Congruence properties modulo powers of 2 for 4-regular partitions

Julia Q. D. Du[a] Dazhao Tang[b]

Submitted: Mar 3, 2023; Accepted: Aug 29, 2024; Published: Sep 20, 2024

Abstract

Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. Congruences properties modulo powers of 2 for $b_4(n)$ have been considered subsequently by Andrews–Hirschhorn–Sellers, Chen, Cui–Gu, Xia, Dai, and Ballantine–Merca. In this paper, we present an approach which can be utilized to prove "self-similar" congruence property satisfied by the generating function of $b_4(n)$. As an immediate consequence, one can obtain dozens of congruence families modulo powers of 2 enjoyed by $b_4(n)$. These results not only generalize some previous results, but also can be viewed as a supplement to Keith and Zanello's comprehensive study of the congruence properties for ℓ-regular partition functions. Finally, we also pose several conjectures on congruence families, internal congruence families and self-similar congruence properties for 4-, 8- and 16-regular partition functions.

Mathematics Subject Classifications: 11P83, 05A17

1 Introduction

Throughout, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),$$

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty.$$

A partition λ of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ such that $\sum_{i=1}^{r}\lambda_i = n$. The numbers λ_i are called the parts

[a] School of Mathematical Sciences, Hebei Key Laboratory of Computational Mathematics and Applications, Hebei Normal University, Shijiazhuang, P. R. China (qddu@hebtu.edu.cn).
[b] Corresponding author. School of Mathematical Sciences, Chongqing Normal University, Chongqing, P. R. China (dazhaotang@sina.com).

of the partition λ. Let $p(n)$ denote the number of partitions of n with the convention that $p(0) = 1$. The generating function of $p(n)$, derived by Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

In 1919, Ramanujan [25] discovered three celebrated partition congruences modulo 5, 7 and 11 satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1}$$
$$p(7n+5) \equiv 0 \pmod 7, \tag{2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{3}$$

Motivated by (1)–(3), many scholars considered congruence properties for various partition functions. Congruence properties of partition functions have motivated a tremendous amount of research for over a century.

For an integer $\ell \geqslant 2$, a partition is called ℓ-regular if all of the parts are not divisible by ℓ. In classical representation theory, ℓ-regular partitions of n parameterize the irreducible ℓ-modular representations of the symmetric group S_n when ℓ is prime [15]. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

In 2009, Andrews [1] introduced the partition function $ped(n)$ while considering the following classical identity of Lebesgue [17]:

$$\sum_{n=0}^{\infty} \left(\prod_{i=1}^{n} \frac{1+q^i}{1-q^i}\right) q^{n(n+1)/2} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} =: \sum_{n=0}^{\infty} ped(n)q^n.$$

Partition-theoretically, $ped(n)$ denotes the number of partitions of n with even parts distinct (and odd parts unrestricted). A slight calculation reveals that there are as many partitions of n with even parts distinct as 4-regular partitions of n, that is, for any $n \geqslant 0$, $ped(n) = b_4(n)$. Using some q-series manipulations, Andrews, Hirschhorn and Sellers [3, Theorem 3.5] proved that for any $\alpha \geqslant 1$ and $n \geqslant 0$,

$$b_4\left(3^{2\alpha+1}n + \frac{17 \times 3^{2\alpha} - 1}{8}\right) \equiv 0 \pmod 2, \tag{4}$$
$$b_4\left(3^{2\alpha+2}n + \frac{11 \times 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod 2, \tag{5}$$
$$b_4\left(3^{2\alpha+2}n + \frac{19 \times 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod 2. \tag{6}$$

With the help of the theory of Hecke eigenforms, Chen [6, p. 941] obtained the following congruence families modulo 4 satisfied by $b_4(n)$:

$$b_4\left(5^{2\alpha+2} + \frac{r \times 5^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{4}, \tag{7}$$

where $\alpha \geqslant 1$, $n \geqslant 0$ and $r \in \{13, 21, 29, 37\}$. Utilizing the p-dissection of Ramanujan's classical theta function $\psi(q)$ (see [10, Theorem 2.1]), Cui and Gu [10, Theorem 3.7] proved that for any $\alpha \geqslant 0$ and $n \geqslant 0$,

$$b_4\left(p^{2\alpha+2}n + \frac{(8i+p) \times p^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{2}, \tag{8}$$

where $p \geqslant 5$ is prime and $1 \leqslant i \leqslant p-1$. It is worth pointing out that (7) is a stronger form of the case $p = 5$ in (8). Later, Xia [29] found that the moduli in some cases in (5) and (6) can be improved. More precisely, he [29, Theorem 1] proved that for any $\alpha \geqslant 1$ and $n \geqslant 0$,

$$b_4\left(3^{4\alpha+4}n + \frac{11 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}, \tag{9}$$

$$b_4\left(3^{4\alpha+4}n + \frac{19 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}. \tag{10}$$

By using the theory of quadratic forms, Dai [12, Theorem 1.2] established an infinite family of congruences modulo 8 for $b_4(n)$. In particular, he derived that for any $\alpha \geqslant 0$ and $n \geqslant 0$,

$$b_4\left(7^{2\alpha+2}n + \frac{r \times 7^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{8}, \tag{11}$$

where $r \in \{13, 20, 27, 34, 41, 48\}$. Other congruence properties for $b_4(n)$ have been considered successively by Gordon and Ono [14], Pennison [23], Lovejoy and Osburn [18], Chen [7, 8], Merca [19], and Cui and Gu [11].

In a recent paper, Ballantine and Merca [4] derived some congruences modulo 16 and 64 for $b_4(n)$ by using the Smoot's `Mathematica` implementation [26] of Radu's algorithm on Ramanujan–Kolberg identities for partition functions. More specifically, they [4, Theorems 5.1 and 5.4] proved that

$$b_4(25n + \alpha) \equiv 0 \pmod{16}, \qquad \alpha \in \{8, 13, 18, 23\}, \tag{12}$$

$$b_4(49n + \beta) \equiv 0 \pmod{64}, \qquad \beta \in \{13, 20, 27, 34, 41, 48\}. \tag{13}$$

Moreover, in a recent paper, Keith and Zanello [16] studied systematically the density of odd values in $b_\ell(n)$, in particular establishing lacunarity modulo 2 for specified coefficients; self-similar congruence properties modulo 2; and congruences families in arithmetic progression. Further, for any $\ell \leqslant 28$, they either established new results of these types where none were known, extend previous ones, or conjectured that such results are impossible. For example, they [16, Theorem 7] proved the following self-similar congruence property for $b_3(n)$:

$$\sum_{n=0}^{\infty} b_3(26n+14)q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{13n} \pmod{2}.$$

Obviously, (12) is a stronger form of the case $\alpha = 0$ in (7), and (13) is a stronger form of the case $\alpha = 0$ in (11). Moreover, both (7) and (11) are the stronger forms of the cases $p = 5$ and $p = 7$ in (8), respectively. Motivated by (4)–(11) and the work of Keith and Zanello, there are two natural questions. One is whether there exist corresponding congruence families which contain (12) and (13) as special cases. The other is whether there are some self-similar congruence properties modulo powers of 2 for $b_4(n)$. In this paper, we consider the following self-similar congruence properties enjoyed by the generating function of $b_4(n)$:

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right)q^n \equiv c_p \sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{2^k},$$

where $p \geqslant 5$ is prime, $k \geqslant 1$ and c_p is a constant depending on p.

Theorem 1. *Let S be defined by*

$$\begin{aligned} S \in \{&(5,4),(7,6),(11,2),(13,4),(19,2),(23,6),(29,4),(31,8),(37,4),\\ &(43,2),(47,7),(53,4),(59,2),(61,4),(67,2),(71,6),(79,7),\\ &(83,2),(101,4),(103,6),(107,2),(109,4),(127,11),(131,2)\}. \end{aligned} \tag{14}$$

Then for any $(p,k) \in S$,

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right)q^n \equiv c_p \sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{2^k}, \tag{15}$$

where c_p is given in the following table:

Table 1: A table of values of c_p

p	5	7	11	13	19	23	29	31	37	43	47	53
c_p	3	9	1	11	1	25	11	33	3	1	49	3
p	59	61	67	71	79	83	101	103	107	109	127	131
c_p	1	11	1	9	81	1	3	41	1	11	129	1

Remark 2. Two remarks on Theorem 1 are in order. First, all powers of the moduli in (15) are best possible. Moreover, the missing pairs of (14) are covered by (8). In other words, the moduli in (15) are always 2 in these cases.

As an immediate consequence of (15), we establish the following infinite families of congruences and internal congruences enjoyed by $b_4(n)$. From this perspective, we give a positive answer to the first question as mentioned before. For a given formal power series $\sum_{n=0}^{\infty} f(n)q^n$, an internal congruence of $f(n)$ is a congruence of the form

$$f(An+B) \equiv \lambda f(Cn+D) \pmod{M},$$

where λ is an integer and M is a positive integer, $An+B$ and $Cn+D$ are certain arithmetic progressions.

Corollary 3. *Let S be defined as in* (14)*. Then for any $(p,k) \in S$, $\alpha \geqslant 0$ and $1 \leqslant i \leqslant p-1$,*

$$b_4\left(p^{2\alpha+2}n + \frac{(8i+p) \times p^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{2^k}. \tag{16}$$

Moreover, for any $n \geqslant 0$,

$$b_4\left(p^2 n + \frac{p^2-1}{8}\right) \equiv c_p b_4(n) \pmod{2^k},$$

where c_p is given in Table 1.

The following theorem provides another self-similar congruence property modulo 4 satisfied by $b_4(n)$. Compared to Theorem 1, the self-similar congruence property in the following theorem is valid for any prime $p \geqslant 5$.

Theorem 4. *Let $p \geqslant 5$ be a prime number such that $p \equiv 3 \pmod 4$. Then*

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right) q^n \equiv \sum_{n=0}^{\infty} b_4(n) q^{pn} \pmod 4. \tag{17}$$

The rest of this paper is organized as follows. In Section 2, we introduce some terminology and collect necessary results in the theory of modular forms. Section 3 gives the proofs of Theorem 1 and Corollary 3. In Section 4, we provide a proof of Theorem 4. We conclude this paper with some remarks, and pose some conjectures on self-similar congruence properties on $b_4(n)$, $b_8(n)$ and $b_{16}(n)$ in Section 5.

2 Preliminaries

We first recall some terminology in the theory of modular forms. The full modular group is given by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\},$$

and for a positive integer N, the congruence subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\colon c \equiv 0 \pmod{N} \right\}.$$

Let γ be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ hereinafter. Let γ act on $\tau \in \mathbb{C} \cup \{\infty\}$ by the linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

This is understood to mean that if $c \neq 0$, then $\gamma(-d/c) = \infty$, and $\gamma\infty = a/c$; if $c = 0$ then $\gamma\infty = \infty$.

Let N, k be positive integers and $\mathbb{H} = \{\tau \in \mathbb{C}\colon \mathrm{Im}(\tau) > 0\}$. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a modular function of weight k for $\Gamma_0(N)$, if it satisfies the following two conditions:

(1) for all $\gamma \in \Gamma_0(N)$, $f(\gamma\tau) = (c\tau + d)^k f(\tau)$;

(2) for any $\gamma \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $q_{w_\gamma} = e^{2\pi i\tau/w_\gamma}$, and

$$w_\gamma = \frac{N}{\gcd(c^2, N)}.$$

In particular, if $n_\gamma \geqslant 0$ for all $\gamma \in \Gamma$, then we call that f is a modular form of weight k for $\Gamma_0(N)$.It is known that if $f_1(\tau)$ and $f_2(\tau)$ are modular functions of weight k_1 and k_2 for $\Gamma_0(N)$, respectively, then $f_1(\tau)f_2(\tau)$ is a modular function of weight $k_1 + k_2$ for $\Gamma_0(N)$.

A modular function with weight 0 for $\Gamma_0(N)$ is referred to as a modular function for $\Gamma_0(N)$. For a modular function $f(\tau)$ of weight k with respect to $\Gamma_0(N)$, the order of $f(\tau)$ at the cusp $a/c \in \mathbb{Q} \cup \{\infty\}$ is defined by

$$\mathrm{ord}_{a/c}(f) = n_\gamma$$

for some $\gamma \in \Gamma$ such that $\gamma\infty = a/c$. It is known that $\mathrm{ord}_{a/c}(f)$ is well-defined, see [13, p. 72].

Radu [24] developed the Ramanujan–Kolberg algorithm to derive Ramanujan–Kolberg identities on a class of partition functions defined in terms of eta-quotients using modular functions for $\Gamma_0(N)$. A description of the Ramanujan–Kolberg algorithm can be found in Paule and Radu [22]. Smoot [26] developed a Mathematica package `RaduRK` to implement Radu's algorithm.

Let the partition function $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta, q^\delta)_\infty^{r_\delta}, \tag{18}$$

where M, δ are positive integers, and r_δ are integers. For any $m \geqslant 1$ and $0 \leqslant t \leqslant m-1$, Radu [24] defined

$$g_{m,t}(\tau) = q^{(t+\ell)/m} \sum_{n=0}^{\infty} a(mn+t)q^n, \tag{19}$$

where

$$\ell = \frac{1}{24} \sum_{\delta|M} \delta r_\delta,$$

and gave a criterion for a function involving $g_{m,t}(\tau)$ to be a modular function with respect to $\Gamma_0(N)$, where N satisfies the following: let $\kappa = \gcd(1-m^2, 24)$,

1. for every prime p, $p\,|\,m$ implies $p\,|\,N$;

2. for every $\delta\,|\,M$ with $r_\delta \neq 0$, $\delta\,|\,M$ implies $\delta\,|\,mN$;

3. $\kappa m N^2 \sum_{\delta|M} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}$;

4. $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$;

5. $\frac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} \,|\, N$;

6. if $2\,|\,m$, then $\kappa N \equiv 0 \pmod{4}$ and $8\,|\,Ns$, or $2\,|\,s$ and $8\,|\,N(1-j)$, where $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$, and $j, s \in \mathbb{Z}$, j is odd.

Given a positive integer n and an integer x, we denote by $[x]_n$ the residue class of x modulo n. Let

$$\mathbb{Z}_n^* = \{[x]_n \in \mathbb{Z}_n \colon \gcd(x,n) = 1\} \qquad \text{and} \qquad \mathbb{S}_n = \{y^2 \colon y \in \mathbb{Z}_n^*\}.$$

Define the set

$$P_m(t) = \left\{ \left[ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \right]_m \colon s \in \mathbb{S}_{24m} \right\}.$$

The Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H}$.

Theorem 5. [24, Theorem 45] *For a partition function $a(n)$ defined as in* (18)*, and integers $m \geqslant 1$, $0 \leqslant t \leqslant m-1$, suppose that N is a positive integer satisfying the conditions* (1)–(6)*. Let*

$$F(\tau) = \prod_{\delta|N} \eta^{s_\delta}(\delta\tau) \prod_{t' \in P_m(t)} g_{m,t'}(\tau),$$

where s_δ are integers. Then $F(\tau)$ is a modular function for $\Gamma_0(N)$ if and only if s_δ satisfy the following

(1) $|P_m(t)| \sum_{\delta|M} r_\delta + \sum_{\delta|N} s_\delta = 0$;

(2) $\sum_{t' \in P_m(t)} \frac{(1-m^2)(24t' + \sum_{\delta|M} \delta r_\delta)}{m} + |P_m(t)| m \sum_{\delta|M} \delta r_\delta + \sum_{\delta|N} \delta s_\delta \equiv 0 \pmod{24}$;

(3) $|P_m(t)| mN \sum_{\delta|M} \frac{r_\delta}{\delta} + \sum_{\delta|N} \frac{N}{\delta} s_\delta \equiv 0 \pmod{24}$;

(4) $\left(\prod_{\delta|M} (m\delta)^{|r_\delta|}\right)^{|P_m(t)|} \prod_{\delta|N} \delta^{|s_\delta|}$ *is a square.*

Radu also gave lower bounds of the orders of $F(\tau)$ at cusps of $\Gamma_0(N)$.

Theorem 6. [24, Theorem 47] *For a partition function $a(n)$ defined as in* (18)*, and integers $m \geqslant 1$, $0 \leqslant t \leqslant m-1$, let*

$$F(\tau) = \prod_{\delta|N} \eta^{s_\delta}(\delta\tau) \prod_{t' \in P_m(t)} g_{m,t'}(\tau),$$

be a modular function for $\Gamma_0(N)$, where s_δ are integers and N satisfies the conditions (1)–(6)*. Let $\{s_1, s_2, \ldots, s_\epsilon\}$ be a complete set of inequivalent cusps of $\Gamma_0(N)$, and for each $1 \leqslant i \leqslant \epsilon$, let $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma$ be such that $\gamma_i \infty = s_i$. Then*

$$\mathrm{ord}_{s_i}(F(\tau)) \geqslant \frac{N}{\gcd(c_i^2, N)} \big(|P_m(t)| p(\gamma_i) + p^*(\gamma_i)\big),$$

where

$$p(\gamma_i) = \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a_i + \kappa\lambda c_i), mc_i)}{\delta m}, \tag{20}$$

and

$$p^*(\gamma_i) = \frac{1}{24} \sum_{\delta|N} s_\delta \frac{\gcd^2(\delta, c_i)}{\delta}. \tag{21}$$

The following theorem of Sturm [27] plays an important role in proving congruences using the theory of modular forms.

Theorem 7. [27, Theorem 1] *Let k be an integer and $g(\tau) = \sum_{n=0}^{\infty} c(n) q^n$ a modular form of weight k for $\Gamma_0(N)$. For any given positive integer u, if $c(n) \equiv 0 \pmod{u}$ holds for all $n \leqslant \frac{k}{12} N \prod_{p|N,\, p \text{ prime}} \left(1 + \frac{1}{p}\right)$, then $c(n) \equiv 0 \pmod{u}$ holds for any $n \geqslant 0$.*

3 Proofs of Theorem 1 and Corollary 3

This section is devoted to the proofs of Theorem 1 and Corollary 3.

The following lemma plays a vital role in the proof of Theorem 1.

Lemma 8. *For any prime $p \geqslant 5$, let $k_1 = \lceil (p^2-1)/(32p) \rceil$ and $k_2 = \lceil (3p^2-3)/(32p^2) \rceil$. Then for any constant c, we have*

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\left(q^{p/8}\frac{\eta(p\tau)}{\eta(4p\tau)}\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n - c\right)$$

is a modular form of weight $12k_1 + 4k_2$ for $\Gamma_0(4p)$.

Proof. Since the generating function of $b_4(n)$ is

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}, \tag{22}$$

taking $M = 4$, $(r_1, r_4) = (-1, 1)$, $m = p$, $t = (p^2-1)/8$ in Theorem 5 yields that $N = 4p$ satisfies the conditions (1)–(6), and for $(s_1, s_2, s_4, s_p, s_{2p}, s_{4p}) = (0, 0, 0, 1, 0, -1)$, we get that

$$F(\tau) = q^{p/8}\frac{\eta(p\tau)}{\eta(4p\tau)}\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n$$

is a modular function for $\Gamma_0(4p)$.

From [9, p. 354], the set of inequivalent cusps of $\Gamma_0(4p)$ is given by

$$\left\{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{p}, \frac{1}{2p}, \infty\right\}.$$

Next we compute the lower bounds of $F(\tau)$ at the cusps of $\Gamma_0(4p)$. For $s = 0$, let

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

We have $\gamma\infty = 0$. In view of (20) and (21), we get

$$p(\gamma) = -\frac{p}{32} \qquad \text{and} \qquad p^*(\gamma) = \frac{1}{32p}.$$

Thus, from Theorem 6, we obtain that

$$\mathrm{ord}_0(F(\tau)) \geqslant -\frac{p^2-1}{8}.$$

In the same vein, we deduce that

$$\operatorname{ord}_{1/2}(F(\tau)) \geqslant 0, \qquad \operatorname{ord}_{1/4}(F(\tau)) \geqslant 0,$$
$$\operatorname{ord}_{1/p}(F(\tau)) \geqslant \frac{p^2-1}{8p}, \qquad \operatorname{ord}_{1/2p}(F(\tau)) \geqslant 0,$$

and

$$\operatorname{ord}_{\infty}(F(\tau)) \geqslant -\frac{p^2-1}{8p}.$$

Thus, we get

$$\operatorname{ord}_0(F(\tau)-c) \geqslant -\frac{p^2-1}{8}, \quad \operatorname{ord}_{1/2}(F(\tau)-c) \geqslant 0, \quad \operatorname{ord}_{1/4}(F(\tau)-c) \geqslant 0,$$
$$\operatorname{ord}_{1/p}(F(\tau)-c) \geqslant 0, \quad \operatorname{ord}_{1/2p}(F(\tau)-c) \geqslant 0, \quad \operatorname{ord}_{\infty}(F(\tau)-c) \geqslant -\frac{p^2-1}{8p}.$$

By Theorem 1.64 and Theorem 1.65 in [21], one easily obtains that

$$F_1(\tau) = \eta^{24}(\tau) \qquad \text{and} \qquad F_2(\tau) = \frac{\eta^{16}(2p\tau)}{\eta^8(p\tau)}$$

are modular forms with weight 12 and 4 for $\Gamma_0(4p)$, respectively, and the orders at the cusps of $\Gamma_0(4p)$ are

$$\begin{aligned}
\operatorname{ord}_0(F_1(\tau)) &= 4p, & \operatorname{ord}_{1/2}(F_1(\tau)) &= p, & \operatorname{ord}_{1/4}(F_1(\tau)) &= p,\\
\operatorname{ord}_{1/p}(F_1(\tau)) &= 4, & \operatorname{ord}_{1/2p}(F_1(\tau)) &= 1, & \operatorname{ord}_{\infty}(F_1(\tau)) &= 1,\\
\operatorname{ord}_0(F_2(\tau)) &= 0, & \operatorname{ord}_{1/2}(F_2(\tau)) &= 1, & \operatorname{ord}_{1/4}(F_2(\tau)) &= 1,\\
\operatorname{ord}_{1/p}(F_2(\tau)) &= 0, & \operatorname{ord}_{1/2p}(F_2(\tau)) &= p, & \operatorname{ord}_{\infty}(F_2(\tau)) &= p.
\end{aligned}$$

Therefore, we obtain that the orders of $F_1^{k_1}(\tau)F_2^{k_2}(\tau)(F(\tau)-c)$ at all cusps of $\Gamma_0(4p)$ are nonnegative. Since both $F(\tau)$ and c are modular functions with weight 0 for $\Gamma_0(4p)$, we conclude that $F_1^{k_1}(\tau)F_2^{k_2}(\tau)(F(\tau)-c)$ is a modular form with weight $12k_1+4k_2$ for $\Gamma_0(4p)$. The proof is therefore complete. $\square$

Now, we turn to prove Theorem 1.

Proof of Theorem 1. By Lemma 8 and Sturm's Theorem, in order to prove

$$\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n \equiv c_p \frac{(q^{4p};q^{4p})_\infty}{(q^p;q^p)_\infty} \pmod{2^k}$$

for some $k \geqslant 1$, we only need to check the coefficients of the first $l_p = 2(p+1)(3k_1+k_2)$ terms of the expansion for

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\left(q^{p/8}\frac{\eta(p\tau)}{\eta(4p\tau)}\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n - c_p\right) \tag{24}$$

Table 2: A table of values of l_p

p	5	7	11	13	19	23	29	31	37	43	47	53
l_p	48	64	96	112	160	192	240	256	532	616	672	756
p	59	61	67	71	79	83	101	103	107	109	127	131
l_p	840	868	1360	1440	1600	1680	2652	2704	2808	2860	3328	4224

are congruent to 0 modulo 2^k. The corresponding l_p of Theorem 1 are displayed in Table 2.

This completes the proof of Theorem 1. $\square$

Remark 9. From Table 2, one can see that the l_p values are growing quite rapidly, so that the time of checking the first l_p coefficients in (24) is also growing quite expeditiously. For example, it only takes less than 25 seconds to check the first l_p coefficients in (24) for $5 \leqslant p \leqslant 23$, and it takes about 1 minute to check the first l_p coefficients in (24) for $p = 29$, 31. However, in order to test the first l_p coefficients in (24) for $p = 37$, it will take about 13 minutes.

Next, we prove Corollary 3.

Proof of Corollary 3. In view of (15), we get for any $(p, k) \in S$,

$$b_4\left(p^2 n + \frac{p^2 - 1}{8}\right) \equiv c_p b_4(n) \pmod{2^k}, \tag{25}$$

and for $1 \leqslant i \leqslant p - 1$,

$$b_4\left(p^2 n + \frac{(8i + p) \times p - 1}{8}\right) \equiv 0 \pmod{2^k}. \tag{26}$$

Iterating $\alpha - 1$ times in (25), we have

$$b_4\left(p^{2\alpha} n + \frac{p^{2\alpha} - 1}{8}\right) \equiv c_p^\alpha b_4(n) \pmod{2^k}.$$

Substituting (26) into the above congruence gives (16). $\square$

4 Proof of Theorem 4

In this section, we give a proof of Theorem 4.

Before stating the proof of Theorem 4, we need to recall Ramanujan's theta function, given by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, \qquad |ab| < 1, \tag{27}$$

where the last identity in (27) is the well-known Jacobi triple product [2, p. 17, Eq. (1.4.8)]. Two important cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{28}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

Replacing q by $-q$ in (28) yields that

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.$$

The following p-dissections for $\varphi(-q)$ and $\psi(q)$ play an important role in the proof of Theorem 4.

Lemma 10. *Let $p \geqslant 5$ be prime. Then*

$$\varphi(-q) = \varphi\big(-q^{p^2}\big) + 2\sum_{j=1}^{(p-1)/2} q^{j^2} f\big(-q^{p^2+2pj}, -q^{p^2-2pj}\big), \tag{29}$$

$$\psi(q) = \sum_{n=0}^{(p-3)/2} q^{k(k+1)/2} f\Big(q^{(p^2+(2k+1)p)/2}, q^{(p^2-(2k+1)p)/2}\Big) + q^{(p^2-1)/8}\psi\big(q^{p^2}\big), \tag{30}$$

Further, for $0 \leqslant k \leqslant (p-3)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Proof. The identity (29) follows immediately from [5, p. 49]. The identity (30) appears in [10, Theorem 2.1]. □

Now, we are in a position to prove Theorem 4.

The proof of Theorem 4. From (22) we find that

$$\sum_{n=0}^{\infty} b_4(n) q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^4} \cdot \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty (q^4;q^4)_\infty} \equiv \varphi(-q^2)\cdot\psi(q) \pmod{4}. \tag{31}$$

For a prime $p \geqslant 5$, $0 \leqslant j \leqslant (p-1)/2$, $0 \leqslant k \leqslant (p-1)/2$, let us consider the following congruence equation

$$2j^2 + \frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod{p},$$

which implies that

$$(4j)^2 + (2k+1)^2 \equiv 0 \pmod{p}.$$

Since $p \equiv 3 \pmod{4}$, then we have $\left(\frac{-1}{p}\right) = -1$, we further get $j = 0$ and $k = (p-1)/2$, where $\left(\frac{\bullet}{\bullet}\right)$ is the Legendre symbol. Substituting (29) and (30) into (31), we find that

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right)q^n \equiv \varphi(-q^{2p})\psi(q^p) \equiv \sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{4}, \tag{32}$$

where we have used (31) in the last congruence. The congruence (17) thus follows.

This completes the proof of Theorem 4. □

5 Final remarks

We conclude this paper with several remarks.

Firstly, the numerical evidence suggests the following self-similar congruence property modulo 4 for $b_4(n)$, which contains Theorem 4 as a special case.

Conjecture 11. Let $p \geqslant 5$ be a prime number such that $p \not\equiv 1, 17 \pmod{24}$. Then

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right)q^n \equiv c_p \sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{4},$$

where $c_p = -1$ or 1.

Secondly, the powerful result of Gordon and Ono [14, Theorem 1] suggests the following identity:

$$\lim_{X\to\infty} \frac{\#\{0 \leqslant n < X \colon b_{2^k}(n) \equiv 0 \pmod{2^m}\}}{X} = 1, \tag{33}$$

where $k \geqslant 1$ and $m \geqslant 1$. Quite recently, Merca [20] established some congruences modulo 4 and 8 for $b_2(n)$ by utilizing the Smoot's `Mathematica` implementation of Radu's algorithm. More precisely, he [20, Theorem 1] proved that for any $n \geqslant 0$ and $1 \leqslant i \leqslant 4$,

$$b_2\big(5(5n+i)+1\big) \equiv 0 \pmod{4}.$$

Using some q-series identities and iterative computations, the second author [28] proved a large number of internal congruences and congruences modulo powers of 2 for $b_2(n)$. For example, he proved that for any $n \geqslant 0$,

$$b_2\left(5^{256}n + \frac{5^{256}-1}{24}\right) \equiv 257\, b_2(n) \pmod{512} \tag{34}$$

and

$$b_2\left(5^{256\alpha}n+\frac{(24i+5)\times 5^{256\alpha-1}-1}{24}\right)\equiv 0 \pmod{512}, \tag{35}$$

where $\alpha \geqslant 1$. Further, the second author conjectured that there are some internal congruence families and congruence families modulo high powers of 2 satisfied by $b_2(n)$. The internal congruence (34) and congruence (35) are just two special cases. Following a similar strategy of proving (15), one can derive the following congruence relations:

$$\sum_{n=0}^{\infty} b_4(125n+78)q^n \equiv 25\sum_{n=0}^{\infty} b_4(n)q^{5n} \pmod{32}, \tag{36}$$

$$\sum_{n=0}^{\infty} b_4(343n+300)q^n \equiv 17\sum_{n=0}^{\infty} b_4(n)q^{7n} \pmod{128}, \tag{37}$$

$$\sum_{n=0}^{\infty} b_4(1331n+1830)q^n \equiv 5\sum_{n=0}^{\infty} b_4(n)q^{11n} \pmod{8}, \tag{38}$$

here we only check the first 96, 352 and 2040 terms, respectively. Based on (36)–(38) and the second author's conjecture on $b_2(n)$ (see [28]), we pose the following conjecture.

Conjecture 12. Let $p \geqslant 5$ be a prime number. If there exist two positive integers c_p and k_p such that

$$\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n \equiv c_p\sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{2^{k_p}},$$

$$\sum_{n=0}^{\infty} b_4\left(pn+\frac{p^2-1}{8}\right)q^n \not\equiv c_p\sum_{n=0}^{\infty} b_4(n)q^{pn} \pmod{2^{k_p+1}},$$

then for any $\alpha \geqslant 1$ and $n \geqslant 0$,

$$b_4\left(p^{2^\alpha}n+\frac{p^{2^\alpha}-1}{8}\right)\equiv \widehat{c}_{p,\alpha}b_4(n) \pmod{2^{k_p+\alpha-1}} \tag{39}$$

and

$$b_4\left(p^{2^\alpha}n+\frac{(8i+p)\times p^{2^\alpha-1}-1}{8}\right)\equiv 0 \pmod{2^{k_p+\alpha-1}}, \tag{40}$$

where $\widehat{c}_{p,k_p}$ is a constant related to p and α.

Thirdly, Cui and Gu [10] also derived the following self-similar congruence properties for $b_8(n)$ and $b_{16}(n)$, namely, for any prime $p \geqslant 5$,

$$\sum_{n=0}^{\infty} b_8\left(pn+\frac{7p^2-7}{24}\right)q^n \equiv \sum_{n=0}^{\infty} b_8(n)q^{pn} \pmod{2}, \qquad \text{if } p\equiv 5 \pmod{6}, \tag{41}$$

$$\sum_{n=0}^{\infty} b_{16}\left(pn+\frac{5p^2-5}{8}\right)q^n \equiv \sum_{n=0}^{\infty} b_{16}(n)q^{pn} \pmod{2}, \qquad \text{if } p\equiv 3 \pmod{4}. \tag{42}$$

Based on numerical evidence, we pose the following conjectures, which can be viewed as the corresponding complements of (41) and (42).

Conjecture 13. Let $p \geqslant 5$ be a prime number.

(i) If $p \not\equiv 7 \pmod{24}$, then

$$\sum_{n=0}^{\infty} b_8\left(pn + \frac{7p^2-7}{24}\right)q^n \equiv \sum_{n=0}^{\infty} b_8(n)q^{pn} \pmod{2}.$$

(ii) If $p \equiv 7 \pmod{24}$, then

$$\sum_{n=0}^{\infty} b_8\left(pn + \frac{7p^2-7}{24}\right)q^n \not\equiv \sum_{n=0}^{\infty} b_8(n)q^{pn} \pmod{2}.$$

(iii) If $p \equiv 13, 17, 19, 23 \pmod{24}$, then

$$\sum_{n=0}^{\infty} b_8\left(pn + \frac{7p^2-7}{24}\right)q^n \equiv \sum_{n=0}^{\infty} b_8(n)q^{pn} \pmod{4}.$$

Conjecture 14. Let $p \geqslant 5$ be a prime number such that $p \equiv 17 \pmod{24}$. Then

$$\sum_{n=0}^{\infty} b_{16}\left(pn + \frac{5p^2-5}{8}\right)q^n \equiv \sum_{n=0}^{\infty} b_{16}(n)q^{pn} \pmod{2}.$$

Finally, it is natural to ask whether there exist some internal congruence families and congruence families similar to (39) and (40) for $b_{2^k}(n)$ with $k \geqslant 3$.

Acknowledgements

The authors would like to acknowledge the anonymous referees for their careful reading of the manuscript and many constructive suggestions, which improved the quality of the paper to a great extent. Julia Q. D. Du was partially supported by the National Natural Science Foundation of China (No. 12201177), the Hebei Natural Science Foundation (Nos. A2024205012 and A2023205045), the Science and Technology Project of Hebei Education Department (No. BJK2023092), the China Scholarship Council, and the Program for Foreign Experts of Hebei Province. Dazhao Tang was partially supported by the National Natural Science Foundation of China (No. 12201093), the Natural Science Foundation Project of Chongqing CSTB (No. CSTB2022NSCQ–MSX0387), the Science and Technology Research Program of Chongqing Municipal Education Commission (No. KJQN202200509), and the Doctoral start-up research Foundation (No. 21XLB038) of Chongqing Normal University.

References

[1] G.E. Andrews, Partitions with distinct evens, in: I.S. Kotsireas, E.V. Zima (Eds.), *Advances in Combinatorial Mathematics*, Springer, Berlin, 2009, pp. 31–37.

[2] G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook, Part II*, (2005), Springer-Verlag, New York.

[3] G.E. Andrews, M.D. Hirschhorn, J.A. Sellers, *Arithmetic properties of partitions with even parts distinct, Ramanujan J.* **23** (2010), no. 1-3, 169–181.

[4] C. Ballantine, M. Merca, 4-regular partitions and the pod function, *Quaes. Math.* **46** (2023), no. 10, 2027–2051.

[5] B.C. Berndt, *Ramanujan's Notebook, Part III*, Springer-Verlag, New York, 1991.

[6] S.-C. Chen, On the number of partitions with distinct even parts, *Discrete Math.* **311** (2011), no. 12, 940–943.

[7] S.-C. Chen, Congruences for the number of k-tuple partitions with distinct even parts, *Discrete Math.* **313** (2013), no. 15, 1565–1568.

[8] S.-C. Chen, The number of partitions with distinct even parts revisited, *Discrete Math.* **346** (2023), no. 7, Paper No. 113398, 8 pp.

[9] K.S. Chua, M.L. Lang, Congruence subgroups associated to the monster, *Experiment. Math.* **13** (2004), no. 3, 343–360.

[10] S.-P. Cui, N.S.S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. in Appl. Math.* **51** (2013), no. 4, 507–523.

[11] S.-P. Cui, N.S.S. Gu, Congruences for ℓ-regular partitions and bipartitions, *Rocky Mountain J. Math.* **50** (2020), no. 2, 513–526.

[12] H. Dai, Congruences for the number of partitions and bipartitions with distinct even parts, *Discrete Math.* **338** (2015), no. 3, 133–138.

[13] F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005.

[14] B. Gordon, K. Ono, Divisibility of certain partition functions by powers of primes, *Ramanujan J.* **1** (1997), no. 1, 25–34.

[15] G. James, A. Kerber, *The Representation of the Symmetric Group*, Addison-Wesley, Reading, 1979.

[16] W.J. Keith, F. Zanello, Parity of the coefficients of certain eta-quotients, *J. Number Theory* **235** (2022), 275–304.

[17] V.A. Lebesgue, Sommation de quelques series, *J. Math. Pures Appl.* **5** (1840), 42–71.

[18] J. Lovejoy, R. Osburn, Quadratic forms and four partition functions modulo 3, *Integers* **11** (2011), A4, 6 pp.

[19] M. Merca, New relations for the number of partitions with distinct even parts, *J. Number Theory* **176** (2017), 1–12.

[20] M. Merca, Ramanujan-type congruences modulo 4 for partitions into distinct parts, *An. Şt. Univ. Ovidius Constanţa* **30** (2022), no. 3, 185–199.

[21] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Regional Conference Series in Mathematics, 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.

[22] P. Paule, S. Radu, Partition analysis, modular functions, and computer algebra. In: A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker, P. Tetali (eds.), *Recent Trends in Combinatorics*, IMA Vol. Math. Appl., 159, pp. 511–543. Springer, Cham, 2016.

[23] D. Pennison, The p^{α}-regular partition function modulo p^{j}, *J. Number Theory* **94** (2002), no. 2, 320–325.

[24] C.-S. Radu, An algorithmic approach to Ramanujan–Kolberg identities, *J. Symbolic Comput.* **68** (2015), no. 1, 225–253.

[25] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, *Proc. Camb. Philos. Soc.* **XIX** (1919), 207–210.

[26] N. A. Smoot, On the computation of identities relating partition numbers in arithmetic progressions with eta quotients: an implementation of Radu's algorithm, *J. Symbolic Comput.* **104** (2021), 276–311.

[27] J. Sturm, On the congruence of modular forms, in: *Number Theory*, pp. 275–280, Berlin, Springer, 1987.

[28] D. Tang, Congruence properties modulo powers of 2 for two partition functions, *J. Ramanujan Math. Soc.* **39** (2024), no. 2, 171–185.

[29] E.X.W. Xia, New infinite families of congruences modulo 8 for partitions with even parts distinct, *Electron. J. Combin.* **21** (2014), no. 4, #P4.8, 10 pp.