ (\subseteq X_2 \setminus A_2)$.

(H4) We have $V(H'_4) \cap X_1 = \emptyset$ and $V(H'_3) \cap X_1 = A_1$.

For each $i \in \{3, 4\}$, the following hold:

(H5) By Lemma 11 (i) and (ii), H'_i is a subforest of G and every component of H'_i contains exactly one vertex in A_{i-2}.
(H6) For each $v \in V(H_i) \setminus Z_{i-1}$, $d_{H'_i}(v) = d_{H_i}(v)$.
(H7) By Lemma 11 (iii) and (H6), $(V(H_i) \setminus (Z_{i-1} \cup Z'_{i-1})) \cap V_2(H'_i) = (V(H_i) \setminus (Z_{i-1} \cup Z'_{i-1})) \cap V_2(H_i) = \emptyset$.
(H8) By Lemma 11 (iv) and (H6), for every $v \in Z'_{i-1}$, if $v \in Z_{i-1}$, then $d_{H'_i}(v) = d_{H_i}(v) + |\{a_v\}| \geqslant 1 + 1$; if $v \notin Z_{i-1}$, then $d_{H'_i}(v) = d_{H_i}(v) \geqslant 2$.
(H9) By Lemma 11 (iv), for every $v \in Z_{i-1} \setminus Z'_{i-1}$, $d_{H'_i}(v) = d_{H_i}(v) + |\{a_v\}| \geqslant 2 + 1$.
(H10) By (A1) and (A'1), for every $a \in A_{i-2}$, $d_{H'_i}(a) \geqslant |\{v \in Z_{i-1} : a = a_v\}| \geqslant 1$.
(H11) For every $v \in D_{i-1}$, $d_{H'_i}(v) = 1$.

Furthermore, we can easily verify some neighborhood structures of vertices in H'_i:

(H12) For every $v \in Z'_{i-1}$, if $v \in Z_{i-1}$, then $N_{H'_i}(v) \subseteq V(F_i) \cup \{a_v\}$; if $v \notin Z_{i-1}$, then by (S8), $N_{H'_i}(v) \subseteq V(F_i) \cup (V(L_{i-1}) \setminus Z'_{i-1})$.
(H13) For every $a \in A_{i-2}$, $N_{H'_i}(a) = \{v \in Z_{i-1} \cup D_{i-1} : a = a_v\}$.

Let
$$A_1^* = \{a \in A_1 : |\{v \in Z_2 : a_v = a\}| = 1\}.$$
For each $a \in A_1^*$, write $\{v \in Z_2 : a_v = a\} = \{c_a\}$. For $a \in A_1^*$, it follows from (A'1) that $N_G(c_a) \cap A_1 = \{a\}$. It follows from (A'2) that

$$|X_1 \setminus A_1| \geqslant d_G(x) - (R' - 1) \geqslant (3RR' - 3R + R' + 2) - (R' - 1) = 3(RR' - R + 1).$$

This together with (A2), (A3), (A'2) and (A'3) leads to

$$\begin{aligned}
&\left| \left(\bigcap_{a \in A_2} N_G(a) \right) \cap \left(\bigcap_{a \in A_1^*} (N_G(a) \cup N_G(c_a)) \right) \cap (X_1 \setminus A_1) \right| - 2|A_2| - |A_1^*| - 3 \\
&\geqslant |X_1 \setminus A_1| - 2(R-1)|A_2| - (R-1)|A_1^*| - 2|A_2| - |A_1^*| - 3 \\
&= |X_1 \setminus A_1| - R(2|A_2| + |A_1^*|) - 3 \\
&\geqslant 3(RR' - R + 1) - R(2(R'-1) + (R'-1)) - 3 \\
&= 0,
\end{aligned}$$

and hence there exist $2|A_2| + |A_1^*| + 3$ vertices y_1, y_2, y_3, w_a $(a \in A_2), w'_a$ $(a \in A_2)$ and z_a $(a \in A_1^*)$ in $X_1 \setminus A_1$ such that

- for each $a \in A_2$, $aw_a, aw'_a \in E(G)$, and
- for each $a \in A_1^*$, there exists a vertex $\tilde{c}_a \in \{a, c_a\}$ such that $\tilde{c}_a z_a \in E(G)$.

Note that $(\bigcap_{a\in A_2} N_G(a))\cap(\bigcap_{a\in A_1^*}(N_G(a)\cup N_G(c_a)))\cap(X_1\setminus A_1) = X_1\setminus A_1$ if $A_2 = A_1^* = \emptyset$. In particular, y_1, y_2 and y_3 are defined no matter whether $A_2\cup A_1^*$ is the empty set or not. Let $B_1 = \{y_1, y_2, y_3\}\cup\{w_a, w'_a : a\in A_2\}\cup\{z_a : a\in A_1^*\}$ and $\tilde{C} = \{\tilde{c}_a : a\in A_1^*\}$. Let $D'_2 = (X_2\setminus(A_2\cup V(L_2)\cup D_2))\cap N_G(B_1)$. For each $v\in D'_2$, let $b_v\in N_G(v)\cap B_1$.

Let $X'_2 = X_2\setminus(A_2\cup V(L_2)\cup D_2\cup D'_2)$. Then no vertex in X'_2 is adjacent to a vertex in $A_1\cup B_1$, and hence $X_1\setminus(A_1\cup B_1)$ dominates X'_2 in G. Let $\mathcal{S}_2 := (F_2, Y_2; L_1, Z_1, Z'_1)$ be an SP-system of $(G, X_1\setminus(A_1\cup B_1), X'_2)$, and let $D_1 = X_1\setminus(A_1\cup B_1\cup V(L_1))$. Let H_2 be an $\mathcal{S}_2$-typical subgraph of G, and let

$$H'_2 = H_2 + \{xv : v\in Z_1\cup D_1\}.$$

Then the following properties, which are similar to (H1)–(H13):

- **(H14)** We have $V(H'_2)\cap(X_4\cup X_3) = \emptyset$.
- **(H15)** We have $V(H'_2)\cap X_2 = X'_2$ $(= X_2\setminus(A_2\cup V(L_2)\cup D_2\cup D'_2))$. This together with (H3) implies that X_2 is the disjoint union of $V(H'_4)\cap X_2$, $V(H'_3)\cap X_2$, D'_2 and $V(H'_2)\cap X_2$.
- **(H16)** We have $V(H'_2)\cap X_1 = V(L_1)\cup D_1$ $(= X_1\setminus(A_1\cup B_1)$. This together with (H4) implies that X_1 is the disjoint union of $V(H'_3)\cap X_1$, B_1 and $V(H'_2)\cap X_1$.
- **(H17)** By Lemma 11 (i) and (ii), H'_2 is a subtree of G.
- **(H18)** For each $v\in V(H_2)\setminus Z_1$, $d_{H'_2}(v) = d_{H_2}(v)$.
- **(H19)** By Lemma 11 (iii) and (H18), $(V(H_2)\setminus(Z_1\cup Z'_1))\cap V_2(H'_2) = (V(H_2)\setminus(Z_1\cup Z'_1))\cap V_2(H_2) = \emptyset$.
- **(H20)** By Lemma 11 (iv) and (H18), for every $v\in Z'_1$, if $v\in Z_1$, then $d_{H'_2}(v) = d_{H_2}(v) + |\{x\}| \geqslant 1+1$; if $v\notin Z_1$, then $d_{H'_2}(v) = d_{H_2}(v)\geqslant 2$.
- **(H21)** By Lemma 11 (iv), for every $v\in Z_1\setminus Z'_1$, $d_{H'_2}(v) = d_{H_2}(v) + |\{x\}|\geqslant 2+1$.
- **(H22)** For every $v\in D_1$, $d_{H'_2}(v) = 1$.
- **(H23)** For every $v\in Z'_1$, if $v\in Z_1$, then $N_{H'_2}(v)\subseteq V(F_2)\cup\{x\}$; if $v\notin Z_1$, then by (S8), $N_{H'_2}(v)\subseteq V(F_2)\cup(V(L_1)\setminus Z'_1)$.

Let

$$\begin{aligned} T = {}& (H'_4\cup H'_3\cup H'_2) + \{aw_a, aw'_a, xw_a : a\in A_2\} + \{xa : a\in A_1\} + \{\tilde{c}_a z_a : a\in A_1^*\} \\ & + \{vb_v : v\in D'_2\} + \{xy_i : y_i : 1\leqslant i\leqslant 3\} \end{aligned}$$

(see Figure 4). Then the following hold:

- **(T1)** By Claim 18 (iii), $X_i = \emptyset$ for all integers $i\geqslant 5$. Hence by (H1), (H2), (H5), (H14)–(H17), T is a spanning tree of G.
- **(T2)** Since $d_T(v) = d_{H'_4}(v)$ for every $v\in V(H_4)\cup D_3$, it follows from (H7)–(H9) and (H11) that $(V(H_4)\cup D_3)\cap V_2(T)\subseteq Z'_3$.
- **(T3)** For every $a\in A_2$, it follows from (H10) that $d_T(a) = d_{H'_4}(a) + |\{w_a, w'_a\}|\geqslant 1+2$.
- **(T4)** For every $a\in A_1^*$, if $\tilde{c}_a = a$, then by (H10), $d_T(\tilde{c}_a) = d_T(a) = d_{H'_3}(a) + |\{z_a, x\}|\geqslant 1+2$; if $\tilde{c}_a = c_a$, then by (H8) and (H9), $d_T(\tilde{c}_a) = d_T(c_a) = d_{H'_3}(c_a) + |\{z_a\}|\geqslant 2+1$. In particular, $\tilde{C}\subseteq V_{\geqslant 3}(T)$.

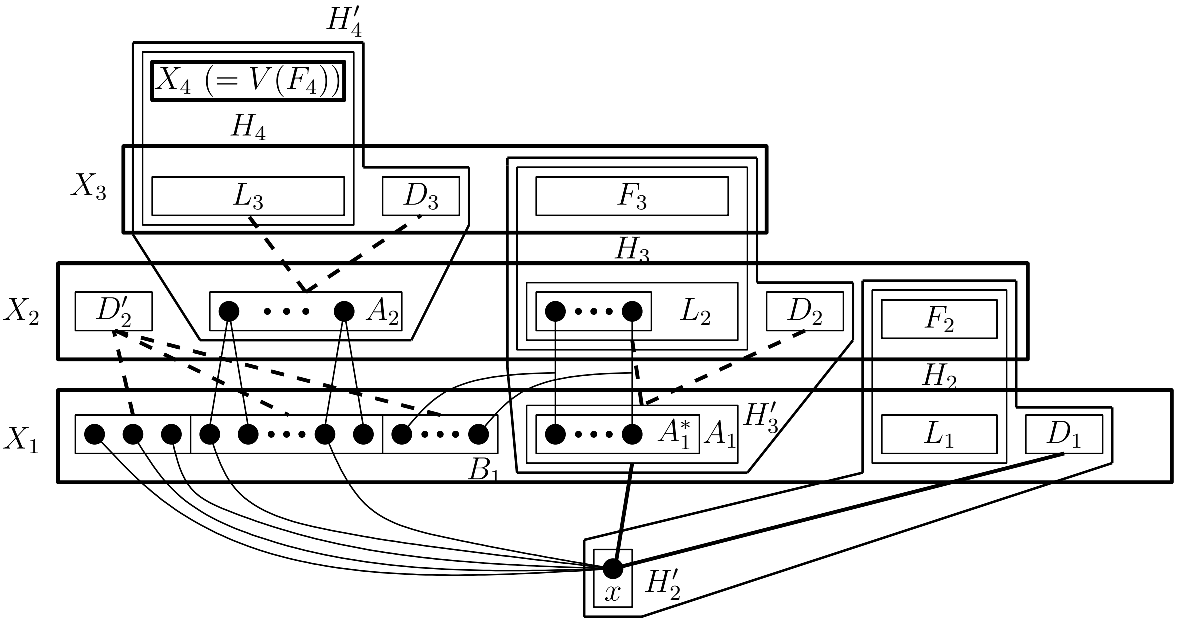

Figure 4: The structure of T, where the dashed lines represent that some edges between two sets are used in T, and the bold lines represent that all edges between two sets are used in T.

(T5) A vertex $v \in V(H_3) \cup D_2$ satisfies $d_T(v) = d_{H_3'}(v)$ if and only if $v \notin \tilde{C}$. Hence by (H7)–(H9), (H11) and (T4), we have $(V(H_3) \cup D_2) \cap V_2(T) \subseteq Z_2' \setminus \tilde{C}$.

(T6) Let $a \in A_1 \setminus A_1^*$. Then by (A'1) and the definition of A_1^*, we have $|\{v \in Z_2 : a_v = a\}| \geqslant 2$. This together with (H10) implies that $d_T(a) = d_{H_3'}(a) + |\{x\}| \geqslant |\{v \in Z_2 : a_v = a\}| + 1 \geqslant 2 + 1$.

(T7) Let $a \in A_1^* \setminus \tilde{C}$. Then by (H10), $d_T(a) = d_{H_3'}(a) + |\{x\}| \geqslant 1 + 1$.

(T8) For every $a \in A_2$, $d_T(w_a) = |\{v \in D_2' : w_a = b_v\}| + |\{a, x\}| \geqslant 2$ and $d_T(w_a') = |\{v \in D_2' : w_a' = b_v\}| + |\{a\}| \geqslant 1$.

(T9) For every $a \in A_1^*$, $d_T(z_a) = |\{v \in D_2' : z_a = b_v\}| + |\{\tilde{c}_a\}| \geqslant 1$.

(T10) For every integer i with $1 \leqslant i \leqslant 3$, $d_T(y_i) = |\{v \in D_2' : y_i = b_v\}| + |\{x\}| \geqslant 1$.

(T11) For every $v \in D_2'$, $d_T(v) = |\{b_v\}| = 1$.

(T12) Since $d_T(v) = d_{H_2'}(v)$ for every $v \in V(H_2) \cup D_1$ $(= V(H_2') \setminus \{x\})$, it follows from (H19)–(H22) that $(V(H_2) \cup D_1) \cap V_2(T) \subseteq Z_1'$.

(T13) We have $d_T(x) \geqslant |\{y_i : 1 \leqslant i \leqslant 3\}| = 3$.

(T14) By (T2) and (T3), $V(H_4') \cap V_2(T) \subseteq Z_3'$. By (T4)–(T7), $V(H_3') \cap V_2(T) \subseteq (Z_2' \setminus \tilde{C}) \cup (A_1^* \setminus \tilde{C})$. By (T8)–(T13), $(B_1 \cup D_2' \cup V(H_2')) \cap V_2(T) \subseteq B_1 \cup Z_1'$. Hence $V_2(T) \subseteq Z_3' \cup (Z_2' \setminus \tilde{C}) \cup (A_1^* \setminus \tilde{C}) \cup B_1 \cup Z_1'$.

(T15) Let $v \in Z_3'$. By (H12), (T2) and (T3), if $v \in Z_3$, then $N_T(v) = N_{H_4'}(v) \subseteq V(F_4) \cup \{a_v\} \subseteq V_{\neq 2}(T)$; if $v \notin Z_3$, then $N_T(v) = N_{H_4'}(v) \subseteq V(F_4) \cup (V(L_3) \setminus Z_3') \subseteq V_{\neq 2}(T)$.

(T16) For every $v \in Z_2' \setminus Z_2$, it follows from (H12) and (T5) that $N_T(v) = N_{H_3'}(v) \subseteq V(F_3) \cup (V(L_2) \setminus Z_2') \subseteq V_{\neq 2}(T)$.

(T17) Let $v \in Z_2 \cap Z_2'$ be a vertex with $a_v \notin A_1^*$. Then it follows from (T6) that $d_T(a_v) \geqslant 3$.

Hence by (H12) and (T5), $N_T(v) = N_{H_3'}(v) \subseteq V(F_3) \cup \{a_v\} \subseteq V_{\neq 2}(T)$.

(T18) Let $v \in Z_2' \setminus \tilde{C}$. We show that $N_T(v) \subseteq V_{\neq 2}(T)$. By (T16) and (T17), we may assume that $v \in Z_2 \cap Z_2'$ and $a_v \in A_1^*$. Since $v \notin \tilde{C}$, it follows from (T4) that $a_v \in \tilde{C} \subseteq V_{\geqslant 3}(T)$. This together with (H12) and (T5) implies that $N_T(v) = N_{H_3'}(v) \subseteq V(F_3) \cup \{a_v\} \subseteq V_{\neq 2}(T)$, as desired.

(T19) Let $a \in A_1^* \setminus \tilde{C}$. Then by (T4), the unique vertex $v \in Z_2$ with $a_v = a$ satisfies $v = \tilde{c}_a$ ($\in \tilde{C} \subseteq V_{\geqslant 3}(T)$). This together with (H13), (T5) and (T13) implies that $N_T(a) = N_{H_3'}(a) \cup \{x\} \subseteq D_2 \cup \{v, x\} \subseteq V_{\neq 2}(T)$.

(T20) For every $a \in A_2$, it follows from (T3), (T11) and (T13) that $N_T(w_a) = \{v \in D_2' : w_a = b_v\} \cup \{a, x\} \subseteq V_{\neq 2}(T)$ and $N_T(w_a') = \{v \in D_2' : w_a = b_v\} \cup \{a\} \subseteq V_{\neq 2}(T)$.

(T21) For every $a \in A_1^*$, it follows from (T4) and (T11) that $N_T(z_a) = \{v \in D_2' : z_a = b_v\} \cup \{\tilde{c}_a\} \subseteq V_{\neq 2}(T)$.

(T22) Let $b \in B_1$. We show that $N_T(b) \subseteq V_{\neq 2}(T)$. By (T20) and (T21), we may assume that $b = y_i$ for an integer i with $1 \leqslant i \leqslant 3$. Then by (T11) and (T13), we have $N_T(b) \subseteq \{v \in D_2' : y_i = b_v\} \cup \{x\} \subseteq V_{\neq 2}(T)$, as desired.

(T23) Let $v \in Z_1'$. By (H23), (T12) and (T13), if $v \in Z_1$, then $N_T(v) = N_{H_2'}(v) \subseteq V(F_2) \cup \{x\} \subseteq V_{\neq 2}(T)$; if $v \notin Z_1$, then $N_T(v) = N_{H_2'}(v) \subseteq V(F_2) \cup (V(L_1) \setminus Z_1') \subseteq V_{\neq 2}(T)$.

By (T14), (T15), (T18), (T19), (T22) and (T23), $V_2(T)$ is an independent set of T. This together with (T1) implies that (i) holds.

We next prove that (ii) holds. It suffices to show that if G is $H_n^{(2)}$-free, then $|V_2(T)| \leqslant 4nR' - 2n + 1$. By (A2) and (A'2),

$$|A_i| \leqslant R' - 1 \text{ for each } i \in \{1, 2\}. \tag{12}$$

Applying Lemma 17 with

$$(X, X', \mathcal{S}, A) \in \{(X_3, X_4, \mathcal{S}_4, A_2), (X_2 \setminus A_2, X_3', \mathcal{S}_3, A_1), (X_1 \setminus (A_1 \cup B_1), X_2', \mathcal{S}_2, \{x\})\},$$

we have

$$|Z_{i-1}'| \leqslant (2n-2)|A_{i-2}| \text{ for each } i \in \{3, 4\}, \text{ and} \tag{13}$$

$$|Z_1'| \leqslant (2n-2)|\{x\}| = 2n - 2. \tag{14}$$

Recall that $B_1 = \{y_1, y_2, y_3\} \cup \{w_a, w_a' : a \in A_2\} \cup \{z_a : a \in A_1^*\}$. Hence by (T14) and (12)–(14),

$$\begin{aligned}
|V_2(T)| &\leqslant |Z_3'| + |Z_2' \setminus \tilde{C}| + |A_1^* \setminus \tilde{C}| + |B_1| + |Z_1'| \\
&\leqslant |Z_3'| + |Z_2'| + |A_1| + (3 + 2|A_2| + |A_1|) + |Z_1'| \\
&\leqslant (2n-2)|A_2| + (2n-2)|A_1| + |A_1| + (3 + 2|A_2| + |A_1|) + (2n-2) \\
&= 2n(|A_2| + |A_1| + 1) + 1 \\
&\leqslant 2n(2(R'-1) + 1) + 1,
\end{aligned}$$

which proves (ii).

This completes the proof of Theorem 15. □

5.2 The "only if" part of Theorems 3 and 4

In this subsection, we prove the following proposition, which gives the "only if" part of Theorems 3 and 4.

Proposition 20. *Let $\mathcal{H}$ be a family of connected graphs.*

(i) *If $\mathcal{H}$ satisfies (F3), then $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(3)}, H_n^{(4)}\}$ for an integer $n \geqslant 2$.*

(ii) *If $\mathcal{H}$ satisfies (F4), then $\mathcal{H} \leqslant \{P_n, H_n^{(1)}, H_n^{(2)}, H_n^{(3)}, H_n^{(4)}\}$ for an integer $n \geqslant 2$.*

Proof. We first suppose that $\mathcal{H}$ satisfies (F3), and show that (i) holds. Let $n_0 = n_0(\mathcal{H})$ be a constant such that every connected $\mathcal{H}$-free graph of order at least n_0 has a spanning tree T such that $V_2(T)$ is an independent set of T. We may assume that $n_0 \geqslant 4$. Let $H \in \{P_{n_0}, H_{n_0}^{(3)}, H_{n_0}^{(4)}\}$, and let T be a spanning tree of H. Then there exist two adjacent vertices u and v of H such that both u and v are cut-vertices of H and $d_H(u) = d_H(v) = 2$. Hence by Lemma 5, $u, v \in V_2(T)$, and in particular, $V_2(T)$ is not an independent set of T. Since H and T are arbitrary, neither P_{n_0} nor $H_{n_0}^{(3)}$ nor $H_{n_0}^{(4)}$ is $\mathcal{H}$-free, and so

$$\mathcal{H} \leqslant \{P_{n_0}, H_{n_0}^{(3)}, H_{n_0}^{(4)}\} \tag{15}$$

Let T' be a spanning tree of $H_{n_0}^{(1)}$. Since T' is connected, $T'' := T'[\{x_i : 1 \leqslant i \leqslant n_0\}]$ is a tree. Then there exists an integer h with $1 \leqslant h \leqslant n_0$ such that $d_T(x_h) = 1$. Furthermore, it follows from Lemma 5 that $y_h \in V_2(T')$. This implies that $x_h y_h \in E(T')$, and hence $d_{T'}(x_h) = d_{T''}(x_h) + |\{y_h\}| = 2$. In particular, $x_h, y_h \in V_2(T')$, and so $V_2(T')$ is not an independent set of T'. Since T' is arbitrary, $H_{n_0}^{(1)}$ is not $\mathcal{H}$-free, and so

$$\mathcal{H} \leqslant \{H_{n_0}^{(1)}\}. \tag{16}$$

By (15) and (16), we obtain $\mathcal{H} \leqslant \{P_{n_0}, H_{n_0}^{(1)}, H_{n_0}^{(3)}, H_{n_0}^{(4)}\}$, which proves (i).

Next we suppose that $\mathcal{H}$ satisfies (F4), and show that (ii) holds. It is clear that $\mathcal{H}$ satisfies (F3). Hence by (i), there exists an integer $n_1 \geqslant 2$ such that

$$\mathcal{H} \leqslant \{P_{n_1}, H_{n_1}^{(1)}, H_{n_1}^{(3)}, H_{n_1}^{(4)}\}. \tag{17}$$

Since $\mathcal{H}$ satisfies (F4), there exist two constants $c'(\mathcal{H})$ and $n'(\mathcal{H})$ such that every connected $\mathcal{H}$-free graph G of order at least $n'(\mathcal{H})$ has a spanning tree T with $|V_2(T)| \leqslant c'(\mathcal{H})$. Let $r = \max\{c'(\mathcal{H}), n'(\mathcal{H}) - 1\}$. Note that r is a constant depending on $\mathcal{H}$ only. Let G be a connected $\mathcal{H}$-free graph. If $|V(G)| \leqslant n'(\mathcal{H}) - 1$, then any spanning trees T of G satisfy $|V_2(T)| \leqslant |V(G)| \leqslant n'(\mathcal{H}) - 1 \leqslant r$; if $|V(G)| \geqslant n'(\mathcal{H})$, then there exists a spanning tree T with $|V_2(T)| \leqslant c'(\mathcal{H}) \leqslant r$. In either case, G has a spanning tree T with $|V_2(T)| \leqslant r$. In particular, $\mathcal{H}$ satisfies (F2). Hence by Proposition 14, there exists an integer $n_2 \geqslant 2$ such that

$$\mathcal{H} \leqslant \{P_{n_2}, H_{n_2}^{(1)}, H_{n_2}^{(2)}\}. \tag{18}$$

Let $n^* = \max\{n_1, n_2\}$. Then by (17) and (18), we obtain $\mathcal{H} \leqslant \{P_{n^*}, H_{n^*}^{(1)}, H_{n^*}^{(2)}, H_{n^*}^{(3)}, H_{n^*}^{(4)}\}$, as desired. □

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