Spectral extremal graphs without intersecting triangles as a minor

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Abstract

Let F_s be the friendship graph obtained from s triangles by sharing a common vertex. For every $s \geqslant 2$ and $n \geqslant 50s^2$, the Turán number of F_s was investigated by Erdős, Füredi, Gould and Gunderson (1995). For sufficiently large n, the F_s-free graphs of order n which attain the maximum spectral radius were firstly characterized by Cioabă, Feng, Tait and Zhang (2020), and later uniquely determined by Zhai, Liu and Xue (2022). Recently, the spectral extremal problems were studied for graphs that do not contain a certain graph H as a minor. For instance, Tait (2019), Zhai and Lin (2022), Chen, Liu and Zhang (2024) solved the case of cliques, bicliques, cliques with some paths removed, respectively. Motivated by these results, we consider the spectral extremal problem for friendship graphs. Let $K_s \vee I_{n-s}$ be the complete split graph, which is the join of a clique of size s with an independent set of size $n-s$. For sufficiently large n, we prove that $K_s \vee I_{n-s}$ is the unique graph that attains the maximal spectral radius over all n-vertex F_s-minor-free graphs.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We write $G - v$ for the graph obtained from G by deleting vertex $v \in V(G)$ and its incident edges, and $G - uv$ for the graph by deleting the edge $uv \in E(G)$. This notation is naturally extended if more than one vertex or edge is deleted. Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E(G)$. As usual, a *complete graph* on n vertices is denoted by K_n, and an *independent set* on n vertices is denoted by I_n. We write P_t for a *path* on t vertices. Commonly, we refer to a path by the nature sequence of its vertices, say $P_t = x_1x_2 \ldots x_t$, and call P_t a path starting from x_1 to x_t. In

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addition, we write C_t for a *cycle* on t vertices, and write $K_{s,t}$ for the *complete bipartite graph* with two parts of sizes s and t. We denote by $G[X,Y]$ the *bipartite subgraph* with bipartition (X,Y) whose edges are that of G between X and Y.

The *adjacency matrix* $A(G)=(a_{ij})$ of G is an $n\times n$ matrix with $a_{ij}=1$ if v_i is adjacent to v_j, and 0 otherwise. The *spectral radius* of G is the largest eigenvalue of $A(G)$, which is denoted by $\rho(G)$. For each vertex v in G, let $N_G(v):=\{u\in V(G): uv\in E(G)\}$ and $N_G[v]:=N_G(v)\cup\{v\}$. The *degree* of v is denoted by $d_G(v)=|N_G(v)|$. Similarly, for each subgraph H of G, let $N_G(H)$ be the set of vertices in $V(G)\setminus V(H)$ that are adjacent to a vertex of H. A *clique* of G is a subset S of $V(G)$ such that the induced subgraph $G[S]$ is a complete subgraph. Let G_1 and G_2 be two vertex-disjoint graphs, and denote by $G_1\cup G_2$ the *union* of G_1 and G_2. For simplicity, we write sG for the vertex-disjoint union of s copies of G. The *join* $G_1\vee G_2$ is obtained from $G_1\cup G_2$ by joining each vertex of G_1 to each vertex of G_2. For graph notation and terminology undefined here, readers are referred to [4].

1.1 Spectral extremal graphs for friendship graphs

A graph G is called H-free if H is not a subgraph of G. In 2010, Nikiforov [35] proposed a spectral extremal problem, which is now known as the Brualdi–Solheid–Turán type problem. More precisely, what is the maximum spectral radius among all n-vertex H-free graphs? In the past few decades, this problem has been investigated by many researchers for various graphs H, such as, the complete graphs [33, 45, 26], the complete bipartite graphs [3, 34], the books and theta graphs [49], the friendship graphs [7, 48, 52, 27], the intersecting odd cycles [25, 13], the intersecting cliques [15, 32], the paths and linear forests [35, 12], the odd wheels [8], the quadrilaterals [33, 46], the hexagons [47] and even cycle [9], the short odd cycles [21, 23, 28, 30], the square of paths [53] and cycles [18], the fan graphs [43]. We refer the readers to [37, 24] for related surveys.

Let F_s be the graph obtained from s triangles by intersecting in exactly one common vertex. In other words, we have $F_s=K_1\vee sK_2$. The graph F_s is also known as the friendship graph because it is the only extremal graph in the famous Friendship Theorem [1, Chapter 43], which asserts that if G is a graph on n vertices such that any two distinct vertices have exactly one common neighbor, then n is odd and G consists of $\frac{n-1}{2}$ triangles intersecting in a common vertex. The extremal problem involving F_s was widely studied in the literature. Tracing back to 1995, Erdős, Füredi, Gould and Gunderson [17] proved the following result.

Theorem 1 (Erdős–Füredi–Gould–Gunderson [17], 1995)**.** *Let $s\geqslant 1$ and $n\geqslant 50s^2$ be positive integers. If G is an F_s-free graph on n vertices, then*

$$e(G)\leqslant \left\lfloor \frac{n^2}{4}\right\rfloor + \begin{cases} s^2-s & \text{if } s \text{ is odd;}\\ s^2-\frac{3}{2}s & \text{if } s \text{ is even.}\end{cases}$$

We write $\mathrm{EX}(n,F_s)$ for the set of n-vertex F_s-free graphs which attain the equality of Theorem 1. Furthermore, the extremal graphs in $\mathrm{EX}(n,F_s)$ were also characterized by Erdős, Füredi, Gould and Gunderson [17]. More precisely, for odd s, the graphs

$G \in \mathrm{EX}(n, F_s)$ are obtained from $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by embedding two vertex-disjoint copies of K_s in any one side. For even s, the extremal graphs are constructed from $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by embedding a graph with $2s-1$ vertices, $s^2 - \frac{3}{2}s$ edges and maximum degree $s-1$ in any one side. Here, we remark that for even s, the embedded graph is a nearly $(s-1)$-regular graph on $2s-1$ vertices with degree sequence $(s-1, \ldots, s-1, s-2)$. It is known [4] that such a graph does exist and it is not unique for every even $s \geqslant 2$.

In 2020, the spectral version of Theorem 1 was firstly studied by Cioabă, Feng, Tait and Zhang [7]. They characterized the spectral extremal F_s-free graphs. More precisely, they proved that for fixed $s \geqslant 2$ and sufficiently large n, if G is an F_s-free graph of order n with maximal spectral radius, then G attains the maximum number of edges over all n-vertex F_s-free graphs.

Theorem 2 (Cioabă–Feng–Tait–Zhang [7], 2020)**.** *Let $s \geqslant 2$ and G be an F_s-free graph on n vertices. For sufficiently large n, if G has the maximal spectral radius, then*

$$G \in \mathrm{EX}(n, F_s).$$

In 2022, Zhai, Liu and Xue [48] provided a further characterization of G and determined the *unique* spectral extremal graph of F_s for sufficiently large n. In other words, they determined the unique embedded subgraph in the extremal graph of $\mathrm{EX}(n, F_s)$. Let H^* be the graph of order $2s-1$ with vertex set $V(H^*) = \{w_0\} \cup A \cup B$ such that $N(w_0) = A$ and $|B| = |A| + 2 = s$. Then we partition A into $A_1 \cup A_2$, and B into $\{u_0\} \cup B_1 \cup B_2$ such that $|A_1| = |A_2| = |B_2| = \frac{s-2}{2}$ and $|B_1| = \frac{s}{2}$. Finally, we join $s-1$ edges from u_0 to $A_1 \cup B_1$, $\frac{s-2}{2}$ independent edges between B_2 and A_2, and some additional edges such that both A and $B_1 \cup B_2$ are cliques.

Theorem 3 (Zhai–Liu–Xue [48], 2022)**.** *Let $s \geqslant 2$ and G be an F_s-free graph with the maximal spectral radius. Then for sufficiently large n, the graph G is obtained from $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by embedding a graph H in the part of size $\lfloor \frac{n}{2} \rfloor$, where $H = K_s \cup K_s$ if s is odd; and $H = H^*$ if s is even.*

We refer the readers to [48] for more details and [44] for a generalization of Theorem 2. Furthermore, Li, Lu and Peng [27] got rid of the condition that n is sufficiently large if $s = 2$. They proved that for every $n \geqslant 7$, the unique n-vertex F_2-free spectral extremal graph is the balanced complete bipartite graph by adding an edge in the vertex part with smaller size. Moreover, it was also proved in [27] that the unique m-edge F_2-free spectral extremal graph is the join of K_2 with an independent set of $\frac{m-1}{2}$ vertices if $m \geqslant 8$. The conditions $n \geqslant 7$ and $m \geqslant 8$ are the best possible. For a general $s \geqslant 3$, Li, Feng and Peng [29] recently provided a new short proof, which avoids the use of triangle removal lemma and shows that the result for F_s holds for every $n \geqslant Cs^4$ with a constant $C > 0$.

1.2 Spectral extremal graphs for minors

A graph H is a *minor* of G if H can be obtained from G by means of a sequence of vertex deletions, edge deletions and edge contractions. A graph G is *H-minor-free* if it does not contain H as a minor. A natural question at the intersection of graph minor theory and Brualdi-Solheid-Turán type problem asks, for a given graph H, what is the maximal spectral radius over all n-vertex graphs which do not contain H as a minor? Indeed, such a problem was recently becoming popular and some elegant results have been published in the following two aspects.

There are two famous conjectures in the study of spectral extremal problems on planar and outerplanar graphs. It is known that a graph is planar if and only if it is $\{K_5, K_{3,3}\}$-minor-free. Moreover, a graph is outerplanar if and only if it is $\{K_4, K_{2,3}\}$-minor-free; see, e.g., [4]. In 1990, Cvetković and Rowlinson [10] conjectured that the join graph $K_1 \vee P_{n-1}$ is the unique graph attaining the maximal spectral radius over all outerplanar graphs of order n. In 1991, Boots and Royle [5], and independently Cao and Vince [11], proposed a spectral problem for planar graphs, which conjectured that $K_2 \vee P_{n-2}$ is the unique graph attaining the maximal spectral radius over all planar graphs of order n. Many scholars contributed to these two conjectures. In particular, Tait and Tobin [39] confirmed these conjectures for sufficiently large n. In 2021, Lin and Ning [22] confirmed Cvetković-Rowlinson conjecture for all $n \geqslant 2$ except for $n = 6$.

Apart from the planar and outerplanar graphs, it is natural to consider the maximal spectral radius for H-minor-free graphs with a specific graph H. In particular, setting H as the complete graph K_r and the complete bipartite graph $K_{s,t}$. In 2004, Hong [20] determined the extremal graph with maximal spectral radius for K_5-minor-free graphs. In 2017, Nikiforov [36] obtained a sharp upper bound on the spectral radius of $K_{2,t}$-minor-free graphs. In 2019, Tait [40] characterized the spectral extremal graphs with no K_r as a minor. In 2022, Zhai and Lin [50] completely determined the spectral extremal graphs for $K_{s,t}$-minor. Recently, Chen, Liu and Zhang [14] presented the spectral extremal graphs for F-minor, where F is obtained from K_r by deleting some disjoint paths.

Comparing with the rich development of the traditional spectral extremal problem (see Subsection 1.1), there are few results on the spectral radius for minor-free graphs, although the spectral problem of minors has risen in popularity in the past few years. Recall that F_s is the friendship graph which consists of s triangles intersecting in a common vertex. As stated in previous subsection, the traditional extremal problem for the friendship graph F_s has recently received extensive attention and investigation; see, e.g., Theorems 1, 2 and 3. Inspired by the results on K_r-minor-free and $K_{s,t}$-minor-free graphs, we shall present one more result on spectral radius for H-minor-free graphs by taking $H = F_s$.

Recall that $K_s \vee I_{n-s}$ is the join graph consisting of a clique on s vertices and an independent set on $n - s$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. For sufficiently large n, we determine the largest spectral radius of a graph over all F_s-minor-free graphs of order n, and we show that $K_s \vee I_{n-s}$ is the unique spectral extremal graph.

Theorem 4. *Let $s \geqslant 1$ be an integer and G be an F_s-minor-free graph of order n. Then for sufficiently large n, we have*

$$\rho(G) \leqslant \rho(K_s \vee I_{n-s}),$$

with equality if and only if $G = K_s \vee I_{n-s}$.

Corollary 5. *Let G be a graph of order n that dose not contain s intersecting cycles as a minor. Then for sufficiently large n, we have*

$$\rho(G) \leqslant \rho(K_s \vee I_{n-s}),$$

with equality if and only if $G = K_s \vee I_{n-s}$.

Proof of Corollary 5. The proof is straightforward. Note that $K_s \vee I_{n-s}$ dose not contain s intersecting cycles as a minor. Since G dose not contain s intersecting cycles as a minor, G is also F_s-minor-free. By Theorem 4, for sufficiently large n, we have $\rho(G) \leqslant \rho(K_s \vee I_{n-s})$ with equality if and only if $G = K_s \vee I_{n-s}$. □

Organization. The rest of this paper is organized as follows. In Section 2, some preliminaries are presented for our purpose. In Section 3, we will give some structural properties of F_s-minor-free graphs. In Section 4, we shall give the details of the proof of Theorem 4. The techniques used in our proof are mainly inspired by Tait [40]. In Section 5, we will propose some spectral problems for interested readers.

2 Preliminary

Mader [31] proved an elegant result on the number of edges in H-minor-free graphs.

Lemma 6 (Mader [31], 1967)**.** *Let G be an n-vertex graph. For every graph H, if G is H-minor-free, then there exists a constant $C > 0$ such that*

$$e(G) \leqslant Cn.$$

We remark that a clear bound on C could be found in [38]. The following lemma has been proved many times in the literature; see, e.g., [41, 42].

Lemma 7. *Let G be a bipartite graph on n vertices with no $K_{s,t}$-minor and vertex partition A and B. Let $|A| = a$ and $|B| = n - a$. Then there is a constant $C > 0$ depending only on s and t such that*

$$e(G) \leqslant Ca + (s-1)n.$$

The following lemma was an implicative result, which can be seen from the proof of [39, Lemma 10] and [40, Claim 3.4] as well.

Lemma 8. *Let H be a graph with no pendant edge and G be a graph with maximum spectral radius among all n-vertex connected H-minor-free graphs. If $\mathbf{x} = (\mathrm{x}_u)_{u \in V(G)}$ is a positive eigenvector with the maximum entry 1 which corresponds to $\rho(G)$, then $\mathrm{x}_u \geqslant \frac{1}{\rho(G)}$ for all $u \in V(G)$.*

3 Structure of graphs without F_s-minor

In this subsection, we will present some lemmas for F_s-minor-free graphs.

Lemma 9. *Let G be an n-vertex F_s-minor-free bipartite graph with vertex partition A and B. If $|A| = a$ and $|B| = n - a$, then there exists a constant $C > 0$ depending only on s such that*

$$e(G) \leqslant Ca + sn.$$

Proof. Suppose that G is F_s-minor-free. Note that the complete bipartite graph $K_{s+1,2s}$ contains s copies of C_4 intersecting in a common vertex. By contracting an edge in each copy of C_4, we can obtain a copy of F_s. So any $K_{s+1,2s}$-minor contains an F_s-minor. This yields that G is $K_{s+1,2s}$-minor-free. Hence, the assertion follows from Lemma 7. $\square$

Lemma 10. *Let G be an n-vertex graph with the maximum spectral radius $\rho(G)$ among all F_s-minor-free graphs. Then $\rho(G) \geqslant \sqrt{s(n-s)}$.*

Proof. Observe that $K_{s,n-s}$ is F_s-minor-free and $\rho(K_{s,n-s}) = \sqrt{s(n-s)}$, as desired. $\square$

Before showing our results, we fix some notions firstly. A subset S of $V(G)$ is called a *fragment* if the induced subgraph $G[S]$ is connected. Distinct fragments S' and S'' are said to be *adjacent* if there exist two vertices $u' \in S'$ and $u'' \in S''$ such that $u'u'' \in E(G)$.

Lemma 11. *Let G be an n-vertex F_s-minor-free graph. If G contains a complete bipartite subgraph $K_{s,(1-\delta)n} = [A, B]$ with $|A| = s$ and $|B| = (1-\delta)n \geqslant 2s$, then*

(i) $G[B]$ is P_2-free, and $|N_G(v) \cap B| \leqslant 1$ for any $v \in V(G)\backslash(A \cup B)$;

(ii) There are at least $(1-2\delta)n$ vertices in B which have no neighbors in $V(G)\backslash(A\cup B)$.

Proof. We first prove that $G[B]$ is P_2-free. In fact, if there exists a path P_2 in $G[B]$, then $G[A \cup B]$ contains a subgraph consisting of one triangle and $s-1$ copies of C_4 by sharing a common vertex. By contracting an edge in each copy of C_4, we observe that F_s is a minor of G, which is a contradiction. Hence, $G[B]$ is P_2-free. Furthermore, we have the following claim.

Claim. If H is a component of $G - (A \cup B)$, then $|N_G(H) \cap B| \leqslant 1$.

Proof of Claim. Suppose that there are two vertices $u, v \in N_G(H) \cap B$. Let G' be obtained from G by contracting $G[\{v\} \cup V(H)]$ to a single vertex v'. Then $K_{s,(1-\delta)n} = [A, (B\backslash\{v\}) \cup \{v'\}]$ and $P_2 = uv'$ are subgraphs of G'. Note that the intersecting subgraph consisting of one triangle and $s-1$ copies of C_4 is contained in the union of $K_{s,(1-\delta)n} = [A, (B\backslash\{v\}) \cup \{v'\}]$ and $P_2 = uv'$. Hence, F_s is a minor of G, which is a contradiction. So the claim holds. $\square$

By Claim, we know that Part (i) holds immediately.

Now let $R = V(G)\backslash(A \cup B)$ and $D = \{v \in B : N_G(v) \cap R = \emptyset\}$. By the definition of R,

$$|R| = n - |A| - |B| = n - s - (1-\delta)n < \delta n,$$

which implies that R has at most δn components. By Claim, $B\backslash D$ has at most δn vertices. Hence,

$$|D| = |B| - |B\backslash D| \geqslant (1-\delta)n - \delta n = (1-2\delta)n.$$

This completes the proof of Part (ii). □

Lemma 12. *Let G be an n-vertex F_s-minor-free graph. Suppose G contains a complete bipartite subgraph $K_{s,(1-\delta)n} = [A, B]$ with $|A| = s$, $|B| = (1-\delta)n$ and $(1-2\delta)n \geqslant 2s+1$. Let G^* be obtained from G by adding edges to A to make it a clique. Then G^* is also F_s-minor-free.*

Proof. Denote by $R = V(G)\backslash(A \cup B)$ and $D = \{v \in B : N_G(v) \cap R = \emptyset\}$. By Lemma 11, $|D| \geqslant (1-2\delta)n$. Suppose that G^* contains an F_s-minor. Then there exist $2s+1$ disjoint fragments $S_0, S_1, \ldots, S_{2s} \subseteq V(G^*) = V(G)$ with the following properties:

(a) There is at least one edge between S_0 and S_i for all $i = 1, \ldots, 2s$.

(b) There is at least one edge between S_{2i-1} and S_{2i} for $i = 1, \ldots, s$.

(c) There is an integer j such that $S_j \cap D \neq \emptyset$, $j \in \{0, 1, \ldots, 2s\}$.

In fact, if $S_0, S_1, \ldots, S_{2s} \subseteq (B \setminus D) \cup R$, then $S_0, S_1, \ldots, S_{2s}$ in G form an F_s-minor, which is a contradiction. Hence, there exists a set S_j such that $S_j \cap (A \cup D) \neq \emptyset$ for $j = 0, 1, \ldots, 2s$. If $S_j \cap D \neq \emptyset$, then we are done. Otherwise, we have $S_j \cap A \neq \emptyset$. Furthermore, we can suppose that $S_i \cap D = \emptyset$ for all $0 \leqslant i \leqslant 2s$. Then choose one vertex $u \in D$ and let $S'_j = S_j \cup \{u\}$. Then $S_0, \ldots, S'_j, \ldots, S_{2s}$ satisfying $(a), (b)$ and (c).

Let

$$f(S_0, S_1, \ldots, S_{2s}) = |\{S_i : S_i \cap D \neq \emptyset \text{ for } i = 0, 1, \ldots, 2s\}|.$$

Hence, we can choose $2s+1$ disjoint fragments $S_0, S_1, \ldots, S_{2s}$ satisfying $(a), (b)$ and (c) such that $f(S_0, S_1, \ldots, S_{2s})$ is as large as possible.

For $i = 0, 1, \ldots, 2s$, if $|S_i \cap D| \geqslant 2$, choose a vertex $u_i \in S_i \cap D$ and let $U_i = (S_i\backslash D) \cup \{u_i\}$. If $|S_i \cap D| \leqslant 1$, let $U_i = S_i$.

Claim 1. For $i = 0, 1, \ldots, 2s$, the induced subgraph $G^*[U_i]$ is connected.

Proof. For any two vertices u, v in U_i, there exists a path P from u to v in $G^*[S_i]$ since $G^*[S_i]$ is connected. If P contains a vertex $w \in (S_i \cap D) \setminus \{u_i\}$, then there exist two vertices w_1, w_2 in P such that $\{ww_1, ww_2\} \subseteq E(P)$. Since $G[B]$ is P_2-free, $w_1, w_2 \in A$. So w_1 is adjacent to w_2. So there is a path in $G^*[S_i]$ from u to v containing no w. Hence, $G^*[U_i]$ is connected. □

Claim 2. If S_i and S_j are adjacent in G^* such that $|U_i \cap D| = 1$ and $|U_j \cap D| \leqslant 1$, then U_i and U_j are adjacent in G^*.

Proof. Suppose on the contrary that there are no edges between U_i and U_j in G^*. Then $S_j \cap A = \emptyset$ and all the edges between S_i and S_j in G^* have one endpoint in $(S_i \cap D) \backslash \{u_i\}$ or $(S_j \cap D) \backslash \{u_j\}$. Hence, $S_j \cap B \neq \emptyset$. We claim that $S_i \cap A = \emptyset$. Otherwise, since there are no edges between U_i and U_j in G^*, we have $S_j \cap B = S_j \cap D$. Then $|U_j \cap D| = 1$. Thus, there is at least one edge between $U_i \cap A = S_i \cap A$ and $U_j \cap D$ in G^*, a contradiction. Suppose uv is an edge with $u \in (S_i \cap D) \backslash \{u_i\}$ and $v \in S_j$, then we have $v \in S_j \cap A$, contradicting to $S_j \cap A = \emptyset$. On the other hand, suppose uv is an edge with $u \in (S_j \cap D) \backslash \{u_j\}$ and $v \in S_i$, then we have $v \in S_i \cap A$, contradicting to $S_i \cap A = \emptyset$. Hence, there is at least one edge between U_i and U_j in G^*. □

Claim 3. If $U_i \cap A \neq \emptyset$, then $U_i \cap D \neq \emptyset$ for $0 \leqslant i \leqslant 2s$.

Proof. By Claims 1 and 2, disjoint fragments $U_0, U_1, \dots, U_{2s}$ satisfy $(a), (b)$ and (c). Now suppose that there exists $0 \leqslant j \leqslant 2s$ such that $U_j \cap A \neq \emptyset$ and $U_j \cap D = \emptyset$. Then choose a vertex $w \in D \backslash \cup_{i=0}^{2s} U_i$ and let $V_j = U_j \cup \{w\}$ and $V_i = U_i$ for $0 \leqslant i \neq j \leqslant 2s$. It is easy to see that the $2s+1$ disjoint fragments $V_0, V_1, \dots, V_{2s}$ satisfy $(a), (b)$ and (c). Moreover,

$$f(V_0, V_1, \dots, V_{2s}) = f(U_0, U_1, \dots, U_{2s}) + 1 = f(S_0, S_1, \dots, S_{2s}) + 1,$$

which contradicts to the choice of $S_0, \dots, S_{2s}$. □

Claim 4. $G[U_i]$ is connected for $0 \leqslant i \leqslant 2s$.

Proof. Since $G^*[U_i]$ is connected, there exists a path P from u to v in $G^*[U_i]$ for any two vertices $u, v \in U_i$. If P contains an edge a_1a_2 with $a_1, a_2 \in A$, then by Claim 3, there exists a vertex $w \in U_i \cap D$. If $w \notin V(P)$, then the edge a_1a_2 of P may be replaced by edges a_1w and a_2w. If $w \in V(P)$, then the subpath of P containing a_1a_2 and w may be replaced by an edge a_1w or wa_2. The above transformations yield a path P' from u to v which contains no edges in $G^*[A]$. So there exists a path from u to v in $G[U_i]$ and thus $G[U_i]$ is connected. □

Claim 5. If S_i and S_j are adjacent in G^*, then U_i and U_j are adjacent in G.

Proof. Suppose that there are no edges between U_i and U_j in G. There must exist two vertices u, v such that $u \in U_i \cap A$ and $v \in U_j \cap A$. By Claim 3, there exists a vertex $w \in U_j \cap D$. Hence, there is one edge uw between U_i and U_j in G. This is a contradiction. □

By Claims 4 and 5, we can see that $U_0, U_1, \dots, U_{2s}$ form an F_s-minor of G, which is a contradiction. This completes the proof. □

4 Proof of Theorem 4

Let G be an n-vertex F_s-minor-free graph with the maximal spectral radius. Without loss of generality, we may assume that G is connected. Indeed, if G is not connected, then we choose G_1 as a component such that $\rho(G_1) = \rho(G)$. (Note that adding an edge between different components of G may lead to an F_s-minor. For example, taking $G = F_{s-1} \cup C_4$.) Since G is F_s-minor-free, we know that the component G_1 is F_s-minor-free. Denote $|G_1| = n_1$. If n_1 is finite, then $\rho(G_1) \leqslant \rho(K_{n_1}) = n_1 - 1$ and so $\rho(G_1) < \sqrt{s(n-s)} < \rho(K_s \vee I_{n-s})$ for sufficiently large n; if n_1 is infinite, that is, $n_1 = \omega(n)$, then the case for connected graphs yields $\rho(G_1) \leqslant \rho(K_s \vee I_{n_1-s}) \leqslant \rho(K_s \vee I_{n-s})$ for sufficiently large n.

Since G is connected, by a result of Perron and Frobenius, we can choose $\mathbf{x} = (\mathrm{x}_u)_{u\in V(G)}$ as a positive eigenvector of G corresponding to the spectral radius $\rho(G)$. We may assume by scaling that the maximum entry of $\mathbf{x}$ is $\mathrm{x}_w = 1$ for some $w \in V(G)$. We will use throughout the section that $e(G) = O(n)$ by Lemma 6. For $0 < \epsilon < 1$, we denote

$$L = \{v \in V(G) : \mathrm{x}_v > \epsilon\}$$

and

$$S = \{v \in V(G) : \mathrm{x}_v \leqslant \epsilon\},$$

where ϵ is a small constant which will be chosen later. Clearly, we have $V(G) = L \cup S$.

The outline of our proof is as follows:

- ♡ Firstly, we show $\rho(G) = \Theta(\sqrt{n})$. Then we will show that $|L| \leqslant O(\sqrt{n})$ by Lemma 6. Thus, we get $|S| = n - |L| = (1 - o(1))n$. Consequently, we obtain $e(L) = O(|L|) \leqslant O(\sqrt{n})$ and $e(S) \leqslant O(n)$. Moreover, we can show that $e(L, S) \leqslant (s + o(1))n$.
- ♡ Secondly, we shall prove that if a vertex has eigenvector entry close to 1, then it has degree close to n; see Claim 2. Furthermore, we will show by induction that there are s vertices in L with eigenvector entry close to 1, and hence its degree close to n; see Claim 3.
- ♡ Moreover, we shall show that these s vertices induce a clique K_s; see Claim 4.
- ♡ Finally, we show that each of the s vertices in the clique actually has degree $n - 1$.

Proof of Theorem 4. By Lemma 6, there is a constant $C_1 := 2C > 0$ such that

$$2e(S) \leqslant 2e(G) \leqslant C_1 n. \tag{1}$$

In addition, by Lemma 10, we obtain

$$\rho(G) \geqslant \sqrt{s(n-s)}. \tag{2}$$

Claim 1. $e(L, S) \leqslant (s + \epsilon)n$ and $2e(L) \leqslant \epsilon n$.

Proof. It is easy to see that

$$\rho(G)\epsilon|L| \;<\; \sum_{v\in L}\rho(G)\mathrm{x}_v = \sum_{v\in L}\sum_{z\in N_G(v)}\mathrm{x}_z \leqslant \sum_{v\in L} d_G(v) \leqslant 2e(G).$$

Then by (1) and (2), it implies that

$$|L| \leqslant \frac{2e(G)}{\epsilon\rho(G)} \leqslant \frac{C_1 n}{\epsilon\sqrt{s(n-s)}} \leqslant \frac{2C_1\sqrt{n}}{\epsilon\sqrt{s}}, \tag{3}$$

where the last inequality holds for sufficiently large n.

By Lemma 9, there is a constant $C_2 > 0$ only depending on s such that

$$e(L,S) \leqslant C_2|L| + sn \leqslant \frac{2C_1C_2\sqrt{n}}{\epsilon\sqrt{s}} + sn \leqslant (s+\epsilon)n \tag{4}$$

as long as n is large enough so that $n \geqslant 4(C_1C_2)^2/(s\epsilon^4)$.

In addition, by (3) and Lemma 6, we have

$$2e(L) \leqslant C_1|L| \leqslant \frac{2C_1^2\sqrt{n}}{\epsilon\sqrt{s}} \leqslant \epsilon n \tag{5}$$

as long as n is sufficiently large. So Claim 1 holds. □

Claim 2. If $u \in L$ is a vertex with $\mathrm{x}_u = 1-\alpha$ for some constant $\alpha \geqslant 0$, then there exists a constant $C_3 > 1$ independent of α and ϵ such that

$$d_G(u) \geqslant [1 - C_3(\alpha+\epsilon)]n.$$

Proof. Clearly, we have

$$\begin{aligned}\rho(G)\sum_{v\in V(G)}\mathrm{x}_v &= \sum_{v\in V(G)}\sum_{z\in N_G(v)}\mathrm{x}_z = \sum_{v\in V(G)} d_G(v)\mathrm{x}_v \leqslant \sum_{v\in L} d_G(v) + \epsilon\sum_{v\in S} d_G(v)\\ &= 2e(L) + \epsilon\cdot 2e(S) + (1+\epsilon)e(L,S),\end{aligned}$$

which implies

$$\sum_{v\in V(G)}\mathrm{x}_v \leqslant \frac{2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S)}{\rho(G)}. \tag{6}$$

Let $N_G^c(u) := V(G)\backslash N_G(u)$. By Lemma 8 and (6),

$$\begin{aligned}|N_G^c(u)|\cdot\frac{1}{\rho(G)} &\leqslant \sum_{v\in N_G^c(u)}\mathrm{x}_v = \sum_{v\in V(G)}\mathrm{x}_v - \sum_{v\in N_G(u)}\mathrm{x}_v = \sum_{v\in V(G)}\mathrm{x}_v - \rho(G)\mathrm{x}_u\\ &\leqslant \frac{2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S)}{\rho(G)} - \rho(G)\mathrm{x}_u.\end{aligned}$$

Furthermore, using (1), (2), (4) and (5), we have

$$\begin{aligned}|N_G^c(u)| &\leqslant 2e(L)+2\epsilon e(S)+(1+\epsilon)e(L,S)-\rho(G)^2\mathrm{x}_u\\ &\leqslant \epsilon n+\epsilon C_1 n+(1+\epsilon)(s+\epsilon)n-s(n-s)(1-\alpha)\\ &= \big[\epsilon(1+C_1)+(1+\epsilon)(s+\epsilon)-s(1-\alpha)\big]n+s^2(1-\alpha)\\ &\leqslant (C_1+s+4)(\alpha+\epsilon)n,\end{aligned}$$

where the last inequality holds as long as $n \geqslant s^2/\epsilon$. Hence,

$$d_G(u)=n-|N_G^c(u)|\geqslant n-(C_1+s+4)(\alpha+\epsilon)n=[1-(C_1+s+4)(\alpha+\epsilon)]n.$$

Denote $C_3 := C_1+s+4>1$, which is independent of α and ϵ. So Claim 2 holds. □

Claim 3. There exist s distinct vertices $v_1,\dots,v_s\in L$ satisfying $\mathrm{x}_{v_i}\geqslant 1-C_4\epsilon$ and $d_G(v_i)\geqslant(1-C_4\epsilon)n$ for every $i=1,\dots,s$, where $C_4>0$ is a constant independent of ϵ and n.

Proof. We shall prove this claim by induction. First of all, setting $v_1=w$, which is a vertex with the largest entry of the eigenvector $\mathbf{x}$, then $\mathrm{x}_{v_1}=1$. Furthermore, by Claim 2, there exists a constant $c_1=C_3>1$ independent of ϵ and n such that $d_G(v_1)\geqslant(1-c_1\epsilon)n$.

Now assume that we have chosen $v_1,\dots,v_k\in L$ satisfying $\mathrm{x}_{v_i}\geqslant 1-c_k\epsilon$ and $d_G(v_i)\geqslant(1-c_k\epsilon)n$ for $1\leqslant i\leqslant k\leqslant s-1$, where c_k is a constant independent of ϵ and n. Our goal is to show that there exist an absolute constant c_{k+1} and a vertex $v_{k+1}\in L\backslash\{v_1,\dots,v_k\}$ such that the degree $d_G(v_{k+1})\geqslant(1-c_{k+1}\epsilon)n$ and the eigen-entry $\mathrm{x}_{v_{k+1}}\geqslant 1-c_{k+1}\epsilon$.

Let $U=\{v_1,\dots,v_k\}$. By (1), (2) and Claim 1, we have

$$\begin{aligned}s(n-s) &\leqslant \rho(G)^2\mathrm{x}_w=\sum_{v\in N(w)}\sum_{z\in N(v)}\mathrm{x}_z\leqslant\sum_{vz\in E(G)}(\mathrm{x}_v+\mathrm{x}_z)\\ &= \sum_{vz\in E(S)}(\mathrm{x}_v+\mathrm{x}_z)+\sum_{vz\in E(L,S)}(\mathrm{x}_v+\mathrm{x}_z)+\sum_{vz\in E(L)}(\mathrm{x}_v+\mathrm{x}_z)\\ &\leqslant 2\epsilon e(S)+2e(L)+\epsilon e(L,S)+\sum_{\substack{uv\in E(U,S)\\ u\in U}}\mathrm{x}_u+\sum_{\substack{uv\in E(L\backslash U,S)\\ u\in L\backslash U}}\mathrm{x}_u.\\ &\leqslant \epsilon C_1 n+\epsilon n+\epsilon(s+\epsilon)n+kn+\sum_{\substack{uv\in E(L\backslash U,S)\\ u\in L\backslash U}}\mathrm{x}_u,\end{aligned}$$

which implies that

$$\sum_{\substack{uv\in E(L\backslash U,S)\\ u\in L\backslash U}}\mathrm{x}_u\geqslant[s-k-\epsilon(C_1+s+2+\epsilon)]n \tag{7}$$

as long as $n\geqslant s^2/\epsilon$. On the other hand, recall that $U\subseteq L$ and $V(G)=L\cup S$, then

$$e(U,S)+e(U,L\backslash U)+2e(U)=\sum_{v\in U}d_G(v)\geqslant k(1-c_k\epsilon)n.$$

We have

$$\begin{aligned}e(U,S) &\geqslant k(1-c_k\epsilon)n - e(U,L\backslash U) - 2e(U)\\ &\geqslant k(1-c_k\epsilon)n - k(|L|-k) - k(k-1)\\ &\geqslant k(1-c_k\epsilon)n - k(\epsilon n-k) - k(k-1)\\ &= k(1-c_k\epsilon-\epsilon)n + k,\end{aligned}$$

where the last inequality holds by (3) for sufficiently large n.

By Claim 1, we have

$$\begin{aligned}e(L\backslash U,S) = e(L,S) - e(U,S) \leqslant (s+\epsilon)n - k(1-c_k\epsilon-\epsilon)n - k\\ < [s+\epsilon-k(1-c_k\epsilon-\epsilon)]n.\end{aligned} \tag{8}$$

Let

$$h(x) = \frac{s-x-\epsilon(C_1+s+2+\epsilon)}{s+\epsilon-x(1-c_k\epsilon-\epsilon)}.$$

It is easy to see that $h(x)$ is decreasing with respect to $1 \leqslant x \leqslant s-1$. Then (7) and (8) imply

$$\begin{aligned}\frac{\sum\limits_{\substack{uv\in E(L\backslash U,S)\\ u\in L\backslash U}} \mathrm{x}_u}{e(L\backslash U,S)} &\geqslant h(k) \geqslant h(s-1) = \frac{1-\epsilon(C_1+s+2+\epsilon)}{1+\epsilon+(s-1)(c_k\epsilon+\epsilon)}\\ &\geqslant 1-(C_1+2s+2)(c_k\epsilon+\epsilon).\end{aligned}$$

Hence, by averaging, there exists a vertex $v_{k+1} \in L\backslash U$ such that

$$\mathrm{x}_{v_{k+1}} \geqslant 1-(C_1+2s+2)(c_k\epsilon+\epsilon).$$

Therefore, setting $\alpha \leqslant (C_1+2s+2)(c_k\epsilon+\epsilon)$ in Claim 2, we get

$$\begin{aligned}d_G(v_{k+1}) &\geqslant [1-C_3((C_1+2s+2)(c_k\epsilon+\epsilon)+\epsilon)]n\\ &\geqslant [1-C_3(C_1+2s+3)(c_k\epsilon+\epsilon)]n\\ &= [1-C_3(C_1+2s+3)(c_k+1)\epsilon]n.\end{aligned}$$

Let $c_{k+1} := C_3(C_1+2s+3)(c_k+1)$. Then c_{k+1} is independent of ϵ and n. Clearly, we have $c_k < c_{k+1}$ since $C_3 > 1$ obtained from Claim 2. Consequently, we get $\mathrm{x}_{v_i} \geqslant 1-c_{k+1}\epsilon$ and $d_G(v_i) \geqslant (1-c_{k+1}\epsilon)n$ for every $i = 1,\dots,k+1$. Hence Claim 3 holds. □

Let $v_1, v_2, \dots, v_s \in L$ be defined in Claim 3. Denote by

$$A := \{v_1, v_2, \dots, v_s\}.$$

The set of common neighbors of vertices of A is denoted by

$$B := \bigcap_{i=1}^{s} N_G(v_i).$$

Finally, the set of remaining vertices of G is denoted by

$$R := V(G)\backslash(A \cup B).$$

Then by $d_G(v_i) \geqslant (1 - C_4\epsilon)n$ for every $i = 1, \dots, s$, we have

$$\begin{aligned}|B| &\geqslant \sum_{i=1}^{s} |N_G(v_i)| - (s-1)\left|\bigcup_{i=1}^{s} N_G(v_i)\right| \\ &\geqslant \sum_{i=1}^{s}(1 - C_4\epsilon)n - (s-1)n \\ &= (1 - C_4 s\epsilon)n\end{aligned}$$

and

$$|R| = n - |A| - |B| \leqslant C_4 s\epsilon n. \tag{9}$$

Claim 4. $A = \{v_1, v_2, \dots, v_s\}$ is a clique in G.

Proof. Clearly, $G[A, B]$ is a complete bipartite graph with $|A| = s$ and $|B| = (1 - \delta)n$, where $\delta \leqslant C_4 s\epsilon$. Moreover, $(1 - 3\delta)n \geqslant 2s + 1$ for sufficiently large n. Since adding edges to a connected graph strictly increases its spectral radius, by Lemma 12 and the maximality of G, we know that A must induce a clique in G. This proves Claim 4. ☐

Claim 5. For every $v \in V(G)\backslash A$, we have $\mathrm{x}_v \leqslant \frac{1}{C_1+3}$.

Proof. On one hand, for any $u \in R$, that is, u is not the common neighbor of vertices of A, we have $|N_G(u) \cap A| \leqslant s - 1$. By Lemma 11 (i), we have $|N_G(u) \cap B| \leqslant 1$. Therefore,

$$|N_G(u) \cap (A \cup B)| = |N_G(u) \cap A| + |N_G(u) \cap B| \leqslant s - 1 + 1 = s. \tag{10}$$

Hence, it follows that

$$\rho(G) \sum_{u \in R} \mathrm{x}_u = \sum_{u \in R} \sum_{w \in N_G(u)} \mathrm{x}_w \leqslant \sum_{u \in R} d_G(u) \leqslant 2e(R) + e(R, A \cup B) \leqslant 2e(R) + s|R|.$$

Note that $G[R]$ is F_s-minor-free. By Lemma 6, we have

$$\sum_{u \in R} \mathrm{x}_u \leqslant \frac{2e(R) + s|R|}{\rho(G)} \leqslant \frac{C_1|R| + s|R|}{\rho(G)} = \frac{(C_1 + s)|R|}{\rho(G)}. \tag{11}$$

On the other hand, for any vertex $u \in B$, by Lemma 11 (i), we have

$$|N_G(u) \cap (A \cup B)| = |N_G(u) \cap A| = s. \tag{12}$$

Let $v \in V(G)\backslash A = R \cup B$ be a fixed vertex. Next, we show that $\mathrm{x}_v \leqslant \frac{1}{C_1+3}$. By (10), (11) and (12), we have $|N_G(v) \cap (A \cup B)| \leqslant s$ and

$$\rho(G)\mathrm{x}_v = \sum_{u \in N_G(v)} \mathrm{x}_u = \sum_{\substack{u \in N_G(v) \\ u \in A \cup B}} \mathrm{x}_u + \sum_{\substack{u \in N_G(v) \\ u \in R}} \mathrm{x}_u \leqslant s + \sum_{u \in R} \mathrm{x}_u \leqslant s + \frac{(C_1 + s)|R|}{\rho(G)},$$

which together with (9) implies

$$\begin{aligned}
\mathrm{x}_v &\leqslant \frac{s}{\rho(G)} + \frac{(C_1+s)|R|}{\rho(G)^2} \leqslant \frac{s}{\sqrt{s(n-s)}} + \frac{(C_1+s)C_4\epsilon n}{n-s} \\
&\leqslant \frac{1}{2(C_1+3)} + \frac{1}{2(C_1+3)} = \frac{1}{C_1+3},
\end{aligned}$$

where the last inequality holds as long as $\epsilon > 0$ is a small constant with $(C_1+s)C_4\epsilon(C_1+3) < \frac{1}{4}$, and n is sufficiently large satisfying $n \geqslant 4s(C_1+3)^2 + s$. So Claim 5 holds. □

Claim 6. The induced subgraph $G[B]$ consists of some isolated vertices.

Proof. By Lemma 11 (i), we know that $G[B]$ does not contain a copy of P_2, and so B is an independent set, that is, $G[B]$ consists of some isolated vertices. □

Claim 7. R is empty, and so $d_G(v) = n-1$ for any $v \in A$.

Proof. Assume that R is not empty. Since $G[R]$ is F_s-minor-free, by Lemma 6, there is a constant C_1 such that $2e(R) \leqslant C_1|R|$. Then the minimum degree of $G[R]$ is at most C_1, and there exists a vertex $v \in R$ such that $d_R(v) = |N_G(v) \cap R| \leqslant C_1$. Now, we can order the vertices of $G[R]$ as follows: $z_1, z_2, \ldots, z_{|R|}$ such that $d_{G[R]}(z_1) \leqslant C_1$ and for every $i = 2, 3, \ldots, |R|$,

$$|N_G(z_i) \cap \{z_{i+1}, \ldots, z_{|R|}\}| \leqslant C_1. \tag{13}$$

In other words, each vertex $z_i \in R$ has at most C_1 neighbors in $\{z_{i+1}, \ldots, z_{|R|}\}$. Recall that $B = \cap_{i=1}^{s} N_G(v_i)$ and $R = V(G) \setminus (A \cup B)$. Any vertex $z_i \in R$ has at least one non-neighbor in A. Moreover, by Lemma 11 (i), each vertex $z_i \in R$ has at most one neighbor in B. We define a new graph G^* as below:

$$\begin{aligned}
G^* := G &- \{z_i z_j \in E(G) : z_i, z_j \in R\} - \{z_i u \in E(G) : z_i \in R, u \in B\} \\
&+ \{z_i v_j \notin E(G) : z_i \in R, v_j \in A\}.
\end{aligned}$$

Clearly, we have $G^* = K_s \vee I_{n-s}$. Since $K_s \vee I_{n-s}$ is F_s-minor-free, we know that G^* is also F_s-minor-free. Using Rayleigh's formula, together with Claims 3 and 5, we obtain

$$\begin{aligned}
\rho(G^*) - \rho(G) &\geqslant \frac{\mathbf{x}^{\mathrm{T}} A(G^*)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{\mathbf{x}^{\mathrm{T}} A\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\geqslant \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \left(\sum_{\substack{z_i v_j \notin E(G) \\ z_i \in R, v_j \in A}} \mathrm{x}_{v_j}\mathrm{x}_{z_i} - \sum_{\substack{z_i z_j \in E(G) \\ z_i, z_j \in R}} \mathrm{x}_{z_i}\mathrm{x}_{z_j} - \sum_{\substack{z_i u \in E(G) \\ z_i \in R, u \in B}} \mathrm{x}_{z_i}\mathrm{x}_u \right) \\
&\geqslant \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \left((1 - C_4\epsilon) \sum_{i=1}^{|R|} \mathrm{x}_{z_i} - \frac{C_1}{C_1+3} \sum_{i=1}^{|R|} \mathrm{x}_{z_i} - \frac{1}{C_1+3} \sum_{i=1}^{|R|} \mathrm{x}_{z_i} \right) \\
&= \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \left(1 - C_4\epsilon - \frac{C_1+1}{C_1+3} \right) \sum_{i=1}^{|R|} \mathrm{x}_{z_i} > 0,
\end{aligned}$$

where the last inequality holds as long as ϵ is a small positive constant so that $\epsilon < \frac{2}{C_4(C_1+3)}$. Consequently, we get a new graph G^*, which is an F_s-minor-free graph and has larger spectral radius than G, a contradiction. Hence, R is empty. This proves Claim 7. $\square$

It follows from Claims 4, 6 and 7 that $G = K_s \vee I_{n-s}$, as needed.

5 Concluding remarks

As we stated in the introduction, the traditional Turán problem and spectral extremal problem for F_s-free graphs were completely studied in Theorems 1, 2 and 3, respectively. In this paper, we have investigated the spectral extremal problem for F_s-minor-free graphs. As we all know, it is challenging and difficult to treat the extremal problem when we forbid bipartite graphs as substructures. We denote by Q_t the graph obtained from t copies of C_4 by intersecting in one vertex. Clearly, Q_t is a bipartite graph. Observe that Q_t contains a vertex class in which each vertex has degree 2. For such a sparse bipartite graph, we know from a result of Füredi [19], or Alon, Krivelevich and Sudakov [2] that $\mathrm{ex}(n, Q_t) = O(n^{3/2})$. It is extremely difficult to determine the exact Turán number of Q_t. In the original version of our manuscript (`arXiv:2301.06008v1`), we have proved the following theorem, which determined the spectral extremal graph among all Q_t-minor-free graphs. To begin with, let M_{n-t} be the graph obtained from an independent set on $n-t$ vertices by embedding a maximal matching. In other words, we have $M_{n-t} = \frac{n-t}{2}K_2$ if $n-t$ is even; and $M_{n-t} = K_1 \cup \frac{n-t-1}{2}K_2$ if $n-t$ is odd.

Theorem 13. *Let $t \geqslant 1$ be an integer and G be a Q_t-minor-free graph of order n. Then for sufficiently large n, we have*

$$\rho(G) \leqslant \rho(K_t \vee M_{n-t}),$$

with equality if and only if $G = K_t \vee M_{n-t}$.

The proof of Theorem 13 can be provided by a similar method as in Theorem 4. Recently, there are several spectral results involving the intersecting odd cycles [25, 13, 43]. Inspired by these results, we proposed the following spectral problem, which is a generalization of Theorem 13 since a Q_t-minor-free graph must be Q_t-free.

Problem 14. Let $t \geqslant 1$ and n be sufficiently large. If G is a Q_t-free graph on n vertices, then

$$\rho(G) \leqslant \rho(K_t \vee M_{n-t}),$$

where the equality holds if and only if $G = K_t \vee M_{n-t}$.

We remark here that the case $t = 1$ reduces to the problem for C_4-free graphs, it was early proved by Nikiforov [33] for odd n, and by Zhai and Wang [46] for even n.

After the submission of this paper, we have learned that Desai [16] confirmed Problem 14. More generally, Desai considered a more general problem. Let $C_{2k_1,2k_2,\dots,2k_t}$ be the graph that consists of t even cycles $C_{2k_1}, C_{2k_2}, \ldots, C_{2k_t}$ sharing a common vertex. For fixed $k_1, \ldots, k_t \geqslant 2$ and sufficiently large n, Desai [16] determined the spectral extremal graph among all n-vertex $C_{2k_1,2k_2,\ldots,2k_t}$-free graphs. We refer the interested readers to [16].

The spectral extremal graphs among F_s-free and/or F_s-minor-free graphs are quite different, while the spectral extremal graphs among Q_t-free and/or Q_t-minor-free graphs are the same. *Hence, a natural question one may ask is that for which type of graphs H, the spectral extremal graph over all H-free graphs is the same as that over all H-minor-free graphs.* To our knowledge, the spectral extremal problems that forbid a particular minor as subgraph are investigated until now for complete graphs [40], complete bipartite graphs [50], the complete graph removing some disjoint paths [14], the intersecting 3-cycles and 4-cycles in the present paper. It is also important for us to consider the spectral problem for H-minor-free graphs when H is other specific graph, such as books, wheels, fans, cycles, intersecting cycles, intersecting cliques or disjoint cliques, etc. For related results, we recommend two newly updated papers [6, 51].

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

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