Hankel determinants of q-Stirling numbers

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Abstract

In this paper, we consider the q-analogue of the Hankel determinants of the Bell numbers and give combinatorial proofs of these results. We show that the Hankel determinants of the q-Stirling numbers can be simplified to a determinant that is almost upper-triangular, and then construct sign-reversing involutions on certain sets of RG-words that give rise to the determinants.
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1 Introduction

The *q-Stirling numbers of the second kind* $S[n, k]$ are defined by

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q \cdot S_q[n-1, k], \text{ for } 1 \leqslant k \leqslant n,$$

with boundary conditions $S_q[n, 0] = S_q[0, n] = \delta_{n,0}$, the usual Kronecker delta function. Here $[k]_q = 1 + q + \cdots + q^{k-1}$ denotes the q-analogue of k. Setting $q = 1$ gives the familiar Stirling numbers of the second kind $S(n, k)$ which enumerate the number of partitions of an n-element set into k nonempty disjoint blocks. See [4, pages 128–129] and [5, Section 3].

Define the *Bell polynomial*

$$B_n(z) = \sum_{k=0}^{n} S(n, k) \cdot z^k.$$

These polynomials are also known as exponential polynomials. Note that $B_n(1)$ is the n-th Bell number. Aigner [1] showed that the Hankel determinant of the Bell numbers is $\det(B_{i+j}(1))_{0 \leqslant i,j \leqslant n} = \prod_{k=0}^{n} k!$. Radoux [16] showed that $\det(B_{i+j}(z))_{0 \leqslant i,j \leqslant n} = \prod_{k=0}^{n} k! \cdot$

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$z^{\binom{n+1}{2}}$. For further study of the Hankel determinant of the Bell numbers, and more generally, of the exponential polynomials, see [5, 8, 16, 17, 19].

In [6], Cigler considered the q-analogue of the Bell polynomials:

$$e_n(z) = \sum_{k=0}^{n} S_q[n,k] \cdot z^k$$

and its Hankel determinant

$$\det \begin{pmatrix} e_s(z) & e_{s+1}(z) & \cdots & e_{s+n}(z) \\ e_{s+1}(z) & e_{s+2}(z) & \cdots & e_{s+n+1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ e_{s+n}(z) & e_{s+n+1}(z) & \cdots & e_{s+2n}(z) \end{pmatrix} = \det(e_{s+i+j}(z))_{0 \leqslant i,j \leqslant n}. \tag{1.1}$$

He computed this determinant for $s = 0, 1, 2$. The results are

$$\det(e_{i+j}(z))_{0\leqslant i,j\leqslant n} = q^{\binom{n+1}{3}} \cdot z^{\binom{n+1}{2}} \cdot \prod_{i=0}^{n} [i]_q!, \tag{1.2}$$

$$\det(e_{i+j+1}(z))_{0\leqslant i,j\leqslant n} = q^{\binom{n+2}{3}} \cdot z^{\binom{n+2}{2}} \prod_{i=0}^{n} [i]_q!, \tag{1.3}$$

$$\det(e_{i+j+2}(z))_{0\leqslant i,j\leqslant n} = q^{\binom{n+2}{3}} \cdot z^{\binom{n+2}{2}} \prod_{i=0}^{n} [i]_q! \cdot \sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot z^k \cdot \frac{[n+1]_q!}{[k]_q!}. \tag{1.4}$$

The original work considered the derivatives of orthogonal polynomials and the properties of the coefficients thereafter. On the other hand, due to various nice combinatorial interpretations of q-Stirling numbers of the second kind, a natural question is can we find combinatorial proofs for these results?

For Eq. (1.2), Ehrenborg had a proof in [9] using the juggling interpretation of the q-Stirling numbers. In [2], the authors gave a combinatorial proof of this result via RG-words. For the other two determinants, no combinatorial proofs are known so far.

The goal for this paper is to give combinatorial proofs of the above identities. Our tool is the RG-word interpretation of q-Stirling numbers.

This paper is organized as follows: In Section 2, we give preliminaries on q-Stirling numbers and RG-words. In Section 3, we show that the determinantal formula can be simplified to a form that is almost upper-triangular. In Sections 4 and 5, we give combinatorial proofs to Eqs. (1.3) and (1.4).

2 Preliminaries

Given a set partition of the n elements $\{1, 2, \ldots, n\}$ into k mutually disjoint nonempty blocks, denote it by $\pi = B_1/B_2/\cdots/B_k$, where the blocks are ordered so that $\min(B_1) <$

$\min(B_2) < \cdots < \min(B_k)$. We encode it using a *restricted growth word*, or *RG-word*, $\boldsymbol{w}(\pi) = w_1 w_2 \cdots w_n$, where $w_i = j$ if the element i occurs in block B_j of π. Restricted growth words are also known as restricted growth functions. Let $[i, j]$ denote the interval of integers $[i, j] = \{k \in \mathbb{P} : i \leqslant k \leqslant j\}$. Recall a *restricted growth function* $f : [1, n] \longrightarrow [1, k]$ is a surjective map which satisfies $f(1) = 1$ and $f(i) \leqslant \max(f(1), f(2), \ldots, f(i-1)) + 1$ for $i = 2, 3, \ldots, n$. Denote by $RG(n, k)$ the set of all RG-words of length n and maximal entry k. Denote by $\boldsymbol{\varepsilon}$ the empty word such that $RG(0, 0) = \{\boldsymbol{\varepsilon}\}$.

One way to obtain the q-Stirling numbers of the second kind is to introduce a weight on RG-words. For $\boldsymbol{w} \in RG(n, k)$, let $m_i = \max(w_1, w_2, \ldots, w_i)$ and form the weight $\mathrm{wt}(\boldsymbol{w}) = \prod_{i=1}^{n} \mathrm{wt}_i(w)$, where $\mathrm{wt}_1(w) = 1$ and for $2 \leqslant i \leqslant n$, let

$$\mathrm{wt}_i(w) = \begin{cases} q^{w_i - 1} & \text{if } m_{i-1} \geqslant w_i, \\ 1 & \text{if } m_{i-1} < w_i. \end{cases} \tag{2.1}$$

Proposition 1. *For $\boldsymbol{w} = w_1 \cdots w_n \in RG(n, k)$ the weight is given by*

$$\mathrm{wt}(\boldsymbol{w}) = q^{\sum\limits_{i=1}^{n}(w_i - 1) - \binom{k}{2}}.$$

Lemma 2. *The q-Stirling numbers of the second kind are given by*

$$S_q[n, k] = \sum_{\boldsymbol{w} \in RG(n,k)} \mathrm{wt}(\boldsymbol{w}) \text{ for } 1 \leqslant k \leqslant n.$$

There is a long history of studying restricted growth functions [13, 14, 15] and q-Stirling numbers of the second kind [2, 4, 10, 12, 15, 20]. See [3, Sections 2 and 3] for details.

For a word $\boldsymbol{w} = w_1 w_2 \cdots w_n$ define the *length* of $\boldsymbol{w}$ to be $|\boldsymbol{w}| = n$. Similarly, define its ls-*weight* to be $\mathrm{ls}(\boldsymbol{w}) = q^{\sum\limits_{i=1}^{n}(w_i - 1)}$. This is a q-generalization of the ls-weight of RG-words defined by Wachs and White [20, Section 2]. Denote by $\max(\boldsymbol{w})$ the maximal entry in $\boldsymbol{w}$.

The *concatenation* of two words $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$. The word $\boldsymbol{v}$ is a *factor* of the word $\boldsymbol{w}$ if one can write $\boldsymbol{w} = v_1 \cdot \boldsymbol{v} \cdot v_2$. A word $\boldsymbol{v} = v_1 v_2 \cdots v_k$ is a *subword* of $\boldsymbol{w}$ if there is a subsequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $w_{i_j} = v_j$ for all $1 \leqslant j \leqslant k$. In other words, a factor of $\boldsymbol{w}$ is a subword consisting of consecutive entries.

For a word $\boldsymbol{w} = w_1 w_2 \cdots w_n \in RG(n, k)$, let $\mathrm{NLRM}(\boldsymbol{w})$ be the set of all entries w_r that are not left-to-right maxima of the word $\boldsymbol{w}$, that is, $w_r \leqslant \max(w_1, w_2, \ldots, w_{r-1})$. Furthermore, for $w_r \in \mathrm{NLRM}(\boldsymbol{w})$ define the *bound* $b(w_r)$ to be $\max(w_1, w_2, \ldots, w_{r-1})$.

3 A Simplified Determinantal Formula

In this section, we show that the determinantal formula (1.1) can be simplified to a form that is almost upper-triangular, and we will use this simplified identity to prove Cigler's results.

Theorem 3. *For any integer $s \geqslant 0$ and $e_n(z) = \sum_{k=0}^{n} S_q[n,k] \cdot z^k$, we have*

$$\det(e_{i+j+s}(z))_{0\leqslant i,j\leqslant n} = \det\left(q^{\binom{i}{2}} \cdot [i]_q! \cdot \sum_{k=i}^{j+s} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+s,k] \cdot z^k\right)_{i,j=0}^{n}.$$

Proof. Let T be the set of all $(n+2)$-tuples $(\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n))$, where σ is a permutation on $n+s$ elements for any fixed non-negative integer s and $\boldsymbol{w}(i) \in RG(i+\sigma(i)+s, k)$. For any $t \in T$, define its weight $\mathrm{wt}(t) = (-1)^\sigma \cdot \prod_{i=0}^{n} \mathrm{wt}(\boldsymbol{w}(i)) \cdot z^{\max(\boldsymbol{w}(i))}$, where $(-1)^\sigma$ is the sign of the permutation σ. Define the weight of the set T as $\mathrm{wt}(T) = \sum_{t\in T} \mathrm{wt}(t)$. The determinant expands as the sum

$$\det\left(e_{i+j+s}(z)\right)_{i,j=0}^{n} = \sum_{t\in T} \mathrm{wt}(t) = \sum_{t\in T} (-1)^\sigma \cdot \prod_{i=0}^{n} \mathrm{wt}(\boldsymbol{w}(i)) \cdot z^{\max(\boldsymbol{w}(i))}.$$

Factor $\boldsymbol{w}(i) = \boldsymbol{u}(i) \cdot \boldsymbol{v}(i)$ where $|\boldsymbol{u}(i)| = \sigma(i) + s$ and $|\boldsymbol{v}(i)| = i$.

Let $S_1 \subseteq T$ be the set of all tuples t where there is a position $r \in [1,i]$ of $\boldsymbol{v}(i) = v(i)_1 v(i)_2 \cdots v(i)_i$ such that $v(i)_r < r$, that is, r is an *anti-exceedance* of $\boldsymbol{v}(i)$.

Let $S_2 \subseteq T - S_1$ be the set of all tuples t where $\max \boldsymbol{v}(i) > \max \boldsymbol{u}(i)$ for some index $i \in [0,n]$.

Let $T_1 = T - S_1 - S_2$ be the set of all remaining tuples. By construction, all entries in $\boldsymbol{v}(i)$ are non-left-to-right maxima of $\boldsymbol{w}(i)$ and $v(i)_r \geqslant r$. In this case, the maximal entry of $\boldsymbol{w}(i)$, say k, must appear in $\boldsymbol{u}(i)$, that is, $\boldsymbol{u}(i) \in RG(\sigma(i)+s, k)$. Furthermore each entry in $\boldsymbol{v}(i)$ contributes a factor of q to $\mathrm{wt}(\boldsymbol{w}(i))$: since $\boldsymbol{v}(i)_r \geqslant r$, this entry can take values in $[r,k]$, contributing a total of $q^{r-1} + q^r + \cdots + q^{k-1} = q^{r-1}[k-r+1]_q$.

Thus the total weight on set T_1 is

$$\begin{aligned} &\sum_{t\in T_1} (-1)^\sigma \prod_{i=0}^{n} \mathrm{wt}(\boldsymbol{u}(i)) \cdot \mathrm{ls}(\boldsymbol{v}(i)) \cdot z^{\max(\boldsymbol{w}(i))} \\ &= \sum_{t\in T_1} (-1)^\sigma \prod_{k=i}^{\sigma(i)+s} S_q[\sigma(i)+s,k] \cdot [k]_q \cdot q[k-1]_q \cdots q^{i-1}[k-i+1]_q \cdot z^k \qquad (3.1) \\ &= \det\left(q^{\binom{i}{2}} [i]_q! \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{k=i}^{j+s} S_q[j+s,k] z^k\right)_{i,j=0}^{n}. \end{aligned}$$

It remains to show that the weights on sets S_1 and S_2 add up to 0. We do this by constructing a sign-reversing involution on the sets respectively.

First we consider the set S_1.

Let $D(k,\ell) \subseteq S_1$ be the set of all tuples $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n))$ where $k \geqslant 1$ is the largest index such that $v(k)_\ell < \ell$ and $v(i)_j \geqslant j$ for all $i \in [1,n]$ and $j < \ell - 1$. Then $S_1 = \bigcup_{1\leqslant \ell \leqslant k \leqslant n} D(k,\ell)$.

Now we construct a sign-reversing involution φ_1 on $D(k,\ell)$ for each $1 \leqslant \ell \leqslant k \leqslant n$.

For $t \in D(k,\ell)$, factor $\boldsymbol{v}(k) = \boldsymbol{\alpha}(k) \cdot y \cdot \boldsymbol{\beta}(k)$ where $|\boldsymbol{\alpha}(k)| = \ell - 1$, $y < \ell$ and $\boldsymbol{\beta}(k)$ is the remaining subword.

Let $\boldsymbol{v}(k-1) = \boldsymbol{\alpha}(k-1) \cdot \boldsymbol{\beta}(k-1)$ where $|\boldsymbol{\alpha}(k-1)| = \ell - 1$. Then define $t' = \varphi_1(t) = (\sigma', \boldsymbol{w}(0)', \boldsymbol{w}(1)', \ldots, \boldsymbol{w}(n)')$ such that $\sigma' = \sigma \circ (k-1, k)$, $\boldsymbol{w}(i)' = \boldsymbol{w}(i)$ for $i \neq k-1, k$, and

$$\boldsymbol{v}(k-1)' = \boldsymbol{\alpha}(k) \cdot \boldsymbol{\beta}(k), \boldsymbol{v}(k)' = \boldsymbol{\alpha}(k-1) \cdot y \cdot \boldsymbol{\beta}(k-1).$$

Moreover, let

$$\boldsymbol{w}(k-1)' = \boldsymbol{u}(k) \cdot \boldsymbol{v}(k-1)', \boldsymbol{w}(k)' = \boldsymbol{u}(k-1) \cdot \boldsymbol{v}(k)'.$$

It is straightforward to check that $t' = \varphi_1(t) \in D(k,\ell)$. Moreover, all non-left-to-right maxima in $\boldsymbol{v}(k-1)'$ and $\boldsymbol{v}(k)'$ are still non-left-to-right maxima of $\boldsymbol{w}(k-1)'$ and $\boldsymbol{w}(k)'$ respectively, and $\max(\boldsymbol{w}(k)) = \max(\boldsymbol{w}(k-1)')$, $\max(\boldsymbol{w}(k-1)) = \max(\boldsymbol{w}(k)')$. Since $(-1)^{\sigma'} = -1 \cdot (-1)^{\sigma}$, the map φ_1 is indeed a sign-reversing involution on $D(k,\ell)$, and hence S_1.

Next we consider the set S_2.

Let $A(k,i) \subseteq S_2$ be the set of all tuples $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n)) \in S_2$ where $k \geqslant 1$ is the smallest index such that $\boldsymbol{w}(k) = \boldsymbol{u}(k) \cdot \boldsymbol{v}(k)$ has a left-to-right maxima in $\boldsymbol{v}(k)$. And the first left-to-right maxima of $\boldsymbol{v}(k)$ is at position $i \in [1,k]$. Thus $S_2 = \bigcup_{1 \leqslant i \leqslant k \leqslant n} A(k,i)$.

Note that for any $\boldsymbol{w}(i) = \boldsymbol{u}(i) \cdot \boldsymbol{v}(i)$, we have $v(i)_r \geqslant r$.

We construct a sign-reversing involution φ_2 on S_2 as follows.

For $t \in A(k,i)$, write $\boldsymbol{v}(k) = \boldsymbol{\alpha}(k) \cdot x \cdot \boldsymbol{\beta}(k)$ where $|\boldsymbol{\alpha}(k)| = i - 1$, $x > \max \boldsymbol{u}(k)$, and $\boldsymbol{\beta}(k)$ is the remaining subword. Assume there are r non-left-to-right maxima of $\boldsymbol{w}(k)$ in $\boldsymbol{v}(k)$, record these entries as $v(k)_{s_1}, v(k)_{s_2}, \ldots, v(k)_{s_r}$. Then there are $k - r$ many left-to-right maxima of $\boldsymbol{w}(k)$ in $\boldsymbol{v}(k)$.

First we claim that $r \leqslant \max \boldsymbol{u}(k)$. To see this, note that the biggest entry in $\boldsymbol{w}(k)$ must be $\max \boldsymbol{u}(k) + k - r$, thus for the last entry of $\boldsymbol{w}(k)$, i.e. $\beta(k)_k$, we have $k \leqslant \beta(k)_k \leqslant \max \boldsymbol{u}(k) + k - r$, hence we obtain the claim.

Now define $t' = \varphi_2(t) = (\sigma', \boldsymbol{w}(0)', \boldsymbol{w}(1)', \ldots, \boldsymbol{w}(n)')$ such that $\sigma' = \sigma \circ (r,k)$ where (r,k) is the transposition, $\boldsymbol{w}(j)' = \boldsymbol{w}(j)$ for $j \neq r, k$. And $\boldsymbol{w}(r)' = \boldsymbol{u}(k) \cdot \boldsymbol{v}(r)'$, $\boldsymbol{w}(k)' = \boldsymbol{u}(r) \cdot \boldsymbol{v}(k)'$, where $\boldsymbol{v}(r)'$ and $\boldsymbol{v}(k)'$ are constructed in three cases.

(1) If $\max \boldsymbol{u}(r) = \max \boldsymbol{u}(k)$, then $\boldsymbol{v}(r)' = \boldsymbol{v}(r)$, and $\boldsymbol{v}(k)' = \boldsymbol{v}(k)$. It is straightforward to check that $t' = \varphi_2(t) \in A(k,i)$ and $\mathrm{wt}(t') = -\mathrm{wt}(t)$.

(2) If $\max \boldsymbol{u}(r) > \max \boldsymbol{u}(k)$, let $d = \max \boldsymbol{u}(r) - \max \boldsymbol{u}(k)$, then let

$$v(r)'_j = \begin{cases} v(r)_j, & \text{if } v(r)_j \leqslant \max \boldsymbol{u}(k), \\ \max \boldsymbol{u}(k), & \text{if } v(r)_j > \max \boldsymbol{u}(k), \end{cases}$$

and

$$v(k)'_s = \begin{cases} v(k)_{s_j}, & \text{if } s = s_j \text{ and } v(r)_j \leqslant \max \boldsymbol{u}(k), \\ v(k)_{s_j} + v(r)_j - \max \boldsymbol{u}(k), & \text{if } s = s_j \text{ and } v(r)_j > \max \boldsymbol{u}(k), \\ v(k)_s + d, & \text{if } v(k)_s \notin \mathrm{NLRM}(\boldsymbol{w}(k)). \end{cases}$$

Since $\max \boldsymbol{u}(k) \geqslant r \geqslant j$ for any $1 \leqslant j \leqslant r$, we have $v(r)'_j \geqslant j$ for all $1 \leqslant j \leqslant r$.

On the other hand, if $v(r)_j > \max \boldsymbol{u}(k)$, we have $v(k)_{s_j} + v(r)_j - \max \boldsymbol{u}(k) > v(k)_{s_j} \geqslant s_j$, and $v(k)_s + d > v(k)_s \geqslant s$, so $v(k)'_s \geqslant s$ for all $1 \leqslant s \leqslant k$.

Moreover, suppose there are ℓ many left-to-right maxima in $\boldsymbol{v}(k)$ of $\boldsymbol{w}(k)$ before the position s_j, then

$$\begin{aligned} v(k)_{s_j} + v(r)_j - \max \boldsymbol{u}(k) &\leqslant v(k)_{s_j} + \max \boldsymbol{u}(r) - \max \boldsymbol{u}(k) \\ &\leqslant \max \boldsymbol{u}(k) + \ell + \max \boldsymbol{u}(r) - \max \boldsymbol{u}(k) \\ &= \max \boldsymbol{u}(r) + \ell, \end{aligned}$$

that is, $v(k)'_{s_j} \in \mathrm{NLRM}(\boldsymbol{w}(k)')$, thus $v(k)'_s$ is a non-left-to-right maxima of $\boldsymbol{w}(k)'$ if and only if $v(k)_s$ is a non-left-to-right maxima of $\boldsymbol{w}(k)$.

Thus it is straightforward to check that $\mathrm{wt}(t) = -\mathrm{wt}(\varphi_2(t))$.

(3) If $\max \boldsymbol{u}(r) < \max \boldsymbol{u}(k)$, let $d = \max \boldsymbol{u}(k) - \max \boldsymbol{u}(r)$, then

$$v(k)'_s = \begin{cases} v(k)_{s_j}, & \text{if } s = s_j \text{ and } v(k)_{s_j} \leqslant b(v(k)_{s_j}) - d, \\ b(v(k)_{s_j}) - d, & \text{if } s = s_j \text{ and } v(k)_{s_j} > b(v(k)_{s_j}) - d, \\ v(k)_s - d, & \text{if } v(k)_s \notin \mathrm{NLRM}(\boldsymbol{w}(k)), \end{cases}$$

and

$$v(r)'_j = \begin{cases} v(r)_j, & \text{if } v(k)_{s_j} \leqslant b(v(k)_{s_j}) - d, \\ v(r)_j + v(k)_{s_j} - b(v(k)_{s_j}) + d, & \text{if } v(k)_{s_j} > b(v(k)_{s_j}) - d. \end{cases}$$

If $v(k)_s \notin \mathrm{NLRM}(\boldsymbol{w}(k))$, then there are $v(k)_s - \max \boldsymbol{u}(k)$ many left-to-right maxima in $\boldsymbol{v}(k)'$ of $\boldsymbol{w}(k)'$ before position s, thus there are $s - v(k)_s + \max \boldsymbol{u}(k)$ many non-left-to-right maxima, so

$$\max \boldsymbol{u}(r) \geqslant r \geqslant s - v(k)_s + \max \boldsymbol{u}(k),$$

thus $v(k)'_s = v(k)_s - d \geqslant s$.

Similarly, if $v(k)_{s_j} > b(v(k)_{s_j}) - d$, there are $s_j - b(v(k)_{s_j}) + \max \boldsymbol{u}(k)$ many non-left-to-right maxima before position s_j, and

$$\max \boldsymbol{u}(r) \geqslant s_j - b(v(k)_{s_j}) + \max \boldsymbol{u}(k),$$

thus $v(k)'_{s_j} = b(v(k)_{s_j}) - d \geqslant s_j$. And $\boldsymbol{v}(k)'$ has no anti-exceedance

Next we check the word $\boldsymbol{w}(r)'$. If $v(k)_{s_j} > b(v(k)_{s_j}) - d$, then $v(r)_j + v(k)_{s_j} - b(v(k)_{s_j}) + d > v(r)_j \geqslant j$ is an exceedance. On the other hand, $v(r)_j \leqslant \max \boldsymbol{u}(r) < \max \boldsymbol{u}(k)$. If $v(k)_{s_j} > b(v(k)_{s_j}) - d$, suppose there are ℓ many left-to-right maxima of $\boldsymbol{w}(k)$ in $\boldsymbol{v}(k)$ before the position s_j, then

$$\begin{aligned} v(r)_j + v(k)_{s_j} - b(v(k)_{s_j}) + d &= v(r)_j + v(k)_{s_j} - \max \boldsymbol{u}(k) - \ell \\ &\quad + \max \boldsymbol{u}(k) - \max \boldsymbol{u}(r) \\ &= v(r)_j + v(k)_{s_j} - \ell - \max \boldsymbol{u}(r) \\ &\leqslant v(k)_{s_j} - \ell \leqslant \max \boldsymbol{u}(k). \end{aligned}$$

In other words, $v(r)'_j$ is a non-left-to-right maxima in any cases. Thus $\varphi_2(t) \in A(k,i)$.

By a similar argument as in case (2), $\mathrm{wt}(t) = -\mathrm{wt}(t')$, hence φ_2 is a sign-reversing involution on $A(k,i)$.

Thus the Hankel determinant is computed by Eq. (3.1). □

Pulling out the factors $[i]_q!$ yields the next result:

Lemma 4.

$$\det(e_{i+j+s}(z))_{0\leqslant i,j\leqslant n} = \prod_{i=0}^{n}[i]_q! \cdot \det\left(\sum_{k=i}^{j+s} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+s,k]\cdot z^k\right)_{i,j=0}^{n}.$$

When $s=0$ observe that the matrix we are taking the determinant of is triangular and has the entries z^i on the diagonal, proving Eq. (1.2).

When $s=1$, the matrix is no longer triangular. It is almost triangular, but with one more sub-diagonal. Eq. (1.3) becomes the following result:

Theorem 5.

$$q^{\binom{n+2}{3}} \cdot z^{\binom{n+2}{2}} = \det\left(\sum_{k=i}^{j+1} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+1,k]\cdot z^k\right)_{i,j=0}^{n}. \tag{3.2}$$

Similarly, for the $s=2$ case, Eq. (1.4) becomes

Theorem 6.

$$q^{\binom{n+2}{3}} \cdot z^{\binom{n+2}{2}} \cdot \sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot z^k \cdot \frac{[n+1]_q!}{[k]_q!} = \det\left(\sum_{k=i}^{j+2} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+2,k]\cdot z^k\right)_{i,j=0}^{n}. \tag{3.3}$$

In the next two sections, we give combinatorial proofs for these two identities.

4 Proof of Theorem 5

Definition 7. For two words $\boldsymbol{u} = u_1u_2\cdots u_i$ and $\boldsymbol{v} = v_1v_2\cdots v_j$, define the order relation $\boldsymbol{u} \leqslant_w \boldsymbol{v}$ if $i \leqslant j$ and $u_k \leqslant v_k$ for all $k = 1,2,\ldots,i$.

Let $\mathrm{Inc}(n,k) \subseteq \mathbb{N}^n$ denote the set of sequences of positive integers of length n that are strictly increasing with maximal entry at most k, that is,

$$\mathrm{Inc}(n,k) = \{\boldsymbol{w} = w_1w_2\cdots w_n : w_1 < w_2 < \cdots < w_n \leqslant k \text{ and } w_i \in \mathbb{Z}^+\}.$$

Definition 8. Let $W(n,i) = \bigcup_{i\leqslant k\leqslant n} RG(n,k) \times \mathrm{Inc}(i,k) \subseteq RG(n+i,k)$ be the set of all words of the form $\boldsymbol{w} = \boldsymbol{u}\cdot\boldsymbol{v}$ where $\boldsymbol{u} \in RG(n,k)$ and $\boldsymbol{v} \in \mathrm{Inc}(i,k)$.

As a remark, $\mathrm{wt}(\boldsymbol{w}) = \mathrm{wt}(\boldsymbol{u}) \cdot \mathrm{ls}(\boldsymbol{v})$.

Proof of Theorem 5. Let T be the set of all $(n+2)$-tuples of the form $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n))$ where σ is a permutation on the $n+1$ elements $\{0, 1, 2, \ldots, n\}$, and $\boldsymbol{w}(i) \in W(\sigma(i)+1, i)$ that factors as $\boldsymbol{w}(i) = \boldsymbol{u}(i) \cdot \boldsymbol{v}(i)$ where $\boldsymbol{u}(i) \in RG(\sigma(i)+1, k)$ and $\boldsymbol{v}(i) \in \mathrm{Inc}(i, k)$ for all $0 \leqslant i \leqslant n$ and $i \leqslant k \leqslant \sigma(i)+1$. Then the determinant expands as the sum

$$\det\left(\sum_{k=i}^{j+1} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+1, k] \cdot z^k\right)_{i,j=0}^{n} = \sum_{t \in T} (-1)^{\sigma} \cdot \prod_{i=0}^{n} \mathrm{wt}(\boldsymbol{w}(i)) \cdot z^{\max(\boldsymbol{w}(i))}.$$

Denote by $k_i = \max \boldsymbol{u}(i)$ and $p_i = \max \boldsymbol{v}(i)$. Note that since $\boldsymbol{v}(i)$ is strictly increasing, p_i is indeed $v(i)_i$, the last entry of $\boldsymbol{v}(i)$. Thus $i \leqslant p_i \leqslant k_i$.

Let $A_1 \subseteq T$ be the subset where there is a least index j in lexicographic order such that $\boldsymbol{v}(j) \not\leqslant_w \boldsymbol{v}(j+1)$. Then factor $\boldsymbol{v}(j) = \boldsymbol{\alpha}(j) \cdot \boldsymbol{\beta}(j)$ and $\boldsymbol{v}(j+1) = \boldsymbol{\alpha}(j+1) \cdot y \cdot \boldsymbol{\beta}(j+1)$ where $\boldsymbol{\alpha}(j) \leqslant_w \boldsymbol{\alpha}(j+1)$ and $y < \beta(j)_1$.

Define a sign-reversing involution ψ_1 on A_1 where $\sigma' = \sigma \circ (j, j+1)$, $\boldsymbol{w}(i)' = \boldsymbol{w}(i)$ for $i \neq j, j+1$. And $\boldsymbol{w}(j)' = \boldsymbol{u}(j+1) \cdot \boldsymbol{\alpha}(j) \cdot \boldsymbol{\beta}(j+1)$, $\boldsymbol{w}(j+1)' = \boldsymbol{u}(j) \cdot \boldsymbol{\alpha}(j+1) \cdot y \cdot \boldsymbol{\beta}(j)$. It is straightforward to check that $\boldsymbol{w}(j)' \in W(\sigma(j+1)+1, j)$, $\boldsymbol{w}(j+1)' \in W(\sigma(j)+1, j+1)$ and $t' = \psi_1(t) = (\sigma', \boldsymbol{w}(0)', \boldsymbol{w}(1)', \ldots, \boldsymbol{w}(n)') \in A_1$. Moreover, since $k'_j = k_{j+1}, k'_{j+1} = k_j$ and $\mathrm{wt}(\boldsymbol{w}(j)) \cdot \mathrm{wt}(\boldsymbol{w}(j+1)) = \mathrm{wt}(\boldsymbol{w}(j)') \cdot \mathrm{wt}(\boldsymbol{w}(j+1)')$, we have $\mathrm{wt}(t) = -\mathrm{wt}(t')$.

Let $A_2 \subseteq T - A_1$ be the set where there exists a least index j in lexicographic order such that $k_j \geqslant p_{j+1}$. Note that in this case, since $\boldsymbol{v}(i) \leqslant_w \boldsymbol{v}(i+1)$ for all i, we have $k_{j+1} \geqslant p_{j+1} > p_j$. Define a sign-reversing involution ψ_2 on A_2 where $\sigma', \boldsymbol{w}(i)'$ for $i \neq j, j+1$ are the same as in the case of A_1, and

$$\boldsymbol{w}(j)' = \boldsymbol{u}(j+1) \cdot \boldsymbol{v}(j), \boldsymbol{w}(j+1)' = \boldsymbol{u}(j) \cdot \boldsymbol{v}(j+1).$$

It is straightforward to that $\mathrm{wt}(t) = -\mathrm{wt}(t')$.

Now let $B^* = T - A_1 - A_2$ be the set of all remaining words. By the constructions above, for $t \in B^*$, we have $k_i < p_{i+1}$ and $\boldsymbol{v}(i) \leqslant_w \boldsymbol{v}(i+1)$ for all i. In particular,

$$1 = k_0 < p_1 \leqslant k_1 < p_2 \leqslant k_2 < p_3 \leqslant k_3 < \cdots < p_n \leqslant k_n \leqslant n+1.$$

So $k_i = p_i = i+1$, and $i+1 = k_i \leqslant \sigma(i)+1$ for all i. Thus $\sigma(i) = i$ and $\sigma = \mathrm{id}$ is the identity permutation.

In this case we have $\boldsymbol{u}(i) = 123 \cdots (i+1)$, $\boldsymbol{v}(i) = 23 \cdots (i+1)$.

Thus

$$B^* = \{(\mathrm{id}, 1, 12 \cdot 2, 123 \cdot 23, \ldots, 123 \cdots (n+1) \cdot 23 \cdots (n+1))\}$$

consists of only one tuple, and

$$\begin{aligned} \mathrm{wt}(B^*) &= q^0 z \cdot q^1 z^2 \cdot q^3 z^3 \cdots q^{\binom{n+1}{2}} z^{n+1} \\ &= q^{\binom{n+2}{3}} z^{\binom{n+2}{2}}. \end{aligned} \tag{4.1}$$

Hence the theorem follows. □

5 Proof of Theorem 6

In this section we consider the case $s = 2$.

Definition 9. For an RG-word $\boldsymbol{u} \in RG(n,k)$, let u_i be a non-left-to-right maxima. If $u_i < b(u_i)$, call it an *inversion*, and if $u_i = b(u_i)$, call it a *repeat*.

First we state a lemma.

Lemma 10. *Let $\sigma \in \mathfrak{S}_n$ be a permutation on $\{0,1,2,\dots,n-1\}$ and $\sigma(i) \geqslant i-1$ for all $i = 0,1,\dots,n-1$. If $\sigma(j) = k \geqslant j$, then $\sigma(\ell) = \ell - 1$ for all $j < \ell \leqslant k$.*

Proof. If $j = k$, the statement is vacuously true.

If $k > j$, since $\sigma(k+1) \geqslant k$, but $\sigma(j) = k$, we must have $\sigma(k+1) \geqslant k+1$, and $\sigma(k+2) \geqslant k+1$, $\sigma(k+3) \geqslant k+2 \geqslant k+1, \dots, \sigma(n-1) \geqslant k+1$. Thus these entries form a permutation on $\{k+1, k+2, \dots, n-1\}$ and $\sigma(s) \leqslant k$ for $s \leqslant k$.

Thus $k > \sigma(k) \geqslant k-1$, and $\sigma(k) = k-1$. Similarly, $\sigma(\ell) = \ell - 1$ for all $j < \ell \leqslant k$. ☐

To prove Theorem 6, we first describe a set K, and then show that the weight on set K gives the formula.

Definition 11. Let $K = \{t = (\mathrm{id}, \boldsymbol{u}(0), \boldsymbol{u}(1)\cdot\boldsymbol{v}(1), \dots, \boldsymbol{u}(n)\cdot\boldsymbol{v}(n))\}$ with the following properties.

(1) There exist an index i such that $\boldsymbol{u}(i) \in RG(i+2, i+2)$, $\boldsymbol{v}(i) \in \mathrm{Inc}(i, p_i)$, where $p_i = i+1$ or $p_i = i+2$.

(2) For $j < i$, $\boldsymbol{u}(j) \in RG(j+2, j+1)$ contains a repeat and $\boldsymbol{v}(j) \in \mathrm{Inc}(j, j+1)$.

(3) For $j > i$, $\boldsymbol{u}(j) = 12\cdots(j+2) \in RG(j+2, j+2)$ and $\boldsymbol{v}(j) \in \mathrm{Inc}(j, j+2)$.

(4) $\boldsymbol{v}(j) \leqslant_w \boldsymbol{v}(j+1)$ for all $j \in [1, n]$

Lemma 12.

$$\det\left(\sum_{k=i}^{j+2} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+2,k] \cdot z^k\right)_{i,j=0}^{n} = \mathrm{wt}(K).$$

Proof. Let T be the set of all $(n+2)$-tuples $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \dots, \boldsymbol{w}(n))$ where σ is a permutation on the $n+1$ elements $\{0,1,2,\dots,n\}$, and $\boldsymbol{w}(i)$ is a word which factors as $\boldsymbol{w}(i) = \boldsymbol{u}(i)\cdot\boldsymbol{v}(i)$ where $\boldsymbol{u}(i) \in RG(\sigma(i)+2, k)$ and $\boldsymbol{v}(i) \in \mathrm{Inc}(i,k)$ for all $0 \leqslant i \leqslant n$ and $i \leqslant k \leqslant \sigma(i)+2$.

The determinant expands as the sum

$$\det\left(\sum_{k=i}^{j+2} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot S_q[j+2,k] \cdot z^k\right)_{i,j=0}^{n} = \sum_{t\in T} (-1)^{\sigma} \cdot \prod_{i=0}^{n} \mathrm{wt}(\boldsymbol{w}(i)) \cdot z^{\max \boldsymbol{w}(i)}.$$

Define k_i, p_i, A_1, A_2 in the same way as in Section 4. Then the sign-reversing involutions ψ_1, ψ_2 still work on the corresponding sets.

Now consider $B_1 = T - A_1 - A_2$. Similarly, we have

$$1 \leqslant k_0 < p_1 \leqslant k_1 < p_2 \leqslant k_2 < \cdots < p_n \leqslant k_n \leqslant n+2.$$

This indicates that the k_i are distinct and $i+1 \leqslant k_i \leqslant i+2$ for all i. In particular, since $k_i \leqslant \sigma(i)+2$, we have $\sigma(i) \geqslant i-1$ for all i. Thus by Lemma 10, if $\sigma(s) = j > s$ for some s, $\sigma(\ell) = \ell - 1$ for all $s < \ell \leqslant j$, which implies that $\boldsymbol{u}(\ell) \in RG(\ell+1, \ell+1)$ for $s+1 \leqslant \ell \leqslant j$, and $k_\ell = \ell+1$. Since all k_i are distinct, $k_s = s+1$.

Let $A_3 \subseteq B_1$ be the set where for $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n)) \in A_3$, there is a $\boldsymbol{w}(j) = \boldsymbol{u}(j) \cdot \boldsymbol{v}(j)$ for a least index j in lexicographic order such that there is an inversion or at least two repeats in $\boldsymbol{u}(j)$.

First we claim that $\sigma(j) \geqslant j$. In fact, if $\sigma(j) = j-1$, by the argument above, there is some $s \leqslant j-1$ with $\sigma(s) = m \geqslant j$ and $k_s = s+1$. Then $\boldsymbol{u}(s) \in RG(m+2, s+1)$ contains $m-s+1 \geqslant 2$ many non-left-to-right maxima, contradiction to the fact that $\boldsymbol{u}(j)$ is the first such word. So $\sigma(j) \geqslant j$.

Define a sign-reversing involution η on A_3 for the following two cases.

Case 1. If the first non-left-to-right maxima in $\boldsymbol{u}(j)$ is an inversion, then let $\boldsymbol{u}(j) = \boldsymbol{\alpha}_1 \cdot x \cdot \boldsymbol{\alpha}_2 \in RG(\sigma(j)+2, j+1)$, where $|\boldsymbol{\alpha}_1| = m$, x is the first inversion in $\boldsymbol{u}(j)$ and $\boldsymbol{\alpha}_2$ is the remaining word. In other words, $\boldsymbol{\alpha}_1 = 12\cdots m$ is increasing and $x < m$.

Let $\boldsymbol{u}(j-1) = \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2$, where $|\boldsymbol{\beta}_1| = m$ and $\boldsymbol{\beta}_2$ is the remaining word.

Note that since $k_j = j+1 \geqslant 2$, we have $j \geqslant 1$. Thus $\boldsymbol{u}(j-1)$ is well-defined.

Since $\boldsymbol{u}(j-1)$ contains at most one non-left-to-right maxima, and this non-left-to-right maxim must be a repeat, we have $\sigma(j-1)+2-k_{j-1} \leqslant 1$. So $\sigma(j-1)+1 \leqslant k_{j-1} < p_j = j+1$, thus $\sigma(j-1) = j-1$ and $\boldsymbol{u}(j-1) \in RG(j+1, j)$ is weakly increasing.

Let $t' = \eta(t) = (\sigma', \boldsymbol{w}(0)', \boldsymbol{w}(1)', \ldots, \boldsymbol{w}(n)')$ where $\sigma' = \sigma \circ (j, j-1)$, $\boldsymbol{w}(i)' = \boldsymbol{w}(i)$ for $i \neq j, j-1$. And $\boldsymbol{w}(j-1)' = \boldsymbol{u}(j-1)' \cdot \boldsymbol{v}(j-1)$, $\boldsymbol{w}(j)' = \boldsymbol{u}(j)' \cdot \boldsymbol{v}(j)$ where

$$\boldsymbol{u}(j-1)' = \boldsymbol{\beta}_1 \cdot x \cdot \boldsymbol{\alpha}_2 \text{ and } \boldsymbol{u}(j)' = \boldsymbol{\alpha}_1 \cdot \boldsymbol{\beta}_2.$$

Since $\boldsymbol{\beta}_1$ is weakly increasing with at most one repeat and $|\boldsymbol{\beta}_1| = m$, the last entry of $\boldsymbol{\beta}_1$ is m (if $\boldsymbol{\beta}_1$ is strictly increasing) or $m-1$ (if $\boldsymbol{\beta}_1$ has a repeat). In either case, x will be an inversion or repeat in $\boldsymbol{u}(j-1)'$, thus $t' \in A_3$, and one can check that $\mathrm{wt}(t) = -\mathrm{wt}(t')$.

Case 2. If the first non-left-to-right maxima in $\boldsymbol{u}(j)$ is a repeat, note that since there are at least two left-to-right maxima in $\boldsymbol{u}(j)$ in this case, $\sigma(j) > j$. Hence $\sigma(j+1) = j$ by Lemma 10.

Let $\boldsymbol{u}(j) = 12\cdots m \cdot m \cdot \boldsymbol{\alpha} \in RG(\sigma(j)+2, j+1)$ where $\boldsymbol{\alpha}$ is the remaining word. In other words, the second occurrence of m is the first repeat in $\boldsymbol{u}(j)$.

Consider $\boldsymbol{u}(j+1) \in RG(j+2, j+2)$, that is, $\boldsymbol{u}(j+1) = 12\cdots(j+2)$. Factor $\boldsymbol{u}(j+1) = 12\cdots(m+1) \cdot \boldsymbol{\beta}$, that is $|\boldsymbol{\beta}| = j+1-m$.

Let $t' = \eta(t)$ where $\sigma' = \sigma \circ (j, j+1)$, $\boldsymbol{w}(i)' = \boldsymbol{w}(i)$ for $i \neq j, j+1$. And $\boldsymbol{w}(j+1)' = \boldsymbol{u}(j+1)' \cdot \boldsymbol{v}(j+1)$, $\boldsymbol{w}(j)' = \boldsymbol{u}(j)' \cdot \boldsymbol{v}(j)$ where

$$\boldsymbol{u}(j+1)' = 12\cdots(m+1) \cdot \boldsymbol{\alpha}' \text{ and } \boldsymbol{u}(j)' = 12\cdots m \cdot m \cdot \boldsymbol{\beta}'.$$

Here

$$\beta_i' = \beta_i - 1 \text{ and } \alpha_i' = \begin{cases} \alpha_i, \text{ if } \alpha_i \in \mathrm{NLRM}(\boldsymbol{u}(j)), \\ \alpha_i + 1, \text{ if } \alpha_i \notin \mathrm{NLRM}(\boldsymbol{u}(j)). \end{cases}$$

It is straightforward to check that $\boldsymbol{u}(j)'$ contains only one non-left-to-right maxima, which is a repeat. And if $\alpha_i \in \mathrm{NLRM}(\boldsymbol{u}(j))$, α_i' must be an inversion in $\boldsymbol{u}(j+1)'$. Thus $t' \in A_3$, and $\mathrm{wt}(t') = -\mathrm{wt}(t)$. Thus η is a sign-reversing involution on A_3.

Let $K = B_1 - A_3$ be the set of all remaining words. For $t = (\sigma, \boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(n)) \in K$, we must have $\sigma(j) \geqslant j$ for all $j \in [0, n]$, that is, $\sigma = \mathrm{id}$ is the identity permutation. And one can check that K satisfy the properties in Definition 11, proving the lemma. $\square$

It remains to compute the weight on K.

Theorem 13.

$$\mathrm{wt}(K) = q^{\binom{n+2}{3}} \cdot z^{\binom{n+2}{2}} \cdot \sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot z^k \cdot \frac{[n+1]_q!}{[k]_q!}. \tag{5.1}$$

Note that Eq. (5.1) differs from Eq. (3.2) only by a factor of $\sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot [n+1]!/[k]! \cdot z^k$. Thus we show that the words in K can be constructed from $t^* = (\mathrm{id}, 1, 12 \cdot 2, 123 \cdot 23, \ldots, 123 \cdots (n+1) \cdot 23 \cdots (n+1)) \in B^*$ in the proof of Theorem 5 and only compute the additional weight created from the construction.

Proof. For any word $t = (\mathrm{id}, \boldsymbol{u}(1) \cdot \boldsymbol{v}(1), \ldots, \boldsymbol{u}(n) \cdot \boldsymbol{v}(n)) \in K$, by Definition 11, there exists an $i \in [0, n]$, such that for $j < i$, $\boldsymbol{u}(j) \in RG(j+2, j+1)$ contains a repeat. This can be obtained from $\boldsymbol{u}_j^* = 12 \cdots (j+1)$ by inserting a repeat. This repeat can be anything from 1 to $j+1$, contributing a total weight of $[j+1]_q$, together this gives $[i]_q!$.

For $j \geqslant i$, each $\boldsymbol{u}(j) = \boldsymbol{u}_j^* \cdot (j+2) = 12 \cdots (j+2)$ contributes an extra z to the weight, which gives a total of z^{n-i+1}.

On the other hand, for $j > i$, since $\boldsymbol{v}(j) \in \mathrm{Inc}(j, j+2)$, it is obtained from $\boldsymbol{v}^*(j) = 23 \cdots (j+1)$ by increasing some of its entries by 1. Note that $\boldsymbol{v}(j) \leqslant_w \boldsymbol{v}(j+1)$ for all j, the last element of $\boldsymbol{v}(i+1)$ has to be $i+3$. We must have that the last two elements of $\boldsymbol{v}(i+2)$ are $(i+3) \cdot (i+4)$, the last three elements of $\boldsymbol{v}(i+3)$ are $(i+3) \cdot (i+4) \cdot (i+5), \ldots$, and $\boldsymbol{v}(n)$ ends with $(i+3) \cdot (i+4) \cdots (n+2)$. All of these elements contribute an extra $q^{1+2+\cdots+n-i} = q^{\binom{n-i+1}{2}}$ to the weight.

Next consider the first i elements of each $\boldsymbol{v}(j)$ where $j \geqslant i$. Denote by $\boldsymbol{v}(j)^{\underline{\boldsymbol{i}}}$ the first i entries of $\boldsymbol{v}(j)$, then $\boldsymbol{v}(j)^{\underline{\boldsymbol{i}}} \in \mathrm{Inc}(i, p_j')$ where $j+1 \leqslant p_j' \leqslant j+2$.

Consider the construction as follows: Given the word $\boldsymbol{v}(j)^{\underline{\boldsymbol{i}}*} = 23 \cdots (i+1)$, construct a non-decreasing sequence of integers $0 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_{n-i+1} \leqslant i$. Add one to the last m_j entries in $\boldsymbol{v}(j)^{\underline{\boldsymbol{i}}*}$ to obtain $\boldsymbol{v}(j)^{\underline{\boldsymbol{i}}}$. Then each word contributes a increase of q^{m_j} to the total weight. Together there are $\begin{bmatrix} n+1 \\ n-i+1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ i \end{bmatrix}_q$ ways to do that.

In summary, the total weight increased is

$$\sum_{i=0}^{n+1} \begin{bmatrix} n+1 \\ i \end{bmatrix}_q [i]_q! q^{\binom{n-i+1}{2}} z^{n-i+1} = \sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot \frac{[n+1]_q!}{[k]_q!} \cdot z^k$$

as desired. $\square$

6 Concluding remarks

In this paper, we give combinatorial proofs of the Hankel determinant Eq. (1.1) for the cases $s = 0, 1, 2$. The closed forms of the determinant for other values of s remain open. Theorem 3 presents a simplified determinantal form for the computation. One natural question is, can we derive closed forms for the determinant for general s? Notice that in our proof of the case $s = 2$, the words are constructed based on the words in the case of $s = 1$, one may expect that the constructions for general s is recursive, that is, the words in the case of s is based on the words in the case of $s - 1$. Thus the idea of the RG-word decomposition may be used for general cases.

There are many other combinatorial structures giving q-Stirling numbers of the second kind such as rook placements [11], 0-1 tableaux [7] and juggling patterns [10], it would also be interesting to find combinatorial proofs of the determinantal formula using these structures.

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