On the matchings-Jack and hypermap-Jack conjectures for labelled matchings and star maps

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Abstract

Introduced by Goulden and Jackson in their 1996 paper, the matchings-Jack conjecture and the hypermap-Jack conjecture (also known as the b-conjecture) are two major open questions relating Jack symmetric functions, the representation theory of the symmetric groups and combinatorial maps. They show that the coefficients in the power sum expansion of some Cauchy sum for Jack symmetric functions and in the logarithm of the same sum interpolate respectively between the structure constants of the class algebra and the double coset algebra of the symmetric group and between the numbers of orientable and locally orientable hypermaps. They further provide some evidence that these two families of coefficients indexed by three partitions of a given integer n and the Jack parameter α are polynomials in $\beta = \alpha - 1$ with non-negative integer coefficients of combinatorial significance. This paper is devoted to the case when one of the three partitions is equal to (n). We exhibit some polynomial properties of both families of coefficients and prove a variation of the hypermap-Jack conjecture and the matchings-Jack conjecture involving labelled hypermaps and matchings in some important cases.

Mathematics Subject Classifications: 05A15, 05C30, 05E05

Keywords: Matchings-Jack and hypermap-Jack conjectures, Jack symmetric functions, matchings, star bipartite maps.

1 Introduction

1.1 Cauchy sums for Jack symmetric functions

For any integer n denote $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p) \vdash n$ an integer partition of $|\lambda| = n$ with $\ell(\lambda) = p$ parts sorted in decreasing order. The set of all integer partitions (including the

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empty one) is denoted $\mathcal{P}$. If $m_i(\lambda)$ is the number of parts of λ that are equal to i, then we may write λ as $[1^{m_1(\lambda)}\, 2^{m_2(\lambda)} \cdots]$ and define $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$, $Aut_\lambda = \prod_i m_i(\lambda)!$. When there is no ambiguity, the one part partition of integer n, $(n) = [n^1]$ is simply denoted n. Given a parameter α, denote $p_\lambda(x)$ and $J^\alpha_\lambda(x)$ the power sum and the **Jack symmetric function** indexed by λ in variable $x = (x_1, x_2, \cdots)$. Jack symmetric functions are orthogonal for the scalar product $\langle \cdot\,, \cdot \rangle_\alpha$ defined by $\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}$. Denote $j_\lambda(\alpha)$ the value of the scalar product $\langle J^\alpha_\lambda, J^\alpha_\mu \rangle_\alpha = j_\lambda(\alpha)\delta_{\lambda,\mu}$. This paper is devoted to the study of the following series for Jack symmetric functions introduced by Goulden and Jackson in [9].

$$\Phi(x, y, z, t, \alpha) = \sum_{\gamma \in \mathcal{P}} \frac{J^\alpha_\gamma(x) J^\alpha_\gamma(y) J^\alpha_\gamma(z) t^{|\gamma|}}{\langle J^\alpha_\gamma, J^\alpha_\gamma \rangle_\alpha},$$
$$\Psi(x, y, z, t, \alpha) = \alpha t \frac{\partial}{\partial t} \log \Phi(x, y, z, t, \alpha).$$

More specifically, we focus on the coefficients $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ in their power sum expansions defined by:

$$\Phi(x, y, z, t, \alpha) = \sum_{n \geqslant 0} t^n \sum_{\lambda,\mu,\nu \vdash n} \alpha^{-\ell(\lambda)} z_\lambda^{-1} a^\lambda_{\mu,\nu}(\alpha) p_\lambda(x) p_\mu(y) p_\nu(z),$$
$$\Psi(x, y, z, t, \alpha) = \sum_{n \geqslant 1} t^n \sum_{\lambda,\mu,\nu \vdash n} h^\lambda_{\mu,\nu}(\alpha) p_\lambda(x) p_\mu(y) p_\nu(z).$$

Goulden and Jackson conjecture that both the $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ may have a strong combinatorial interpretation. In particular thanks to exhaustive computations of the coefficients they show that the $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ are polynomials in $\beta = \alpha - 1$ with non-negative integer coefficients and of degree at most $n - \min\{\ell(\mu), \ell(\nu)\}$ for all $\lambda, \mu, \nu \vdash n \leqslant 8$. They conjecture this property for arbitrary λ, μ, ν and prove it in the limit cases $\lambda = [1^n]$ and $\lambda = [1^{n-2}2^1]$. Moreover, for λ, μ, ν partitions of a given integer n, they make the stronger suggestion that the coefficients in the powers of β in $a^\lambda_{\mu,\nu}(\alpha)$ count certain sets of *matchings* i.e. fixpoint-free involutions of the symmetric group on $2n$ elements (the *matchings-Jack conjecture*) and that the coefficients in the powers of β in $h^\lambda_{\mu,\nu}(\alpha)$ count certain sets of *locally orientable hypermaps* i.e. connected face-bicolored graphs embedded in a locally orientable surface (the *hypermap-Jack conjecture* or *b-conjecture*). We look at the case $\mu = (n)$ and study a variant of the conjectures involving labelled objects defined in the following section. In this specific case the two conjectures are related. Indeed, one has:

$$\Psi(x, y, z, t, \alpha) = \alpha t \frac{\partial}{\partial t} \sum_{k \geqslant 0} (-1)^{k+1} \sum_{\mu^1 \cdots \mu^k \in \mathcal{P} \setminus \emptyset} \prod_i p_{\mu^i}(y) t^{|\mu^i|} \sum_{\lambda,\nu \vdash |\mu^i|} z_\lambda^{-1} \alpha^{-\ell(\lambda)} a^\lambda_{\mu^i \nu}(\alpha) p_\lambda(x) p_\nu(z),$$

which implies that

$$[p_n(y)]\Psi(x,y,z,t,\alpha) = \alpha t\frac{\partial}{\partial t}t^n \sum_{\lambda,\nu\vdash n} z_\lambda^{-1}\alpha^{-\ell(\lambda)}a_{n\nu}^\lambda(\alpha)p_\lambda(x)p_\nu(z).$$

As a result, the following formula holds:

$$h_{n\nu}^\lambda(\alpha) = \alpha n z_\lambda^{-1}\alpha^{-\ell(\lambda)}a_{n\nu}^\lambda(\alpha). \tag{1}$$

However, because of the difference in the combinatorial objects involved in the two conjectures, they do not seem to be equivalent.

Remark 1. According to the definition of the coefficients $a_{\mu,\nu}^\lambda(\alpha)$, Equation (1) can be rewritten as $h_{n\nu}^\lambda(\alpha) = a_{\lambda\nu}^n(\alpha)$.

1.2 Combinatorial background

1.2.1 Matchings

Given a non-negative integer n and a set of $2n$ vertices $V_n = \{1, \widehat{1}, \cdots, n, \widehat{n}\}$ we call a **matching** on V_n a set of n non-adjacent edges such that all the vertices are the endpoint of one edge. Given two matchings δ_1 and δ_2, the graph induced by the vertices in V_n and the $2n$ edges of $\delta_1 \cup \delta_2$ is composed of cycles $L_1, \ldots, L_p$ of even length $2\varepsilon_1, \cdots, 2\varepsilon_p$ for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \vdash n$ and we denote $\Lambda(\delta_1, \delta_2) = \varepsilon$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_p)$ of n, define two canonical matchings $\mathbf{g}_n$ and $\mathbf{b}_\lambda$. The matching $\mathbf{g}_n$ is obtained by drawing a gray colored edge between vertices i and $\widehat{i} = \mathbf{g}_n(i)$ for $i = 1, \cdots, n$. The matching $\mathbf{b}_\lambda$ is obtained by drawing a black colored edge between vertices $\widehat{i}$ and $\mathbf{b}_\lambda(\widehat{i})$ for $i = 1, \cdots, n$ where $\mathbf{b}_\lambda(\widehat{i}) = 1 + \sum_{k=1}^{l-1}\lambda_k$ if $i = \sum_{k=1}^{l}\lambda_k$ for some $1 \leqslant l \leqslant p$ and $\mathbf{b}_\lambda(\widehat{i}) = i+1$ otherwise. Obviously $\Lambda(\mathbf{g}_n, \mathbf{b}_\lambda) = \lambda$. Denote by $\mathcal{G}_{\mu,\nu}^\lambda$ the set of all the matchings δ on V_n such that $\Lambda(\mathbf{g}_n, \delta) = \mu$ and $\Lambda(\mathbf{b}_\lambda, \delta) = \nu$. A matching δ in which all edges are of the type $\{i, \widehat{j}\}$ is called **bipartite**. This graph model is closely linked to the structure constants of two classical algebras.

- The **class algebra** is the center of the group algebra $\mathbb{C}S_n$. For $\lambda \vdash n$, denote by C_λ the formal sum of all permutations with cycle type λ. The set $\{C_\lambda \mid \lambda \vdash n\}$ is a basis of the class algebra.

- The **double coset algebra** is the Hecke algebra of the Gelfand pair (S_{2n}, B_n) where B_n is the centralizer of $f_\star = (1\widehat{1})(2\widehat{2})\ldots(n\widehat{n})$ in S_{2n}. For $\lambda \vdash n$, denote by K_λ the double coset consisting of all the permutations $\omega \in S_{2n}$ such that $f_\star \circ \omega \circ f_\star \circ \omega^{-1}$ has a cycle type $\lambda\lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$. The set $\{K_\lambda \mid \lambda \vdash n\}$ is a basis of the double coset algebra ([2, 26]).

We define the structure constants of these algebras by

$$c_{\mu\nu}^\lambda = [C_\lambda]C_\mu C_\nu \quad \text{and} \quad b_{\mu\nu}^\lambda = [K_\lambda]K_\mu K_\nu, \qquad \lambda, \mu, \nu \vdash n. \tag{2}$$

Proposition 1 ([9], Proposition 4.1; [12], Lemma 3.2). Using the notation above:

$$b^{\lambda}_{\mu\nu}/|B_n| = |\mathcal{G}^{\lambda}_{\mu\nu}| \text{ and } c^{\lambda}_{\mu\nu} = |\{\delta \in \mathcal{G}^{\lambda}_{\mu\nu} \mid \delta \text{ is bipartite}\}|.$$

This paper is focused on the case $\mu = (n)$. For $\lambda, \nu \vdash n$ we consider the set of **labelled matchings** $\widetilde{\mathcal{G}}^{\lambda}_{\nu}$ i.e. the tuples $\delta = (\bar{\delta}, \sigma_2, \cdots)$ composed of a matching $\bar{\delta} \in \mathcal{G}^{\lambda}_{n,\nu}$ and permutations σ_i on the $m_i(\nu)$ cycles of length $2i$ in $\mathbf{b}_\lambda \cup \bar{\delta}$ for all $i > 1$ (the cycles of length 2 are not labelled). We call cycles of length 4 and 6 **squares** and **hexagons** respectively. Clearly,

$$|\widetilde{\mathcal{G}}^{\lambda}_{\nu}| = \frac{Aut_\nu}{m_1(\nu)!}|\mathcal{G}^{\lambda}_{n,\nu}|.$$

Example 1. Figure 1 depicts a labelled matching from $\widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$ with three labelled squares: $\widehat{1}\widehat{2}34$, $\widehat{2}\widehat{3}65$ and $\widehat{4}\widehat{1}\widehat{5}6$.

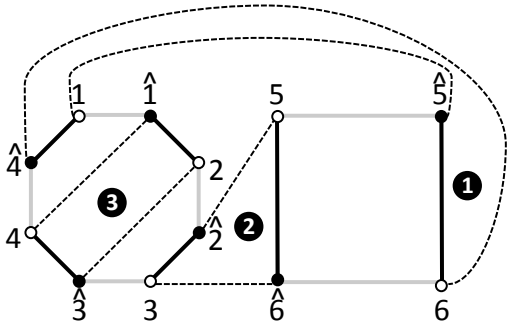


Figure 1: A labelled matching from $\widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$ with three labelled squares.

1.2.2 Locally orientable hypermaps and bipartite maps

As stated in introduction hypermaps are connected face-bicolored graphs embedded in a locally orientable surface. Hypermaps are in one-to-one correspondance with **bipartite maps** [31] and may arguably be viewed as such. Locally orientable bipartite maps are defined up to homeomorphism as connected bipartite graphs with black and white vertices. Each edge is composed of two edge-sides both connecting the two incident vertices. This graph is embedded in a **surface**, i.e. a two-dimensional manifold such that if we cut the graph from the surface, the remaining part consists of connected components called **faces or cells**, each homeomorphic to an open disk. The map can also be represented (not in a unique way) as a **ribbon graph** on the plane keeping the incidence order of the edges around each vertex. In such a representation, two edge-sides can be parallel or cross in the middle. Further, we define a **corner** as the area around a vertex delimited by two consecutive edges. We say that a bipartite map is **orientable** if it is embedded in an orientable surface (sphere, torus, bretzel, ...), i.e. if a consistent concept of clockwise rotation can be defined on the surface in a continuous manner. Otherwise the bipartite

map is embedded in a non-orientable surface (projective plane, Klein bottle, ...) and is said to be **non-orientable**. We consider only **rooted bipartite maps**, i.e. bipartite maps with a distinguished vertex and a distinguished edge-side or equivalently a distinguished corner. More details about maps can be found in [**?**].
The **degree** of a face, a white vertex or a black vertex is half the number of edge-sides incident to it or equivalently the number of incident corners (for faces count only incident corners around white vertices). Bipartite maps are also classified according to a triple of integer partitions that give respectively the degree distribution of the faces, the degree distribution of the white vertices, and the degree distribution of the black vertices. For any integer n and partitions λ, μ and ν of n, denote $\mathcal{L}^{\lambda}_{\mu,\nu}$ and $l^{\lambda}_{\mu,\nu}$ (resp. $\mathcal{M}^{\lambda}_{\mu,\nu}$ and $m^{\lambda}_{\mu,\nu}$) the set and the number of locally orientable bipartite maps (resp. orientable) of face degree distribution λ, white vertices degree distribution μ and black vertices degree distribution ν. When $\mu = (n)$, the map has only one white (root) vertex. We call it a **star bipartite map**.

Remark 2. Star bipartite maps are in natural bijection with **unicellular bipartite maps**, i.e. bipartite maps with only one face but an arbitrary number of white vertices. While unicellular maps received a more significant attention in previous papers (see e.g. [11, 22, 27, 29]), it is much more convenient to work with multicellular star maps for our purpose.

Example 2. Two star bipartite maps are depicted on Figure 2. The leftmost (resp. rightmost) one is orientable (resp. non-orientable) and has a face degree distribution $\lambda = (4, 1, 1)$ (resp. $\lambda = (4)$).



Figure 2: Examples of star bipartite maps embedded in the torus (left) and the Klein bottle (right).

Remark 3. When $\nu = [2^m]$ for some integer m, bipartite reduce to classical (non-bipartite maps). The reduction is obtained by connecting the two edges incident to each black vertex and removing all the black vertices. Figure 3 gives an example of a non-bipartite star map represented as a ribbon graph.

In this paper, we look at **labelled star bipartite maps**, i.e. bipartite maps where the $\ell(\nu)$ black vertices are labelled by integers $1, \cdots, \ell(\nu)$ such that the vertex incident to the root is labelled 1. Denote d_i the degree of the black vertex indexed i, we further

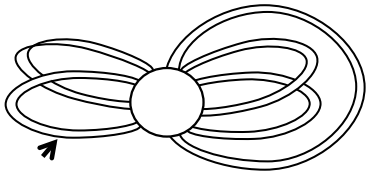


Figure 3: Example of a non-bipartite star map.

assume that the edges incident to the black vertex indexed i are labelled with

$$\sum_{1\leqslant j<i} d_j + 1, \sum_{1\leqslant j<i} d_j + 2, \cdots, \sum_{1\leqslant j\leqslant i} d_j$$

with the additional condition that the edge containing the root edge-side (incident to the black vertex indexed 1) is labelled 1. For $\lambda, \nu \vdash n$, denote $\widetilde{\mathcal{L}}_\nu^\lambda$ the set of labelled star maps with face degree distribution λ and black vertices degree distribution ν. We focus on the special case $\nu = [k^m]$. Clearly,

$$|\widetilde{\mathcal{L}}_{[k^m]}^\lambda| = (m-1)!k!^{m-1}(k-1)!l_{n,[k^m]}^\lambda = \frac{m!k!^m}{n} l_{n,[k^m]}^\lambda.$$

Example 3. The two ribbon graphs depicted on Figure 4 are labelled star maps with $\nu = [3^m]$ (left-hand side) and $\nu = [2^m]$ (right-hand side).

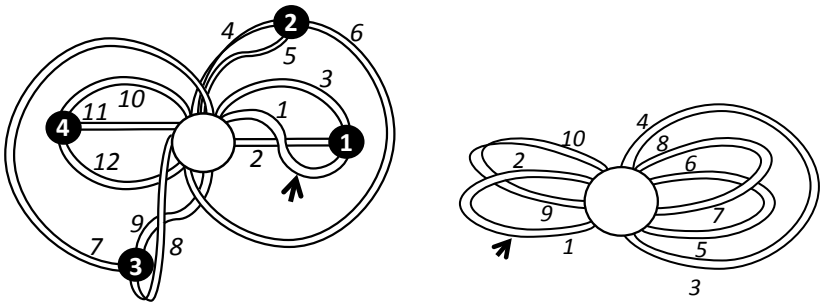


Figure 4: Examples of a labelled star bipartite map (left) and star non-bipartite map (right).

1.3 Relation to the matchings-Jack and hypermap-Jack conjectures

One can show (see e.g. [9], [12], [30]) that the numbers $c_{\mu,\nu}^\lambda$, $b_{\mu,\nu}^\lambda$ and numbers of hypermaps are linked to the coefficients $a_{\mu,\nu}^\lambda(\alpha)$ and $h_{\mu,\nu}^\lambda(\alpha)$:

$$\begin{aligned} a_{\mu,\nu}^\lambda(1) &= c_{\mu,\nu}^\lambda \quad \text{and} \quad a_{\mu,\nu}^\lambda(2) = \frac{1}{|B_n|} b_{\mu,\nu}^\lambda, \\ h_{\mu,\nu}^\lambda(1) &= m_{\mu,\nu}^\lambda \quad \text{and} \quad h_{\mu,\nu}^\lambda(2) = l_{\mu,\nu}^\lambda. \end{aligned}$$

For general values of α Goulden and Jackson conjecture the following relations between $a^{\lambda}_{\mu,\nu}(\alpha)$ (resp. $h^{\lambda}_{\mu,\nu}(\alpha)$) and sets of matchings (resp. hypermaps).

Conjecture 1 (Matchings-Jack conjecture, [9], conjecture 4.2). For $\lambda, \mu, \nu \vdash n$ there exists a function $\mathrm{wt}_\lambda \colon \mathcal{G}^{\lambda}_{\mu,\nu} \to \{0, 1, \cdots, n - \min\{\ell(\mu), \ell(\nu)\}\}$ such that

$$a^{\lambda}_{\mu,\nu}(\beta + 1) = \sum_{\delta \in \mathcal{G}^{\lambda}_{\mu,\nu}} \beta^{\mathrm{wt}_\lambda(\delta)}$$

and $\mathrm{wt}_\lambda(\delta) = 0 \iff \delta$ is bipartite.

Conjecture 2 (Hypermap-Jack conjecture, [9], conjecture 6.3). For $\lambda, \mu, \nu \vdash n$ there exists a function $\vartheta \colon \mathcal{L}^{\lambda}_{\mu,\nu} \to \{0, 1, \cdots, n - \ell(\lambda) - \ell(\mu) - \ell(\nu) + 2\}$ such that

$$h^{\lambda}_{\mu,\nu}(\beta + 1) = \sum_{M \in \mathcal{L}^{\lambda}_{\mu,\nu}} \beta^{\vartheta(M)}$$

and $\vartheta(M) = 0 \iff M$ is orientable.

2 Main results

We use linear operators for Jack symmetric functions to derive a new formula for the coefficients $a^{\lambda}_{n,\nu}(\alpha)$ for general λ and ν which shows their polynomial properties and, as a consequence of Equation (1), the polynomial properties of the coefficients $h^{\lambda}_{n,\nu}(\alpha)$. Making this formula explicit and using some bijective constructions for labelled star bipartite maps and matchings, we show a variant of the matchings-Jack and the hypermap-Jack conjectures for labelled objects in some important cases.

Denote D_α, the Laplace-Beltrami operator. Namely,

$$D_\alpha = \frac{(\alpha - 1)}{2} \sum_i i(i-1) p_i \frac{\partial}{\partial p_i} + \frac{\alpha}{2} \sum_{i,j} ij p_{i+j} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + \frac{1}{2} \sum_{i,j} (i + j) p_i p_j \frac{\partial}{\partial p_{i+j}}.$$

and let Δ and $\{\Omega_k\}_{k \geqslant 1}$ be the operators on symmetric functions defined by

$$\Delta = [D_\alpha, [D_\alpha, p_1/\alpha]], \quad \Omega_1 = [D_\alpha, p_1/\alpha], \quad \Omega_{k+1} = [\Delta, \Omega_k].$$

where $[\cdot, \cdot]$ stands for the Lie bracket. Our main result can be stated as follows

Theorem 1. For any integer n and $\lambda, \nu \vdash n$, the coefficient $a^{\lambda}_{n,\nu}(\alpha)$ satisfies:

$$Aut_\nu \sum_{\lambda \vdash n} z_\lambda^{-1} \alpha^{-\ell(\lambda)} a^{\lambda}_{n,\nu}(\alpha) p_\lambda = \frac{1}{\prod_{i \geqslant 1} \nu_i!} \left(\prod_{i \geqslant 2} \Omega_{\nu_i} \right) \Delta^{\nu_1 - 1}(p_1/\alpha). \tag{3}$$

As a consequence of Theorem 1, we have the following polynomial properties.

Corollary 1. For $\lambda, \nu \vdash n$, $Aut_\nu |C_\lambda| a^{\lambda}_{n,\nu}(\alpha)$ and $Aut_\nu \prod_{i \geqslant 1} \nu_i! h^{\lambda}_{n,\nu}(\alpha)$ are polynomials in α with integer coefficients of respective degrees at most $n - \ell(\nu)$ and $n + 1 - \ell(\lambda) - \ell(\nu)$.

Remark 4. Our definition of the Laplace-Beltrami D_α operator can be found e.g. in [21, VI.4, example 3]. It differs a little from the one used by Stanley in [25] but can be defined on symmetric functions with an infinite number of indeterminates and provides the good properties we need.

Explicit computation of operators Ω_k for $k = 1, 2, 3$ and Δ, allows us to show:

Theorem 2. For $\lambda, \nu \vdash n$, define $\widetilde{a}^\lambda_{n,\nu}(\alpha) = (Aut_\nu/m_1(\nu)!)a^\lambda_{n,\nu}(\alpha)$. If at most one part of ν is strictly greater than 3, there exists a function $\mathrm{wt}\colon \widetilde{\mathcal{G}}^\lambda_\nu \to \{0, 1, 2, \cdots, n - \ell(\nu)\}$ such that

$$\widetilde{a}^\lambda_{n,\nu}(\beta + 1) = \sum_{\delta \in \widetilde{\mathcal{G}}^\lambda_\nu} \beta^{\mathrm{wt}(\delta)}$$

and $\mathrm{wt}(\delta) = 0 \iff \delta$ is bipartite.

Remark 5. The inequality $\mathrm{wt}(\delta) \leqslant n - \ell(\nu)$ in Theorem 2 follows from Corollary 1 while Conjecture 1 assumes the weaker upper bound $\mathrm{wt}(\delta) \leqslant n - 1$.

Remark 6. As an immediate corollary to Theorem 2, the stronger form of Conjecture 1 is true for $\lambda, \mu, \nu \vdash n$, with $\mu = (n)$, at most one part of ν is strictly greater than 3 and $m_2(\nu), m_3(\nu) \in \{0, 1\}$.

Remark 7. Let $\rho \vdash m$ be a given integer partition. If Conjecture 1 were proved to be true for the coefficients $a^{\lambda'}_{m,\rho}(\beta + 1)$ for all $\lambda' \vdash m$ then Theorem 2 remains true if one replaces $\widetilde{a}^\lambda_{n,\nu}(\beta + 1)$ by $\widetilde{a}^{\lambda''}_{n+m,\rho\cup\nu}(\beta + 1)$ with $\lambda'' \vdash n + m$ and ν satisfies the conditions of Theorem 2.

Theorem 3. For $\lambda \vdash n$ and integers k and m with $n = km$, define $\widetilde{h}^\lambda_{n,[k^m]}(\alpha) = \frac{m!k!^m}{n} h^\lambda_{n,[k^m]}(\alpha)$. For all $k \in \{1, 2, 3, n\}$ there exists a function $\vartheta\colon \widetilde{\mathcal{L}}^\lambda_{[k^m]} \to \{0, 1, 2, \cdots, n + 1 - \ell(\lambda) - m\}$ such that

$$\widetilde{h}^\lambda_{n,[k^m]}(\beta + 1) = \sum_{M \in \widetilde{\mathcal{L}}^\lambda_{[k^m]}} \beta^{\vartheta(M)}$$

and $\vartheta(M) = 0 \iff M$ is orientable.

Remark 8. The focus on labelled objects and this variant of the matchings-Jack and the hypermap-Jack conjectures is motivated by the coefficients Aut_ν and $\prod_{i \geqslant 1} \nu_i!$ that appear in Equation (3).

Remark 9. One can notice that we consider all the partitions ν with any number of parts 1, 2 and 3 in Theorem 2 but only the partitions of the type $\nu = [k^m]$ in Theorem 3. This is due to the existence of a distinguished (root) edge-side in the star maps of the later theorem that prevents the extension of our methods to less symmetric cases.

3 Background and prior works

The following sections provide some relevant background regarding the computation of $c^\lambda_{\mu,\nu}$, $b^\lambda_{\mu,\nu}$, $m^\lambda_{\mu,\nu}$ and $l^\lambda_{\mu,\nu}$, i.e. the computation of the coefficients $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ in the classical cases $\alpha \in \{1, 2\}$ and known results for these coefficients with general α.

3.1 Classical enumeration results for matchings and maps

Except for special cases no closed formulas are known for the coefficients $c^{\lambda}_{\mu,\nu}$, $b^{\lambda}_{\mu,\nu}$, $m^{\lambda}_{\mu,\nu}$ and $l^{\lambda}_{\mu,\nu}$. Prior works on the subjects are usually focused on the case $\lambda = (n)$. With this particular parameter, one has $c^{n}_{\mu,\nu} = m^{n}_{\mu,\nu}$ and $l^{n}_{\mu,\nu} = \widetilde{b}^{n}_{\mu,\nu}$. Using an inductive argument Bédard and Goupil [1] first found a formula for $c^{n}_{\lambda,\mu}$ in the case $\ell(\lambda)+\ell(\mu) = n+1$, which was later reproved by Goulden and Jackson [10] via a bijection with a set of ordered rooted bicolored trees. Later, using characters of the symmetric group and a combinatorial development, Goupil and Schaeffer [11] derived an expression for the connection coefficients $c^{n}_{\mu,\nu}$ in the general case as a sum of positive terms (see Biane [3] for a succinct algebraic derivation; and Poulalhon and Schaeffer [24], and Irving [13] for further generalisations). Closed form formulas of the expansion of the generating series for the $c^{n}_{\lambda,\mu}$ and $b^{n}_{\lambda,\mu}$ and their generalisations in the monomial basis were provided by Morales and Vassilieva and Vassilieva using bijective constructions for hypermaps in [22], [29] and [27]. Equivalent results using purely algebraic methods are provided in [28].

3.2 Prior results on the Matchings-Jack conjecture and the Hypermap-Jack conjecture

While the matchings-Jack and the hypermap-Jack conjecture are still open in the general case, some special cases and weakened forms have been solved over the past decade. In particular, Brown and Jackson in [4] prove that for any partition $\mu \vdash 2m$, $\sum_{\lambda} h^{\lambda}_{\mu,[2^m]}(\beta+1)$ satisfies a weaker form of the hypermaps-Jack conjecture. Later on, in his PhD thesis ([15]), Lacroix defines a *measure of non-orientability* ϑ for hypermaps and focuses on a stronger form of the result of Brown and Jackson. He shows that

$$\sum_{\ell(\lambda)=r} h^{\lambda}_{\mu,[2^m]}(\beta+1) = \sum_{M \in \bigcup_{\ell(\lambda)=r} \mathcal{L}^{\lambda}_{\mu,[2^m]}} \beta^{\vartheta(M)}.$$

In particular he proves the hypermap-Jack conjecture for $h^{n}_{\mu,[2^m]}(\beta+1)$. Finally, Dolega in [6] shows that

$$h^{n}_{\mu,\nu}(\beta+1) = \sum_{M \in \mathcal{L}^{n}_{\mu,\nu}} \beta^{\vartheta(M)}$$

holds true when either β is restricted to the values $\beta \in \{-1, 0, 1\}$ or β is general but $\ell(\mu)+\ell(\nu) \geqslant n-3$. Recently Chapuy and Dolega [5] provides a generalisation of the method of Lacroix to prove the conjecture for the coefficients

$$\sum_{\ell(\lambda)=r} h^{\lambda}_{\mu,\nu}(\beta+1)$$

with general ν.

Except the limit cases $\lambda = [1^n], [2, 1^{n-2}]$ already covered by Goulden and Jackson [9], the matchings-Jack conjecture has been proved by Kanunnikov and Vassiliveva [14] in

the case $\mu = \nu = (n)$. More precisely the authors introduce a weight function wt_λ for matchings in $\mathcal{G}^\lambda_{n,n}$

$$a^\lambda_{n,n}(\beta+1) = \sum_{\delta \in \mathcal{G}^\lambda_{n,n}} \beta^{\mathrm{wt}_\lambda(\delta)},$$

besides, $\mathrm{wt}(\delta) = 0$ iff δ is bipartite.

In [8] and [7] Dolega and Feray focus only on the polynomiality part of the conjectures and show that the $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ are polynomials in α with **rational** coefficients for arbitrary partitions λ, μ, ν. See also [30] for a proof of the polynomiality with non-negative integer coefficients of a multi-indexed variation of $a^\lambda_{\mu,\nu}(\alpha)$ in some important special cases.

4 Proof of Theorem 1 and Corollary 1

We provide an algebraic proof of Theorem 1 and Corollary 1 using classical operators for Jack symmetric functions and their properties.

4.1 Properties of Jack symmetric functions

In order to prove Theorem 1, we need to recall some known properties of Jack symmetric functions.

Pieri formulas

For $\rho \vdash n+1$ and integer $1 \leqslant i \leqslant \ell(\rho)$ define the partition $\rho_{(i)}$ of n (if it exists) obtained by replacing ρ_i in ρ by $\rho_i - 1$ and keeping all the other parts as in ρ. Similarly, for $\gamma \vdash n$ and integer $1 \leqslant i \leqslant \ell(\gamma)+1$ we define the partition $\gamma^{(i)}$ of $n+1$ (if it exists) obtained by replacing γ_i in γ by $\gamma_i + 1$ and keeping all the other parts as in γ. Finally, recall the notation for the scalar product $j_\lambda(\alpha) = \langle J^\alpha_\lambda, J^\alpha_\lambda \rangle_\alpha$. In [18] Lassalle shows the following Pieri formulas.

$$p_1 J^\alpha_\gamma = \sum_{i=1}^{\ell(\gamma)+1} c_i(\gamma) J^\alpha_{\gamma^{(i)}}, \tag{4}$$

$$p_1^\perp J^\alpha_\rho = \alpha \frac{\partial}{\partial p_1} J^\alpha_\rho = \sum_{i=1}^{\ell(\rho)} \frac{j_\rho(\alpha)}{j_{\rho_{(i)}}(\alpha)} c_i(\rho_{(i)}) J^\alpha_{\rho_{(i)}}, \tag{5}$$

where for any operator A, $A^\perp$ is the adjoint operator of A for the scalar product $\langle \cdot, \cdot \rangle_\alpha$ and the numbers $c_i(\gamma)$ are equal to

$$c_i(\gamma) = \frac{1}{\alpha\gamma_i + \ell(\gamma) - i + 2} \prod_{\substack{j=1 \\ j \neq i}}^{\ell(\gamma)+1} \frac{\alpha(\gamma_i - \gamma_j) + j - i + 1}{\alpha(\gamma_i - \gamma_j) + j - i}.$$

Remark 10. The coefficients $c_i(\gamma)$ may also be defined as

$$c_i(\gamma) = \alpha \binom{\gamma^{(i)}}{\gamma} \frac{j_\gamma(\alpha)}{j_{\gamma^{(i)}}(\alpha)}.$$

where the $\binom{\lambda}{\mu}$ are some generalised binomial coefficients of Lassalle such that for two partitions, $\mu \subseteq \lambda$ and a fixed number of variables N:

$$\frac{J^\alpha_\lambda(1+x_1, 1+x_2, \ldots, 1+x_N)}{J^\alpha_\lambda(1^N)} = \sum_{\mu\subseteq\lambda} \binom{\lambda}{\mu} \frac{J^\alpha_\mu(x_1, x_2, \ldots, x_N)}{J^\alpha_\mu(1^N)}.$$

Details about the existence and properties of these binomial coefficients can be found in [18, 19, 23].

Power sum expansion and Laplace Beltrami operator

For $\lambda, \mu \vdash n$, denote $\theta^\lambda_\mu(\alpha)$ the coefficient of p_μ in the power sum expansion of J^α_λ. Namely,

$$J^\alpha_\lambda = \sum_{\mu\vdash n} \theta^\lambda_\mu(\alpha) p_\mu.$$

As shown in [20, 21] Jack symmetric functions are eigenfunctions of D_α and satisfy

$$D_\alpha J^\alpha_\lambda = \theta^\lambda_{[1^{|\lambda|-2}2^1]}(\alpha) J^\alpha_\lambda. \tag{6}$$

Furthermore, according to [14, Lemma 2], for any partition γ of some integer n

$$\theta^{\gamma^{(i)}}_{n+1}(\alpha) = \theta^\gamma_n(\alpha) \left(\theta^{\gamma^{(i)}}_{[1^{n-1}2]}(\alpha) - \theta^\gamma_{[1^{n-2}2]}(\alpha)\right). \tag{7}$$

Finally, for integers $a, b \geqslant 0$, the following relation holds ([14, Equation (30)]):

$$\sum_{i=1}^{\ell(\gamma)+1} c_i(\gamma) \left(\theta^{\gamma^{(i)}}_{[1^{n-1}2]}\right)^a \left(\theta^\gamma_{[1^{n-2}2]}\right)^b J^\alpha_{\gamma^{(i)}} = D^a_\alpha p_1 D^b_\alpha J^\alpha_\gamma \tag{8}$$

Operators

Following [17], denote also the two conjugate operators E_2 and $E_2^\perp$ defined by

$$E_2 = [D_\alpha, p_1/\alpha] = \sum_{i\geqslant 1} i p_{i+1} \frac{\partial}{\partial p_i},$$
$$E_2^\perp = [p_1^\perp/\alpha, D_\alpha] = \sum_{i\geqslant 1} (i+1) p_i \frac{\partial}{\partial p_{i+1}}.$$

We show in [14, Theorem 5] that for x and y indeterminates the following relation for Jack symmetric functions holds

$$\sum_{\rho\vdash n+1} \frac{\theta^\rho_{n+1}(\alpha) J^\alpha_\rho(x) E_2^\perp J^\alpha_\rho(y)}{j_\rho(\alpha)} = \sum_{\gamma\vdash n} \frac{\theta^\gamma_n(\alpha) J^\alpha_\gamma(y) \Delta J^\alpha_\gamma(x)}{j_\gamma(\alpha)}. \tag{9}$$

4.2 Proof of Theorem 1

We can now proceed with the proof of Theorem 1. The first step is to show the following lemma.

Lemma 1. Let x and y be two indeterminates. Jack symmetric functions satisfy

$$\sum_{\rho\vdash n+1} \frac{\theta^{\rho}_{n+1}(\alpha) J^{\alpha}_{\rho}(x) p_1^{\perp} J^{\alpha}_{\rho}(y)}{j_{\rho}(\alpha)} = \alpha \sum_{\gamma\vdash n} \frac{\theta^{\gamma}_{n}(\alpha) J^{\alpha}_{\gamma}(y) E_2 J^{\alpha}_{\gamma}(x)}{j_{\gamma}(\alpha)}. \tag{10}$$

Proof. Start with the second Pieri formula and then apply the known identities above. For brevity, we omit parameter α in Jack symmetric functions and their coefficients in the power sum basis.

$$\begin{aligned}
\sum_{\rho\vdash n+1} \frac{\theta^{\rho}_{n+1} J_{\rho}(x) p_1^{\perp} J_{\rho}(y)}{j_{\rho}} &\overset{(5)}{=} \sum_{\rho\vdash n+1} \sum_{i=1}^{\ell(\rho)} \frac{\theta^{\rho}_{n+1} J_{\rho}(x) c_i(\rho_{(i)}) J_{\rho_{(i)}}(y)}{j_{\rho_{(i)}}}, \\
&= \sum_{\gamma\vdash n} \sum_{i=1}^{\ell(\gamma)+1} \frac{\theta^{\gamma^{(i)}}_{n+1} J_{\gamma^{(i)}}(x) c_i(\gamma) J_{\gamma}(y)}{j_{\gamma}}, \\
&\overset{(7)}{=} \sum_{\gamma\vdash n} \sum_{i=1}^{\ell(\gamma)+1} \frac{c_i(\gamma) \left(\theta^{\gamma^{(i)}}_{[1^{n-1}2]} - \theta^{\gamma}_{[1^{n-2}2]}\right) \theta^{\gamma}_{n} J_{\gamma^{(i)}}(x) J_{\gamma}(y)}{j_{\gamma}}, \\
&\overset{(8)}{=} \sum_{\gamma\vdash n} \frac{\theta^{\gamma}_{n} J_{\gamma}(y) (D_{\alpha} p_1 - p_1 D_{\alpha}) J_{\gamma}(x)}{j_{\gamma}}, \\
&= \alpha \sum_{\gamma\vdash n} \frac{\theta^{\gamma}_{n} J_{\gamma}(y) E_2 J_{\gamma}(x)}{j_{\gamma}}.
\end{aligned}$$

□

The key element of the proof of Theorem 1 is the following result.

Theorem 4. For any integer $k \geqslant 1$ denote Π_k the operator defined by:

$$\Pi_1 = \frac{1}{\alpha} p_1^{\perp} = \frac{\partial}{\partial p_1}, \quad \Pi_{k+1} = [\Pi_k, E_2^{\perp}].$$

Given two indeterminates x and y, the following identity holds:

$$\sum_{\rho\vdash n+k} \frac{\theta^{\rho}_{n+k}(\alpha) J^{\alpha}_{\rho}(x) \Pi_k J^{\alpha}_{\rho}(y)}{j_{\rho}(\alpha)} = \sum_{\gamma\vdash n} \frac{\theta^{\gamma}_{n}(\alpha) J^{\alpha}_{\gamma}(y) \Omega_k J^{\alpha}_{\gamma}(x)}{j_{\gamma}(\alpha)}. \tag{11}$$

Proof. In the case $k = 1$ Theorem 4 reduces to Equation (10). Assume the property is

true for some $k \geqslant 1$. We have (reference to parameter α is also removed)

$$\begin{aligned}
&\sum_{\rho\vdash n+k+1} \frac{\theta^\rho_{n+k+1} J_\rho(x)\Pi_{k+1}J_\rho(y)}{j_\rho} \\
&= \sum_{\rho\vdash n+k+1} \frac{\theta^\rho_{n+k+1} J_\rho(x)[\Pi_k, E_2^\perp]J_\rho(y)}{j_\rho} \\
&= \Pi_k \sum_{\rho\vdash n+k+1} \frac{\theta^\rho_{n+k+1} J_\rho(x)E_2^\perp J_\rho(y)}{j_\rho} - E_2^\perp \sum_{\rho\vdash n+k+1} \frac{\theta^\rho_{n+k+1} J_\rho(x)\Pi_k J_\rho(y)}{j_\rho} \\
&= \Pi_k \sum_{\rho\vdash n+k} \frac{\theta^\rho_{n+k} \Delta J_\rho(x) J_\rho(y)}{j_\rho} - E_2^\perp \sum_{\gamma\vdash n+1} \frac{\theta^\gamma_{n+1} J_\gamma(y)\Omega_k J_\gamma(x)}{j_\gamma}
\end{aligned}$$

$$\begin{aligned}
&= \Delta \sum_{\rho\vdash n+k} \frac{\theta^\rho_{n+k} J_\rho(x)\Pi_k J_\rho(y)}{j_\rho} - \Omega_k \sum_{\gamma\vdash n+1} \frac{\theta^\gamma_{n+1} E_2^\perp J_\gamma(y) J_\gamma(x)}{j_\gamma} \\
&= \Delta \sum_{\gamma\vdash n} \frac{\theta^\gamma_{n} \Omega_k J_\gamma(x) J_\gamma(y)}{j_\gamma} - \Omega_k \sum_{\gamma\vdash n} \frac{\theta^\gamma_{n} J_\gamma(y) \Delta J_\gamma(x)}{j_\gamma} \\
&= \sum_{\gamma\vdash n} \frac{\theta^\gamma_{n} \Delta\Omega_k J_\gamma(x) J_\gamma(y)}{j_\gamma} - \sum_{\gamma\vdash n} \frac{\theta^\gamma_{n} J_\gamma(y)\Omega_k \Delta J_\gamma(x)}{j_\gamma} \\
&= \sum_{\gamma\vdash n} \frac{\theta^\gamma_{n} [\Delta, \Omega_k] J_\gamma(x) J_\gamma(y)}{j_\gamma},
\end{aligned}$$

where the fourth and the sixth line are both obtained by applying Equation (9) and the recurrence hypothesis. As a result the property is true for $k+1$. □

We end the proof of Theorem 1 by noticing that

$$\Pi_k = k! \frac{\partial}{\partial p_k}.$$

For an arbitrary integer partition $\nu = (\nu_1, \ldots, \nu_p)$ of n, rewrite Equation (11) with n instead of $n+k$ and ν_p instead of k and extract the coefficient in $p_{\nu\setminus\nu_p}(y)$:

$$m_{\nu_p}(\nu) \sum_{\lambda\vdash n} z_\lambda^{-1} \alpha^{-\ell(\lambda)} a^\lambda_{n,\nu} p_\lambda(x) = \Omega_{\nu_p} \sum_{\rho\vdash n-\nu_p} \frac{z_\rho^{-1}\alpha^{-\ell(\rho)}}{\nu_p!} a^\rho_{n-\nu_p,\nu\setminus\nu_p} p_\rho(x). \tag{12}$$

Iterating the equation above for $\nu_2, \ldots, \nu_{p-1}$ and, then, applying $\nu_1 - 1$ times Equation (9) yields the desired formula.

4.3 Proof of Corollary 1

Note that

$$[D_\alpha, p_1/\alpha] = E_2 = \sum_{i\geqslant 1} i p_{i+1} \frac{\partial}{\partial p_i}.$$

As a result it is clear from the definition of operators $\{\Omega_k\}_k$ and Δ that, for any integer partition ν, the coefficients in the power sum expansion of $\left(\prod_{i\geqslant 2}\Omega_{\nu_i}\right)\Delta^{\nu_1-1}(p_1)$ are polynomial in α with (possibly negative) integer coefficients. Denote for any $\lambda, \nu \vdash n$ the integers $\{g^i_{\lambda,\nu}\}_{i\geqslant 0}$ such that

$$Aut_\nu |C_\lambda| \alpha^{-\ell(\lambda)} a^\lambda_{n,\nu}(\alpha) = \frac{1}{\alpha}\binom{n}{\nu}[p_\lambda]\left(\prod_{i\geqslant 2}\Omega_{\nu_i}\right)\Delta^{\nu_1-1}(p_1) = \frac{1}{\alpha}\sum_{i\geqslant 0} g^i_{\lambda,\nu}\alpha^i,$$

where $\binom{n}{\nu} = n!/(\nu_1!\nu_2!\dots)$ is the classical multinomial coefficient. But, according to [30, Thm. 5],

$$\alpha^{\ell(\lambda)} a^\lambda_{n,\nu}(\alpha^{-1}) = (-\alpha)^{-n+1+\ell(\lambda)+\ell(\nu)} \alpha^{-\ell(\lambda)} a^\lambda_{n,\nu}(\alpha).$$

Replacing the coefficients $a^\lambda_{n,\nu}(\alpha)$ and $a^\lambda_{n,\nu}(\alpha^{-1})$ by their expressions in terms of $\{g^i_{\lambda,\nu}\}_{i\geqslant 0}$ yields

$$\begin{aligned}
\alpha \sum_{i\geqslant 0} g^i_{\lambda,\nu}\alpha^{-i} &= (-\alpha)^{-n+1+\ell(\lambda)+\ell(\nu)} \frac{1}{\alpha}\sum_{i\geqslant 0} g^i_{\lambda,\nu}\alpha^i, \\
\sum_{i\geqslant 0} g^i_{\lambda,\nu}\alpha^{n+1-\ell(\lambda)-\ell(\nu)-i} &= (-1)^{-n+1+\ell(\lambda)+\ell(\nu)} \sum_{i\geqslant 0} g^i_{\lambda,\nu}\alpha^i.
\end{aligned}$$

Equating the coefficients in α^i in the equation above shows that $i > n+1-\ell(\lambda)-\ell(\nu)$ implies that $g^i_{\lambda,\nu} = 0$. As a consequence, $Aut_\nu|C_\lambda|\alpha^{1-\ell(\lambda)} a^\lambda_{n,\nu}(\alpha)$ is a polynomial in α with integer coefficients of degree at most $n+1-\ell(\lambda)-\ell(\nu)$ and, finally, $Aut_\nu|C_\lambda| a^\lambda_{n,\nu}(\alpha)$ is a polynomial in α with integer coefficients of degree at most $n-\ell(\nu)$.
Using this result together with Equation (1) shows that $Aut_\nu(n-1)! h^\lambda_{n,\nu}(\alpha)$ is a polynomial in α with integer coefficients of degree at most $n+1-\ell(\nu)-\ell(\lambda)$.

5 Star maps and matching decomposition

While Theorem 1 allows us to demonstrate most of the polynomial properties of $a^\lambda_{n,\nu}(\alpha)$ and $h^\lambda_{n,\nu}(\alpha)$, it is not enough to prove the matchings-Jack and the hypermap-Jack conjectures. In particular, it is not clear from the definition of operators $\{\Omega_k\}_k$ that the coefficients of the expansion of $a^\lambda_{n,\nu}(\alpha)$ and $h^\lambda_{n,\nu}(\alpha)$ in $\beta = \alpha - 1$ are non-negative. We overcome this issue by proving a combinatorial interpretation of Operators Ω_k and Δ in terms of maps and matching decompositions. More precisely we show that the application of Operator Ω_k corresponds to the addition of a black vertex of degree k in star maps and a cycle of length $2k$ in matchings. Previously, in [14], we provided an interpretation

of the application of operator Δ in terms of edges addition for matchings with only one cycle. In this paper, we give the interpretation of Δ in terms of edge addition in the case of maps with exactly one white and one black vertex. We proceed as follows. First we introduce the combinatorial decomposition involved. Then we focus on the interpretation of Ω_2 for matchings and Δ for maps. Finally we provide more general constructions in order to prove Theorems 2 and 3.

5.1 Edge deletion procedure for matchings

Recall the definition of labelled matchings from section 1.2.1 and their induced graph. We look at a recursive decomposition of matchings consisting in:

- removing the smallest labelled cycle (vertices and edges) with greatest label in the corresponding induced graph
- properly reconnecting vertices that lost connectedness in the procedure

Keeping track of the removed vertices' labels we make the decomposition reversible. An example of this decomposition with a cycle of length 4 is depicted on Figure 5.

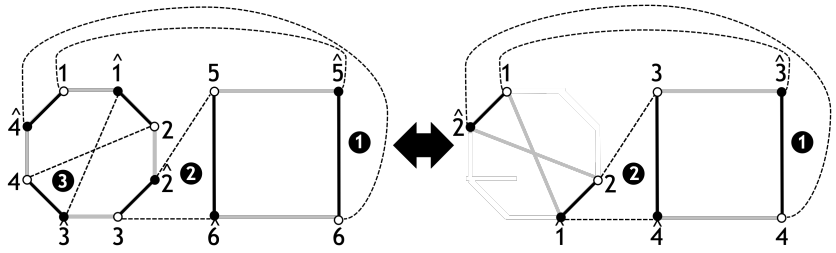


Figure 5: Application of the multi-edge deletion procedure to a labelled matching $\delta \in \widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$. The resulting labelled matching δ' belongs to $\widetilde{\mathcal{G}}^{(2,2)}_{[2^2]}$

We proceed with a formal definition.

Definition 1 (Multi-edge deletion procedure). For any integer partitions $\lambda, \nu \vdash n$ such that the smallest part of ν is equal to $i \geqslant 1$ (i.e. $m_j(\nu) = 0$ for $j < i$) we associate to a labelled matching $\delta \in \widetilde{\mathcal{G}}^{\lambda}_{\nu}$, a labelled matching $\delta' \in \widetilde{\mathcal{G}}^{\lambda'}_{\nu\setminus\{i\}}$. The partition λ' is obtained by deleting successively all the edges $(u, \delta(u))$ according to the procedure below in the labelled graph $\Gamma_{\lambda,\nu}(\delta)$ induced by $\mathbf{g}_n$, $\mathbf{b}_\lambda$ and δ where u runs over all the vertices belonging to the cycle of length $2i$ of $\mathbf{b}_\lambda \cup \bar{\delta}$ with the greatest label $m_i(\nu)$. In the case $i = 1$ the cycles are not labelled, and we delete at once all the edges belonging to the cycles of length 2 in $\mathbf{b}_\lambda \cup \bar{\delta}$.

(i) Delete the vertices u, $\delta(u)$ and all their incident edges.

(ii) Draw a new gray edge between vertices $\widehat{u}$ and $\mathbf{g}_n \circ \delta(u)$.

(iii) If $\delta(u) \neq \mathbf{b}_\lambda(u)$ (i.e. $i \neq 1$) draw a new black edge between $\mathbf{b}_\lambda(u)$ and $\mathbf{b}_\lambda \circ \delta(u)$.

After deletion of the considered set of edges, relabel the vertices in the graph with $(1, \widehat{1}, \cdots, n-i, \widehat{n-i})$ (for i>1 and $(1, \widehat{1}, \cdots, n-m_1(\nu), \widehat{n-m_1(\nu)})$ otherwise) in some canonical way. The cycles' labels are not modified. As a result of this procedure, one gets new (labelled) matchings $\mathbf{g}_{n-1}$, $\mathbf{b}_{\lambda'}$ and δ'.

Remark 11. It is easy to show that the resulting graph does not depend on the order of deletion of the edges within a given cycle of length $2i$ of $\mathbf{b}_\lambda \cup \bar{\delta}$. Furthermore, one can show that thanks to iteration of item (ii) $\Lambda(\mathbf{g}_{n-i}, \delta') = (n-i)$. Finally, note that the procedure may imply that some vertices with a non-hat index in $\Gamma_{\lambda,\nu}(\delta)$ are eventually relabelled with a hat index and vice-versa.

Example 4. Figure 5 illustrates the application of the edge removal procedure to a labelled matchings in $\widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$. Note that among the remaining vertices, the one indexed $\widehat{2}$ (resp. the one indexed 3) in the original labelled matching on the left-hand side is relabelled with a non-hat (resp. hat) index.

We have the following immediate lemma.

Lemma 2. Fix integer partitions $\lambda, \nu \vdash n$, integer $i > 1$ with $m_j(\nu) = 0$ for $j < i$ and a given set E of i non-adjacent edges on the set of vertices $V_n = \{1, \widehat{1}, \cdots, n, \widehat{n}\}$ such that the biggest connected component of $E \cup \mathbf{b}_\lambda$ is a cycle of length $2i$. The multi-edge deletion procedure provides a natural bijection φ_E between the set of labelled matchings $\delta \in \widetilde{\mathcal{G}}^{\lambda}_{\nu}$ such that E is contained in the cycle of size $2i$ with the maximum label in $\mathbf{b}_\lambda \cup \bar{\delta}$ and the set of labelled matchings $\bigcup_{\lambda'} \widetilde{\mathcal{G}}^{\lambda'}_{\nu\setminus\{i\}}$ for some set of integer partitions $\lambda' \vdash n-i$. For $i = 1$ we define φ_E for any set of k edges $E = \bigcup_{t=1}^{k}(u_t, \mathbf{b}_\lambda(u_t))$.

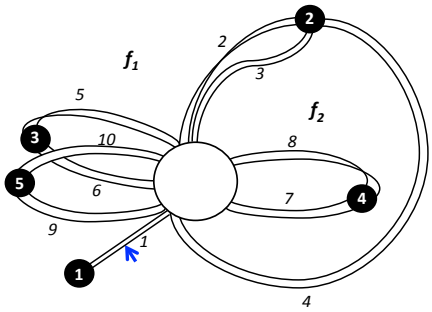
5.2 Root edge deletion and measure of non-orientability for labelled star bipartite maps

Lacroix in [15, Definition 4.1] introduces both a recursive decomposition and a so called **measure of non-orientability** for maps. Essentially it consists in deleting the root edge (i.e. the edge containing the distinguished edge-side) and rerooting the map in some canonical way. Depending on the properties of the recursively deleted edges, an integer is increased or not at each step. This is the measure of non-orientability. We adapt this definition to the case of labelled star bipartite maps. We begin with an illustration on Figure 6 and proceed with the formal definition.

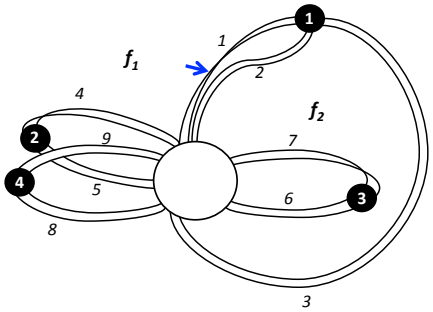
In what follows, we name **a leaf** an edge connecting a black vertex of degree 1 and the white vertex.

Definition 2 (Measure of non-orientability for labelled star bipartite map). To any labelled star bipartite map M of face distribution $\lambda \neq \emptyset$, we associate a labelled star bipartite map M' of face distribution λ' obtained by

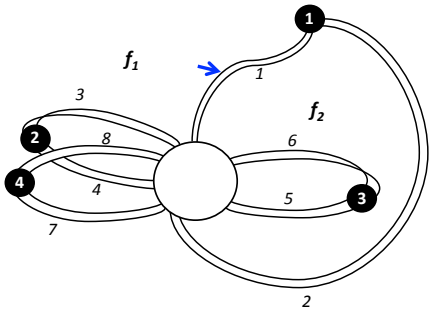
- deleting the edge containing the root edge-side,
- defining the new root as the edge-side of the edge labelled 2 in M on the same side as the previous root using the local order around the white vertex



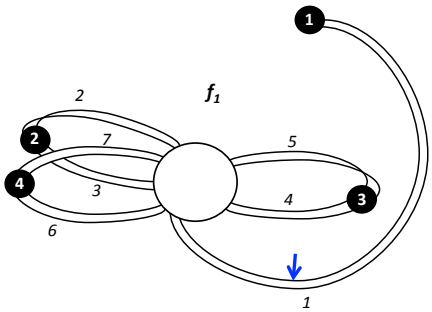
(a) A labelled star bipartite map with two faces f_1 and f_2.



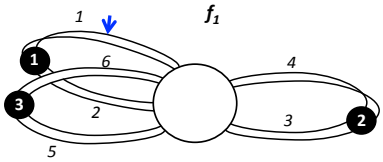
(b) Leaf deletion.



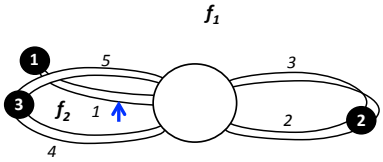
(c) Cross border deletion.



(d) Border deletion. The two faces are merged into one single face f_1.



(e) Leaf deletion.



(f) Handle deletion. A new face f_2 is created.

Figure 6: Iteration of the root deletion process. The type of the deleted edge and the impact on the number of faces is mentioned below each figure.

- relabelling all the remaining edges by its label in M minus 1.
- if the edge containing the root in M is a leaf, the black vertex attached to it is deleted as well. Relabel all the black vertices by its label in M minus 1.

The procedure described above is called the **root deletion process**. Following Lacroix, define recursively the function ϑ on labelled star maps as:

- If the deleted edge of M is a leaf then $\vartheta(M) = \vartheta(M')$
- Otherwise, it is not a leaf. We have $|\ell(\lambda') - \ell(\lambda)| \leqslant 1$ and:
  - if $\ell(\lambda') = \ell(\lambda)$ the deleted edge is a **cross-border** and $\vartheta(M) = 1 + \vartheta(M')$,
  - if $\ell(\lambda') = \ell(\lambda) - 1$ the deleted edge is a **border** and $\vartheta(M) = \vartheta(M')$,
  - if $\ell(\lambda') = \ell(\lambda) + 1$ the deleted edge is a **handle**. In this case, there is a second bipartite map $\tau(M)$ obtained from M by twisting the ribbon associated with the edge containing the root edge-side. The root of $\tau(M)$ is also a handle and deleting it from $\tau(M)$ also produces M'. Define

    $$\{\vartheta(M), \vartheta(\tau(M))\} = \{\vartheta(M'), 1 + \vartheta(M')\}.$$

    At most one of M and $\tau(M)$ is orientable, and any canonical choice such that if M is orientable, then $\vartheta(M) = 0$ and $\vartheta(\tau(M)) = 1$ is acceptable.

If M is the empty map of face distribution $\lambda = \emptyset$, define $\vartheta(M) = 0$.

Remark 12. Some details about the topological meaning of the three types of edges (cross-border, border or handle) can be found in [15, Remark 4.3]. Note that in the case of star bipartite maps, the connectivity of the map is not altered after deletion of an edge and such maps do not contain bridges (except the degenerate case of leaves).

Example 5. Figure 6 shows the first iterations of the root deletion process applied to the labelled star hypermap 6(a).

Remark 13. Note that the value of function ϑ depends on the labels of the edges. As an example the respective values of ϑ on the two labelled star bipartite maps depicted on Figure 7 are 3 and 4.

6 Combinatorial interpretation of Operators Ω_2 and Δ

We proceed with a partial proof of Theorems 2 and 3 by working out two cases. Our aim is to provide the reader with the relevant insights before stating a more general solution in the next section. In both cases, we proceed as follows:

(i) we provide a more explicit forms of the operators involved,

(ii) use this explicit form to derive recurrence formulas for the coefficients $\widetilde{a}^{\lambda}_{\mu,\nu}$ and $\widetilde{h}^{\lambda}_{\mu,\nu}$,

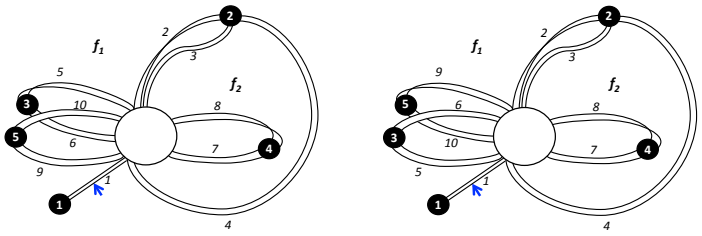


Figure 7: Two labelled star bipartite maps with the same underlying unlabelled map but different values of ϑ.

(iii) use the combinatorial decompositions of the previous section to show that the right-hand sides of the main equations of theorems 2 and 3 fulfil the same recurrence relation,

(iv) deduce Theorems 2 and 3.

6.1 Recurrence relations for the coefficients $\widetilde{a}^{\boldsymbol{\lambda}}_{\boldsymbol{n,\nu}}$ and $\widetilde{h}^{\boldsymbol{\lambda}}_{\boldsymbol{n,\mu}}$

Define when applicable for $i, j \geqslant 1$ the operations on partitions:

$$\begin{aligned}
\lambda_{\downarrow(i)} &= \lambda \setminus \{i\} \cup \{i-1\}, & \lambda_{\Downarrow(i)} &= \lambda \setminus \{i\} \cup \{i-2\},\\
\lambda_{\downarrow(i,j)} &= \lambda \setminus \{i,j\} \cup \{i+j-1\}, & \lambda_{\Downarrow(i,j)} &= \lambda \setminus \{i,j\} \cup \{i+j-2\},\\
\lambda^{\uparrow(i,j)} &= \lambda \setminus \{i+j+1\} \cup \{i,j\}, & \lambda^{\Uparrow(i,j)} &= \lambda \setminus \{i+j+2\} \cup \{i,j\}.
\end{aligned}$$

We prove the following recurrence relations.

Lemma 3. For λ integer partition of $n \geqslant 2$, the number $\widetilde{h}^{\lambda}_{n,n}(\alpha)$ satisfies

$$\begin{aligned}
\widetilde{h}^{\lambda}_{n,n}(\alpha) = \sum_{i\geqslant 1} \Big[& (\alpha-1)(i-1)^2 m_{i-1}(\lambda_{\downarrow(i)}) \widetilde{h}^{\lambda_{\downarrow(i)}}_{n-1,n-1}(\alpha)\\
& + \alpha \sum_{i,d\geqslant 1} (i-1-d) d m_{i-1-d,d}(\lambda^{\uparrow(i-1-d,d)}) \widetilde{h}^{\lambda^{\uparrow(i-1-d,d)}}_{n-1,n-1}(\alpha)\\
& + \sum_{i,j\geqslant 1} (i+j-1) m_{i+j-1}(\lambda_{\downarrow(i,j)})\ \widetilde{h}^{\lambda_{\downarrow(i,j)}}_{n-1,n-1}(\alpha) \Big].
\end{aligned}$$

Proof. Using Theorem 1 in [14] in the case $\nu = (n)$ one gets the following formula for the coefficients $a^{\lambda}_{n,n}(\alpha)$.

$$\begin{aligned}
n a^{\lambda}_{n,n}(\alpha) = \sum_i \lambda_i \Big[& (\alpha-1)(\lambda_i-1) a^{\lambda_{\downarrow(\lambda_i)}}_{n-1,n-1}(\alpha)\\
& + \sum_{d=1}^{\lambda_i-2} a^{\lambda^{\uparrow(\lambda_i-1-d,d)}}_{n-1,n-1}(\alpha) + \alpha \sum_{j\neq i} \lambda_j\ a^{\lambda_{\downarrow(\lambda_i,\lambda_j)}}_{n-1,n-1}(\alpha) \Big]. \qquad (13)
\end{aligned}$$

Using Equation (1) one can rewrite Equation (13) in terms of the $h^{\lambda}_{n,n}$ as

$$\begin{aligned}(n-1)h^{\lambda}_{n,n}(\alpha) = \sum_{i\geqslant 1}\Big[&(\alpha-1)(i-1)^2 m_{i-1}(\lambda_{\downarrow(i)})h^{\lambda_{\downarrow(i)}}_{n-1,n-1}(\alpha)\\ &+\alpha\sum_{i,d\geqslant 1}(i-1-d)d\,m_{i-1-d,d}(\lambda^{\uparrow(i-1-d,d)})h^{\lambda^{\uparrow(i-1-d,d)}}_{n-1,n-1}(\alpha)\\ &+\sum_{i,j\geqslant 1}(i+j-1)m_{i+j-1}(\lambda_{\downarrow(i,j)})\; h^{\lambda_{\downarrow(i,j)}}_{n-1,n-1}(\alpha)\Big].\end{aligned}$$

Multiplying both sides by $(n-2)!$ yields the desired result. □

Lemma 4. For any integer $n \geqslant 0, l \geqslant 1$ and any partitions $\rho \vdash n$, $\lambda \vdash n+2l$ such that $1,2 \notin \rho$, the following formula is true:

$$\begin{aligned}\widetilde{a}^{\lambda}_{n+2l,\rho\cup[2^l]}(\alpha) &= (\alpha-1)\sum_{i:\ \lambda_i>2}\frac{\lambda_i(\lambda_i-1)}{2}\widetilde{a}^{\lambda_{\Downarrow\lambda_i}}_{n+2l-2,\rho\cup[2^{l-1}]}(\alpha)\\ &+\alpha\sum_{i<j,\lambda_i+\lambda_j>2}\lambda_i\lambda_j\widetilde{a}^{\lambda_{\Downarrow(\lambda_i,\lambda_j)}}_{n+2l-2,\rho\cup[2^{l-1}]}(\alpha)+\frac{1}{2}\sum_i\lambda_i\sum_{d=1}^{\lambda_i-3}\widetilde{a}^{\lambda^{\Uparrow(\lambda_i-2-d,d)}}_{n+2l-2,\rho\cup[2^{l-1}]}(\alpha).\end{aligned}\tag{14}$$

Proof. Operator Ω_2 admits the following explicit expression.

$$\begin{aligned}\Omega_2 =&(\alpha-1)\sum_i(i-1)(i-2)p_i\frac{\partial}{\partial p_{i-2}}\\ &+\sum_{i,j}(i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}}+\alpha\sum_{i,j}ijp_{i+j+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}.\end{aligned}\tag{15}$$

The proof of Equation (15) uses elementary computations on operators detailed in Appendix 9. Using Equations (15) and (12) we find

$$\frac{2la^{\lambda}_{n+2l,\rho\cup[2^l]}(\alpha)}{z_\lambda\alpha^{\ell(\lambda)}} = [p_\lambda(x)]\Omega_2\left(\sum_{\varepsilon\vdash n+2l-2}\frac{a^{\varepsilon}_{n+2l-2,\rho\cup[2^{l-1}]}(\alpha)}{z_\varepsilon\alpha^{\ell(\varepsilon)}}p_\varepsilon(x)\right).\tag{16}$$

For each of the summands S of Ω_2 in Equation (15), there exists a partition $\varepsilon \vdash n+2l-2$ such that $p_\lambda(x)$ contribute to $S(p_\varepsilon(x))$. All the possible cases are presented in the following table where $m_k = m_k(\lambda)$.

Summand of Ω_2	ε	$\frac{z_\lambda}{z_\varepsilon}$	$\frac{\alpha^{\ell(\lambda)}}{\alpha^{\ell(\varepsilon)}}$
$(\alpha-1)(i-1)(i-2)p_i\frac{\partial}{\partial p_{i-2}}$	$\lambda_{\Downarrow(i)}$, $i>2$	$\frac{im_i}{(i-2)(m_{i-2}+1)}$	1
$(i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}}$	$\lambda_{\Downarrow(i,j)}$	$\frac{ijm_{i,j}}{(i+j-2)(m_{i+j-2}+1)}$	α
$\alpha ijp_{i+j+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$	$\lambda^{\Uparrow(i,j)}$	$\frac{(i+j+2)m_{i+j+2}}{ij(m_i+1)(m_j+1+\delta_{i,j})}$	α^{-1}

Putting everything together and rewriting the integer partitions in terms of $(\lambda_1, \ldots, \lambda_p)$ instead of $[1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots]$ and multiplying both sides by $(l-1)!$ yields the desired formula. □

6.2 Bijective construction for labelled matching: square removal

We proceed with a partial proof of Theorem 2 by looking at parts of size 2 in the partition ν of coefficients $\widetilde{a}^{\lambda}_{n,\nu}$. To this end we define a weight function wt for labelled matchings such that the polynomial $\mathfrak{S}^{\lambda}_{\nu}(\beta)$ defined as

$$\mathfrak{S}^{\lambda}_{\nu}(\beta) = \sum_{\delta \in \widetilde{\mathcal{G}}^{\lambda}_{\nu}} \beta^{\mathrm{wt}(\delta)}$$

satisfies the same recurrence relation as the coefficient $\widetilde{a}^{\lambda}_{n,\nu}(1+\beta)$. We base our construction on the multi-edge deletion procedure for labelled matchings of Definition 1. More precisely, recall the bijection φ_E defined in Lemma 2. Given integer partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_p)$ and an integer $l \geqslant 1$, let δ be a labelled matching in $\widetilde{\mathcal{G}}^{\lambda}_{[2^l]}$. Denote $\square_l$ the cycle of length 4 (square) in $\mathbf{b}_\lambda \cup \bar{\delta}$ with label l and E the two edges of $\square_l \cap \bar{\delta}$. Further denote $L_1, L_2, \cdots L_p$ the cycles of respective length $2\lambda_1, 2\lambda_2, \cdots, 2\lambda_p$ in $\mathbf{g}_{|\lambda|} \cup \mathbf{b}_\lambda$. Clearly $\varphi_E(\delta) \in \mathcal{G}^{\lambda'}_{[2^{l-1}]}$ for some $\lambda' \vdash 2l-2$. The two edges of E are either both bipartite or both non-bipartite. We call $\square_l$ bipartite in the former case and non-bipartite in the latter. We define

$$\mathrm{wt}(\delta) := \begin{cases} \mathrm{wt}(\varphi_E(\delta)) & \text{if } \square_l \text{ is bipartite,} \\ \mathrm{wt}(\varphi_E(\delta)) + 1 & \text{otherwise.} \end{cases} \tag{17}$$

We look at the following three cases (see Figure 8 for an illustration).

- $\square_l$ is not bipartite and the edges of $\square_l \backslash E$ lie inside L_i. In this case, one can show that $\lambda_i > 2$, $\lambda' = \lambda_{\Downarrow \lambda_i}$ and for each such E, φ_E is a bijection between the set of labelled matchings in $\widetilde{\mathcal{G}}^{\lambda}_{[2^l]}$ with $E = \square_l \cap \mathbf{b}_\lambda$ and $\widetilde{\mathcal{G}}^{\lambda_{\Downarrow \lambda_i}}_{[2^{l-1}]}$. In this case $\mathrm{wt}(\delta) = 1 + \mathrm{wt}(\varphi_E(\delta))$.

- $\square_l$ is bipartite and the edges of $\square_l \setminus E$ lie inside L_i. In this case there are $u, v \in [n]$ such that $u < v$ and $E = \{(u, \delta(u)), (v, \delta(v))\}$. We have $\lambda_i > 3$, $\lambda' = \lambda^{\Uparrow(\lambda_i - d_E - 2, d_E)}$ with $d_E = \mathbf{g}_{|\lambda|} \circ \delta(u) - u$ and φ_E is a bijection between such labelled matchings in $\widetilde{\mathcal{G}}^{\lambda}_{[2^l]}$ and $\widetilde{\mathcal{G}}^{\lambda^{\Uparrow(\lambda_i - d_E - 2, d_E)}}_{[2^{l-1}]}$. In this case $\mathrm{wt}(\delta) = \mathrm{wt}(\varphi_E(\delta))$.

- The edges of $\square_l \setminus E$ lie in two different cycles, namely L_i and L_j. In this case, $\lambda_i + \lambda_j > 2$, $\lambda' = \lambda_{\Downarrow(\lambda_i, \lambda_j)}$ and φ_E is a bijection between such labelled matchings in $\widetilde{\mathcal{G}}^{\lambda}_{[2^l]}$ and $\widetilde{\mathcal{G}}^{\lambda_{\Downarrow(\lambda_i, \lambda_j)}}_{[2^{l-1}]}$. In this case $\mathrm{wt}(\delta) = \mathrm{wt}(\varphi_E(\delta))$ if the edges of E are bipartite and $\mathrm{wt}(\delta) = 1 + \mathrm{wt}(\varphi_E(\delta))$ otherwise.

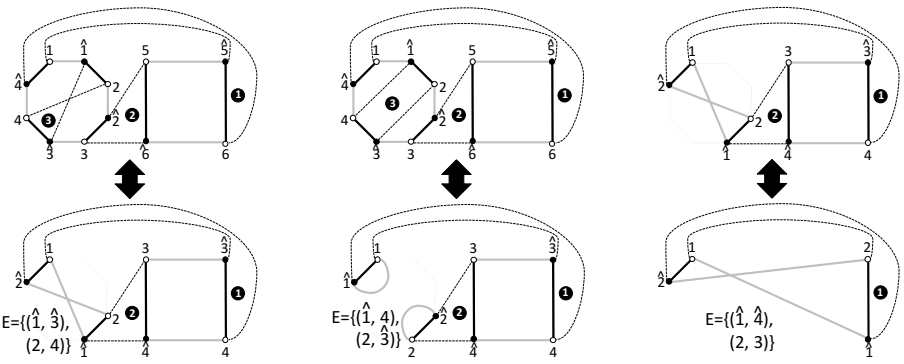
Using these three bijections, one gets:

$$\mathfrak{S}^{\lambda}_{[2^l]}(\beta) = \sum_{\substack{i\geqslant 1\\ \square_l\setminus E\subset L_i\\ E \text{ non-bipartite}}} \sum_{\varphi_E(\delta)\in\widetilde{\mathcal{G}}^{\lambda\Downarrow(\lambda_i)}_{[2^{l-1}]}} \beta^{1+\mathrm{wt}(\varphi_E(\delta))}$$
$$+ \sum_{\substack{i\geqslant 1\\ \square_l\setminus E\subset L_i\\ E \text{ bipartite}}} \sum_{\substack{i\geqslant 1\\ \varphi_E(\delta)\in\widetilde{\mathcal{G}}^{\Uparrow(\lambda_i-d_E-2,d_E)}_{[2^{l-1}]}}} \beta^{\mathrm{wt}(\varphi_E(\delta))}$$
$$+ \sum_{\substack{i<j\\ \square_l\setminus E=(e,f)\\ e\in L_i, f\in L_j}} \sum_{\varphi_E(\delta)\in\widetilde{\mathcal{G}}^{\lambda\Downarrow(\lambda_i,\lambda_j)}_{[2^{l-1}]}} \beta^{\begin{cases}\mathrm{wt}(\varphi_E(\delta)) & e \text{ is bipartite}\\ 1+wt(\varphi_E(\delta)) & otherwise\end{cases}}$$

$$= \sum_{i\geqslant 1}\binom{\lambda_i}{2} \sum_{\delta'\in\widetilde{\mathcal{G}}^{\lambda\Downarrow(\lambda_i)}_{[2^{l-1}]}} \beta^{1+\mathrm{wt}(\delta')} + \frac{1}{2}\sum_{i,d\geqslant 1}\lambda_i \sum_{\delta'\in\widetilde{\mathcal{G}}^{\Uparrow(\lambda_i-d-2,d)}_{[2^{l-1}]}} \beta^{\mathrm{wt}(\delta')}$$
$$+ \sum_{\substack{i<j\\ \lambda_i+\lambda_j>2}} \lambda_i\lambda_j \sum_{\delta'\in\widetilde{\mathcal{G}}^{\lambda\Downarrow(\lambda_i,\lambda_j)}_{[2^{l-1}]}} \left(\beta^{\mathrm{wt}(\delta')} + \beta^{1+\mathrm{wt}(\delta')}\right)$$

As a result, combining the fact that the initial condition $l = 1$ is already proven in [14] and the recurrence for coefficients $\widetilde{a}^{\lambda}_{2l,[2^l]}$ from Equation (14),

$$\mathfrak{S}^{\lambda}_{[2^l]}(\beta) = \widetilde{a}^{\lambda}_{2l,[2^l]}(1+\beta).$$



(a) First bijection. (b) Second bijection. (c) Third bijection.

Figure 8: Examples of application of the three bijections for labelled matchings.

6.3 Bijective constructions for labelled star bipartite maps: edge removal in two-vertex maps

We proceed with a partial proof of Theorem 3 and use the edge removal procedure and the measure of non-orientabiliy ϑ described in Definition 2 to prove that the polynomials $\Sigma^{\lambda}_{[k^m]}(\beta)$ defined as

$$\Sigma^{\lambda}_{[k^m]}(\beta) = \sum_{M \in \widetilde{\mathcal{L}}^{\lambda}_{[k^m]}} \beta^{\vartheta(M)}$$

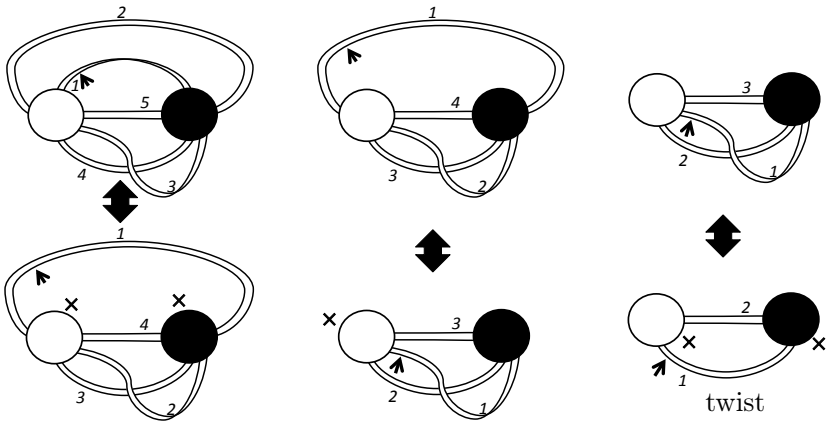
satisfy the same recurrence relation as the coefficients $\widetilde{h}^{\lambda}_{n,[k^m]}(1+\beta)$. Although it shares some similarities with the proof of Theorem 2, the differences in the combinatorial objects involved (bipartite maps instead of matchings) make the proof of Theorem 3 independent. We look at the recurrence relation satisfied by $\Sigma^{\lambda}_{n}(\beta)$. In the case $n = 1$, $\Sigma^{\lambda}_{1}(\beta) = 1 = \widetilde{h}^{1}_{1,1}(\beta+1)$. If $n > 1$, star maps in $\widetilde{\mathcal{L}}^{\lambda}_{n}$ do not contain any leaf and we may split these maps into three sets according to the type (cross border, border or handle) of their edge containing the root edge-side that we delete to get the following bijective constructions.

- (Cross border) The set of labelled star map with one black vertex, face distribution λ and a cross border root incident to a face of degree i is in bijection with the set of labelled star map with (i) one black vertex, (ii) face distribution $\lambda_{\downarrow(i)}$, (iii) one marked corner around the white vertex incident to a face of degree $i-1$, (iv) one marked corner around the black vertex incident to the same face.

- (Border) The set of labelled star maps with one black vertex, face distribution λ and a border root incident to both a face of degree i and a face of degree j is in bijection with the set of labelled star maps with (i) one black vertex, (ii) face distribution $\lambda_{\downarrow(i,j)}$, (iii) one marked corner around the white vertex incident to a face of degree $i+j-1$ (once the position around the white vertex is chosen there is only one position around the black vertex such that connecting these two positions with a border cuts the face of degree $i+j-1$ into two faces of degree i and j).

- (Handle) The set of labelled star maps with one black vertex, face distribution λ and a handle root incident to a face of degree i such that removing the root yields a face of degree d and one of degree $i-1-d$ is in bijection with the set of labelled star maps with (i) one black vertex, (ii) face distribution $\lambda^{\uparrow(i-1-d,d)}$, (iii) one marked corner around the white vertex incident to a face of degree $i-1-d$, (iv) one marked corner around the black vertex incident to a face of degree d and (v) a type for the removed root: twist or untwist (as noted above, twisting the ribbon of a handle root yields another map).

Example 6. Figure 9 illustrates the three bijections described above.



(a) First bijection. (b) Second bijection. (c) Third bijection.

Figure 9: Examples of application of the three bijections for labelled star bipartite maps.

As a consequence, one gets

$$\begin{aligned}\Sigma_n^\lambda(\beta) = &\sum_{i\geqslant 1}(i-1)^2 m_{i-1}(\lambda_{\downarrow(i)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda_{\downarrow(i)}}} \beta^{1+\vartheta(M')} \\ &+ \sum_{i,j\geqslant 1}(i+j-1) m_{i+j-1}(\lambda_{\downarrow(i,j)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda_{\downarrow(i,j)}}} \beta^{\vartheta(M')} \\ &+ \sum_{i,d\geqslant 1}(i-d-1) d m_{i-d-1,d}(\lambda^{\uparrow(i-1-d,d)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda^{\uparrow(i-1-d,d)}}} \left(\beta^{\vartheta(M')} + \beta^{1+\vartheta(M')}\right)\end{aligned}$$

As a conclusion, for any integer $n \geqslant 1$, $\Sigma_n^\lambda(\beta) = \widetilde{h}_{n,n}^\lambda(\beta+1)$.

7 Full proof of Theorem 2

We proceed with a complete proof of Theorem 2. To this end we extend the weight function wt of Section 6.2 to labelled matchings such that for any partitions λ and ν with at most one part of ν strictly greater than 3. As in Section 6.2, we show that the polynomial $\mathfrak{S}_\nu^\lambda(\beta)$ defined as

$$\mathfrak{S}_\nu^\lambda(\beta) = \sum_{\delta\in\widetilde{\mathcal{G}}_\nu^\lambda} \beta^{\mathrm{wt}(\delta)}$$

satisfies the same recurrence relation as the coefficient $\widetilde{a}_{n,\nu}^\lambda(1+\beta)$. In what follows, for two partitions ρ_1 and ρ_2 we denote $\rho_1 \cup \rho_2$ the partition obtained by reordering in decreasing order the union of the parts of ρ_1 and the parts of ρ_2.

7.1 More recurrence relations for the coefficients $\widetilde{a}^{\lambda}_{n,\nu}(\alpha)$

In order to show that $\mathfrak{S}^{\lambda}_{\nu}(\beta)$ and $\widetilde{a}^{\lambda}_{n,\nu}(1+\beta)$ follows the same recurrence relation, we first explicit the recurrence relation for the $\widetilde{a}^{\lambda}_{n,\nu}(1+\beta)$. We show the following lemmas.

Lemma 5. Given two integers n and k, a partition $\lambda = (\lambda_1, \dots, \lambda_p) \vdash n+k$ and a tuple of integers $\kappa = (k_1, \dots, k_p)$ such that $k_1 + \ldots + k_p = k$, $k_1, \dots, k_p \geqslant 0$ and $k_i < \lambda_i$ for all i, we denote $\lambda - \kappa$ the reordering in decreasing order of the non-negative integers $(\lambda_1 - k_1, \dots, \lambda_p - k_p)$. Clearly $\lambda - \kappa \vdash n$. Given an integer partition $\rho \vdash n$ such that $m_1(\rho) = 0$, we write the coefficient $a^{\lambda}_{n+k,\rho\cup[1^k]}(\alpha)$ as a sum of the $a^{\lambda-\kappa}_{n,\rho}(\alpha)$'s.

$$\widetilde{a}^{\lambda}_{n+k,\rho\cup[1^k]}(\alpha) = \sum_{\substack{\kappa=(k_1,\dots,k_p)\\ 0\leqslant k_i<\lambda_i,\\ \sum_i k_i = k}} \binom{\lambda_1}{k_1} \dots \binom{\lambda_p}{k_p} \widetilde{a}^{\lambda-\kappa}_{n,\rho}(\alpha). \tag{18}$$

If $p > n$ then this sum is empty and $\widetilde{a}^{\lambda}_{n+k,\rho\cup[1^k]}(\alpha) = 0$.

Lemma 6. For any integers i, j, k and an integer partition λ, when applicable define

$$\begin{aligned}
\lambda_{\Vdash(i)} &= \lambda \setminus \{i\} \cup \{i-3\},\\
\lambda_{\Vdash(i,j)} &= \lambda \setminus \{i,j\} \cup \{i+j-3\},\\
\lambda^{\Vdots(i,j)} &= \lambda \setminus \{i+j+3\} \cup \{i,j\},\\
\lambda^{\Vdots(i,j,k)} &= \lambda \setminus \{i+j+k+3\} \cup \{i,j,k\},\\
\lambda_{\Vdash(i,j,k)} &= \lambda \setminus \{i,j,k\} \cup \{i+j+k-3\}.
\end{aligned}$$

Let $n \geqslant 0, m \geqslant 1$ be integer such that $n+3m > 3$ and $\rho \vdash n$, $\lambda \vdash n+3m$ be partitions such that $1, 2, 3 \notin \rho$. The following recurrence formula holds (reference to α in the coefficients a is removed for the sake of clarity).

$$\begin{aligned}
\widetilde{a}^{\lambda}_{n+3m,\rho\cup[3^m]} &= (2\alpha^2 - 3\alpha + 2) \sum_{\lambda_i > 3} \binom{\lambda_i}{3} \widetilde{a}^{\lambda_{\Vdash\lambda_i}}_{n+3m-3,\rho\cup[3^{m-1}]}\\
&\quad + 2\alpha^2 \sum_{\substack{i<j<k\\ \lambda_i+\lambda_j+\lambda_k>3}} \lambda_i\lambda_j\lambda_k \widetilde{a}^{\lambda_{\Vdash(\lambda_i,\lambda_j,\lambda_k)}}_{n+3m-3,\rho\cup[3^{m-1}]}\\
&\quad + \frac{1}{3}\sum_i \lambda_i \sum_{d,f\geqslant 1} \widetilde{a}^{\lambda^{\Vdots(\lambda_i-3-d-f,d,f)}}_{n+3m-3,\rho\cup[3^{m-1}]}\\
&\quad + (\alpha-1)\alpha \sum_{\substack{i<j\\ \lambda_i+\lambda_j>3}} \lambda_i\lambda_j(\lambda_i+\lambda_j-2) \widetilde{a}^{\lambda_{\Vdash(\lambda_i,\lambda_j)}}_{n+3m-3,\rho\cup[3^{m-1}]}\\
&\quad + \alpha \sum_{i<j} \lambda_i\lambda_j \sum_{d=1}^{\lambda_i+\lambda_j-4} \widetilde{a}^{\lambda^{\shortmid(\lambda_i,\lambda_j)\Vdots(\lambda_i+\lambda_j-3-d,d)}}_{n+3m-3,\rho\cup[3^{m-1}]}\\
&\quad + (\alpha-1)\sum_i \binom{\lambda_i}{2} \sum_{d=1}^{\lambda_i-4} \widetilde{a}^{\lambda^{\Vdots(d,\lambda_i-3-d)}}_{n+3m-3,\rho\cup[3^{m-1}]}.
\end{aligned} \tag{19}$$

We provide a full proof of Lemma 5.

Proof of Lemma 5. First notice that

$$\Omega_1 = \sum_{i \geqslant 1} i p_{i+1} \frac{\partial}{\partial p_i}. \tag{20}$$

Then, rewrite Equation (12) with $\nu_p = 1$:

$$k \sum_{\lambda \vdash n+k} z_\lambda^{-1} \alpha^{-\ell(\lambda)} a^\lambda_{n+k,\rho\cup[1^k]}(\alpha) p_\lambda(x) = \sum_{i \geqslant 1} i p_{i+1} \frac{\partial}{\partial p_i} \sum_{\tau \vdash n+k-1} z_\tau^{-1} \alpha^{-\ell(\tau)} a^\tau_{n+k-1,\rho\cup[1^{k-1}]}(\alpha) p_\tau(x).$$

Recall $\lambda_{\downarrow(i)} = \lambda \setminus \{i\} \cup \{i-1\}$. Extracting the coefficient in $p_\lambda(x)$ yields

$$k a^\lambda_{n+k,\rho\cup[1^k]}(\alpha) = \sum_{i\colon \lambda_i > 1} \lambda_i a^{\lambda_{\downarrow(\lambda_i)}}_{n+k-1,\rho\cup[1^{k-1}]}(\alpha). \tag{21}$$

In the case $p \leqslant n$, there exists $\binom{k}{k_1,\ldots,k_p} = \frac{k!}{k_1!\ldots k_p!}$ ways to turn λ into $\lambda - \kappa$ in k steps for fixed κ (each step is a decrease of some λ_i by one). Therefore the reduction of the quantity $k! a^\lambda_{n+k,\rho\cup[1^k]}(\alpha)$ step by step yields:

$$k! a^\lambda_{n+k,\rho\cup[1^k]}(\alpha) = \sum_{\substack{\kappa=(k_1,\ldots,k_p)\\ 0\leqslant k_i<\lambda_i,\\ \sum_i k_i = k}} (\lambda_1)_{k_1} \ldots (\lambda_p)_{k_p} \frac{k!}{k_1!\ldots k_p!} a^{\lambda-\kappa}_{n,\rho}(\alpha)$$

where for positive integers N and s, $(N)_s := N(N-1)\ldots(N-s+1)$ if $s \geqslant 1$ and $(N)_0 = 1$. Dividing both sides by $k!$ we get Equation (18). If $p > n$ then the reduction process terminates after $n+k-p < k$ steps and we get that $a^\lambda_{n+k,\rho\cup[1^k]}(\alpha)$ is proportional to $a^{[1^p]}_{p,\rho\cup[1^{p-n}]}(\alpha) = 0$ as per [9, lemma 3.3]. □

Proof of Lemma 6 (sketch). The proof of Lemma 6 is similar to the one of Lemma 4. In a first step, compute (left to the reader):

$$\begin{aligned}\Omega_3 =& (2\alpha^2 - 3\alpha + 2) \sum_i (i-1)(i-2)(i-3) p_i \frac{\partial}{\partial p_{i-3}} \\ &+ 3(\alpha - 1) \sum_{i,j} (i+j-2)(i+j-3) p_i p_j \frac{\partial}{\partial p_{i+j-3}}\end{aligned}$$

$$+ 3\alpha(\alpha-1)\sum_{i,j} ij(i+j+2)p_{i+j+3}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$+ 3\alpha\sum_{i,j,k} ijp_{i+j-k+3}p_k\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$+ 2\alpha^2\sum_{i,j,k} ijkp_{i+j+k+3}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_k}$$
$$+ 2\sum_{i,j,k}(i+j+k-3)p_ip_jp_k\frac{\partial}{\partial p_{i+j+k-3}}. \tag{22}$$

Finally use the connection between the $\widetilde{a}^{\lambda}_{n,\nu}(1+\beta)$ and Operator Ω_k. □

7.2 Recursive combinatorial construction of the weight function for labelled matchings

Assume the partition ν is of the form $\nu = [1^k 2^l 3^m n^1]$ where $n > 3$. When $k = l = m = 0$, the definition of the weight function wt and the proof of equality

$$\widetilde{a}^{\lambda}_{n,n}(1+\beta) = \mathfrak{S}^{\lambda}_n(\beta)$$

is provided in [14]. We proceed by defining a function wt in the cases $i = 1, 2, 3$ such that $\mathrm{wt}(\delta) = 0$ if and only if δ is a bipartite labelled matching and by proving that if the equality

$$\widetilde{a}^{\lambda}_{|\rho|,\rho}(1+\beta) = \mathfrak{S}^{\lambda}_{\rho}(\beta) \tag{23}$$

is true for any partition λ and for some partition ρ such that $|\lambda| = |\rho|$ and all the parts of ρ are strictly greater than i then the same equality is true if we replace ρ by $\rho \cup [i^j]$ where $j \geqslant 0$. It is more convenient to give the proof first for $i = 1$ and then increase the value of i.

7.2.1 Case $i = 1$

Assume Equation (23) is true for some partition ρ such that $1 \notin \rho$. We show that this equality remains true if we replace ρ by $\rho \cup [1^k]$.
Indeed, let δ be a labelled matching of $\widetilde{\mathcal{G}}^{\lambda}_{\rho\cup[1^k]}$. The edges $E = \bigcup_{t=1}^{k}(u_t, \mathbf{b}_\lambda(u_t))$ of $\bar{\delta}$ belonging to cycles of length 2 in $\mathbf{b}_\lambda \cup \bar{\delta}$ are always bipartite (they link vertices u and $\mathbf{b}_\lambda(u)$ for some index u). If we further suppose that the two vertices of such an edge belong to a cycle of length $2\lambda_i$ in $\mathbf{b}_\lambda \cup \mathbf{g}_n$, one obtains a labelled matching $\delta' \in \widetilde{\mathcal{G}}^{\lambda_{\setminus(\lambda_i)}}_{\rho\cup[1^{k-1}]}$ by removing these two vertices and the edge. Deleting all the edges in E one gets a labelled matching $\varphi_E(\delta) \in \widetilde{\mathcal{G}}^{\lambda'}_{\rho}$ for some partition $\lambda' \vdash |\rho|$ such that $\lambda' = \lambda - \kappa$ for some tuple $\kappa = (k_1, \ldots, k_p)$ with $\sum_i k_i = k$ and $0 \leqslant k_i < \lambda_i$ for all $i \in \{1, \ldots, p\}$. Define

$$\mathrm{wt}(\delta) = \mathrm{wt}(\varphi_E(\delta)).$$

For every $\kappa = (k_1, \cdots, k_p)$ such that $0 \leqslant k_i < \lambda_i$ and $\sum_i k_i = k$ there exist $\binom{\lambda_1}{k_1} \dots \binom{\lambda_p}{k_p}$ valid sets E of $k_1, \cdots, k_p$ edges belonging to the cycles of length $\lambda_1, \cdots, \lambda_p$ in $\mathbf{b}_\lambda \cup \mathbf{g}_{|\rho|+k}$. As a result,

$$\begin{aligned}
\mathfrak{S}^\lambda_{\rho\cup[1^k]}(\beta) = \sum_{\delta\in\widetilde{\mathcal{G}}^\lambda_{\rho\cup[1^k]}} \beta^{\mathrm{wt}(\delta)} &= \sum_{\substack{\kappa=(k_1,\dots,k_p)\\ 0\leqslant k_i<\lambda_i,\\ \sum_i k_i=k}} \sum_{\substack{\delta\in\widetilde{\mathcal{G}}^\lambda_{\rho\cup[1^k]}\\ \Lambda(\varphi_E(\delta),\mathbf{b}_\lambda\setminus E)=\lambda-\kappa}} \beta^{\mathrm{wt}(\varphi_E(\delta))} \\
&= \sum_{\substack{\kappa=(k_1,\dots,k_p)\\ 0\leqslant k_i<\lambda_i,\\ \sum_i k_i=k}} \binom{\lambda_1}{k_1}\cdots\binom{\lambda_p}{k_p} \sum_{\delta'\in\widetilde{\mathcal{G}}^{\lambda-\kappa}_\rho} \beta^{\mathrm{wt}(\delta')} \\
&= \sum_{\substack{\kappa=(k_1,\dots,k_p)\\ 0\leqslant k_i<\lambda_i,\\ \sum_i k_i=k}} \binom{\lambda_1}{k_1}\cdots\binom{\lambda_p}{k_p} \widetilde{a}^{\lambda-\kappa}_{|\rho|,\rho}(1+\beta) \\
&= \widetilde{a}^\lambda_{|\rho|+k,\rho\cup[1^k]}(1+\beta),
\end{aligned}$$

where the last equality is given by Equation (18) and the previous one is the recurrence hypothesis.

7.2.2 Case $i = 2$

The case $i = 2$ is already covered in Section 6.2.

7.2.3 Case $i = 3$

Similarly, as in the case of 2-parts we define the weight function by induction on m. Assume $m > 0$ and fix the notation:

- $\lambda = (\lambda_1, \dots, \lambda_p) \vdash n + 3m$, $1, 2, 3 \notin \rho \vdash n$, $\nu' = \rho \cup [3^{m-1}] \vdash N' = n + 3m - 3$, $\mathbf{b} = \mathbf{b}_\lambda$, $\mathbf{g} = \mathbf{g}_{n+3m}$;
- The graph $\mathbf{b} \cup \mathbf{g}$ is composed of cycles $L_1, \dots, L_p$ of length $2\lambda_1, \dots, 2\lambda_p$;
- $\delta \in \widetilde{\mathcal{G}}^\lambda_{\rho\cup[3^m]}$ is a labelled matching with the m-th hexagon $\hexagon_m \subset \delta \cup \mathbf{b}$;
- $E = \hexagon_m \cap \delta = \{\{\widehat{t}, \mathbf{b}(\widehat{t})\}, \{\widehat{u}, \mathbf{b}(\widehat{u})\}, \{\widehat{v}, \mathbf{b}(\widehat{v})\}\}$;
- φ_E is the bijection between labelled matchings from Lemma 2.

Suppose that the function wt is well-defined for $m - 1$ hexagons, i.e.

$$\widetilde{a}^{\lambda'}_{N',\nu'}(\beta + 1) = \mathfrak{S}^{\lambda'}_{\nu'}(\beta)$$

for all $\lambda' \vdash N'$.

Definition 3. We define the value $\mathrm{wt}(\delta)$ for all $\delta \in \widetilde{\mathcal{G}}^{\lambda}_{\rho\cup[3^m]}$ in each of the following possible cases.

Case 1: $E \subset L_i$ for some i. In this case we consider that $t < u < v$.

Case 2: $\{\widehat{t}, \mathbf{b}(\widehat{t})\}, \{\widehat{v}, \mathbf{b}(\widehat{v})\} \in L_i$ and $\{\widehat{u}, \mathbf{b}(\widehat{u})\} \in L_j$ for some $i \neq j$. In this case we consider that $t < v$.

Case 3: $\{\widehat{t}, \mathbf{b}(\widehat{t})\} \in L_i$, $\{\widehat{u}, \mathbf{b}(\widehat{u})\} \in L_j$, $\{\widehat{v}, \mathbf{b}(\widehat{v})\} \in L_k$ for some $i < j < k$.

Define

$$\mathrm{wt}(\delta) := \mathrm{wt}(\varphi_E(\delta)) + w(\delta)$$

where the **additional weight** $w(\delta) \in \{0, 1, 2\}$ is stated in the following table in which all the 8 possible types of the hexagon $\hexagon_m$ are considered.

Type T	Edges of $\delta \cap \hexagon_m$	$w(\delta)$
1	$\{\widehat{t}, \mathbf{b}(\widehat{u})\}, \{\widehat{u}, \mathbf{b}(\widehat{v})\}, \{\widehat{v}, \mathbf{b}(\widehat{t})\}$	0
2	$\{\widehat{t}, \widehat{u}\}, \{\mathbf{b}(\widehat{t}), \widehat{v}\}, \{\mathbf{b}(\widehat{u}), \mathbf{b}(\widehat{v})\}$	1
3	$\{\widehat{t}, \mathbf{b}(\widehat{u})\}, \{\widehat{u}, \widehat{v}\}, \{\mathbf{b}(\widehat{v}), \mathbf{b}(\widehat{t})\}$	2
4	$\{\widehat{t}, \widehat{v}\}, \{\mathbf{b}(\widehat{t}), \mathbf{b}(\widehat{u})\}, \{\widehat{u}, \mathbf{b}(\widehat{v})\}$	2
5	$\{\widehat{t}, \widehat{u}\}, \{\mathbf{b}(\widehat{t}), \mathbf{b}(\widehat{v})\}, \{\mathbf{b}(\widehat{u}), \widehat{v}\}$	1
6	$\{\widehat{t}, \widehat{v}\}, \{\mathbf{b}(\widehat{t}), \widehat{u}\}, \{\mathbf{b}(\widehat{u}), \mathbf{b}(\widehat{v})\}$	1
7	$\{\widehat{t}, \mathbf{b}(\widehat{v})\}, \{\mathbf{b}(\widehat{t}), \mathbf{b}(\widehat{u})\}, \{\widehat{u}, \widehat{v}\}$	1
8	$\{\widehat{t}, \mathbf{b}(\widehat{v})\}, \{\widehat{u}, \mathbf{b}(\widehat{t})\}, \{\widehat{v}, \mathbf{b}(\widehat{u})\}$	0

Clearly, $\varphi_E(\delta) \in \mathcal{G}^{\lambda'}_{\nu'}$ for some $\lambda' \vdash N'$. The partition λ' depends on the type of $\hexagon_m$ and is stated in the following table.

T	**Case 1**	**Case 2**	**Case 3**
1	$\lambda_{\uplus\lambda_i}$	$\lambda\backslash(\lambda_i,\lambda_j)\cup(d, \lambda_i+\lambda_j-3-d)$, $d = v-t-1$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
2	$\lambda_{\uplus\lambda_i}$	$\lambda\backslash(\lambda_i,\lambda_j)\cup(d, \lambda_i+\lambda_j-3-d)$, $d = v-t-1$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
3	$\lambda_{\uplus\lambda_i}$	$\lambda_{\uplus(\lambda_i,\lambda_j)}$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
4	$\lambda_{\uplus\lambda_i}$	$\lambda_{\uplus(\lambda_i,\lambda_j)}$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
5	$\lambda^{\uplus(d,\lambda_i-3-d)}$, $d = v-u-1$	$\lambda_{\uplus(\lambda_i,\lambda_j)}$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
6	$\lambda^{\uplus(d,\lambda_i-3-d)}$, $d = u-t-1$	$\lambda_{\uplus(\lambda_i,\lambda_j)}$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
7	$\lambda^{\uplus(d,\lambda_i-3-d)}$, $d = v-t-2$	$\lambda\backslash(\lambda_i,\lambda_j)\cup(d, \lambda_i+\lambda_j-3-d)$, $d = v-t-1$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$
8	$\lambda^{\uplus(d,f,\lambda_i-3-d-f)}$, $d = u-t-1, f = v-u-1$	$\lambda\backslash(\lambda_i,\lambda_j)\cup(d, \lambda_i+\lambda_j-3-d)$, $d = v-t-1$	$\lambda_{\uplus(\lambda_i,\lambda_j,\lambda_k)}$

Remark 14. In some cases there are some extra conditions on t, u, v. For example, in Case 1 we have $u-t, v-u, \lambda_i - v + t > 1$ for type 8, i. e. the edges of E are non-neighboring edges in L_i.

We see that $w(\delta) = 0$ iff $\circlearrowright_m$ is bipartite (i. e. all three edges of E are bipartite) and now we show that $\mathfrak{S}^{\lambda}_{\rho\cup[3^m]}(\beta)$ and the quantity $\widetilde{a}^{\lambda}_{N,\rho\cup[3^m]}(\beta+1)$ satisfy the same recurrence relation. Multiplying both sides of Equation (19) by $(m-1)!$ we rewrite it as

$$\widetilde{a}^{\lambda}_{N,\rho\cup[3^m]} = (2\beta^2+\beta+1)\sum_{\lambda_i>3}\binom{\lambda_i}{3}\widetilde{a}^{\lambda\setminus\lambda_i}_{N',\nu'} \tag{24}$$

$$+ (2\beta^2+4\beta+2)\sum_{\substack{i<j<k\\ \lambda_i+\lambda_j+\lambda_k>3}}\lambda_i\lambda_j\lambda_k\widetilde{a}^{\lambda\setminus(\lambda_i,\lambda_j,\lambda_k)}_{N',\nu'} \tag{25}$$

$$+ \frac{1}{3}\sum_i \lambda_i \sum_{d,f\geqslant 1}\widetilde{a}^{\lambda\setminus\lambda_i\cup(f,d,\lambda_i-3-d-f)}_{N',\nu'} \tag{26}$$

$$+ (\beta^2+\beta)\sum_{\substack{i<j\\ \lambda_i+\lambda_j>3}}\lambda_i\lambda_j(\lambda_i+\lambda_j-2)\widetilde{a}^{\lambda\setminus(\lambda_i,\lambda_j)}_{N',\nu'} \tag{27}$$

$$+ (\beta+1)\sum_{i<j}\lambda_i\lambda_j\sum_{d=1}^{\lambda_i+\lambda_j-4}\widetilde{a}^{\lambda\setminus(\lambda_i,\lambda_j)\cup(d,\lambda_i+\lambda_j-3-d)}_{N',\nu'} + \tag{28}$$

$$+ \beta\sum_i\binom{\lambda_i}{2}\sum_{d=1}^{\lambda_i-4}\widetilde{a}^{\lambda\setminus(d,\lambda_i-3-d)}_{N',\nu'}. \tag{29}$$

Now we split the sum $\sum_\delta \beta^{\mathrm{wt}(\delta)}$ into three parts according to the three cases from definition 3:

$$\sum_\delta \beta^{\mathrm{wt}(\delta)} = \left(\sum_i\sum_{e,f,h\in L_i} + \sum_{i\neq j}\sum_{\substack{e,f\in L_i\\ h\in L_j}} + \sum_{i<j<k}\sum_{\substack{e\in L_i\\ f\in L_j,h\in L_k}}\right)\sum_{\delta: E=\{e,f,h\}}\beta^{\mathrm{wt}(\delta)}.$$

We carry on to split these sums according to the way to join ends of E. Then we group the summands according to the type of λ'. Consider all three cases separately.

Case 1: $e,f,h\in L_i$. In this case $\lambda_i>3$ else $3\in\Lambda(\mathbf{g},\delta)=(N)$. Depending on the way to join ends of E the polygon L_i either stays a polygon or splits into two or three polygons. We have

$$\sum_{e,f,h\in L_i}\sum_{\substack{\delta:\\ E=\{e,f,h\}}}\beta^{\mathrm{wt}(\delta)} = \sum_{\substack{t,u,v\in L_i\\ t<u<v}}\left(\sum_{\delta:\ \mathrm{T1–T4}} + \sum_{\delta:\ \mathrm{T5–T7}} + \sum_{\delta:\ \mathrm{T8}}\right)\beta^{\mathrm{wt}(\delta)}.$$

- For types T1–T4 we have $\lambda' = \lambda_{\Vdots\lambda_i}$ (see Figure 10) and, according to definition 3,

$$\sum_{\substack{t,u,v\in L_i\\ t<u<v}} \sum_{\delta\colon \text{T1–T4}} \beta^{\mathrm{wt}(\delta)} = \sum_{\substack{t,u,v\in L_i\\ t<u<v}} \sum_{\delta'\in\widetilde{\mathcal{G}}_{\nu'}^{\lambda_{\Vdots\lambda_i}}} (2\beta^2+\beta+1)\beta^{\mathrm{wt}(\delta')}$$
$$= \binom{\lambda_i}{3}(2\beta^2+\beta+1)\widetilde{a}_{N',\nu'}^{\lambda_{\Vdots\lambda_i}}.$$

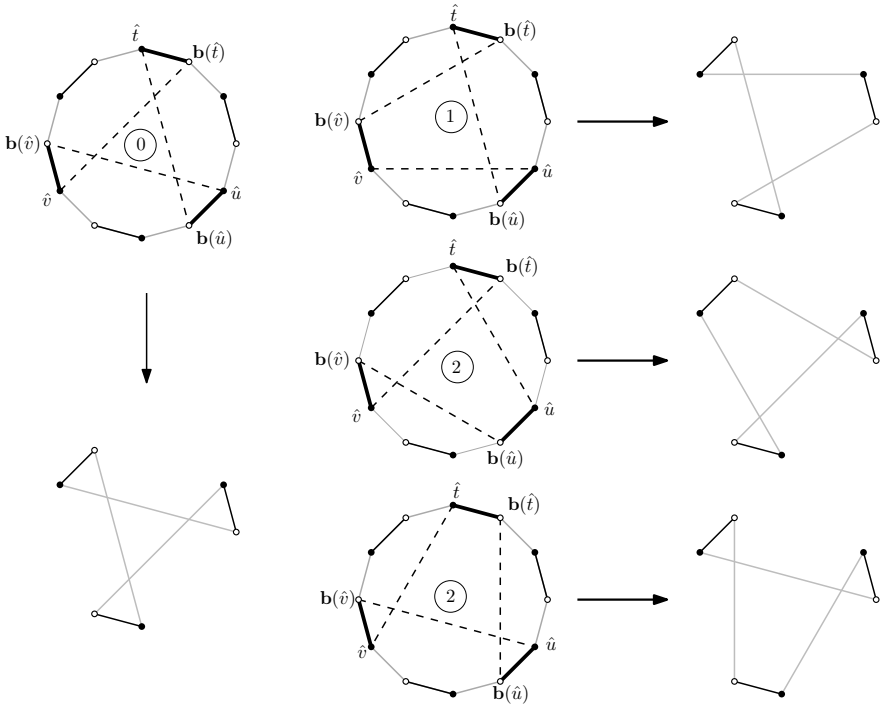


Figure 10: Hexagons of types T1–T4 in Case 1 contributing to summand (24).

Now we numerate vertices of L_i by residues modulo λ_i and their duplicates with a hat.

- For hexagons of types T5–T7, the sequence of vertices is of the kind

$$(x, \widehat{x+d}, x+d+1, y+1, \widehat{y}, \widehat{x-1})$$

for some $x \in \{1,\dots,\lambda_i\}$, $d \in \{1,\dots,\lambda_i-4\}$, $y \in \{x+d+1,\dots,x-2+\lambda_i\}$. Besides $\lambda' = \lambda^{\Vdots(d,\lambda_i-3-d)}$ in this case, see Figure 11. Therefore we have

$$\sum_{\substack{t,u,v\in L_i\\ t<u<v}} \sum_{\delta\colon \text{T5–T7}} \beta^{\mathrm{wt}(\delta)} = \sum_{x=1}^{\lambda_i} \sum_{d=1}^{\lambda_i-4} \sum_{y=x+d+1}^{x-2+\lambda_i} \sum_{\delta'\in\mathcal{G}_{\nu'}^{\lambda^{\Vdots(d,\lambda_i-3-d)}}} \beta^{\mathrm{wt}(\delta')+1}$$
$$= \beta\lambda_i \sum_{d=1}^{\lambda_i-4} (\lambda_i-d-2)\widetilde{a}_{N',\nu'}^{\lambda^{\Vdots(d,\lambda_i-3-d)}}.$$

Since this expression is invariant under replacing d by $\lambda_i - 3 - d$, we obtain

$$\frac{1}{2}\beta\lambda_i \sum_{d=1}^{\lambda_i-4} (\lambda_i - d - 2 + d + 1)\widetilde{a}^{\lambda^{\text{⫛}(d,\lambda_i-3-d)}}_{N',\nu'} = \beta\binom{\lambda_i}{2} \sum_{d=1}^{\lambda_i-4} \widetilde{a}^{\lambda^{\text{⫛}(d,\lambda_i-3-d)}}_{N',\nu'}.$$

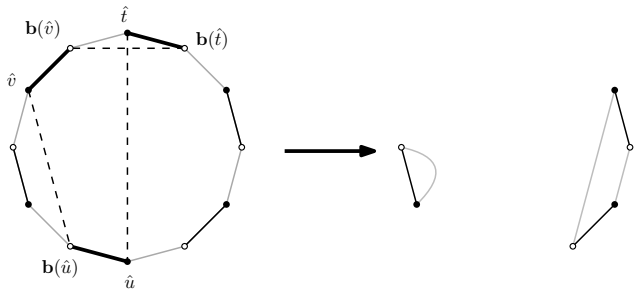


Figure 11: A hexagon of type T5 in Case 1 contributing to summand (29).

- For hexagons of type T8, the sequence of vertices is of the kind

$$(\widehat{x-1}, x, \widehat{x+d}, x+d+1, \widehat{x+d+f+1}, x+d+f+2)$$

for some triple (x, d, f) such that $d, f, \lambda_i - 3 - d - f \geqslant 1$ and $x \in \{1, \ldots, \lambda_i\}$. In this case $\lambda' = \lambda^{\text{⫛}(d,f,\lambda_i-3-d-f)}$, see figure 12. Since the triples

$$(x+d+1, f, \lambda_i - 3 - d - f) \text{ and } (x+d+f+2, \lambda_i - 3 - d - f, d)$$

give the same hexagon, we see that

$$\begin{aligned}\sum_{\substack{t,u,v\in L_i\\ t<u<v}} \sum_{\delta\colon \text{T8}} \beta^{\mathrm{wt}(\delta)} &= \frac{1}{3}\sum_{x=1}^{\lambda_i}\sum_{d,f} \sum_{\delta'\in\widetilde{\mathcal{G}}^{\lambda^{\text{⫛}(d,f,\lambda_i-3-d-f)}}_{\nu'}} \beta^{\mathrm{wt}(\delta')} \\ &= \frac{1}{3}\lambda_i \sum_{d,f} \widetilde{a}^{\lambda^{\text{⫛}(d,f,\lambda_i-3-d-f)}}_{N',\nu'}\end{aligned}$$

Summarizing by all i such that $\lambda_i > 3$ yields summands (24), (29) and (26) respectively.

Case 2: $e, h \in L_i$, $f \in L_j$ for some $i \neq j$. In this case the cycles L_i and L_j either union into a cycle of length $2(\lambda_i + \lambda_2 - 3)$ or regroup into two cycles of that summary length.

- For hexagons of types T3–T6, we have $\lambda' = \lambda_{\text{⫼}(\lambda_i,\lambda_j)}$ therefore

$$\begin{aligned}\sum_{\{e,h\}\subset L_i} \sum_{f\in L_j} \sum_{\delta\colon \text{T3–T6}} \beta^{\mathrm{wt}(\delta)} &= \sum_{\{e,h\}\subset L_i} \sum_{f\in L_j} \sum_{\delta'\in\widetilde{\mathcal{G}}^{\lambda_{\text{⫼}(\lambda_i,\lambda_j)}}_{\nu'}} (2\beta^2+2\beta)\beta^{\mathrm{wt}(\delta')} \\ &= (2\beta^2+2\beta)\binom{\lambda_i}{2}\lambda_j \widetilde{a}^{\lambda_{\text{⫼}(\lambda_i,\lambda_j)}}_{N',\nu'}.\end{aligned}$$

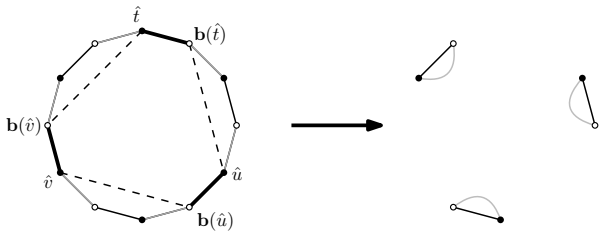


Figure 12: A hexagon of type T8 in Case 1 contributing to summand (26).

Summarizing this by all $i \neq j$ yields summand (27) as

$$\sum_{i\neq j} \binom{\lambda_i}{2}\lambda_j = \frac{1}{2}\sum_{i\neq j}\left(\binom{\lambda_i}{2}\lambda_j + \binom{\lambda_j}{2}\lambda_i\right) = \frac{1}{2}\sum_{i<j}\lambda_i\lambda_j(\lambda_i+\lambda_j-2).$$

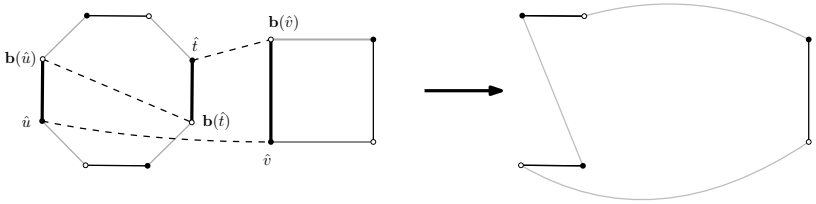


Figure 13: Example of a hexagon of type T7 in Case 2 contributing to summand (27).

- In other cases we numerate vertices of L_i by residues as above. Each bipartite hexagon (of type T1 and T8) can be uniquely described by the sequence of vertices of the kind $(\widehat{x-1}, x, \widehat{x+d}, x+d, \widehat{y}, \mathbf{b}(\widehat{y}))$ where $x \in \{1,\dots,\lambda_i\}$, $d \in \{1,\dots,\lambda_i-2\}$ and $\widehat{y} \in L_j$ ($d \neq \lambda_i - 1$ else $\{x-1, \widehat{x-1}\} \in \delta \cap \mathbf{g}$ and $1 \in \Lambda(\mathbf{g},\delta)$). In this case $\lambda' = \lambda \setminus (\lambda_i,\lambda_j) \cup (d, \lambda_i+\lambda_j-3-d)$ therefore

$$\sum_{\{e,h\}\subset L_i}\sum_{f\in L_j}\sum_{\delta\colon \text{T1,T8}} \beta^{\mathrm{wt}(\delta)} = \sum_{x=1}^{\lambda_i}\sum_{y=1}^{\lambda_j}\sum_{d=1}^{\lambda_i-2}\sum_{\delta'\in\widetilde{\mathcal{G}}_{\nu'}^{\lambda\setminus(\lambda_i,\lambda_j)\cup(d,\lambda_i+\lambda_j-3-d)}} \beta^{\mathrm{wt}(\delta')}$$
$$= \lambda_i\lambda_j\sum_{d=1}^{\lambda_i-2}\widetilde{a}_{N',\nu'}^{\lambda\setminus(\lambda_i,\lambda_j)\cup(d,\lambda_i+\lambda_j-3-d)}.$$

The same result multiplied by β is for types T2 and T7 which differ from the case of bipartite hexagons only by the order of vertices $\widehat{y}, \mathbf{b}(\widehat{y}) \in L_j$: we have the sequence of vertices $(\widehat{x-1}, x, \widehat{x+d}, x+d, \mathbf{b}(\widehat{y}), \widehat{y})$ for hexagons of types T2 and T7. We write the sum by d in the symmetrical way using the following: if

$a_{ij}(d) = a_{ji}(d) = a_{ij}(\lambda_i + \lambda_j - 3 - d)$ then

$$2\sum_{d=1}^{\lambda_i-2} a_{ij}(d) = \sum_{d=1}^{\lambda_i-2} a_{ij}(d) + \sum_{d=1}^{\lambda_j-2} a_{ij}(\lambda_i+\lambda_j-3-d)$$
$$= \sum_{d=1}^{\lambda_i-2} a_{ij}(d) + \sum_{d=\lambda_i-1}^{\lambda_i+\lambda_j-4} a_{ij}(d) = \sum_{d=1}^{\lambda_i+\lambda_j-4} a_{ij}(d)$$

Finally we summarise for $i \neq j$:

$$\sum_{i\neq j} \lambda_i\lambda_j \sum_{d=1}^{\lambda_i-2} \widetilde{a}_{N',\nu'}^{\lambda\setminus(\lambda_i,\lambda_j)\cup(d,\lambda_i+\lambda_j-3-d)} = \sum_{i<j} \lambda_i\lambda_j \sum_{d=1}^{\lambda_i+\lambda_j-4} \widetilde{a}_{N',\nu'}^{\lambda\setminus(\lambda_i,\lambda_j)\cup(d,\lambda_i+\lambda_j-3-d)}.$$

As a result we have the summand (28).

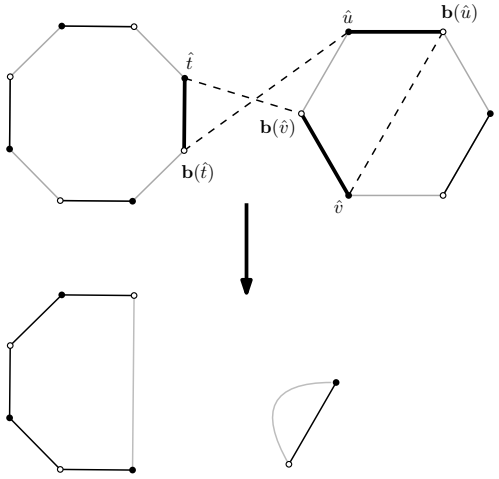


Figure 14: Example of a hexagon of type T8 in Case 2 contributing to summand (28).

Case 3: $e \in L_i$, $f \in L_j$, $h \in L_k$ for some $i < j < k$. In this case $\lambda_i + \lambda_j + \lambda_k > 3$ else $3 \in \Lambda(\mathbf{g}, \delta) = (N)$. As $\lambda' = \lambda_{\Vdash(\lambda_i,\lambda_j,\lambda_k)}$ then

$$\sum_{\substack{e\in L_i\\ f\in L_j, h\in L_k}} \sum_{\delta: E=\{e,f,h\}} \beta^{\mathrm{wt}(\delta)} = \sum_{\substack{e\in L_i\\ f\in L_j, h\in L_k}} \sum_{\delta'\in\mathcal{G}_{\nu'}^{\lambda_{\Vdash(\lambda_i,\lambda_j,\lambda_k)}}} (2\beta^2+4\beta+2)\beta^{\mathrm{wt}(\delta')}$$
$$= (2\beta^2+4\beta+2)\lambda_i\lambda_j\lambda_k \widetilde{a}_{N',\nu'}^{\lambda_{\Vdash(\lambda_i,\lambda_j,\lambda_k)}}.$$

This is summand (25).

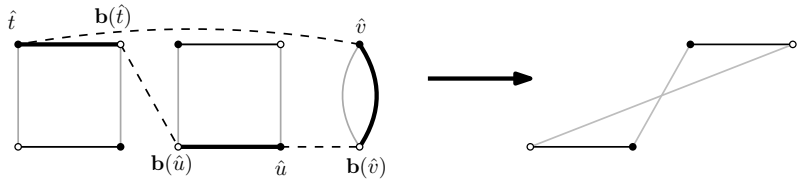


Figure 15: Example of a hexagon of type T4 in Case 3 contributing to summand (25).

7.2.4 Final remarks

We finish the proof of Theorem 2 by noting that the inequality $\mathrm{wt}(\delta) \leqslant N - \ell(\nu)$ holds for all $\lambda, \nu \vdash N$ and $\delta \in \widetilde{\mathcal{G}}_\nu^\lambda$. Indeed, by assumption it holds for $k = l = m = 0$. As one can see from the proof the adding weight of δ is increasing no more than by $d-1$ when we add a part $d \in \{1,2,3\}$ to the partition ν.

8 Full proof of Theorem 3

This section is dedicated to the full proof of Theorem 3. We provide some bijective constructions on labelled star bipartite maps to prove that the polynomials $\Sigma^\lambda_{[k^m]}(\beta)$ defined as

$$\Sigma^\lambda_{[k^m]}(\beta) = \sum_{M \in \widetilde{\mathcal{L}}^\lambda_{[k^m]}} \beta^{\vartheta(M)}$$

satisfy the same recurrence relation as the coefficients $\widetilde{h}^\lambda_{n,[k^m]}(1+\beta)$ where ϑ is the measure of non-orientability of Definition 2. Recall that the case $k = n$ is already covered in Section 6.3. In a first step we state the recurrence formulas that satisfy the $\widetilde{h}^\lambda_{n,[k^m]}(1+\beta)$.

8.1 Recurrence relations for the coefficients $\widetilde{h}^\lambda_{n,[k^m]}(1+\beta)$

We have the following lemmas (the proofs are extensions of the ones in Section 7.1 and are not detailed here).

Lemma 7. For λ integer partition of $2m$ with $m \geqslant 2$, the number $\widetilde{h}^\lambda_{2m,[2^m]}(\alpha)$ satisfies

$$\begin{aligned}\widetilde{h}^\lambda_{2m,[2^m]}(\alpha) = (\alpha-1)\sum_{i\geqslant 1}(i-1)(i-2)m_{i-2}(\lambda_{\Downarrow(i)})\widetilde{h}^{\lambda_{\Downarrow(i)}}_{2m-2,[2^{m-1}]}(\alpha)\\ + \sum_{i,j\geqslant 1}(i+j-2)m_{i+j-2}(\lambda_{\Downarrow(i,j)})\widetilde{h}^{\lambda_{\Downarrow(i,j)}}_{2m-2,[2^{m-1}]}(\alpha)\\ + \alpha\sum_{i,d\geqslant 1}(i-2-d)dm_{i-2-d,d}(\lambda^{\Mapsto(i-2-d,d)})\widetilde{h}^{\lambda^{\Mapsto(i-2-d,d)}}_{2m-2,[2^{m-1}]}(\alpha).\end{aligned}$$

Lemma 8. Recall the notation for partition operations described at the beginning of Lemma 6. For integer $m > 1$ and $\lambda \vdash 3m$, the coefficient $\widetilde{h}^{\lambda}_{3m,[3^m]}(\alpha)$ satisfies

$$\begin{aligned}
&\widetilde{h}^{\lambda}_{3m,[3^m]} = \\
&3\widetilde{h}^{3}_{3,3} \sum_i \binom{i-1}{3} m_{i-3}(\lambda_{\Vbar(i)}) \widetilde{h}^{\lambda_{\Vbar(i)}}_{3m-3,[3^{m-1}]} \\
&+ \widetilde{h}^{(2,1)}_{3,3} \frac{1}{2} \sum_{i,j} (i+j-2)(i+j-3) m_{i+j-3}(\lambda_{\Vbar(i,j)}) \widetilde{h}^{\lambda_{\Vbar(i,j)}}_{3m-3,[3^{m-1}]} \\
&+ 6\alpha \widetilde{h}^{2}_{2,2} \frac{1}{2} \sum_{i,d} (i-1)(i-d-3) d m_{i-3-d,d}(\lambda^{\tbar(i-3-d,d)}) \widetilde{h}^{\lambda^{\tbar(i-3-d,d)}}_{3m-3,[3^{m-1}]}
\end{aligned}$$

$$\begin{aligned}
&+ 6\alpha \widetilde{h}^{(1,1)}_{2,2} \frac{1}{2} \sum_{i,j,d} (i+j-3-d) d m_{i+j-d-3,d}(\lambda^{\dbar(i,j)\tbar(i+j-3-d,d)}) \widetilde{h}^{\lambda^{\dbar(i,j)\tbar(i+j-3-d,d)}}_{3m-3,[3^{m-1}]} \\
&+ 2\alpha^2 \widetilde{h}^{1}_{1,1} \sum_{i,d,f} (i-3-d-f) d f m_{i-3-d-f,d,f}(\lambda^{\tbar(i-3-d-f,d,f)}) \widetilde{h}^{\lambda^{\tbar(i-3-d-f,d,f)}}_{3m-3,[3^{m-1}]} \\
&+ \widetilde{h}^{(1,1,1)}_{3,3} \sum_{i,j,k} (i+j+k-3) m_{i+j+k-3}(\lambda_{\Vbar(i,j,k)}) \widetilde{h}^{\lambda_{\Vbar(i,j,k)}}_{3m-3,[3^{m-1}]}.
\end{aligned}$$

The reference to α has been removed in the coefficients $\widetilde{h}$ for the sake of clarity.

8.2 Bijective constructions for star bipartite maps

We proceed with the proof of Theorem 3 in the cases $k \in \{1, 2, 3\}$ (recall that we already covered the case $k = n$).

8.2.1 Case $k = 1$

For $k = 1$, it is easy to show ([9]) that $\widetilde{h}^{\lambda}_{n,[1^n]} = 0$ if $\lambda \neq (n)$. Using Equation (21) for $\lambda = (n)$ and $\rho = \emptyset$, one gets :

$$a^{n}_{n,[1^n]} = a^{n-1}_{n-1,[1^{n-1}]} = a^{1}_{1,1} = 1$$

Then according to Equation (1),

$$h^{n}_{n,[1^n]} = a^{n}_{n,[1^n]} = 1$$

Finally there are $(n-1)!$ labelled star bipartite maps of face degree distribution (n) and black vertex degree distribution $[1^n]$ that correspond to the $(n-1)!$ possible labelling of the only (unlabelled) star bipartite map with one white vertex and n leaves. Furthermore for any such labelled star map M, $\vartheta(M) = 0$ and

$$\Sigma^{n}_{[1^n]}(\beta) = (n-1)! = \widetilde{h}^{n}_{n,[1^n]}(\beta + 1).$$

8.2.2 Case $k = 2$

As noticed in Remark 3, bipartite maps with black vertices degree distribution equal to $[2^m]$ for some integer m reduce to non-bipartite maps by removing the black vertices. The resulting non-bipartite maps is a labelled **monopole**, i.e. a labelled map composed of a single (white) vertex and m-edges with two consecutive labels and twice incident to the vertex. The type of the edges is the same as in bipartite maps, i.e. an edge can be a cross-border, a border or a handle (but never a leaf). By abuse of notations, we also denote ϑ the induced measure of non-orientability for labelled monopoles and $\widetilde{\mathcal{L}}^{\lambda}_{[2^m]}$ the set of such maps with m edges. Since the two edges incident to the same black vertex in the initial bipartite map have consecutive labels and deleting one of the two edges makes the second one become a leaf, two iterations of the computation of ϑ in the initial bipartite map is equivalent to one iteration in the resulting non-bipartite map.
According to Section 6.3 in the case $m = 1$, we have

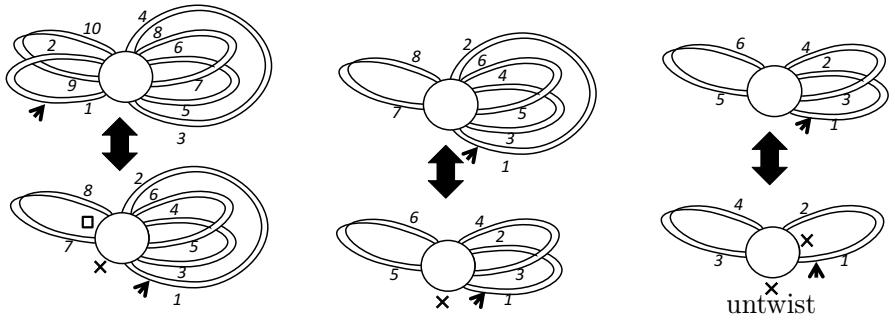
$$\Sigma_2^{(1,1)}(\beta) = \widetilde{h}_{2,2}^{(1,1)}(\beta+1) \quad \text{and} \quad \Sigma_2^2(\beta) = \widetilde{h}_{2,2}^2(\beta+1)$$

Suppose now $m > 1$. As in section 6.3, we split $\widetilde{\mathcal{L}}^{\lambda}_{[2^m]}$ into three subsets depending on the type of the root.

- (Cross border) The set of labelled monopoles with face distribution λ and a cross border root incident to a face of degree i is in bijection with the set of decorated labelled monopoles with (i) face distribution $\lambda_{\Downarrow(i)}$, (ii) two distinct corners incident to a face of degree $i-2$ ($m_{i-2}(\lambda_{\Downarrow(i)})(i-1)(i-2)$ possible choices). one marked corner corresponds to the edge-side labelled 1 in the preimage of the decorated monopole, the other to the edge-side labelled 2.

- (Border) The set of labelled monopoles with face distribution λ and a border root incident to both a face of size i and a face of size j is in bijection with the set of decorated labelled monopoles with (i) face distribution $\lambda_{\Downarrow(i,j)}$, (ii) one marked corner incident to a face of degree $i + j - 2$ (once this first position around the vertex is chosen there is only one more position around the vertex such that drawing a border by connecting these two positions cut the face of size $i+j-2$ into a face of degree i and a face of degree j). We take the convention that the chosen corner corresponds to the root half edge and the other position of the other half edge in the preimage of the decorated monopole.

- (Handle) The set of labelled monopoles with face distribution λ and a handle root incident to a face of size i such that removing the root yields a face of degree d and one of degree $i-2-d$ is in bijection with the set of decorated labelled monopoles with (i) face distribution $\lambda^{\Uparrow(i-2-d,d)}$, (ii) one marked corner around the vertex incident to a face of degree $i - 2 - d$, (iii) one marked corner around the vertex incident to a face of degree d, (we assume without loss of generality that the root half edge of the deleted edge is incident to the face of degree $i - 2 - d$), (iv) and a type for the deleted root : twist or untwist.



(a) First bijection. (b) Second bijection. (c) Third bijection.

Figure 16: Examples of application of the three bijections for labelled star bipartite maps for $k = 2$.

Example 7. Figure 16 illustrates the three bijections described above.

As a consequence, one gets

$$\begin{aligned}\Sigma^{\lambda}_{[2^m]}(\beta) = &\sum_{i\geqslant 1}(i-1)(i-2)m_{i-2}(\lambda_{\Yup(i)}) \sum_{M'\in\widetilde{\mathcal{L}}^{\lambda_{\Yup(i)}}_{[2^{m-1}]}} \beta^{1+\vartheta(M')}\\ &+\sum_{i,j\geqslant 1}(i+j-2)m_{i+j-2}(\lambda_{\Yup(i,j)}) \sum_{M'\in\widetilde{\mathcal{L}}^{\lambda_{\Yup(i,j)}}_{[2^{m-1}]}} \beta^{\vartheta(M')}\\ &+\sum_{i,d\geqslant 1}(i-2-d)dm_{i-d-2,d}(\lambda^{\Tilde{\Yup}(i-2-d,d)}) \sum_{M'\in\widetilde{\mathcal{L}}^{\lambda^{\Tilde{\Yup}(i-2-d,d)}}_{[2^{m-1}]}} \left(\beta^{\vartheta(M')}+\beta^{1+\vartheta(M')}\right)\end{aligned}$$

As a conclusion, for any integer $m \geqslant 1$,

$$\Sigma^{\lambda}_{[2^m]}(\beta) = \widetilde{h}^{\lambda}_{2m,[2^m]}(\beta+1).$$

8.3 Case $k = 3$

As a final case, we show Theorem 3 in the case $k = 3$. We focus on labelled star bipartite maps that contain only black vertices of degree 3. We call these maps **labelled star trivalent maps**. First use Section 6.3, to show that

$$\Sigma^{(1,1,1)}_3(\beta) = \widetilde{h}^{(1,1,1)}_{3,3}(\beta+1) \quad , \quad \Sigma^{(2,1)}_3(\beta) = \widetilde{h}^{(2,1)}_{3,3}(\beta+1)$$
$$\text{and} \quad \Sigma^{3}_3(\beta) = \widetilde{h}^{3}_{3,3}(\beta+1)$$

Then, we build a bijection between labelled star trivalent maps in $\widetilde{\mathcal{L}}^{\lambda}_{[3^m]}$ and some decorated labelled star trivalent maps with $m-1$ black vertices. To this end, we iterate three times the root deletion process. As the edges incident to the black vertex labelled 1 (incident to the root) are labelled with $1, 2$ and 3, exactly these three edges are deleted

after three iterations of the root deletion process and the resulting bipartite map is a labelled star trivalent map in $\widetilde{\mathcal{L}}^{\lambda'}_{[3^{m-1}]}$.
We look at all the possible configurations of the subset of the labelled trivalent map composed of the white vertex, the black vertex incident to the root and the three edges incident to this black vertex. We call this subset the **root submap**. Delete the root submap σ_M of a star trivalent map M of face degree distribution λ but keep a mark at the incidence positions around the white vertex of all of its three edges. Denote M' the resulting star trivalent map with three marks and λ' its resulting face degree distribution. There are 6 possible cases that we split first according to the incidence of the marks to the various faces of M'.

a All the marks are incident to the same face of M'. In this case, it is easy to show that $\vartheta(M) = \vartheta(M') + \vartheta(\sigma_M)$. We have three sub-cases depending on the number of faces in σ_M.

a.1 σ_M has exactly one face of degree 3. In this case, $\lambda' = \lambda_{\Yvcentermath1\yng(i)}$ for some i and there is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of couples composed of a labelled star map with one black vertex of face distribution (3) and a labelled star trivalent map of face degree distribution $\lambda_{\Yvcentermath1\yng(i)}$ with (i) one marked face of degree $i-3$, (ii) three corners among $i-1$ around the white vertex within the marked face and (iii) one distinguished position among the three marks. The distinguished position is used to locate the root of the star map of face degree distribution (3) within the star trivalent map.

a.2 σ_M has exactly two faces of degree $(2,1)$. In this case, $\lambda' = \lambda_{\Yvcentermath1\yng(i,j)}$ for some $i \leqslant j$ and there is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of couples composed of a labelled star map with one black vertex of face distribution $(2,1)$ and a labelled star trivalent map of face degree distribution $\lambda_{\Yvcentermath1\yng(i,j)}$ with (i) one marked face of degree $i+j-3$, (ii) two differentiated corners among $i+j-2$ around the white vertex within the marked face.

a.3 σ_M has three faces of degree $(1,1,1)$. In this case, $\lambda' = \lambda_{\Yvcentermath1\yng(i,j,k)}$ for some i,j,k. We assume that i is the degree of the face incident to the root of σ_M and the next edge of σ_M moving counterclockwise around the white vertex and that k is the degree of the face incident to the root of σ_M and the next edge of σ_M moving clockwise around the white vertex. There is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of couples composed of a labelled star maps with one black vertex of face distribution $(1,1,1)$ and a labelled star trivalent map of face degree distribution $\lambda_{\Yvcentermath1\yng(i,j,k)}$ with (i) one marked face of degree $i+j+k-3$, (ii) one marked corners among $i+j+k-3$ around the white vertex within the marked face. The corner locates the root of the star map of face degree distribution $(1,1,1)$. Then it is easy to show that i and k give exactly the required information to locate the two other edges.

b Two of the marks are incident to one face of M' and the third one to a second distinct face of M'. In this case, there is a handle edge in σ_M whose contribution to $\vartheta(M)$ is not its contribution to $\vartheta(\sigma_M)$ (this edge may not be a handle at all in the submap σ_M). Denote σ'_M the reduction of σ_M obtained by deletion of this handle. We have $\vartheta(M) = \vartheta(M') + \vartheta(\sigma'_M) + \varepsilon$ where $\varepsilon \in \{0, 1\}$ depending on the contribution of the handle (recall that by twisting the handle one get a distinct map with the other value for ε). We have two sub-cases depending on the number of faces in σ'_M.

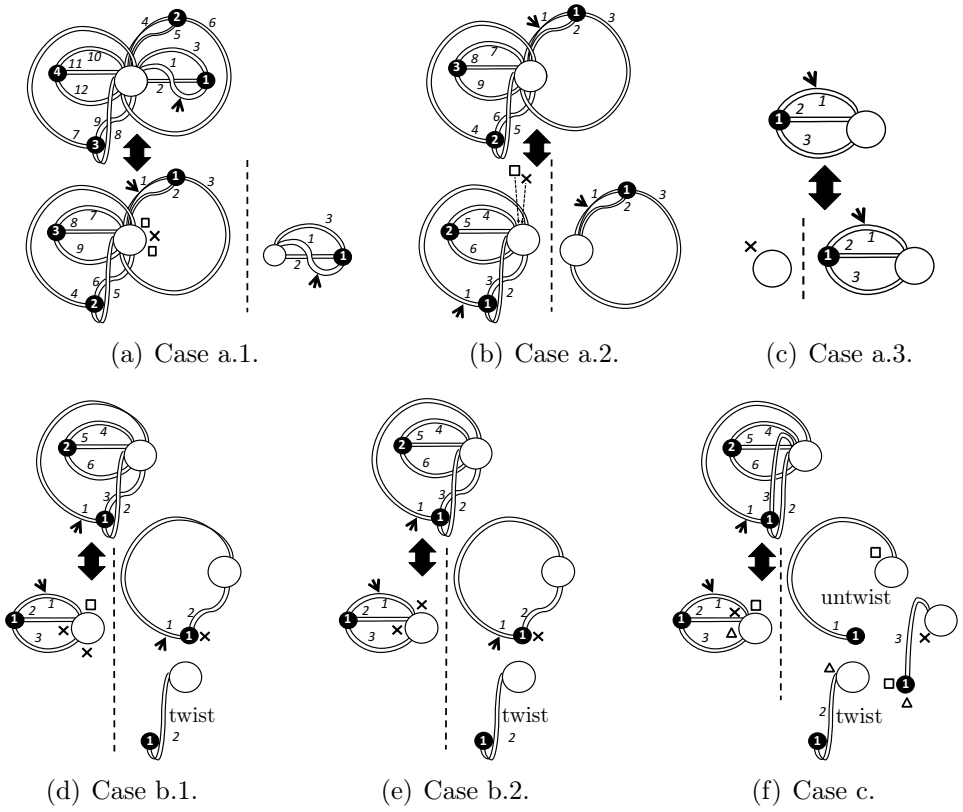
b.1 σ'_M has exactly one face of degree 2. In this case, $\lambda' = \lambda^{\text{Ⅲ}(i-3-d,d)}$ for some i and d with $d \leqslant i-3-d$. There is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of couples composed of a labelled star map with one black vertex of face distribution (2) and a labelled star trivalent map of face degree distribution $\lambda^{\text{Ⅲ}(i-3-d,d)}$ with (i) one marked face of degree $i-3-d$, (ii) one marked face of degree d, (iii) one marked corner around the white vertex within the marked face of degree $i-3-d$, (iv) one marked corner around the white vertex within the marked face of degree d, (v) a distinguished corner around the white vertex within the face of degree d or the face of degree $i-3-d$ (there are $i-1$ possibilities after adding the two first marks), (vi) one position around the white vertex within σ'_M (2 possibilities), (vii) an index for the handle edge (3 possible values) and (viii) an indication whether the handle is twisted or not. The two marks belonging to the same face of M' locate the edges of σ'_M, its root being incident to the distinguished one. The other mark locates the incidence of the handle to the white vertex. The mark within σ'_M locates the incidence of the handle to the black vertex labelled 1.

b.2 σ'_M has two faces of degree $(1, 1)$. In this case, $\lambda' = \lambda^{\text{Ⅱ}(i,j)\text{Ⅲ}(i+j-3-d,d)}$ for some i, j and d. There is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of couples composed of a labelled star map with one black vertex of face distribution $(1, 1)$ and a labelled star trivalent map of face degree distribution $\lambda^{\text{Ⅱ}(i,j)\text{Ⅲ}(i+j-3-d,d)}$ with (i) one marked face of degree $i-3-d$, (ii) one marked face of degree d, (iii) one marked corner around the white vertex within the marked face of degree $i-3-d$, (iii) one marked corner around the white vertex within the marked face of degree d, (iv) one position around the white vertex within σ'_M (2 possibilities), (vi) an index for the handle edge (3 possible values) and (vii) an indication whether the handle is twisted or not.

c The three marks are incident to three disctinct faces of M'. All the edges of σ_M are handles, and we have $\vartheta(M) = \vartheta(M') + \varepsilon_1 + \varepsilon_2$ where $\varepsilon_i \in \{0, 1\}$ depending on the contribution of the two handles with the smallest index.
In this case, $\lambda' = \lambda_{\text{Ⅲ}(i-3-d-f,d,f)}$ for some i, d, f. We assume that (i) $i-3-d-f$ is the degree of the face of M' incident to the mark corresponding to the root of σ_M (ii) d is the degree of the face of M' incident to the mark corresponding to the

edge labelled 2 in σ_M and (iii) f is the degree of the face of M' incident to the mark corresponding to the edge labelled 3 in σ_M. There is a bijection between the set of such labelled star trivalent maps of face degree distribution λ and the set of labelled star trivalent maps of face degree distribution $\lambda_{\uplus(i-3-d-f,d,f)}$ with (i) one marked face of degree $i-3-d-f$, (ii) one marked face of degree d, (iii) one marked face of degree f, (iv) one marked corners around the white vertex within the marked face of degree $i-3-d-f$, (v) one marked corners around the white vertex within the marked face of degree d, (vi) one marked corners around the white vertex within the marked face of degree f, (vii) an indication whether the two handles labelled 1 and 2 are twisted or not, (viii) an indication whether the order of the edges in σ_M moving clockwise around the black vertex is 132 or 123.

Example 8. Figure 17 illustrates the six bijections described above.



(a) Case a.1. (b) Case a.2. (c) Case a.3.

(d) Case b.1. (e) Case b.2. (f) Case c.

Figure 17: Illustration of the six bijections for labelled star trivalent maps.

As a consequence, one gets

$$\Sigma^{\lambda}_{[3^m]}(\beta) = 3\sum_i \binom{i-1}{3} m_{i-3}(\lambda_{\Vert\Vert(i)}) \sum_{\substack{\sigma_M\in\widetilde{\mathcal{L}}_3^3\\ M'\in\widetilde{\mathcal{L}}^{\lambda_{\Vert\Vert(i)}}_{[3^{m-1}]}}} \beta^{\vartheta(\sigma_M)+\vartheta(M')}$$

$$+\frac{1}{2}\sum_{i,j}(i+j-2)(i+j-3)m_{i+j-3}(\lambda_{\Vert\Vert(i,j)}) \sum_{\substack{\sigma_M\in\widetilde{\mathcal{L}}_3^{(2,1)}\\ M'\in\widetilde{\mathcal{L}}^{\lambda_{\Vert\Vert(i,j)}}_{[3^{m-1}]}}} \beta^{\vartheta(\sigma_M)+\vartheta(M')}$$

$$+\sum_{i,j,k}(i+j+k-3)m_{i+j+k-3}(\lambda_{\Vert\Vert(i,j,k)}) \sum_{\substack{\sigma_M\in\widetilde{\mathcal{L}}_3^{(1,1,1)}\\ M'\in\widetilde{\mathcal{L}}^{\lambda_{\Vert\Vert(i,j,k)}}_{[3^{m-1}]}}} \beta^{\vartheta(\sigma_M)+\vartheta(M')}$$

$$+6\frac{1}{2}\sum_{i,d}(i-1)(i-d-3)dm_{i-3-d,d}(\lambda^{\Vert\Pi(i-3-d,d)}) \sum_{\substack{\sigma'_M\in\widetilde{\mathcal{L}}_2^{(2)}\\ M'\in\widetilde{\mathcal{L}}^{\Vert\Pi(i-3-d,d)}_{[3^{m-1}]}}} (1+\beta)\beta^{\vartheta(\sigma'_M)+\vartheta(M')}$$

$$+6\frac{1}{2}\sum_{i,j,d}(i+j-3-d)dm_{i+j-d-3,d}(\lambda^{\downarrow(i,j)\Vert(i+j-3-d,d)}) \sum_{\substack{\sigma'_M\in\widetilde{\mathcal{L}}_2^{(1,1)}\\ M'\in\widetilde{\mathcal{L}}^{\downarrow(i,j)\Vert(i+j-3-d,d)}_{[3^{m-1}]}}} (1+\beta)\beta^{\vartheta(\sigma'_M)+\vartheta(M')}$$

$$+2\sum_{i,d,f}(i-3-d-f)dfm_{i-3-d-f,d,f}(\lambda^{\Vert\Pi(i-3-d-f,d,f)}) \sum_{\substack{\sigma'_M\in\widetilde{\mathcal{L}}_2^{(1,1)}\\ M'\in\widetilde{\mathcal{L}}^{\Vert\Pi(i-3-d-f,d,f)}_{[3^{m-1}]}}} (1+\beta)^2\beta^{\vartheta(M')}.$$

As a conclusion, for any integer $m \geqslant 1$,

$$\Sigma^{\lambda}_{[3^m]}(\beta) = \widetilde{h}^{\lambda}_{3m,[3^m]}(\beta+1).$$

9 Appendix: computation of operator Ω_k

We provide the computation of operator Ω_2. The computation of Ω_3 is performed in a similar way but is much more cumbersome and is not detailed here. Denote $\zeta_k = kp_{k+1}\frac{\partial}{\partial p_k}$ and write

$$\Omega_2 = [\Delta, \Omega_1] = (\alpha-1)\sum_{i,k} B_{i,k} + \alpha\sum_{i,j,k} A_{i,j,k} + \sum_{i,j,k}\Gamma_{i,j,k}$$

where

$$\begin{aligned} B_{i,k} &= \left[(i-1)^2 p_i \frac{\partial}{\partial p_{i-1}}, \zeta_k\right],\\ A_{i,j,k} &= \left[(i+j-1)p_i p_j \frac{\partial}{\partial p_{i+j-1}}, \zeta_k\right],\\ \Gamma_{i,j,k} &= \left[ijp_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}, \zeta_k\right]. \end{aligned}$$

The non-zero terms are:

$$\begin{aligned}B_{i,i} = \left[(i-1)^2 p_i \frac{\partial}{\partial p_{i-1}}, i p_{i+1} \frac{\partial}{\partial p_i}\right] &= (i-1)^2 p_i \frac{\partial}{\partial p_{i-1}} i p_{i+1} \frac{\partial}{\partial p_i} \\ &\quad - i p_{i+1} \frac{\partial}{\partial p_i}\left((i-1)^2 p_i \frac{\partial}{\partial p_{i-1}}\right) \\ &\quad - i(i-1)^2 p_i p_{i+1} \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_{i-1}} = i(i-1)^2 p_{i+1}\frac{\partial}{\partial p_{i-1}}.\end{aligned}$$

$$\begin{aligned}B_{i,i-2} = \left[(i-1)^2 p_i \frac{\partial}{\partial p_{i-1}}, (i-2) p_{i-1} \frac{\partial}{\partial p_{i-2}}\right] &= (i-1)^2 p_i \frac{\partial}{\partial p_{i-1}}\left((i-2) p_{i-1}\frac{\partial}{\partial p_{i-2}}\right) \\ &\quad + (i-2)(i-1)^2 p_i p_{i-1}\frac{\partial}{\partial p_{i-2}}\frac{\partial}{\partial p_{i-1}} \\ &\quad - (i-1)^2(i-2) p_i p_{i-1}\frac{\partial}{\partial p_{i-1}}\frac{\partial}{\partial p_{i-2}} \\ &= (i-1)^2(i-2) p_i \frac{\partial}{\partial p_{i-2}}.\end{aligned}$$

$$\begin{aligned}A_{i,j,i-1} = \left[ij p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}, (i-1) p_i \frac{\partial}{\partial p_{i-1}}\right] &= ij(i-1) p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\left(p_i \frac{\partial}{\partial p_{i-1}}\right) \\ &\quad + ij(i-1) p_{i+j+1} p_i \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_{i-1}} \\ &\quad - ij(i-1) p_{i+j+1} p_i \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_{i-1}} \\ &= ij(i-1) p_{i+j+1}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_{i-1}}.\end{aligned}$$

Similarly $A_{i,j,j-1} = ij(i-1)p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_{j-1}}$.

$$\begin{aligned}A_{i,j,i+j+1} &= \left[ij p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}, (i+j+1) p_{i+j+2}\frac{\partial}{\partial p_{(}i+j+1)}\right] \\ &= ij(i+j+1) p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} p_{j+i+2}\frac{\partial}{\partial p_{j+i+1}} \\ &\quad - ij(i+j+1) p_{j+i+2}\frac{\partial}{\partial p_{j+i+1}}\left(p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right) \\ &\quad - ij(i+j+1) p_{i+j+1} p_{j+i+2}\frac{\partial}{\partial p_{j+i+1}}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} \\ &= -ij(i+j+1) p_{j+i+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}.\end{aligned}$$

$$\Gamma_{i,j,i} = \left[(i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}, ip_{i+1}\frac{\partial}{\partial p_i}\right] = (i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}ip_{i+1}\frac{\partial}{\partial p_i}$$
$$- ip_{i+1}\frac{\partial}{\partial p_i}\left((i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}\right)$$
$$- (i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}ip_{i+1}\frac{\partial}{\partial p_i}$$
$$= -(i+j-1)ip_{i+1}p_j\frac{\partial}{\partial p_{i+j-1}}.$$

Similarly $\Gamma_{i,j,j} = -(i+j-1)ip_{j+1}p_i\frac{\partial}{\partial p_{i+j-1}}$.

$$\Gamma_{i,j,i+j-2} = \left[(i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}, (i+j-2)p_{i+j-1}\frac{\partial}{\partial p_{i+j-2}}\right]$$
$$= (i+j-1)p_ip_j\frac{\partial}{\partial p_{i+j-1}}\left((i+j-2)p_{i+j-1}\frac{\partial}{\partial p_{i+j-2}}\right)$$
$$+ (i+j-1)(i+j-2)p_{i+j-1}p_ip_j\frac{\partial}{\partial p_{i+j-2}}\frac{\partial}{\partial p_{i+j-1}}$$
$$- (i+j-1)(i+j-2)p_{i+j-1}p_ip_j\frac{\partial}{\partial p_{i+j-2}}\frac{\partial}{\partial p_{i+j-1}}$$
$$= (i+j-1)(i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}}.$$

Summing up all the non-zeros A terms yields:

$$\sum_{i,j}A_{i,j,i-1} + \sum_{i,j}A_{i,j,j-1} + \sum_{i,j}A_{i,j,i+j+1} = \sum_{i,j\geqslant 1}ij(j-1)p_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_{j-1}}$$
$$+ \sum_{i,j\geqslant 1}ij(i-1)p_{i+j+1}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_{i-1}}$$
$$- \sum_{i,j\geqslant 1}ij(i+j+1)p_{j+i+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$= \sum_{i,j\geqslant 1}ij(i+1)p_{i+j+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$+ \sum_{i,j\geqslant 1}ij(j+1)p_{i+j+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$- \sum_{i,j\geqslant 1}ij(i+j+1)p_{j+i+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$
$$= \boxed{\sum_{i,j\geqslant 1}ijp_{j+i+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}}.$$

Summing up all the non-zeros B terms yields:

$$\begin{aligned}\sum_i B_{i,i} + \sum_i B_{i,i-2} &= \sum_{i\geqslant 1}(i-2)(i-1)^2 p_i \frac{\partial}{\partial p_{i-2}} - \sum_{i\geqslant 1} i(i-1)^2 p_{i+1}\frac{\partial}{\partial p_{i-1}} \\ &= \sum_{i\geqslant 1}(i-1)i^2 p_{i+1}\frac{\partial}{\partial p_{i-1}} - \sum_{i\geqslant 1} i(i-1)^2 p_{i+1}\frac{\partial}{\partial p_{i-1}} \\ &= \boxed{\sum_{i\geqslant 1}(i-1)ip_{i+1}\frac{\partial}{\partial p_{i-1}}}.\end{aligned}$$

Finally, summing up all the non-zeros Γ terms yields:

$$\begin{aligned}\sum_{i,j}\Gamma_{i,j,i+j-2} + \sum_{i,j}\Gamma_{i,j,i} + \sum_{i,j}\Gamma_{i,j,j} &= \sum_{i,j\geqslant 1}(i+j-1)(i+j-2)p_i p_j \frac{\partial}{\partial p_{i+j-2}} \\ &\quad - \sum_{i,j\geqslant 1}(i+j-1)ip_{i+1}p_j\frac{\partial}{\partial p_{i+j-1}} \\ &\quad - \sum_{i,j\geqslant 1} j(i+j-1)p_{j+1}p_i\frac{\partial}{\partial p_{i+j-1}} \\ &= \sum_{i,j\geqslant 1}(i+j-1)(i+j-2)p_i p_j \frac{\partial}{\partial p_{i+j-2}} \\ &\quad - \sum_{i,j\geqslant 1}(i+j-2)(i-1)p_i p_j\frac{\partial}{\partial p_{i+j-2}} \\ &\quad - \sum_{i,j\geqslant 1}(j-1)(i+j-2)p_j p_i\frac{\partial}{\partial p_{i+j-2}} \\ &= \boxed{\sum_{i,j\geqslant 1}(i+j-2)p_i p_j\frac{\partial}{\partial p_{i+j-2}}}.\end{aligned}$$

As a consequence

$$\Omega_2 = (\alpha-1)\sum_{i\geqslant 1}(i-1)ip_{i+1}\frac{\partial}{\partial p_{i-1}} + \sum_{i,j\geqslant 1}(i+j-2)p_i p_j\frac{\partial}{\partial p_{i+j-2}} + \alpha\sum_{i,j\geqslant 1} ijp_{j+i+2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$$

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