

# A localized approach to generalized Turán problems

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## Abstract

Generalized Turán problems ask for the maximum number of copies of a graph  $H$  in an  $n$ -vertex,  $F$ -free graph, denoted by  $\text{ex}(n, H, F)$ . We show how to extend the new, localized approach of Bradač, Malec, and Tompkins to generalized Turán problems. We weight the copies of  $H$  (typically taking  $H = K_t$ ), instead of the edges, based on the size of the largest clique, path, or star containing the vertices of the copy of  $H$ , and in each case prove a tight upper bound on the sum of the weights. The generalized edge Turán number  $\text{mex}(m, H, F)$  is the maximum number of copies of a graph  $H$  in an  $m$ -edge,  $F$ -free graph. A consequence of our new localized theorems is an asymptotic determination of  $\text{ex}(n, H, K_{1,r})$  for every  $H$  having at least one dominating vertex and  $\text{mex}(m, H, K_{1,r})$  for every  $H$  having at least two dominating vertices.

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## 1 Introduction

Extremal graph theory is often considered the study of how easily measured global graph parameters, such as the numbers of vertices and edges in a graph, influence its local substructures [10]. An archetypical result due to Turán describes which size cliques a graph is guaranteed to contain based on its order and size:

**Theorem** (Turán [25]). *Let  $n, r \geq 1$  be integers. If  $G$  is a  $K_{r+1}$ -free graph on  $n$  vertices (that is, no subgraph of  $G$  is isomorphic to the complete graph  $K_{r+1}$ ), then  $G$  contains at most  $\frac{n^2}{2}(1 - \frac{1}{r})$  edges. This is denoted*

$$\text{ex}(n, K_{r+1}) \leq \frac{n^2}{2} \left(1 - \frac{1}{r}\right).$$

*Furthermore, the  $K_{r+1}$ -free graph on  $n$  vertices with the greatest number of edges is the Turán graph,  $T_r(n)$ , in which the vertices of the graph are partitioned into  $r$  parts of sizes*

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as close to equal as possible, and vertices are adjacent if and only if they are in different parts.

Turán's theorem has been generalized by many authors. In 2015, Alon and Shikhelman [1], expanding on several sporadic results (e.g. [6, 17, 18]), introduced the generalized Turán number  $\text{ex}(n, H, F)$ , which denotes the greatest number of subgraphs of an  $F$ -free graph on  $n$  vertices that are isomorphic to  $H$ . When maximizing the number of copies of  $H \neq K_2$ , it is possible to fix the number of edges instead of the number of vertices. Radcliffe and Uzzell [24] introduced the generalized edge Turán number,  $\text{mex}(m, H, F)$ , which denotes the greatest number of subgraphs that are isomorphic to  $H$  in an  $F$ -free graph on  $m$  edges.

The quantities  $\text{ex}(n, H, F)$  and  $\text{mex}(m, H, F)$  have motivated many interesting results; see [9, 8, 14, 15, 16, 23] for an (incomplete) sample. However, these problems often stretch the notion of “easily measurable” properties on which extremal graph theory is based. Though it is not known to be NP-complete, counting the number of subgraphs of  $G$  that are isomorphic to  $H$  is considered a challenging computation problem. Therefore, practically, it may be difficult to determine whether a specific  $G$  contains a forbidden  $F$  even when the exact value of  $\text{ex}(n, H, F)$  is known.

Recently, Bradač [7], based on a conjecture of Balogh and Lidický, gave a fundamentally different generalization of Turán's theorem:

**Theorem** (Bradač [7]). *Let  $G$  be a graph on  $n$  vertices. For each edge  $e \in E(G)$ , define its weight  $w(e)$  as*

$$w(e) = \frac{k}{2(k-1)}$$

where  $k$  is the size of the largest clique in  $G$  containing  $e$ . Then

$$\sum_{e \in E(G)} w(e) \leq n^2/4.$$

This theorem has since been applied to Ramsey-Turán problems in [3, 4].

Bradač's result generalizes Turán's theorem in the following sense: if it is known that  $G$  is  $K_{r+1}$ -free, then, noting  $w(e)$  is decreasing in  $k$ , we see  $w(e) \geq r/(2(r-1))$  for every edge  $e$ . Thus

$$|E(G)| \cdot \frac{r}{(2(r-1))} \leq \sum_{e \in E(G)} w(e) \leq \frac{n^2}{4} \implies |E(G)| \leq \frac{n^2}{2} \left(1 - \frac{1}{r}\right).$$

The novelty of Bradač's result is in the local nature of the weight function. Rather than counting the total number of edges in the entire graph, a global property, the weight we assign to each edge depends only on the neighborhoods of the vertices of that edge which may be computed more efficiently.

Inspired by Bradač's result, Malec and Tompkins [22] investigated other results which could be “localized” in a similar fashion. In addition to giving a new proof of Bradač's

result, they proved a local version of another celebrated extremal graph theory result of Erdős and Gallai, as well as the LYMB inequality (a generalization of Sperner’s theorem on boolean lattices), a generalization of the Erdős-Ko-Rado theorem, and a theorem of Erdős and Szekeres on sequences.

Both Turán’s theorem and the theorem of Erdős and Gallai considered by Malec and Tompkins tell us something about the graph based on the number of edges it contains. Keeping in mind that an edge is a clique containing two vertices, natural generalizations of both results to cliques of larger size have been investigated. In this article, we show that these results, too, admit localized generalizations. In fact, we prove several local results by weighting the cliques or other subgraphs of  $G$  of a given size based on the maximum size of various substructures that contain them. In some cases these “generalizations” actually establish new extremal results.

Our results follow a general framework. Each theorem concerns a target subgraph  $H$ , which in many cases is a clique of some fixed size, and a family  $\mathcal{F} = \{F_1, F_2, \dots\}$  of graphs in which  $F_i \subseteq F_{i+1}$ . We first establish a size function which, given a copy of  $H$  in  $G$ , returns the largest  $F_i$  such that some subgraph of  $G$  isomorphic to  $F_i$  contains  $H$  in some meaningful way. Then we define a weight function which depends only on the size function of  $H$  and prove a bound on the sum of the weights of every copy of  $H$  in  $G$ . In each case we show that the weight function is a decreasing function of the size function, so a global upper bound on the size function implies a lower bound on the sum of the weights, and we recover a “non-localized” theorem.

The rest of this paper is arranged as follows. We begin with some notational conventions and preliminary results in Section 2. Then in Section 3, we weight  $t$ -cliques in  $G$  by the size of the largest clique containing them to generalize Zykov’s theorem, itself a direct extension of Turán’s theorem. We weight  $t$ -cliques by the longest path containing their vertices in Section 4, considering graphs of fixed order in Section 4.1 and graphs of fixed size in Section 4.2. We weight a broad class of graphs, including cliques, by the size of the largest star containing their vertices in Section 5.1, in which we prove a family of novel generalized Turán and edge Turán results. We also give a hypergraph version of one of these localized results in Section 5.2. We conclude with some open questions in Section 6.

## 2 Preliminaries

### 2.1 Notation

In addition to standard graph theoretic notation (see [5], for example), we establish the following conventions. We define the path graph  $P_n$  to have  $n$  vertices and  $n - 1$  edges and the star graph  $S_r$  to have  $r$  leaves (and thus  $r + 1$  total vertices). Given graphs  $G$  and  $H$ , we let  $\mathcal{N}(H, G)$  denote the number of subgraphs of  $G$  that are isomorphic to  $H$ . If  $\mathcal{N}(H, G) = 0$ , we say  $G$  is  $H$ -free. As mentioned in the introduction, if  $H$  and  $F$  are graphs, then

$$\text{ex}(n, H, F) = \max\{\mathcal{N}(H, G) : |V(G)| = n \text{ and } G \text{ is } F\text{-free}\}$$

and

$$\text{mex}(m, H, F) = \max\{\mathcal{N}(H, G) : |E(G)| = m \text{ and } G \text{ is } F\text{-free}\}.$$

We also have need to refer to the copies of  $H$  in  $G$ . When we do so for cliques, we refer to the sets of vertices that span a complete subgraph and write

$$\mathcal{K}^t(G) = \{S \subseteq V(G) : G[S] \cong K_t\}.$$

For a more general graph  $H$ , we write  $\mathcal{H}(G)$  to refer to the set of (not necessarily induced) subgraphs of  $G$  that are isomorphic to  $H$ .

The size function and weight function(s) in each section are denoted by the notation shown below.

|                    | Cliques    | Paths       | Stars (graphs) | Stars (hypergraphs) |
|--------------------|------------|-------------|----------------|---------------------|
| Section            | 3          | 4           | 5.1            | 5.2                 |
| Size function      | $\alpha_G$ | $\beta_G$   | $\theta_G$     | $x$                 |
| Weight function(s) | $w_G$      | $p_G, p'_G$ | $s_G^u$        | $s$                 |

## 2.2 Generalized binomial coefficients

We use generalized binomial coefficients when working with paths in Section 4 and hypergraphs in Section 5. For a real number  $x \geq k - 1$  and a natural number  $k$ , the *generalized binomial coefficient*  $\binom{x}{k}$  is defined as  $(x)(x-1) \cdots (x-k+1)/k!$ . When  $x < k - 1$ , we set  $\binom{x}{k} = 0$ . The function  $\binom{x}{k}$  is weakly increasing for all real numbers  $x$  and strictly increasing on  $x \geq k - 1$ .

**Observation 1.** For all  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ , we have

$$2^n \binom{x}{n} \leq \binom{2x}{n},$$

with strict inequality when  $2x + 1 > n \geq 2$ .

*Proof.* The right side is always non-negative and is positive for  $2x > n - 1$ . When  $x \leq n - 1$ , the left side is zero. When  $x > n - 1$ , the inequality is equivalent to

$$(2x)(2x-2) \cdots (2x-2n+2) \leq (2x)(2x-1) \cdots (2x-n+1),$$

so is strict when  $n \geq 2$ . □

In Section 4 we also use the following observation and theorem.

**Observation 2.** Let  $G$  be a graph having at least one edge. Write  $|E(G)|$  in the form  $\binom{x}{2}$ , where  $x \geq 2$  is real. Then  $|V(G)| \geq x$ .

*Proof.* If  $|V(G)| < x$  then  $|V(G)| \leq \lceil x \rceil - 1$ , so  $|E(G)| \leq \binom{\lceil x \rceil - 1}{2} < \binom{x}{2}$ . Here we used that the generalized binomial coefficient  $\binom{y}{2}$  is strictly increasing for all  $y \geq 1$ . □

**Theorem 3** (Lovász [20]). *Let  $t \geq 2$ . Let  $G$  be a graph. Write the number of edges of  $G$  in the form  $\binom{x}{2}$ , where  $x \geq 1$  is real. Then  $\mathcal{N}(K_t, G) \leq \binom{x}{t}$ .*

We also use generalized binomial coefficients in Section 5.2, where we introduce hypergraph definitions and notation before stating Theorem 17, a bound on the number of  $t$ -cliques in a  $q$ -uniform hypergraph based on the number of edges, which is a version of Lovász's approximate form of the Kruskal-Katona theorem. Theorem 3 is the special case of Theorem 17 corresponding to graphs.

### 3 Weighting by Maximum Clique Size

In this section we prove the following theorem which extends the localized version of Turán's theorem in [22] by assigning weights to cliques of any size, rather than just edges. Then we show that it simultaneously extends a different extension of Turán's theorem due to Zykov.

**Theorem 4.** *Let  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define*

$$\alpha_G(T) = \max\{k : T \subseteq V(S) \text{ for some } S \subseteq G \text{ s.t. } S \cong K_k\} \quad \text{and} \quad w_G(T) = \frac{\alpha_G(T)^t}{\binom{\alpha_G(T)}{t}}.$$

*Then  $w_G(T)$  is well-defined and decreasing in  $\alpha_G(T)$ , and*

$$w(G) = \sum_{T \in \mathcal{K}^t(G)} w_G(T) \leq n^t,$$

*with equality if and only if  $G$  is a balanced multipartite graph with at least  $t$  parts.*

Note that by setting  $t = 2$ , we recover the result which is Theorem 1 of both [7] and [22].

$$\sum_{T \in \mathcal{K}^2(G)} \frac{\alpha_G(e)^2}{\binom{\alpha_G(e)}{2}} = \sum_{e \in E(G)} \frac{2\alpha_G(e)}{\alpha_G(e) - 1} \leq n^2 \implies \sum_{e \in E(G)} \frac{\alpha_G(e)}{\alpha_G(e) - 1} \leq \frac{n^2}{2}.$$

*Proof.* First,  $\alpha_G(T) \geq t$  because  $G[T] \cong K_t$ , and therefore  $w_G(T)$  is well-defined. We observe  $w_G(T)$  is decreasing in  $\alpha_G(T)$  as we can write

$$w_G(T) = t! \cdot \frac{\alpha_G(T)^t}{\prod_{i=0}^{t-1} (\alpha_G(T) - i)} = t! \cdot \prod_{i=1}^{t-1} \left(1 + \frac{i}{\alpha_G(T) - i}\right),$$

which is a non-empty product of functions decreasing in  $\alpha_G(T)$ .

Let  $G$  be a graph on  $n \geq 3$  vertices. Note that if  $G$  contains an edge  $e$  that is not contained in any  $t$ -clique, then  $e$  also is not contained in any larger clique, so  $\mathcal{K}^t(G - e) = \mathcal{K}^t(G)$ ,  $w_{G-e}(T) = w_G(T)$  for every  $T \in \mathcal{K}^t(G)$ , and  $w(G) = w(G - e)$ . Therefore we may assume that every edge in  $G$  is contained in a  $t$ -clique. First assume there is  $r \geq t$  such

that  $G$  is complete  $r$ -partite with (non-empty) parts  $A_1, \dots, A_r$ . Then for each  $T \in \mathcal{K}^t(G)$  we have  $\alpha_G(T) = r$  and  $w_G(T) = r^t / \binom{r}{t}$ . This gives

$$\begin{aligned} w(G) &= \frac{r^t}{\binom{r}{t}} \mathcal{N}(K_t, G) \\ &= \frac{r^t}{\binom{r}{t}} \sum_{S \in \binom{[r]}{t}} \prod_{s \in S} |A_s| \end{aligned}$$

To bound this sum, we relax the condition that the parts have integral sizes. By symmetry, the real-valued polynomial function

$$f(x_1, \dots, x_r) = \sum_{S \in \binom{[r]}{t}} \prod_{i \in S} x_i$$

has a unique maximum when each  $x_i = n/r$ , in which case each of the  $\binom{r}{t}$  terms is  $(n/r)^t$ , and thus

$$w(G) \leq \frac{r^t}{\binom{r}{t}} \cdot \binom{r}{t} \cdot \left(\frac{n}{r}\right)^t = n^t$$

with equality if and only if  $|A_i| = n/r$ .

Thus we may assume  $G$  is not complete multipartite. We use a technique introduced by Zykov [27] sometimes called Zykov symmetrization. Suppose there are  $x, y, z \in V(G)$  such that  $x \sim z$  but  $x \not\sim y \not\sim z$ . We show that as long as such vertices exist, we can find  $G'$  on  $n$  vertices such that  $w(G') > w(G)$ . If no such vertices exist, then  $x \not\sim y$  and  $y \not\sim z$  implies  $x \not\sim z$ , which is to say nonadjacent vertices can be partitioned into equivalence classes and therefore  $G$  is complete multipartite. Thus producing such a  $G'$  reduces this case to the complete multipartite case, and furthermore proves any graph that meets the bound in the theorem must be complete multipartite.

For convenience, for  $v \in V(G)$ , define

$$w_G(v) = \sum_{\substack{T \in \mathcal{K}^t(G) \\ \text{s.t. } v \in T}} w_G(T).$$

Assume without loss of generality that  $w_G(x) \geq w_G(z)$  and consider two cases.

**Case 4.1.**  $w_G(x) > w_G(y)$ .

Introduce a new vertex  $x'$ , add edges so that  $N(x') = N(x)$ , and let  $G' = G - y + x'$ . Consider a clique  $K'$  (of any size) that is present in  $G'$  but not in  $G$ . Such a clique must contain  $x'$  and so must be contained in  $N_{G'}[x'] \cong N_G[x]$ . Therefore any  $T \in \mathcal{K}^t(G)$  contained in  $K'$  is also contained in some clique  $K \subseteq V(G)$  of the same size. Thus every  $T \in \mathcal{K}^t(G) \cap \mathcal{K}^t(G')$  has  $\alpha_G(T) \geq \alpha_{G'}(T)$ . As  $w_G(T)$  is decreasing in  $\alpha_G(T)$ , we have

$w_G(T) \leq w_{G'}(T)$ . Then

$$\begin{aligned}
\sum_{T \in \mathcal{K}^t(G')} w_{G'}(T) &= \sum_{T \in \mathcal{K}^t(G') \setminus \mathcal{K}^t(G)} w_{G'}(T) + \sum_{T \in \mathcal{K}^t(G') \cap \mathcal{K}^t(G)} w_{G'}(T) \\
&\geq \sum_{T \in \mathcal{K}^t(G') \setminus \mathcal{K}^t(G)} w_{G'}(T) + \sum_{T \in \mathcal{K}^t(G-y)} w_G(T) \\
&= w_{G'}(x') + \left( \sum_{T \in \mathcal{K}^t(G)} w_G(T) - w_G(y) \right) \\
&= w_G(x) + \sum_{T \in \mathcal{K}^t(G)} w_G(T) - w_G(y) \\
&> \sum_{T \in \mathcal{K}^t(G)} w_G(T).
\end{aligned}$$

**Case 4.2.**  $w_G(x) \leq w_G(y)$ .

In this case we introduce two new vertices,  $y'$  and  $y''$ , add edges such that  $N(y'') = N(y') = N(y)$ , and define  $G'' = G - x - z + y' + y''$ . As before, every clique  $K'$  that is in  $G''$  but not in  $G$  must contain  $y'$  or  $y''$  (but not both as  $y' \not\sim y''$ ). Thus once again  $K'$  must be contained in  $N_{G''}[y'] \cong N_G[y]$  or  $N_{G''}[y''] \cong N_G[y]$ , so every  $T \in \mathcal{K}^t(G) \cap \mathcal{K}^t(G'')$  has  $\alpha_G(T) \geq \alpha_{G''}(T)$  and  $w_G(T) \leq w_{G''}(T)$ . Therefore

$$\begin{aligned}
\sum_{T \in \mathcal{K}^t(G'')} w_{G''}(T) &= \sum_{T \in \mathcal{K}^t(G'') \setminus \mathcal{K}^t(G)} w_{G''}(T) + \sum_{T \in \mathcal{K}^t(G'') \cap \mathcal{K}^t(G)} w_{G''}(T) \\
&\geq \sum_{T \in \mathcal{K}^t(G'') \setminus \mathcal{K}^t(G)} w_{G''}(T) + \sum_{T \in \mathcal{K}^t(G-x-z)} w_G(T) \\
&= w_{G''}(y') + w_{G''}(y'') + \left( \sum_{T \in \mathcal{K}^t(G)} w_G(T) - w_G(x) - w_G(z) + \sum_{\substack{T \in \mathcal{K}^t(G) \\ \text{s.t. } x, z \in T}} w_G(T) \right) \\
&= 2w_G(y) + \sum_{T \in \mathcal{K}^t(G)} w_G(T) - w_G(x) - w_G(z) + \sum_{\substack{T \in \mathcal{K}^t(G) \\ \text{s.t. } x, z \in T}} w_G(T) \\
&\geq \sum_{T \in \mathcal{K}^t(G)} w_G(T) + \sum_{\substack{T \in \mathcal{K}^t(G) \\ \text{s.t. } x, z \in T}} w_G(T) \\
&> \sum_{T \in \mathcal{K}^t(G)} w_G(T),
\end{aligned}$$

where the last step holds because  $x \sim z$ , and our initial assumption guarantees every edge is contained in a  $t$ -clique.  $\square$

One can also view Theorem 4 as a generalization of a theorem of Zykov [27] (and independently Erdős [11]). In what is now considered the first generalized Turán result, Zykov proved that, among  $K_{r+1}$ -free graphs, the Turán graph  $T_r(n)$  maximizes not only the number of edges but the number of cliques of any size.

**Theorem 5** (Zykov [27]). *Let  $G$  be a  $K_{r+1}$ -free graph on  $n$  vertices. Then for any  $t \geq 1$ ,*

$$\mathcal{N}(K_t, G) \leq \binom{r}{t} \left(\frac{n}{r}\right)^t.$$

In fact, Zykov actually proved the stronger result that  $\text{ex}(n, K_t, K_{r+1}) = \mathcal{N}(K_t, T_r(n))$ , though these results agree when  $r \mid n$ .

We can prove Theorem 5 as a consequence of Theorem 4:

*Proof of Theorem 5.* Let  $G$  be a an  $n$ -vertex,  $K_{r+1}$ -free graph. Then for each  $T \in \mathcal{K}^t(G)$  we have  $\alpha_G(T) \leq r$  and, as  $w_G(T)$  is decreasing in  $\alpha_G(T)$ ,  $w_G(T) \geq r^t / \binom{r}{t}$ . Thus by Theorem 4,

$$\mathcal{N}(K_t, G) \cdot \frac{r^t}{\binom{r}{t}} = \sum_{T \in \mathcal{K}^t(G)} \frac{r^t}{\binom{r}{t}} \leq \sum_{T \in \mathcal{K}^t(G)} w_G(T) \leq n^t$$

and so

$$\mathcal{N}(K_t, G) \leq \frac{\binom{r}{t}}{r^t} \cdot n^t = \binom{r}{t} \left(\frac{n}{r}\right)^t. \quad \square$$

## 4 Weighting by Maximum Path Length

In 1959, Erdős and Gallai [12] proved that every graph with  $n$  vertices and  $m$  edges contains a path of length at least  $2m/n$ , and as a consequence  $\text{ex}(n, K_2, P_{r+1}) \leq \frac{(r-1)n}{2}$ , which, when  $r \mid n$ , is achieved by a disjoint union of copies of  $K_r$ . Luo [21] determined  $\text{ex}(n, K_t, P_{r+1})$  asymptotically, and Chakraborti and Chen [9] then determined  $\text{mex}(m, K_t, P_{r+1})$ . Malec and Tompkins [22] gave a local version of the Erdős-Gallai result. To extend their result to the  $t$ -clique versions, considering both graphs of fixed order and graphs of fixed size, we define a more general size function for paths.

**Definition 6.** Let  $G$  be a graph on  $n$  vertices and  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define

$$\beta_G(T) = \max\{k : T \subseteq V(S) \text{ for some } S \subseteq G \text{ s.t. } S \cong P_{k+1}\}.$$

### 4.1 Paths in graphs of fixed order

In this section we extend a result of Malec and Tompkins [22, Theorem 2] to cliques, which we will demonstrate also generalizes a result of Luo [21].

**Theorem 7.** *Let  $G$  be a graph on  $n$  vertices and  $t \geq 2$ . Define*

$$p_G(T) = \frac{1}{\binom{\beta_G(T)}{t-1}}.$$

*Then  $p_G(T)$  is well-defined and decreasing in  $\beta_G(T)$ , and*

$$\sum_{T \in \mathcal{K}^t(G)} p_G(T) \leq \frac{n}{t},$$

*with equality if and only if  $G$  is a disjoint union of complete graphs of order at least  $t$ .*

We note that setting  $t = 2$  does not quite recover [22, Theorem 2] as our weight function is not equivalent at  $t = 2$ . Malec and Tompkins define  $\beta_{MT}(e)$  to be the longest path containing the edge  $e$  as a subgraph. This definition does not extend to larger cliques as paths do not contain them as subgraphs. Instead, we merely require that all vertices of the clique appear in the path. Thus when  $t = 2$ , unlike for Malec and Tompkins, the vertices of our edge may occur in the path without the edge being part of the path.

*Proof.* First, we note any ordering of the vertices in any  $T \in \mathcal{K}^t(G)$  is a path of length  $t - 1$ , so  $\beta_G(T) \geq t - 1$  and  $\binom{\beta_G(T)}{t-1} > 0$ . Therefore  $p_G(T)$  is well-defined and decreasing in  $\beta_G(T)$ .

We proceed by induction on  $n$ . When  $1 \leq n \leq t - 1$  we have

$$\sum_{T \in \mathcal{K}^t(G)} p_G(T) = 0 < \frac{n}{t}.$$

If  $G$  is not connected, let  $C_1, \dots, C_q$  be the components of  $G$ . Applying the inductive hypothesis to each component, we have

$$\begin{aligned} \sum_{T \in \mathcal{K}^t(G)} p_G(T) &= \sum_{i=1}^q \sum_{T \in \mathcal{K}^t(C_i)} p_G(T) \\ &= \sum_{i=1}^q \sum_{T \in \mathcal{K}^t(C_i)} p_{C_i}(T) \\ &\leq \sum_{i=1}^q \frac{|V(C_i)|}{t} \quad \text{by induction} \\ &= \frac{n}{t}. \end{aligned}$$

Therefore we may assume  $G$  is connected and  $n \geq t$ . Let  $r$  be the length of a longest path  $P \cong P_{r+1}$  in  $G$ .

**Case 7.1.** There exists a cycle  $C$  containing the vertices of  $P$ .

Suppose for the sake of contradiction that there is a vertex  $u$  that is not on the cycle  $C$ . Since  $G$  is connected, there is a path from  $u$  to  $C$  and then around  $C$ , which is longer than  $P$ , contradicting that  $P$  is a longest path. Therefore every vertex of  $G$  is on  $C$ . Each  $T \in \mathcal{K}^t(G)$  is contained in a path of length  $n - 1$  and has  $p_G(T) = 1/\binom{n-1}{t-1}$ .

Hence,

$$\sum_{T \in \mathcal{K}^t(G)} p_G(T) = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{n-1}{t-1}} = \frac{\mathcal{N}(K_t, G)}{\binom{n-1}{t-1}} \leq \frac{\binom{n}{t}}{\binom{n-1}{t-1}} = \frac{n}{t},$$

and equality implies  $G \cong K_n$ , with  $n \geq t$ .

**Case 7.2.** There does not exist a cycle  $C$  containing the vertices of  $P$ .

Let  $v$  and  $w$  be the endpoints of  $P$ . Then  $\{v, w\} \notin E(G)$ . Label the vertices of  $P$  in order as  $(v = u_1, u_2, \dots, u_{r+1} = w)$ . Let  $V = \{i : v \sim u_i\}$  and  $W = \{i : w \sim u_{i-1}\}$ . If  $i \in V \cap W$  then  $(u_i, u_1, u_2, \dots, u_{i-1}, u_{r+1}, u_r, \dots, u_i)$  is a cycle containing the vertices of  $P$ , so  $V \cap W = \emptyset$ . Since  $V, W \subseteq \{2, \dots, r+1\}$ , we have  $d(v) + d(w) = |V| + |W| = |V \cup W| \leq r$ . Assume, without loss of generality, that  $d(v) \leq r/2$ .

Let  $R(v)$  be the set of  $t$ -cliques of  $G$  that contain  $v$ . Then  $|R(v)| \leq \binom{d(v)}{t-1}$ . The fact that  $P$  is a longest path implies both that  $\beta_G(T) \leq r$  and that every  $t$ -clique  $T$  in  $R(v)$  is contained in  $V(P)$  (so  $\beta_G(T) \geq r$ ), and therefore  $p_G(T) = 1/\binom{r}{t-1}$  for every  $T \in R(v)$ . For any  $T \in \mathcal{K}^t(G - v)$ , we see  $\beta_{G-v}(T) \leq \beta_G(T)$  and therefore, as  $p_G$  is decreasing in  $\beta_G(T)$ ,  $p_{G-v}(T) \geq p_G(T)$ .

Applying the inductive hypothesis to  $G - v$ , we get

$$\begin{aligned}
\sum_{T \in \mathcal{K}^t(G)} p_G(T) &= \sum_{T \in \mathcal{K}^t(G-v)} p_G(T) + \sum_{T \in R(v)} p_G(T) \\
&\leq \sum_{T \in \mathcal{K}^t(G-v)} p_{G-v}(T) + \sum_{T \in R(v)} p_G(T) \\
&\leq \frac{n-1}{t} + \frac{|R(v)|}{\binom{r}{t-1}} \\
&\leq \frac{n-1}{t} + \frac{\binom{d(v)}{t-1}}{\binom{r}{t-1}} \\
&\leq \frac{n-1}{t} + \frac{\binom{r/2}{t-1}}{\binom{r}{t-1}} \\
&\leq \frac{n-1}{t} + \frac{(1/2)^{t-1} \binom{r}{t-1}}{\binom{r}{t-1}} && \text{by Observation 1} \\
&= \frac{n-1}{t} + \frac{1}{2^{t-1}} \\
&\leq \frac{n-1}{t} + \frac{1}{t} \\
&= \frac{n}{t}
\end{aligned}$$

as  $2^{t-1} \geq t$  when  $t \geq 2$ . When  $t \geq 3$ , we see  $2^{t-1} > t$  and thus if equality holds, no component is in Case 7.2, so every component is in Case 7.1, and every component is a clique.

When  $t = 2$ , the claim that equality holds only for cliques follows from [22, Theorem 2]. Though, as noted, our definition of  $\beta_G(T)$  does not quite match their  $\beta_{MT}(e)$ , we have  $\beta_{MT}(e) \leq \beta_G(e)$  as any path containing an edge also contains the vertices of that edge. Because

$$p_G(T) = \frac{1}{\binom{\beta_G(T)}{2-1}} = \frac{1}{\beta_G(T)}$$

is decreasing and matches Malec and Tompkins' weight, we see

$$\sum_{T \in \mathcal{K}^2(G)} p_G(T) = \sum_{e \in E(G)} \frac{1}{\beta_G(e)} \leq \sum_{e \in E(G)} \frac{1}{\beta_{MT}(e)}.$$

Therefore if equality holds in our result, it also holds in Malec and Tompkins' result, and by Theorem 2 in [22],  $G$  is a disjoint union of complete graphs.  $\square$

In 2018, Luo [21] extended Erdős and Gallai's theorem by showing disjoint unions of cliques maximize the number of cliques of any size.

**Theorem 8** (Luo [21]). *Let  $G$  be a  $P_{r+1}$ -free graph on  $n$  vertices. Then for any  $t \leq r$ ,*

$$\mathcal{N}(K_t, G) \leq \frac{n}{r} \binom{r}{t}.$$

*Furthermore, equality holds if and only if  $r \mid n$  and  $G$  is a disjoint union of copies of  $K_r$ .*

We note that Theorem 7 generalizes Theorem 8:

*Proof.* Let  $G$  be a  $P_{r+1}$ -free graph on  $n$  vertices. In such a graph  $G$ , every  $T \in \mathcal{K}^t(G)$  has  $\beta_G(T) \leq r - 1$ , so  $p_G(T) \geq \frac{1}{\binom{r-1}{t-1}}$ . By Theorem 7,

$$\frac{\mathcal{N}(K_t, G)}{\binom{r-1}{t-1}} = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{r-1}{t-1}} \leq \sum_{T \in \mathcal{K}^t(G)} p_G(T) \leq \frac{n}{t},$$

so

$$\mathcal{N}(K_t, G) \leq \frac{n}{t} \binom{r-1}{t-1} = \frac{n}{r} \binom{r}{t}.$$

By Theorem 7, equality implies that every component is a  $P_{r+1}$ -free clique, and that every  $t$ -clique is contained in a  $P_r$ , so  $G$  is a disjoint union of copies of  $K_r$ .  $\square$

## 4.2 Paths in graphs of fixed size

Chakraborti and Chen [9] asked and answered the edge variant of this question by determining  $\text{mex}(m, K_t, P_{r+1})$  exactly for all values of the parameters  $m$ ,  $t$ , and  $r$ . We prove a localized form of their result, then show that it implies an asymptotic determination of  $\text{mex}(m, K_t, P_{r+1})$ . The proof of Theorem 9 will be very similar in structure to the proof of Theorem 7 on paths in graphs of fixed order.

**Theorem 9.** *Let  $t \geq 3$ . Let  $G$  be a graph having  $m$  edges. For each  $T \in \mathcal{K}^t(G)$ , define*

$$p'_G(T) = \frac{1}{\binom{\beta_G(T)-1}{t-2}}.$$

Then  $p'_G(T)$  is well-defined and decreasing in  $\beta_G(T)$ , and

$$\sum_{T \in \mathcal{K}^t(G)} p'_G(T) \leq \frac{m}{\binom{t}{2}},$$

with equality if and only if  $G$  is a disjoint union of complete graphs of order at least  $t$  and any number of isolated vertices.

*Proof.* As before, any ordering of the vertices in any  $T \in \mathcal{K}^t(G)$  is a path of length  $t - 1$ , so  $\beta_G(T) \geq t - 1$  and  $\binom{\beta_G(T)-1}{t-2} > 0$ . Therefore  $p'_G(T)$  is well-defined and decreasing in  $\beta_G(T)$ .

We proceed by induction on  $m$ . When  $1 \leq m \leq \binom{t}{2} - 1$  we have

$$\sum_{T \in \mathcal{K}^t(G)} p'_G(T) = 0 < \frac{m}{\binom{t}{2}}.$$

If  $G$  is not connected, let  $C_1, \dots, C_q$  be the components of  $G$ . Applying the inductive hypothesis to each component, we have

$$\begin{aligned} \sum_{T \in \mathcal{K}^t(G)} p'_G(T) &= \sum_{i=1}^q \sum_{T \in \mathcal{K}^t(C_i)} p'_G(T) \\ &= \sum_{i=1}^q \sum_{T \in \mathcal{K}^t(C_i)} p'_{C_i}(T) \\ &\leq \sum_{i=1}^q \frac{|E(C_i)|}{\binom{t}{2}} \quad \text{by induction} \\ &= \frac{m}{\binom{t}{2}}. \end{aligned}$$

Therefore we may assume  $G$  is connected and  $m \geq \binom{t}{2}$ . Let  $r$  be the length of a longest path  $P \cong P_{r+1}$  in  $G$ .

**Case 9.1.** There exists a cycle  $C$  containing the vertices of  $P$ .

Suppose for the sake of contradiction that there is a vertex  $u$  that is not on the cycle  $C$ . Since  $G$  is connected, there is a path from  $u$  to  $C$  and then around  $C$ , which is longer than  $P$ , contradicting that  $P$  is a longest path. Therefore every vertex of  $G$  is on  $C$ . Each  $T \in \mathcal{K}^t(G)$  is contained in a path of length  $|V(G)| - 1$ , and has  $p'_G(T) = 1/\binom{|V(G)|-2}{t-2}$ . Let  $x \geq t$  be a real number satisfying  $m = \binom{x}{2}$ . Then we have  $|V(G)| \geq x$  by Observation 2 and, by Theorem 3,  $\mathcal{N}(K_t, G) \leq \binom{x}{t}$ . Hence,

$$\sum_{T \in \mathcal{K}^t(G)} p'_G(T) = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{|V(G)|-2}{t-2}} = \frac{\mathcal{N}(K_t, G)}{\binom{|V(G)|-2}{t-2}} \leq \frac{\binom{x}{t}}{\binom{x-2}{t-2}} = \frac{\binom{x}{2}}{\binom{t}{2}} = \frac{m}{\binom{t}{2}},$$

and equality implies  $\binom{|V(G)|-2}{t-2} = \binom{x-2}{t-2}$  with  $x - 2 \geq (t - 2) - 1$ , so  $x = |V(G)|$ . Then  $\mathcal{N}(K_t, G) = \binom{|V(G)|}{t}$ , so  $G \cong K_x$  (and  $x \geq t$ ).

**Case 9.2.** There does not exist a cycle  $C$  containing the vertices of  $P$ .

Let  $v$  and  $w$  be the endpoints of  $P$ . Then  $\{v, w\} \notin E(G)$ . Label the vertices of  $P$  in order as  $(v = u_1, u_2, \dots, u_{r+1} = w)$ . Let  $V = \{i : v \sim u_i\}$  and  $W = \{i : w \sim u_{i-1}\}$ . If  $i \in V \cap W$  then  $(u_i, u_1, u_2, \dots, u_{i-1}, u_{r+1}, u_r, \dots, u_i)$  is a cycle containing the vertices of  $P$ . Because  $V, W \subseteq \{2, \dots, r+1\}$ , we have  $d(v) + d(w) = |V| + |W| = |V \cup W| \leq r$ . Assume, without loss of generality, that  $d(v) \leq r/2$ .

Let  $R(v)$  be the set of  $t$ -cliques of  $G$  that contain  $v$ . Then  $|R(v)| \leq \binom{d(v)}{t-1}$ . Since  $P$  is a longest path,  $v$  and  $w$  have no neighbors outside of  $P$ , and so every  $t$ -clique in  $R(v)$  is contained in  $V(P)$ . Every  $T \in R(v)$  has  $p'_G(T) = 1/\binom{r-1}{t-2}$  because it is contained in  $P$  (and there are no longer paths). As noted above, we have  $\beta_{G-v}(T) \leq \beta_G(T)$  for any  $T \in \mathcal{K}^t(G-v)$  and as  $p'_G$  is also decreasing in  $\beta_G(T)$ ,  $p'_{G-v}(T) \geq p'_G(T)$ . Applying the inductive hypothesis to  $G-v$ , we get

$$\begin{aligned}
\sum_{T \in \mathcal{K}^t(G)} p'_G(T) &= \sum_{T \in \mathcal{K}^t(G-v)} p'_G(T) + \sum_{T \in R(v)} p'_G(T) \\
&\leq \sum_{T \in \mathcal{K}^t(G-v)} p'_{G-v}(T) + \sum_{T \in R(v)} p'_G(T) \\
&\leq \frac{m - d(v)}{\binom{t}{2}} + \frac{|R(v)|}{\binom{r-1}{t-2}} \\
&\leq \frac{m - d(v)}{\binom{t}{2}} + \frac{\binom{d(v)}{t-1}}{\binom{r-1}{t-2}} \\
&= \frac{m - d(v)}{\binom{t}{2}} + \frac{\binom{d(v)-1}{t-2}}{\binom{r-1}{t-2}} \cdot \frac{d(v)}{t-1} \\
&\leq \frac{m - d(v)}{\binom{t}{2}} + \frac{\binom{r/2-1}{t-2}}{\binom{r-1}{t-2}} \cdot \frac{d(v)}{t-1} \\
&\leq \frac{m - d(v)}{\binom{t}{2}} + \frac{(1/2)^{t-2} \binom{r-2}{t-2}}{\binom{r-1}{t-2}} \cdot \frac{d(v)}{t-1} \quad \text{by Observation 1} \\
&= \frac{m - d(v)}{\binom{t}{2}} + \frac{r-t+1}{r-1} \cdot \frac{1}{2^{t-1}} \cdot \frac{d(v)}{\frac{t-1}{2}} \\
&< \frac{m - d(v)}{\binom{t}{2}} + \frac{d(v)}{\binom{t}{2}} \\
&= \frac{m}{\binom{t}{2}}
\end{aligned}$$

as  $2^{t-1} > t$  and  $\frac{r-t+1}{r-1} < 1$  when  $t \geq 3$ . Thus, if equality holds, no component is in Case 9.2, so every component is in Case 9.1, and every component is a clique.  $\square$

As mentioned, Chakraborti and Chen [9] determined  $\text{mex}(m, K_t, P_{r+1})$  exactly, from which one can derive the following weaker result:

**Theorem 10** (Chakraborti and Chen [9]). *For any  $3 \leq t \leq r$ , if  $G$  is a  $P_{r+1}$ -free graph with  $m$  edges, then*

$$\mathcal{N}(K_t, G) \leq \frac{m}{\binom{r}{2}} \cdot \binom{r}{t}.$$

*Furthermore, equality holds if and only if  $\binom{r}{2}$  divides  $m$ , and  $G$  is a disjoint union of copies of  $K_r$ .*

We use Theorem 9 to prove Theorem 10:

*Proof.* Let  $G$  be a  $P_{r+1}$ -free graph. Then for each  $T \in \mathcal{K}^t(G)$ , we have  $\beta_G(T) \leq r-1$  and as  $p'_G(T)$  is decreasing in  $\beta_G$ ,

$$\mathcal{N}(K_t, G) \cdot \frac{1}{\binom{r-2}{t-2}} \leq \sum_{T \in \mathcal{K}^t(G)} p'_G(T) \leq \frac{m}{\binom{2}{2}}$$

and therefore

$$\mathcal{N}(K_t, G) \leq \frac{m}{\binom{r-2}{t-2}} \cdot \binom{r-2}{t-2} = \frac{m}{\binom{r}{2}} \cdot \binom{r}{t}.$$

Equality implies  $G$  is a disjoint union of  $P_{r+1}$ -free complete graphs, and every  $t$ -clique is contained in a  $P_r$ , so  $G$  is a disjoint union of copies of  $K_r$ .  $\square$

## 5 Weighting by Maximum Star Size

### 5.1 Graphs

In this section we consider generalized extremal problems of the form  $\text{ex}(n, H, S_r)$ , forbidding the star with  $r$  leaves, or equivalently requiring that the maximum degree is at most  $r-1$ . Unlike in previous sections where  $H$  was a clique, here we consider a broader range of graphs  $H$ . We use  $\mathcal{H}(G)$  to denote the set of (not necessarily induced) subgraphs of  $G$  isomorphic to  $H$ .

Given a graph  $G$  and a non-empty set of vertices  $U \subseteq V(G)$ , the *common neighborhood* of  $U$  in  $G$  is the set of vertices of  $G$  adjacent to each vertex in  $U$ , or equivalently the intersection of the neighbor sets of each vertex of  $U$ . The common degree of  $U$ , which we denote by  $\text{cd}_G(U)$ , is the size of the common neighborhood. Note that  $U$  is disjoint from its common neighborhood.

Denote the collection of dominating vertices of a graph  $G$  by  $\text{Dom}(G)$ , and let  $\text{dom}(G) = |\text{Dom}(G)|$ . Note that for any  $U \subseteq \text{Dom}(G)$ ,  $U$  is a clique, and the common neighborhood of  $U$  is  $V(G) \setminus U$ . Additionally, if  $U, U' \subseteq \text{Dom}(G)$  and  $|U| = |U'|$ , then  $G - U \cong G - U'$ . For  $u \in \text{dom}(G)$ , we write  $G^{\downarrow u}$  to denote the graph that results from removing  $u$  dominating vertices from  $G$ .

The main result of this section, Theorem 11, provides a template for localized bounds on the number of copies of  $H$  in a graph based on the number of sets of dominating vertices of a given size contained in  $H$ . We will focus on the cases where these sets have size one or two, but we provide the theorem in full generality.

**Theorem 11.** Let  $H$  be a graph on  $t$  vertices with  $\text{dom}(H) \geq 1$ . For every graph  $G$  and each  $T \in \mathcal{H}(G)$ , define

$$\theta_G(T) = \max\{k : \exists v \in V(T) \subseteq V(S) \text{ s.t. } V(S) \subseteq N[v] \text{ and } S_k \cong S \subseteq G\},$$

or, equivalently,  $\theta_G(T) = \max\{d_G(v) : v \in \text{Dom}(T)\}$ , and for each  $1 \leq u \leq \text{dom}(H)$ , define

$$s_G^u(T) = \frac{1}{\binom{\theta_G(T)-u+1}{t-u}}.$$

Then for any  $1 \leq u \leq \text{dom}(H)$ ,  $s_G^u(T)$  is well-defined and decreasing in  $\theta_G(T)$ , and

$$\sum_{T \in \mathcal{H}(G)} s_G^u(T) \leq \frac{\mathcal{N}(H^{\downarrow u}, K_{t-u})}{\binom{\text{dom}(H)}{u}} \cdot \mathcal{N}(K_u, G).$$

Furthermore, equality holds

- if and only if  $G$  has minimum degree at least  $t-1$ , if  $H = S_{t-1}$  for some  $t \geq 3$ ,
- if and only if  $G$  contains no isolated vertices and every component of  $G$  is regular, if  $u = 1$  and  $H = S_1$ , and
- if and only if every component of  $G$  that contains a  $u$ -clique is a complete graph on at least  $t$  vertices, if  $u \geq 2$  or  $H$  is not a star.

*Proof.* First, in any  $T \in \mathcal{H}(G)$ , there is a dominating vertex  $v$  of  $T$  which is the center of a star with (at least)  $t-1$  leaves, so  $\theta_G(T) \geq t-1$  and  $\binom{\theta_G(T)-u+1}{t-u} > 0$ . Therefore  $s_G^u(T)$  is well-defined and decreasing on  $\theta_G(T)$ .

Let  $T \in \mathcal{H}(G)$  and  $U \subseteq \text{Dom}(T)$  such that  $|U| = u$ . Recall  $U$  is a clique. Thus for any  $v \in U$ , the vertices in the common neighborhood of  $U$  in  $G$ , together with the vertices  $U \setminus \{v\}$ , are each adjacent to  $v$ , forming a copy of  $S_{\text{cd}_G(U)+u-1}$  whose center is  $v \in V(T)$  and which contains all vertices of  $T$  (because  $v \in U \subseteq \text{Dom}(T)$ ). Therefore we have  $\theta_G(T) \geq \text{cd}_G(U) + u - 1$ , or  $\theta_G(T) - u + 1 \geq \text{cd}_G(U)$ .

For each  $T \in \mathcal{H}(G)$ , there are  $\binom{\text{dom}(H)}{u}$  sets  $U \in \mathcal{K}^u(G)$  such that each vertex of  $U$  is dominating in  $T$ . For each  $U \in \mathcal{K}^u(G)$ , the number of copies  $T$  of  $H$  in  $G$  for which  $U \subseteq \text{Dom}(T)$  is at most  $\binom{\text{cd}_G(U)}{t-u} \mathcal{N}(H^{\downarrow u}, K_{t-u})$ : we choose  $t-u$  vertices from the common neighborhood of  $U$  in  $G$  to fill out  $T$  and then choose how to embed the vertices of  $H^{\downarrow u}$ , which can be done in at most as many ways as embedding them into a clique of the same

size. Therefore

$$\begin{aligned}
\binom{\text{dom}(H)}{u} \sum_{T \in \mathcal{H}(G)} s_G^u(T) &= \sum_{U \in \mathcal{K}^u(G)} \sum_{\substack{T \in \mathcal{H}(G) \\ U \subseteq \text{Dom}(T)}} s_G^u(T) \\
&= \sum_{U \in \mathcal{K}^u(G)} \sum_{\substack{T \in \mathcal{H}(G) \\ U \subseteq \text{Dom}(T)}} \frac{1}{\binom{\theta_G(T)-u+1}{t-u}} \\
&\leq \sum_{U \in \mathcal{K}^u(G)} \sum_{\substack{T \in \mathcal{H}(G) \\ U \subseteq \text{Dom}(T)}} \frac{1}{\binom{\text{cd}_G(U)}{t-u}} \tag{1}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{U \in \mathcal{K}^u(G)} \mathcal{N}(H^{\downarrow u}, K_{t-u}) \frac{\binom{\text{cd}_G(U)}{t-u}}{\binom{\text{cd}_G(U)}{t-u}} \tag{2} \\
&= \mathcal{N}(H^{\downarrow u}, K_{t-u}) \mathcal{N}(K_u, G).
\end{aligned}$$

which establishes the inequality for all graphs  $G$ .

We now consider when equality holds in the bound. For ease of discussion, if  $U$  is contained in a copy  $T$  of  $H$  such that  $U \subseteq \text{Dom}(T)$ , we say that  $T$  is an extension of  $U$ . Equality in the bound requires equality in equations (1) and (2). In order for equality to hold in (1), every pair  $(U, T)$  where  $T$  is an extension of  $U$  must satisfy  $\text{cd}_G(U) = \theta_G(T) - u + 1$ . Equality in (2) requires two conditions: first, for every  $U \in \mathcal{K}^u(G)$ ,  $G$  must contain an extension of  $U$  as otherwise this choice of  $U$  contributes

$$\sum_{\substack{T \in \mathcal{H}(G) \\ U \subseteq \text{Dom}(T)}} \frac{1}{\binom{\text{cd}_G(U)}{t-u}} = 0$$

instead of

$$\sum_{\substack{T \in \mathcal{H}(G) \\ U \subseteq \text{Dom}(T)}} \frac{1}{\binom{\text{cd}_G(U)}{t-u}} = \mathcal{N}(H^{\downarrow u}, K_{t-u}) > 0.$$

In particular, if equality holds in (2),  $\text{cd}_G(U) \geq t - u$  for every  $U \in \mathcal{K}^u(G)$ . Additionally, every  $(t - u)$ -set in the common neighborhood of  $U$  must contain  $\mathcal{N}(H^{\downarrow u}, K_{t-u})$  copies of  $H^{\downarrow u}$ , or equivalently, the common neighborhood of  $U$  must contain the same number of copies of  $H^{\downarrow u}$  as a clique of the same size.

For each of the three cases delineated in the theorem, we consider the conditions under which equality holds.

**Case 11.1.** Suppose  $H = S_{t-1}$  for some  $t \geq 3$ .

In this case we have  $\text{dom}(H) = 1$  and thus  $u = 1$ . This means that for any  $U \in \mathcal{K}^u(G)$ , there is  $v \in V(G)$  such that  $U = \{v\}$ , and we have  $\text{cd}_G(U) = d_G(v)$ . For any copy  $T$  of  $H$  in  $G$ , there is a unique  $U \subseteq \text{Dom}(T)$  of size  $u = 1$ , and if  $U = \{v\}$ , then  $\theta_G(T) = d_G(v) = \text{cd}_G(U)$ . We conclude that equality holds in (1) for every graph  $G$ . If  $w$

is the center of  $H = S_{t-1}$ , then  $H - w = H^{\downarrow 1}$  is an independent set, so  $\mathcal{N}(H^{\downarrow 1}, K_{t-1}) = 1$ . For any graph  $G$  and any  $v \in V(G)$ , if  $d_G(v) \geq t - 1$ , then the number of copies  $T$  of  $H$  with  $v$  at the center is exactly  $\binom{d_G(v)}{t-1} = \binom{\text{cd}_G(U)}{t-1}$  and equality holds in (2). If  $G$  has minimum degree at least  $t - 1$  then equality holds for every vertex, so equality holds in the bound. On the other hand, if  $d_G(v) < t - 1$  for any  $v \in V(G)$ , then there are no extensions of  $\{v\}$  and equality does not hold in (2). As equality in the bound requires equality for each  $U \in \mathcal{K}^u(G)$ , we conclude equality holds in this case if and only if  $d_G(v) \geq t - 1$  for every  $v \in V(G)$ , which is to say  $G$  has minimum degree at least  $t - 1$ .

**Case 11.2.** Suppose  $u = 1$  and  $H = S_1$ .

This case is similar to Case 11.1: we have  $u = 1$ , so the set of  $u$ -cliques of  $G$  corresponds to the vertices of  $G$ ,  $H^{\downarrow 1}$  is an independent set so  $\mathcal{N}(H^{\downarrow 1}, K_{t-1}) = 1$ , and equality holds in (2) if and only if each vertex has degree at least  $t - 1 = 2 - 1$ , i.e.  $G$  contains no isolated vertices. The difference between the cases is that  $\text{dom}(H) = 2$  when  $H = S_1$ , and therefore rather than being determined by the degree of a single vertex,

$$\theta_G(T = \{v, v'\}) = \max\{d_G(v), d_G(v')\}.$$

This introduces a circumstance in which equality may not hold in (1): if  $T = \{v, v'\}$  and  $d_G(v) > d_G(v')$ , then for  $U = \{v'\}$ ,

$$\frac{1}{\binom{\theta_G(T)}{t-u}} = \frac{1}{d_G(v)} < \frac{1}{d_G(v')} = \frac{1}{\binom{\text{cd}_G(U)}{t-u}}.$$

Thus equality in (1) holds if and only if  $d_G(v) = d_G(v')$  for every  $T = \{v, v'\}$ , which is to say each component of  $G$  is regular. We conclude that a graph  $G$  meets the bound if and only if  $G$  contains no isolated vertices and each component of  $G$  is regular.

**Case 11.3.** Suppose  $u \geq 2$  or that  $H$  is not a star.

First we prove the given conditions are sufficient to achieve the bound. When  $G$  is a disjoint union of complete graphs on at least  $t$  vertices and any number of components without  $u$ -cliques, let  $K_r$ ,  $r \geq t \geq u$ , be a component of  $G$  that contains a  $u$ -clique. Then  $\theta_G(T) = r - 1$  for every  $T \in \mathcal{H}(K_r)$ , and  $s_G^u(T) = 1/\binom{r-u}{t-u}$ . The number of copies of  $H$  in this component is  $\binom{r}{u} \binom{r-u}{t-u} \mathcal{N}(H^{\downarrow u}, K_{t-u}) / \binom{\text{dom}(H)}{u}$ , which can be seen by first choosing a  $u$ -clique of the  $K_r$  to act as a selected set of  $u$  dominating vertices of  $H$ , then choosing  $t - u$  of the  $r - u$  other vertices in the same component, then choosing an embedding of  $H^{\downarrow u}$  into those vertices. Each copy of  $H$  is counted this way once for each choice of selected set of  $u$  dominating vertices of  $H$ . Therefore

$$\begin{aligned} \sum_{T \in \mathcal{H}(K_r)} s_G^u(T) &= \frac{1}{\binom{r-u}{t-u}} \binom{r}{u} \binom{r-u}{t-u} \mathcal{N}(H^{\downarrow u}, K_{t-u}) / \binom{\text{dom}(H)}{u} \\ &= \frac{\mathcal{N}(H^{\downarrow u}, K_{t-u})}{\binom{\text{dom}(H)}{u}} \mathcal{N}(K_u, K_r). \end{aligned}$$

Note that if a component  $C$  of  $G$  contains no  $u$ -cliques, it also contains no copies of  $T$ , so we have

$$\sum_{T \in \mathcal{H}(C)} s_C^u(T) = 0 = \mathcal{N}(K_u, C),$$

and thus  $C$  does not contribute to either side of the bound. If  $C_1, \dots, C_k$  are the components of  $G$  containing  $u$ -cliques, then

$$\begin{aligned} \sum_{T \in \mathcal{H}(G)} s_G^u(T) &= \sum_{i=1}^k \sum_{T \in \mathcal{H}(C_i)} s_{C_i}^u(T) \\ &= \frac{\mathcal{N}(H^{\downarrow u}, K_{t-u})}{\binom{\text{dom}(H)}{u}} \sum_{i=1}^k \mathcal{N}(K_u, C_i) \\ &= \frac{\mathcal{N}(H^{\downarrow u}, K_{t-u})}{\binom{\text{dom}(H)}{u}} \mathcal{N}(K_u, G). \end{aligned}$$

We conclude that any graph  $G$  in which every component that contains a  $u$ -clique is a complete graph on at least  $t$  vertices meets the bound.

Now if  $G$  is a graph meeting the bound, we prove every component of  $G$  containing a  $u$ -clique must be a clique containing at least  $t$  vertices. We begin with three claims.

**Claim 1:** If  $G$  is a graph meeting the bound and  $U \in \mathcal{K}^u(G)$ , then each vertex in  $V(G) \setminus U$  that is adjacent to some  $v \in U$  must be in the common neighborhood of  $U$ .

Suppose otherwise: there is  $v \in U$  and  $x$  not in the common neighborhood of  $U$  such that  $x \sim v$ . As  $G$  meets the bound, equality holds in (2), so we may assume there exists an extension  $T$  of  $U$ . Note that  $T \subseteq N[v]$ . As  $v$  is adjacent to the common neighborhood of  $U$ , each other vertex of  $U$ , and  $x$ , we see

$$\theta_G(T) = \max\{d_G(w) : w \in \text{Dom}(T)\} \geq d_G(v) \geq \text{cd}_G(U) + u - 1 + 1 > \text{cd}_G(U) + u - 1$$

and equality does not hold in (1), contradicting that  $G$  met the bound.

**Claim 2:** If  $u \geq 2$ ,  $G$  is a graph meeting the bound, and  $U \in \mathcal{K}^u(G)$ , then  $U$  and its common neighborhood form a clique in  $G$ .

Suppose  $v, w \in U$  and that  $x$  and  $y$  are vertices in the common neighborhood of  $U$  such that  $x \not\sim y$ . Define  $U' = U - v + x$ . If there are no extensions  $T$  of  $U'$ , then equality will not hold in (2), so assume such a  $T$  exists. Then  $y$  is not in the common neighborhood of  $U'$ , as  $y \not\sim x$ , but  $y \sim w \in U \cap U'$ , so by the contrapositive of Claim 1, equality does not hold.

**Claim 3:** If  $H^{\downarrow u}$  contains at least one edge,  $G$  is a graph meeting the bound, and  $U \in \mathcal{K}^u(G)$ , then  $U$  and its common neighborhood form a clique in  $G$ .

We prove the contrapositive. Assume that some  $U \in \mathcal{K}^u(G)$  and its common neighborhood do not form a clique in  $G$ . Let  $a$  and  $b$  be non-adjacent vertices in the common neighborhood of  $U$  and let  $\{x, y\}$  be an edge in  $H^{\downarrow u}$ . If  $G$  contains no extensions  $T$  of

$U$ , then equality does not hold in (2), so we may assume such an extension does exist, and therefore  $\text{cd}_G(U) \geq t - u = |V(H^{\downarrow u})|$  as the common neighborhood of  $U$  contains at least  $V(T) \setminus U$ . Therefore there exists an injective function  $f$  from  $V(H^{\downarrow u})$  to the common neighborhood of  $U$  such that  $f(x) = a$  and  $f(y) = b$ . Note that  $f$  is not a homomorphism, demonstrating that not every injective function from  $H^{\downarrow u}$  to the common neighborhood of  $U$  is an injective homomorphism. By contrast, every injective function from  $H^{\downarrow u}$  to a clique is an injective homomorphism. This means there are fewer injective homomorphisms from  $H^{\downarrow u}$  into the common neighborhood of  $U$  than into a clique of the same size. For any graphs  $P$  and  $Q$ , the number of injective homomorphisms from  $P$  to  $Q$  is equal to the number of automorphisms of  $P$  times the number of copies of  $P$  in  $Q$ . Therefore there are also fewer copies of  $H^{\downarrow u}$  in the common neighborhood of  $U$  than there are in a clique of the same size, so as mentioned when discussing the conditions under which equality holds in (2),  $G$  does not meet the bound.

We are now prepared to show that  $G$  meets the bound only if every component of  $G$  containing a  $u$ -clique is a complete graph containing at least  $t$  vertices. If  $G$  is disconnected, we consider each component separately, and if a component  $C$  does not contain a  $u$ -clique, we have seen  $C$  does not contribute to either side of the bound and need not be considered. Therefore we may assume  $G$  is connected and contains a  $u$ -clique. Either  $u \geq 2$ , in which case Claim 2 applies, or  $u = 1$  and  $H$  is not a star. In this case, the graph that remains after removing any one dominating vertex contains an edge, so Claim 3 applies. In either case, any  $u$ -clique forms a clique with its common neighborhood. Applying Claim 1 to each  $u$ -set of this clique, no vertex outside of this clique can be adjacent to any vertex contained in the clique, so  $G$  is a complete graph. In order for equality to hold in (2), there must be at least one extension  $T$  of  $U$ , which requires this complete graph to have at least  $t$  vertices.  $\square$

### 5.1.1 Weighting $t$ -cliques by maximum star size

By taking  $H = K_t$  and  $u \in \{1, 2\}$  in Theorem 11 we obtain the following corollaries. We note that Proposition 1 in [22] is the case  $t = 2$  of Corollary 12.

**Corollary 12.** *For every  $n$ -vertex graph  $G$  and every clique size  $t \geq 1$ ,*

$$\sum_{T \in \mathcal{K}^t(G)} s_G^1(T) = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)}{t-1}} \leq \frac{n}{t}.$$

*When  $t = 2$ , equality holds if and only if  $G$  contains no isolated vertices and each component of  $G$  is regular. When  $t \geq 3$ , equality holds if and only if  $G$  is a disjoint union of complete graphs of order at least  $t$ .*

*Proof.* Let  $G$  be a graph on  $n$  vertices. We set  $H = K_t$  and  $u = 1$  in Theorem 11, so  $s_G^1(T) = \frac{1}{\binom{\theta_G(T)}{t-1}}$ . Any vertex  $v$  of  $K_t$  is a dominating vertex of  $K_t$ , so by Theorem 11 we have

$$\sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)}{t-1}} \leq \frac{\mathcal{N}(K_t^{\downarrow 1}, K_{t-1})}{\binom{\text{dom}(K_t)}{1}} \cdot \mathcal{N}(K_1, G) = \frac{n}{t}.$$

When  $t = 2$ ,  $H = K_2 = S_1$ , so the equality condition matches the  $u = 1$  and  $H = S_1$  case in Theorem 11. When  $t \geq 3$ , the equality condition holds as cliques on  $t \geq 3$  vertices are not stars and any non-empty component contains a 1-clique, i.e. a vertex.  $\square$

Note that taking  $t = 1$  in Corollary 12 gives the true, if trivial, statement that an  $n$ -vertex graph  $G$  has at most  $n$  vertices, so equality holds for all graphs.

**Corollary 13.** *For every  $m$ -edge graph  $G$  and every clique size  $t \geq 2$ ,*

$$\sum_{T \in \mathcal{K}^t(G)} s_G^2(T) = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)-1}{t-2}} \leq \frac{m}{\binom{t}{2}}.$$

*When  $t \geq 3$ , equality holds if and only if  $G$  is a disjoint union of complete graphs of order at least  $t$  and any number of isolated vertices.*

*Proof.* Let  $G$  be a graph on  $m$  edges. We set  $H = K_t$  and  $u = 2$  in Theorem 11, so  $s_G^2(T) = \frac{1}{\binom{\theta_G(T)-1}{t-2}}$ . Any two vertices of  $K_t$  are dominating vertices of  $K_t$ , so by Theorem 11 we have

$$\sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)-1}{t-2}} \leq \frac{\mathcal{N}(K_t^{\downarrow 2}, K_{t-2})}{\binom{\text{dom}(K_t)}{2}} \cdot \mathcal{N}(K_2, G) = \frac{m}{\binom{t}{2}}.$$

As above, the equality condition holds as cliques on  $t \geq 3$  vertices are not stars. As edges are 2-cliques, all other components must be isolated vertices.  $\square$

Similar to Corollary 12, taking  $t = 2$  in Corollary 13 gives a true but trivial statement, namely that an  $m$ -edge graph has at most  $m$  edges, so equality holds for all graphs.

Corollary 12 and Corollary 13 in turn imply the following two known theorems on the maximum number of  $t$ -cliques in bounded-degree graphs having a given number of vertices or a given number of edges, respectively. These theorems have also been proved using essentially the same argument as the one used in [26] to give an upper bound on the total number of cliques of all sizes. The first theorem determines  $\text{ex}(n, K_t, S_r)$  asymptotically.

**Corollary 14** (Wood [26]). *Let  $t \geq 1$  and  $G$  be a graph on  $n$  vertices having  $\Delta(G) \leq r - 1$ . Then*

$$\mathcal{N}(K_t, G) \leq \frac{n}{r} \binom{r}{t}.$$

*For  $t \geq 3$ , equality holds if and only if  $r \mid n$  and  $G$  is a disjoint union of copies of  $K_r$ .*

*Proof.* Let  $G$  be a graph on  $n$  vertices with maximum degree at most  $r - 1$ . Then  $\theta_G(T) \leq r - 1$  for every  $T \in \mathcal{K}^t(G)$ . By Corollary 12 we have

$$\frac{\mathcal{N}(K_t, G)}{\binom{r-1}{t-1}} = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{r-1}{t-1}} \leq \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)}{t-1}} \leq \frac{n}{t},$$

so  $\mathcal{N}(K_t, G) \leq \frac{n}{t} \binom{r-1}{t-1} = \frac{n}{r} \binom{r}{t}$ . If  $t \geq 3$  and equality holds, then every component is a clique. As  $G$  has maximum degree at most  $r - 1$ , these cliques have at most  $r$  vertices.

If some  $G$  meeting the bound had a component  $C$  with  $a < r$  vertices, then  $G - C$  would be a graph on  $n - a$  vertices with  $\frac{n}{r} \binom{r}{t} - \binom{a}{t}$  copies of  $K_t$ . As  $t \geq 3$ , we have

$$a \binom{r}{t} = \frac{ar}{t} \binom{r-1}{t-1} > \frac{ar}{t} \binom{a-1}{t-1} = r \binom{a}{t}$$

and thus  $\frac{a}{r} \binom{r}{t} > \binom{a}{t}$ , so

$$\frac{n}{r} \binom{r}{t} - \binom{a}{t} > \frac{n}{r} \binom{r}{t} - \frac{a}{r} \binom{r}{t} = \frac{n-a}{r} \binom{r}{t}$$

which contradicts the bound.  $\square$

If we instead fix the number of edges, we determine  $\text{mex}(m, K_t, S_r)$  asymptotically.

**Corollary 15** (Wood [26]). *Let  $t \geq 2$  and  $G$  be a graph on  $m$  edges with  $\Delta(G) \leq r - 1$ . Then*

$$\mathcal{N}(K_t, G) \leq \frac{m}{\binom{r}{2}} \binom{r}{t}.$$

*For  $t \geq 3$ , equality holds if and only if  $\binom{r}{2} \mid m$  and  $G$  is a disjoint union of copies of  $K_r$  with any number of isolated vertices.*

*Proof.* Let  $G$  be a graph on  $m$  edges with maximum degree at most  $r - 1$ . Again  $\theta_G(T) \leq r - 1$  for every  $T \in \mathcal{K}^t(G)$ . By Corollary 13 we have

$$\frac{\mathcal{N}(K_t, G)}{\binom{r-2}{t-2}} = \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{r-2}{t-2}} \leq \sum_{T \in \mathcal{K}^t(G)} \frac{1}{\binom{\theta_G(T)-1}{t-2}} \leq \frac{m}{\binom{r}{2}},$$

so  $\mathcal{N}(K_t, G) \leq \frac{m}{\binom{r}{2}} \binom{r-2}{t-2} = \frac{m}{\binom{r}{2}} \binom{r}{t}$ . If  $t \geq 3$  and equality holds, then every component is a clique. As  $G$  has maximum degree at most  $r - 1$ , these cliques have at most  $r$  vertices. Suppose some  $G$  meeting the bound has a component  $C$  with  $a < \binom{r}{2}$  edges. Then  $a = \binom{x}{2}$  for some real number  $x < r$ , and  $G - C$  is a graph on  $m - a$  edges with at least  $\frac{m}{\binom{r}{2}} \binom{r}{t} - \binom{x}{t}$  copies of  $K_t$  by Theorem 3. As  $t \geq 3$ , we have

$$a \binom{r}{t} = \binom{x}{2} \binom{r}{t} = \frac{x(x-1)r(r-1)}{2t(t-1)} \binom{r-2}{t-2} > \frac{x(x-1)r(r-1)}{2t(t-1)} \binom{x-2}{t-2} = \binom{r}{2} \binom{x}{t}$$

and thus  $\frac{a}{\binom{r}{2}} \binom{r}{t} > \binom{x}{t}$ , so

$$\frac{m}{\binom{r}{2}} \binom{r}{t} - \binom{x}{t} > \frac{m}{\binom{r}{2}} \binom{r}{t} - \frac{a}{\binom{r}{2}} \binom{r}{t} = \frac{m-a}{\binom{r}{2}} \binom{r}{t}$$

which contradicts the bound.  $\square$

### 5.1.2 Weighting copies of $H$ by maximum star size

As mentioned, we can apply Theorem 11 to a broader class of graphs  $H$  than just cliques. This allows us to prove novel asymptotic results on  $\text{ex}(n, H, S_r)$  for any  $H$  with at least one dominating vertex and on  $\text{mex}(n, H, S_r)$  for  $H$  with at least two dominating vertices:

**Theorem 16.** *Let  $H$  be a graph on  $t$  vertices.*

(i) *If  $H$  has at least one dominating vertex, then*

$$\text{ex}(n, H, S_r) = (1 - o(1))\mathcal{N}(H, \lfloor \frac{n}{r} \rfloor K_r),$$

and

(ii) *if  $H$  has at least two dominating vertices, then*

$$\text{mex}(m, H, S_r) = (1 - o(1))\mathcal{N}(H, \lfloor \frac{m}{\binom{r}{2}} \rfloor K_r).$$

*Proof.* Let  $G$  be an  $S_r$ -free graph on  $n$  vertices and  $m$  edges. Then for each  $T \in \mathcal{H}(G)$ , we have  $\theta_G(T) \leq r - 1$  and thus  $s_G^1(T) \geq 1/\binom{r-1}{t-1}$  and  $s_G^2(T) \geq 1/\binom{r-2}{t-2}$ . Therefore, as long as  $\text{dom}(H) \geq 1$ , let  $v$  be a dominating vertex and apply Theorem 11 with  $u = 1$  to get

$$\mathcal{N}(H, G) \cdot \frac{1}{\binom{r-1}{t-1}} \leq \sum_{T \in \mathcal{H}(G)} s_G^1(T) \leq \frac{\mathcal{N}(H^{\downarrow 1}, K_{t-1})}{\text{dom}(H)} \cdot n$$

so that

$$\mathcal{N}(H, G) \leq n \binom{r-1}{t-1} \frac{\mathcal{N}(H^{\downarrow 1}, K_{t-1})}{\text{dom}(H)} = (1 - o(1))\mathcal{N}(H, \lfloor \frac{n}{r} \rfloor K_r),$$

as, when  $r \mid n$ , we can count copies of  $H$  in  $\lfloor \frac{n}{r} \rfloor K_r$  by first choosing a vertex  $v$  of  $G$  to act as a selected dominating vertex in  $H$ , then choosing  $t - 1$  of the  $r - 1$  other vertices in the same component, then choosing an embedding of  $H^{\downarrow 1}$  into those vertices (which is independent of the choice of  $v$ ). Each copy of  $H$  is counted this way once for each choice of selected dominating vertex in  $H$ . The asymptotic factor allows for  $n$  not divisible by  $r$ . Furthermore, as long as  $\text{dom}(H) \geq 2$ , let  $v$  and  $w$  both be dominating vertices and apply Theorem 11 with  $u = 2$  to get

$$\mathcal{N}(H, G) \cdot \frac{1}{\binom{r-2}{t-2}} \leq \sum_{T \in \mathcal{H}(G)} s_G^2(T) \leq \frac{\mathcal{N}(H^{\downarrow 2}, K_{t-1})}{\text{dom}(H)} \cdot m$$

so that

$$\mathcal{N}(H, G) \leq m \binom{r-2}{t-2} \frac{\mathcal{N}(H^{\downarrow 2}, K_{t-1})}{\text{dom}(H)} = (1 - o(1))\mathcal{N}(H, \lfloor \frac{m}{\binom{r}{2}} \rfloor K_r),$$

where similarly the asymptotic factor allows for  $m$  not divisible by  $\binom{r}{2}$ .

In both cases we achieve a matching lower bound by taking as many disjoint copies of  $K_r$  as possible and making the remaining vertices independent or making the remaining edges a matching. (For the values of  $n$  and  $m$  when we have some remaining vertices or edges, a better lower bound is given by forming a clique with the remaining vertices or a colex graph with the remaining edges.)  $\square$

Notice that cliques, stars, and all connected threshold graphs have at least one dominating vertex so are included as possible graphs  $H$  in part (i) of Theorem 16.

## 5.2 Hypergraphs

We now consider localized bounds for hypergraphs of bounded degree. Recall that a hypergraph is  $q$ -uniform if every edge is a set of  $q$  vertices. The degree of a set of vertices  $I$ , denoted by  $d(I)$ , is the number of edges  $E$  that contain  $I$ . Letting  $i = |I|$ , the neighborhood of  $I$  is the  $(q - i)$ -uniform hypergraph  $\{E \setminus I : I \subset E \in E(\mathcal{H})\}$ . For a  $q$ -uniform hypergraph  $\mathcal{H}$  and  $1 \leq i < q$ , we write  $\Delta_i(\mathcal{H})$  for the maximum degree  $d(I)$  over all sets  $I$  of  $i$  vertices.

For  $t \geq q$ , we denote by  $K_t^{(q)}$  the complete  $q$ -uniform hypergraph on  $t$  vertices. We write  $\mathcal{K}^t(\mathcal{H})$  for the set of  $t$ -cliques in  $\mathcal{H}$ , i.e.,  $\mathcal{K}^t(\mathcal{H}) = \{S \subseteq V(\mathcal{H}) : \mathcal{H}[S] \cong K_t^{(q)}\}$ . We use the following upper bound on  $\mathcal{N}(K_t^{(q)}, \mathcal{H})$ , which is proved in [19, Theorem 32] as an immediate consequence of Lovász' approximate version of the Kruskal-Katona theorem.

**Theorem 17** (Lovász [20]). *Let  $q, t \in \mathbb{N}$  with  $t \geq q$ . Let  $\mathcal{H}$  be a  $q$ -uniform hypergraph. Write the number of edges of  $\mathcal{H}$  in the form  $\binom{x}{q}$ , where  $x \geq q - 1$  is real. Then  $\mathcal{N}(K_t^{(q)}, \mathcal{H}) \leq \binom{x}{t}$ .*

The following theorem generalizes Corollary 12 to  $q$ -uniform hypergraphs. Note that when  $q = 2$ , we have  $i = 1$  and  $x(I) = d(I) + i = d(v) + 1$  for  $I = \{v\}$ , and so the function  $s(T)$  in the following theorem can be thought of as an extension of the function  $s_G^1(T)$  of Theorem 11 to hypergraphs.

**Theorem 18.** *Let  $t \geq q > i \geq 1$  and suppose  $\mathcal{H}$  is a  $q$ -uniform hypergraph on  $n$  vertices. For each  $I \in \binom{V(\mathcal{H})}{i}$ , define  $x(I) \geq q - i - 1$  by the equation  $d(I) = \binom{x(I) - i}{q - i}$ , and, for each  $T \in \mathcal{K}^t(\mathcal{H})$ , define*

$$x(T) = \max \left\{ x(I) : I \in \binom{T}{i} \right\} \quad \text{and} \quad s(T) = \frac{1}{\binom{x(T) - i}{t - i}}.$$

*Then  $s(T)$  is well-defined and decreasing as a function of  $x(T)$ ,*

$$\sum_{T \in \mathcal{K}^t(\mathcal{H})} s(T) \leq \frac{\binom{n}{i}}{\binom{t}{i}},$$

*and there is an infinite family of hypergraphs that achieve the bound.*

*Proof.* Let  $I \in \binom{V(\mathcal{H})}{i}$ . For every  $T \in \mathcal{K}^t(I)$ , we have  $x(T) \geq x(I)$  by definition. If  $\mathcal{K}^t(I)$  is nonempty, then  $d(I) \geq \binom{t - i}{q - i}$  and  $x(I) \geq t$ . Therefore every  $T \in \mathcal{K}^t(\mathcal{H})$  has  $x(T) \geq t$ , so  $s(T)$  is a decreasing function of  $x(T)$ . Hence  $T \in \mathcal{K}^t(I)$  implies  $s(T) \leq 1 / \binom{x(I) - i}{t - i}$ .

Therefore

$$\begin{aligned}
\binom{t}{i} \sum_{T \in \mathcal{K}^t(\mathcal{H})} s(T) &= \sum_{I \in \binom{V(\mathcal{H})}{i}} \sum_{T \in \mathcal{K}^t(I)} s(T) \\
&\leq \sum_{I \in \binom{V(\mathcal{H})}{i}} \sum_{T \in \mathcal{K}^t(I)} \frac{1}{\binom{x(I)-i}{t-i}} \\
&\leq \sum_{I \in \binom{V(\mathcal{H})}{i}} \frac{\binom{x(I)-i}{t-i}}{\binom{x(I)-i}{t-i}} \\
&= \binom{n}{i},
\end{aligned}$$

where the second inequality follows from applying Theorem 17 to the neighborhood of  $I$ .

Design theory provides an infinite family of graphs that meet this bound; we direct the reader to [19] for more information on such hypergraphs. If  $\mathcal{H}$  is a  $q$ -shadow of a Steiner system  $S(i, r, n)$  for some  $r$  then by [19, Lemma 38(b)] we have  $x(I) = r$  for every  $I$  and  $x(T) = r$  for every  $T$ , so  $s(T) = \frac{1}{\binom{r-i}{t-i}}$ . By [19, Lemma 38(a)] we have

$$\mathcal{N}(K_t^{(q)}, \mathcal{H}) = \binom{r}{t} \frac{\binom{n}{i}}{\binom{r}{i}}. \text{ Therefore}$$

$$\sum_{T \in \mathcal{K}^t(\mathcal{H})} s(T) = \frac{\binom{r}{t} \binom{n}{i}}{\binom{r}{i} \binom{r-i}{t-i}} = \frac{\binom{n}{i}}{\binom{t}{i}}. \quad \square$$

It seems interesting and challenging to characterize all of the extremal  $q$ -graphs in Theorem 18. See [19, Theorem 43] for a related characterization of the extremal  $q$ -graphs in the non-localized theorem.

As a corollary of Theorem 18 we obtain the following theorem of Radcliffe and the first author on maximizing the number of  $t$ -cliques among bounded-degree  $q$ -uniform hypergraphs.

**Theorem 19** (Kirsch and Radcliffe [19]). *Let  $1 \leq i < q \leq t$  and suppose  $\mathcal{H}$  is an  $q$ -uniform hypergraph on  $n$  vertices such that  $\Delta_i(\mathcal{H}) \leq \binom{x-i}{q-i}$  for some real number  $x \geq q$ . Then*

$$\mathcal{N}(K_t^{(q)}, \mathcal{H}) \leq \frac{\binom{n}{i}}{\binom{x}{i}} \binom{x}{t}.$$

*Proof using Theorem 18.* The condition  $\Delta_i(\mathcal{H}) \leq \binom{x-i}{q-i}$  implies that  $x(I) \leq x$  for every  $I \in \binom{V(\mathcal{H})}{i}$ , so  $x(T) = \max\{x(I) : I \in \binom{T}{i}\} \leq x$  for every  $T \in \mathcal{K}^t(\mathcal{H})$ . Theorem 18 gives

$$\frac{\mathcal{N}(K_t^{(q)}, \mathcal{H})}{\binom{x-i}{t-i}} = \sum_{T \in \mathcal{K}^t(\mathcal{H})} \frac{1}{\binom{x-i}{t-i}} \leq \sum_{T \in \mathcal{K}^t(\mathcal{H})} s(T) \leq \frac{\binom{n}{i}}{\binom{t}{i}},$$

$$\text{so } \mathcal{N}(K_t^{(q)}, \mathcal{H}) \leq \frac{\binom{n}{i}}{\binom{t}{i}} \binom{x-i}{t-i} = \frac{\binom{n}{i}}{\binom{x}{i}} \binom{x}{t}. \quad \square$$

## 6 Open Problems

We briefly mention a few additional instances of problems that we believe are amenable to localized extensions.

The following conjecture is a localized form of a theorem of Frohmader [13], as phrased in [19, Theorem 8], on maximizing the number of  $t$ -cliques among  $m$ -edge,  $K_{r+1}$ -free graphs.

**Conjecture 20.** Let  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define

$$\alpha_G(T) = \max\{k : T \subseteq V(S) \text{ for some } S \subseteq G \text{ s.t. } S \cong K_k\} \quad \text{and} \quad w'_G(T) = \frac{\binom{\alpha_G(T)}{2}^{t/2}}{\binom{\alpha_G(T)}{t}}.$$

For every  $m$ -edge graph  $G$ ,

$$\sum_{T \in \mathcal{K}^t(G)} w'_G(T) \leq m^{t/2}.$$

After a preprint of this paper was made available, Aragão and Souza [2] announced a proof of a generalization of Conjecture 20.

Many extremal results on paths, beginning with the results of Erdős and Gallai [12], are consequences of extremal theorems regarding cycles. While the family of cycle graphs  $\{C_3, C_4, \dots\}$  does not have the subgraph inclusion property shared by cliques, paths, and stars, these results consider graphs of bounded circumference (that is, maximum cycle length). The techniques in this paper often bounded a weight function by arguing a maximal structure could not be extended; cycles do not allow such arguments, which could make proving localized results more difficult. Nevertheless, we provide the following weight function and conjectures based on results of Luo [21] and Chakraborti and Chen [8], respectively.

**Definition 21.** Let  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define

$$\gamma_G(T) = \max\{k : T \subseteq V(S) \text{ for some } S \subseteq G \text{ s.t. } S \cong C_k\}.$$

**Conjecture 22.** Let  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define

$$c_G(T) = \frac{\gamma_G(T) - 1}{\binom{\gamma_G(T)}{t}}.$$

Then  $c_G(T)$  is well-defined and decreasing in  $\gamma_G(T)$ , and

$$\sum_{T \in \mathcal{K}^t(G)} c_G(T) \leq n - 1,$$

with equality if and only if each 2-connected component of  $G$  is a complete graph of order at least  $t$ .

**Conjecture 23.** Let  $t \geq 2$ . For each  $T \in \mathcal{K}^t(G)$ , define

$$c'_G(T) = \frac{\binom{\gamma_G(T)}{2}}{\binom{\gamma_G(T)}{t}}.$$

Then  $c'_G(T)$  is well-defined and decreasing in  $\gamma_G(T)$ , and

$$\sum_{T \in \mathcal{K}^t(G)} c'_G(T) \leq m,$$

with equality if and only if each 2-connected component of  $G$  is a complete graph of order at least  $t$  and any number of isolated vertices.

It may be possible to generalize Corollary 13 to hypergraphs. We make the following conjecture as a localized version of Theorem 51 in [19], analogously to Theorem 18.

**Conjecture 24.** Let  $t \geq q > i \geq 1$  and suppose  $\mathcal{H}$  is a  $q$ -uniform hypergraph on  $m$  edges. For each  $I \in \binom{V(\mathcal{H})}{i}$ , define  $x(I) \geq q - i - 1$  by the equation  $d(I) = \binom{x(I)-i}{q-i}$ , and, for each  $T \in \mathcal{K}^t(\mathcal{H})$ , define

$$x(T) = \max \left\{ x(I) : I \in \binom{T}{i} \right\} \quad \text{and} \quad s'(T) = \frac{1}{\binom{x(T)-q}{t-q}}.$$

Then

$$\sum_{T \in \mathcal{K}^t(\mathcal{H})} s'(T) \leq \frac{m}{\binom{t}{q}}.$$

Finally, we used Theorem 18 to obtain new asymptotically tight bounds on  $\text{ex}(n, H, S_r)$  when  $H$  has at least one dominating vertex and on  $\text{mex}(m, H, S_r)$  when  $H$  has at least two dominating vertices. It may be possible to prove similar results for hypergraphs.

**Question 25.** Can Theorem 16 (or Theorem 11) be generalized to the setting of  $q$ -uniform hypergraphs with bounded maximum  $i$ -degree, perhaps with  $i = 1$  or  $i = q - 1$ , in such a way as to obtain new generalized Turán-type results for hypergraphs?

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