Hamiltonians of Bipartite Walks

Qiuting Chen[a] Chris Godsil[a] Mariia Sobchuk[a] Hanmeng Zhan[b]

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Abstract

In this paper, we introduce a discrete quantum walk model called bipartite walks. Bipartite walks include many known discrete quantum walk models, like Grover's walks, vertex-face walks. For the transition matrix of a quantum walk, there is a Hamiltonian associated with it. We will study the Hamiltonians of the bipartite walks. Let S be a skew-symmetric matrix. We are mainly interested in the Hamiltonians of the form iS. We show that the Hamiltonian can be written as iS if and only if the adjacency matrix of the bipartite graph is invertible. We show that Grover's walks and vertex-face walks are special cases of bipartite walks. Via the Hamiltonians, phenomena of bipartite walks lead to phenomena of continuous walks. We show in detail how we use bipartite walks on paths to construct universal perfect state transfer in continuous walks.

Mathematics Subject Classifications: 05C90, 05E99

1 Introduction

Quantum walks are a quantum mechanical analogue of classical random walks. They provide a powerful tool for the study and development of quantum algorithms [4, 10]. Based on how time evolves, a quantum walk can be either continuous or discrete. For discrete quantum walks, there are several models that have been proposed and studied [1, 8, 10]. In this paper, the walks we focus on are called bipartite walks; they generalize many known models such as Grover's walks and vertex-face walks.

We turn to a description of bipartite walks. A discrete quantum walk is given by a unitary operator U on a complex vector space $\mathbb{C}^n$. We refer to U as the *transition matrix* of a discrete quantum walk. The state of the underlying quantum system is a unit vector in $\mathbb{C}^n$. If the initial state is z, then after k steps of the walk, the state is $U^k z$. This is a unit vector, and so the squared absolute values of its entries sum to 1. The outcome

[a]Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Ontario, Canada (q243chen@uwaterloo.ca, cgodsil@uwaterloo.ca,msobchuk@uwaterloo.ca).
[b]Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada (hzhan@wpi.edu)

of a measurement after k steps is an element i of $\{1,\ldots,n\}$, and the probabilty that the result is i is $|(U^k z)_i|^2$.

In our case, the state space is the space of complex functions on the edges of a bipartite graph G. We assume that X and Y are the two colour classes of G and using these we construct two partitions of $E(G)$. For the first partition, π_0, two edges are in the same cell if they have a vertex in common, and that vertex is in X. For the second partition π_1, two edges are in the same cell if they have a vertex in common, and that vertex is in Y. Each of these partitions determines a projection, namely the projection onto the functions on $E(G)$ that are constant on the cells of π_0 and π_1. We denote these projections by P and Q respectively. Then $2P-I$ and $2Q-I$ are unitary. (Geometrically they are reflections.) We define the transition matrix of the bipartite walk on G by

$$U := (2P-I)(2Q-I).$$

For each unitary matrix U, there are Hermitian matrices H such that

$$U = \exp(iH). \tag{1.1}$$

(We refer to H as a *Hamiltonian* of U.) Our goal in this paper is to study the Hamiltonians of bipartite walks.

For the discrete quantum walk governed by the unitary matrix U, by Equation 1.1, we know that there is a Hamiltonian H associated with it. When there is a real skew-symmetric S such that the Hamiltonian H is of the form $H = iS$, it can be viewed as the skew-adjacency matrix of an oriented weighted graph, which we call *the H-digraph*.

So far, most studies of the bipartite walk have been limited to the transition matrix and the behaviors of the walk [7, 9, 10]. In this paper, we study Hamiltonians of bipartite walks and H-digraphs associated with it. Spectral properties of the transition matrix is the main tool we exploit to study the Hamiltonian of U.

Consider bipartite walk defined on G. Let S be a skew-symmetric matrix. We are mainly interested in the case when the Hamiltonian H can be written as $H = iS$, which is not always true. We prove that the Hamiltonian H is of the form $H = iS$ if and only if the adjacency matrix of G is invertible.

We look into the bipartite walks defined on paths and even cycles. When G is a path on n vertices, the transition matrix of the bipartite walk is a permutation matrix. When $n \geqslant 4$ is even, the associated H-digraph is a weighted oriented K_{n-1}. When $n \equiv 3 \pmod 4$, the associated H-digraph is two copies of a weighted oriented $K_{\frac{n-1}{2}}$. Similar results can also be proved for the bipartite walk on even cycles.

There is a second class of quantum walks: *continuous quantum walks*. Here the state space is the space of complex functions on the vertices of a graph G. The walk is specified by a Hermitian matrix H with rows and columns indexed by the vertices of G (for example, the adjcency matrix of G). We then define transition matrices $U(t)$ by

$$U(t) := \exp(itH), \quad (t \in \mathbb{R}).$$

If the initial state of the walk is given by the unit vector z, the state at time t is $U(t)z$.

Studying the Hamiltonian of bipartite walks helps us to construct examples of continuous walks with desired properties. Consider continuous quantum walk on a graph G and the Hamiltonian is the adjacency matrix of G. If the walk has perfect state transfer between every pair of vertices of G, the walk has universal perfect state transfer. This is a rare and interesting phenomenon. Using the properties of bipartite walks on paths and cycles, we find a way to weight the edges of complete graphs such that the resulting weighted graph has universal perfect state transfer. This demonstrates how we can use the Hamiltonian and bipartite walks to construct some interesting but previously hard-to-find phenomenon in continuous walks.

Konno et al. in [7] introduce a family of discrete-time quantum walks, called two-partition model, which is based on two equivalence-class partitions of the computational basis. The two partitions used in the two-partition model do not necessarily give us two reflections. Bipartite walks are a special case of the two-partition model introduced by Konno et al. in [7]. Note that the paper by Konno et al. focuses on showing the unitary equivalence between the members of two-partition model while we study the Hamiltonian of the transition matrix of the bipartite walk in this paper.

On the other hand, many of the most commonly used discrete walks can be formulated as bipartite walks. This is one of the reasons why we choose to study bipartite walks. We will give a constructive proof to show that Grover's walk can be viewed as a special case of bipartite walk in Section 3. Besides Grover's walk, vertex-face walk model can also be viewed as a special case of bipartite walk. In Section 6, we show the equivalence relations between bipartite walks and vertex-face walks. The Hamiltonians obtained from vertex-face walks have some interesting properties, which have been studied extensively in [11]. Here we introduce those properties and rephrase them from perspective of bipartite walk in Section 6 and Section 7.

2 Preliminaries

We have described bipartite walks in previous section. Here we give a more detailed description of how we build the transition matrix of the bipartite walk on a given graph. Main purpose of this section is to set up the notations for later use.

Let G be a bipartite graph with two parts C_0, C_1. Now we define two partitions of the edges of G, denoted by π_0, π_1 respectively. If two edges have the same end x in C_0, then they belong to the same cell of π_0. Similarly, if two edges have the same end y in C_1, then they belong to the same cell of π_1.

Given a matrix M, we *normalize* it by scaling each column of M to a unit vector. Let P_0, P_1 be characteristic matrix of π_0, π_1 respectively and let $\widehat{P}_0, \widehat{P}_1$ denote the normalized P_0, P_1 respectively.

Let

$$P = \widehat{P}_0\widehat{P}_0^T, \quad Q = \widehat{P}_1\widehat{P}_1^T$$

be the projections onto the vectors that is constant on the cells of π_0, π_1 respectively. We

define the transition matrix of the bipartite walk over G to be

$$U = \left(2\widehat{P}_0\widehat{P}_0^T - I\right)\left(2\widehat{P}_1\widehat{P}_1^T - I\right) = (2P - I)(2Q - I).$$

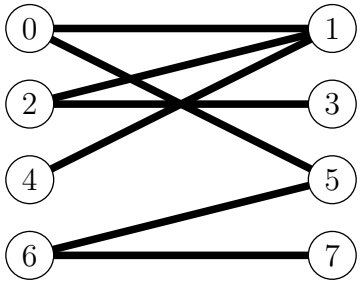


Figure 1: Bipartite graph on 8 vertices.

Now consider the bipartite graph G in Figure 1 as an example. We define a bipartite walk on G. The two parts of G are $C_0 = \{0, 2, 4, 6\}$ and $C_2 = \{1, 3, 5, 7\}$. For the partitions π_0, π_1, the edge $(0, 1), (0, 5)$ are in the same cell in π_0 and Edge $(0, 1), (2, 1), (4, 1)$ are in the same cell in π_1. We have that

$$\hat{P}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \hat{P}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and hence, the corresponding projections are

$$P = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The transition matrix of the bipartite walk on G is

$$U = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let C denote the characteristic matrix of the incidence relation between π_0, π_1 with its rows indexed by the cells of π_1 and its columns indexed by the cells of π_0 such that

$$C_{i,j} = 1$$

if there is an edge that belongs to both c_i in π_1 and c_j in π_0. Then we have that

$$C = P_1^T P_0$$

and normalized C is

$$\hat{C} = \widehat{P}_1^T \widehat{P}_0.$$

The adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} \mathbf{0} & C \\ C^T & \mathbf{0} \end{pmatrix}.$$

The incidence matrix and the normalized incidence matrix of the bipartite graph in Figure 1 are

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

3 Grover's walks are a special case

Grover's walk is a well-studied discrete quantum walk model. We are going to show that Grover's walk is a special case of bipartite walk model. That is, given a graph G, the transition matrix of the Grover's walk on G is the same as the transition matrix of the bipartite walk on the subdivision graph of G.

First, we define the transition matrix of Grover's walks. Given a undirected graph G, we can give directions to the edges of G such that the arc set of the directed G, denoted by $\overrightarrow{G}$, is

$$\mathcal{A} = \big\{(a,b),(b,a) \mid \{a,b\} \in E(G)\big\}.$$

Let $\alpha = (x,y)$ be an arc in $\mathcal{A}$, we say x is the head of α, denoted by $o(\alpha)$ and y is the tail of α, denoted by $t(\alpha)$. When $\alpha = (x,y)$, we define $\alpha^{-1} = (y,x)$. Define a matrix $D \in \mathbb{C}^{V(G)\times\mathcal{A}}$ such that

$$D_{x,\alpha} = \frac{1}{\sqrt{\deg(x)}}\delta_{x,t(\alpha)}.$$

So, we know that $D^*D \in \mathbb{C}^{\mathcal{A}\times\mathcal{A}}$ with

$$(D^*D)_{\alpha,\beta} = \begin{cases} \frac{1}{\deg(t(\alpha))} & \text{if } t(\alpha) = t(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

Let $R \in \mathbb{C}^{\mathcal{A}\times\mathcal{A}}$ denote the arc-reversal matrix, i.e.,

$$R_{\alpha,\beta} = \delta_{\alpha,\beta^{-1}}.$$

The transition matrix of the Grover's walk defined on G is

$$U_{\mathrm{GW}} = R(2D^*D - I).$$

Given a graph G, we define a new graph by subdividing every edge of G exactly once and we call the resulting graph *the subdivision graph* of G, denoted by $S(G)$. Now we are going to show the transition matrix of the Grover's walk defined on G is exactly the same as the transition matrix of the bipartite walk defined on $S(G)$.

Given a graph G, its subdivision graph $S(G)$ is a bipartite graph with parts

$$C_0 = V(S(G))\backslash V(G) = \{a_0, a_1, \ldots, a_m\},$$
$$C_1 = V(G) = \{v_1, v_2, \ldots, v_n\}.$$

Note that every edge $(v_i,v_j) \in E(G)$ is subdivided into two edges $(v_i,a_s),(v_j,a_s)$ in $S(G)$ for some $a_s \in C_0$. We also have that for every edge $(v_i,v_j) \in E(G)$, it contributes two arcs $(v_i,v_j),(v_j,v_i)$ in $\mathcal{A}$ of $\overrightarrow{G}$. Define $\eta : E(S(G)) \to \mathcal{A}$ to be

$$\eta(v_i,a_s) = (v_i,v_j),$$

where $(v_i,a_s),(v_j,a_s)$ are obtained from the edge (v_i,v_j) of G. This is not hard to see that η is a bijection.

Following the notations and the construction in Section 2, the transition matrix of the bipartite walk on $S(G)$ is

$$U_{\mathrm{BW}} = (2P - I)(2Q - I).$$

Note that rows and columns of U_{BW} are indexed by the edges of $S(G)$ and rows and columns of U_{GW} are indexed by arcs of $\overrightarrow{G}$. For each vertex $v_i \in C_1$, we have

$\deg_{S(G)}(v_i) = \deg_G(v_i)$ and $C_1 = V(G)$. Then using the bijection η, we can index edges of $S(G)$ and arcs of $\overrightarrow{G}$ such that

$$2Q - I = 2D^*D - I.$$

Also, note that for each vertex $a_i \in C_0$, we have $\deg_{S(G)}(a_i) = 2$. Two edges $(v_i, a_s), (v_j, a_s)$ of $S(G)$ share a vertex $a_s \in C_0$ if and only if $(v_i, v_j) \in E(G)$. We index rows and columns of P using the same indexing as we do for Q and we also index rows and columns of R use the same indexing as we do for D^*D. Consequently,

$$2P - I = R.$$

Thus, the transition matrix U_{BW} defined on $S(G)$ equals the transition matrix U_{GW} on G.

4 Spectrum of the transition matrix U

Spectral properties of the transition matrix U are the main machinery that we use to analyse the Hamiltonian of U. In this section, we present a complete characterization on the eigenvalues and eigenspaces of U. All the statements presented here are proved in [11] by Zhan in detail, so in this paper we omit the proofs. Note that here we use the same notations as defined before and so,

$$P = \widehat{P}_0\widehat{P}_0^T, \quad Q = \widehat{P}_1\widehat{P}_1^T, \quad \hat{C} = \widehat{P}_1^T\widehat{P}_0$$

and

$$U = (2P - I)(2Q - I).$$

Theorem 1 (Lemma **2.3.5** in [11])**.** *Let P, Q be projections on $\mathbb{C}^m$. The 1-eigenspace of U is*

$$(\mathrm{Col}(P) \cap \mathrm{Col}(Q)) \oplus (\ker(P) \cap \ker(Q))$$

and it has dimension

$$m - \mathrm{rk}(P) - \mathrm{rk}(Q) + 2\dim\left(\mathrm{Col}(P) \cap \mathrm{Col}(Q)\right).$$

Moreover,

$$\mathrm{Col}(P) \cap \mathrm{Col}(Q) = \mathrm{span}\{\mathbf{1}\}.$$

Theorem 2 (Lemma **2.3.6** in [11])**.** *The (-1)-eigenspace for U is*

$$(\mathrm{Col}(P) \cap \ker(Q)) \oplus (\ker(P) \cap \mathrm{Col}(Q))$$

and its dimension is

$$|C_0| + |C_1| - 2\,\mathrm{rk}(C).$$

Theorem 3 (Lemma **2.3.7** in [11]). *Let $\mu \in (0,1)$ be an eigenvalue of $\hat{C}\hat{C}^T$. Choose θ such that*

$$\cos\theta = 2\mu - 1.$$

The map

$$y \mapsto (\cos\theta + 1)\,\widehat{P}_1 y - \left(e^{i\theta} + 1\right)\widehat{P}_0\hat{C}^T y$$

is an isomorphism from μ-eigenspace of $\hat{C}\hat{C}^T$ to the $e^{i\theta}$-eigenspace of U, and the map

$$y \mapsto (\cos\theta + 1)\,\widehat{P}_1 y - \left(e^{-i\theta} + 1\right)\widehat{P}_0\hat{C}^T y$$

is an isomorphism from μ-eigenspace of $\hat{C}\hat{C}^T$ to the $e^{-i\theta}$-eigenspace of U.

Corollary 4 (Corollary **5.2.5** in [11]). *Let $\mu \in (0,1)$ be an eigenvalue of $\hat{C}\hat{C}^T$. Choose θ such that $\cos\theta = 2\mu - 1$. Let E_μ be the orthogonal projection onto the μ-eigenspace of $\hat{C}\hat{C}^T$. Set*

$$W := \widehat{P}_1 E_\mu \widehat{P}_1^T.$$

Then the $e^{i\theta}$-eigenmatrix of U is

$$\frac{1}{\sin^2(\theta)}\left((\cos\theta+1)W - (e^{i\theta}+1)PW - (e^{-i\theta}+1)WP + 2PWP\right),$$

and the $e^{-i\theta}$-eigenmatrix of U is

$$\frac{1}{\sin^2(\theta)}\left((\cos\theta+1)W - (e^{-i\theta}+1)PW - (e^{i\theta}+1)WP + 2PWP\right).$$

5 Hamiltonians

For every unitary matrix U, there exist Hermitian matrices H such that

$$U = \exp(iH).$$

We call such H a *Hamiltonian* of U. Since U is unitary, it has spectral decomposition

$$U = \sum_r e^{i\theta_r} E_r = \exp(iH),$$

and we can write

$$H = -i\sum_r \log(e^{i\theta_r})E_{\theta_r} = \sum_r \theta_r E_{\theta_r}.$$

For each eigenvalue $e^{i\theta_r}$ of U, we have that

$$\log(e^{i\theta_r}) = \log(e^{i\theta_r + 2k_r\pi})$$

for non-zero integer k_r and so, the choice of H is not unique. That is, the Hamiltonian of U is

$$H = \sum_{\theta_r}(\theta_r + 2k_r\pi)E_{\theta_r},$$

for any non-zero integer k_r. Note that k_r are not necessarily equal for all the θ_r.

Let S be a real skew-symmetric matrix and S can be viewed as the skew-adjacency matrix of a weighted oriented graph. When $H = iS$, we define the *H-digraph* to be the weighted oriented graph whose skew-adjacency matrix is S. This paper focuses on the case when the Hamiltonian can be written as $H = iS$ and studies the associated H-digraph.

For each eigenvalue $e^{i\theta_r}$ of U, if $-\pi < \theta_r \leqslant \pi$ and $k_r = 0$, the resulting unique Hamiltonian is called *principal Hamiltonian*. Let H_0 be the principal Hamiltonian. In general, if there is a real skew-symmetric S_0 such that $H_0 = iS_0$, the choice

$$H = H_0 + \sum_r 2k_r \pi E_{\theta_r}$$

for non-constant k_r, cannot be written as $H = iS$ for a real skew-symmetric S.

Unless explicitly stated otherwise, we take the principal Hamiltonian to be the Hamiltonian of U. Later in Corollary 6, we will show that there is a real skew-symmetric S such that $H = iS$ if and only if the adjacency matrix of the bipartite graph $A(G)$ is invertible.

Theorem 5. *Let U be the transition matrix of the bipartite walk on a bipartite graph G. Let H be the Hamiltonian of U and let E_{-1} be the projection onto the (-1)-eigenspace of U. Then there is a real skew-symmetric matrix S such that*

$$H = iS + \pi E_{-1},$$

Proof. Using the spectral decomposition

$$U = \sum_r e^{i\theta_r} E_r = \exp(iH),$$

we can write

$$H = -i \sum_r \log(e^{i\theta_r}) E_r = \sum_r \theta_r E_{\theta_r},$$

where $-\pi < \theta_r \leqslant \pi$. It follows that the 1-eigenspace of U corresponds to the 0-eigenspace of H and the (-1)-eigenspace of U corresponds to the π-eigenspace of H and $e^{i\theta_r}$-eigenspace gives θ_r-eigenspace of H.

Since G is bipartite, the adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} \mathbf{0} & C \\ C^T & \mathbf{0} \end{pmatrix}$$

for some 01-matrix C. Let $\hat{C}$ be denoted the normalized version of C and let $\mu \in (0, 1)$ be an eigenvalue of $\hat{C}\hat{C}^T$. Choose θ such that $\cos\theta = 2\mu - 1$. Let F_μ be the orthogonal projection onto the μ-eigenspace of $\hat{C}\hat{C}^T$. Set

$$W := \widehat{P}_1 F_\mu \widehat{P}_1^T.$$

By Corollary 4, we have that

$$\begin{aligned} H &= \sum_{\theta_r \neq \{1,-1\}} \theta_r \left(E_{\theta_r} - E_{-\theta_r}\right) + \pi \cdot E_{-1} \\ &= \sum_{\theta_r \neq \{1,-1\}} \theta_r \left(-\frac{2i}{\sin(\theta)}(PW - WP)\right) + \pi \cdot E_{-1}. \end{aligned}$$

Since $\hat{C}\hat{C}^T$ is real and symmetric, we know that the orthogonal projection onto its μ-eigenspace F_μ is real and symmetric. It follows that $W = \widehat{P}_1 F_\mu \widehat{P}_1^T$ is real and symmetric. So the matrix $PW - WP$ is real. Set

$$S = \sum_{\theta_r \neq \{1,-1\}} \theta_r \left(-\frac{2}{\sin(\theta_r)}(PW - WP)\right)$$

and we know that S is skew-symmetric. □

Corollary 6. *Let U be the transition matrix of the bipartite walk on a bipartite graph G. Let S be a real skew-symmetric matrix. Then the Hamiltonian H of U is of the form $H = iS$ if and only if $A(G)$ is invertible.*

Proof. Since P, Q are real matrices, it follows from Theorem 2 that E_{-1}, the projection onto the (-1)-eigenspace of U, is a real matrix. By Theorem 5, there is a real skew-symmetric matrix S such that

$$H = iS + \pi E_{-1}.$$

So to prove this corollary, it is sufficient to prove that $E_{-1} = 0$ if and only if $A(G)$ is invertible.

Now consider the (-1)-eigenvalue of U. From Theorem 2 we know that the dimension of (-1)-eigenspace of U is

$$|C_0| + |C_1| - 2\,\mathrm{rk}(C).$$

This implies that $E_{-1} = 0$ if and only if

$$|C_0| + |C_1| - 2\,\mathrm{rk}(C) = 0.$$

Since $\mathrm{rk}(P_0) = |C_0|$ and $\mathrm{rk}(P_1) = |C_1|$ and $C = P_1^T P_0$, we get that

$$\mathrm{rk}\, C \leqslant \min\{|C_0|, |C_1|\}.$$

Thus, $E_{-1} = 0$ if and only if $\mathrm{rk}(P_0) = \mathrm{rk}(P_1) = \mathrm{rk}(C)$, which is equivalent to requiring that C is invertible. Therefore we can conclude that there is a real skew-symmetric S such that $H = iS$ if and only if $A(G)$ is invertible. □

Let $E_{\theta_r}, E_{-\theta_r}$ be the corresponding eigenprojections of eigenvalue $e^{i\theta_r}, e^{-i\theta_r}$ of U. Since E_{θ_r} are Hermitian, we have that

$$E_{\theta_r} = \overline{E_{-\theta_r}}.$$

It follows that when $A(G)$ is invertible, the Hamiltonian

$$H = \sum_r \theta_r \left(E_{\theta_r} - \overline{E_{\theta_r}}. \right)$$

has zero diagonal, which implies that the H-digraph has no loops.

We have proved that when -1 is an eigenvalue of U, there is no skew-symmetric matrix S such that its Hamiltonian is in the form $H = iS$. So when U has eigenvalue -1, we consider instead the Hamiltonian of U^2 and the H-digraph obtained from the Hamiltonian of U^2.

6 Vertex-Face walks

Bipartite walks can be used to generalize many known walk models and one of them is the vertex-face walk. Here we show that vertex-face walk can be viewed as a special case of bipartite walk. That is, given a embedding $\mathcal{M}$ of a graph, the transition matrix of the bipartite walk defined on the vertex-face incidence graph of $\mathcal{M}$ is the same as the transition matrix of vertex-face walk defined on $\mathcal{M}$. As shown in [11], the Hamiltonian raised from vertex-face walk has many interesting properties, some of which will be presented using the bipartite walk language in this section and the next section.

A *surface* is a connected compact Hausdorff topological space S which is locally homeomorphic to an open disc in the plane. A simple arc in S is the image of a one-to-one continuous function $f : [0, 1] \to S$. The arc $\alpha = f([0, 1])$ is said to join its endpoint $f(0)$ and $f(1)$ and we use $(f(0), f(1))$ to denote the arc α. A graph G is embedded in S if the vertices of G are distinct elements of S and every edge of G is a simple arc connecting in S the two vertices which it joins in G, such that its interior is disjoint from other edges and vertices. An embedding of G in S is an isomorphism of G with a graph $\mathcal{M}$ embedded in S. In this case, $\mathcal{M}$ is a representation of G in S. The component of $S\backslash\mathcal{M}$ are the *faces* of the embedding. When each face of the embedding is a cycle, we say the embedding is *circular*. In this paper, when we say the embedding of G in S, we refer to $\mathcal{M}$. All the definitions and terms regarding graph embeddings used in this paper can be found in [6].

Zhan [11, Section 5.1] defines a new model of discrete quantum walk, the *vertex-face walk*. Let $\mathcal{M}$ be a circular embedding of graph G on an orientable surface. Note that here the tail of the arc (a, b) is vertex a. Let M denote the arc-face incidence matrix, i.e., M is a 01-matrix with its rows and columns indexed by the arcs and faces of the the embedding $\mathcal{M}$ and

$$M_{i,j} = 1$$

if the arc i is on the face j. In a similar fashion, we define the arc-tail incidence matrix, which is denoted by N. The transition matrix of vertex-face walk on $\mathcal{M}$ is

$$U := \left(2\widehat{M}\widehat{M}^T - I\right)\left(2\widehat{N}\widehat{N}^T - I\right),$$

where $\widehat{M}, \widehat{N}$ is the matrices obtained from M, N respectively by scaling each column to a unit vector.

The face-vertex incidence graph X of the embedding $\mathcal{M}$ is a bipartite graph and two parts X_0, X_1 of X are labelled by the faces and the vertices of $\mathcal{M}$ respectively, i.e.,

$$X_0 = \{f_0, f_1, \ldots, f_n\}, \quad X_1 = \{v_0, v_1, \ldots, v_m\}.$$

One vertex f_i in X_0 is adjacent to a vertex v_j in X_1 if the corresponding vertex v_j is on the corresponding face f_i in $\mathcal{M}$. The graph in Figure 2b is the face-vertex incidence graph of the circular embedding of K_4 shown in Figure 2a. We can view the vertex-face walk on the circular embedding $\mathcal{M}$ as a bipartite walk by considering the bipartite walk over the face-vertex incidence graph of $\mathcal{M}$.

Now we show that the transition matrix of vertex-face walk on $\mathcal{M}$ is the same as the transition matrix of the bipartite walk on the face-vertex incidence graph G of $\mathcal{M}$. Since $\mathcal{M}$ is a circular orientable embedding, the edges of the face-vertex incidence graph G correspond to arcs of the embedding $\mathcal{M}$. To see this, we construct a bijection between between arcs of $\mathcal{M}$ and edges of G. Let (a, b) be an arc in $\mathcal{M}$ and since $\mathcal{M}$ is circular embedding on an orientable surface, there is a unique face f_i that contains arc (a, b). The map τ:

$$(a, b) \mapsto (a, f_i)$$

gives the desired bijection.

The arc-face incidence matrix M of the embedding $\mathcal{M}$ is exactly the characteristic matrix of the edge-partition matrix P_0 of the vertex-face incidence graph based on X_0. The rows of M are indexed by the arcs of $\mathcal{M}$ with the ordering $\sigma_{\mathcal{A}} = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ and the columns of M are index by the faces of $\mathcal{M}$ with the ordering $\sigma_{\mathcal{F}} = \{f_0, f_1, \ldots, f_n\}$. Now we define a ordering of edges of G:

$$\tau(\sigma_{\mathcal{A}}) = \{\tau(\alpha_0), \tau(\alpha_1), \ldots, \tau(\alpha_m)\}.$$

Let the rows of P_0 be indexed by the edges of G according to the ordering $\tau(\sigma)$ and let the columns of P_0 be indexed by vertices of X_0 with the ordering where vertices are in order with the faces in the ordering of $\sigma_{\mathcal{F}}$. Then we have that

$$M = P_0.$$

Now we show that the arc-tail incidence matrix N of the embedding $\mathcal{M}$ is exactly the characteristic matrix P_1 of the edge-partition matrix of the vertex-face incidence graph according to X_1. We index the rows of N using the same ordering $\sigma_{\mathcal{A}}$ as in M. The columns of N are indexed by vertices of $\mathcal{M}$ according to the order $\sigma_{\mathcal{V}}$. Note that every vertex is a tail of some arc α of $\mathcal{M}$. Now we index the rows of P_1 according to the order $\sigma_{\tau(\mathcal{A})}$. Let $\mathrm{id} : V(\mathcal{M}) \to V(G)$ be

$$v_i \mapsto v_i.$$

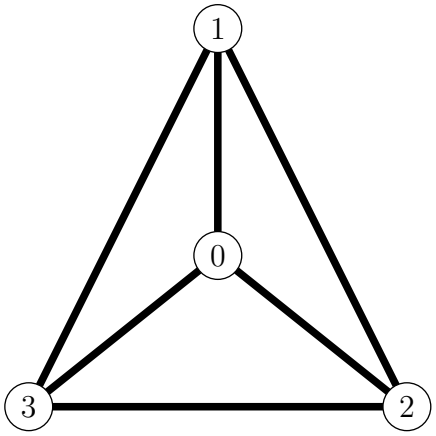
Then we index the columns of P_1 according to $\mathrm{id}\,(\sigma_{\mathcal{V}})$. Then we have that

$$N = P_1.$$

Hence, using the same notations as in Section 2, we have shown that

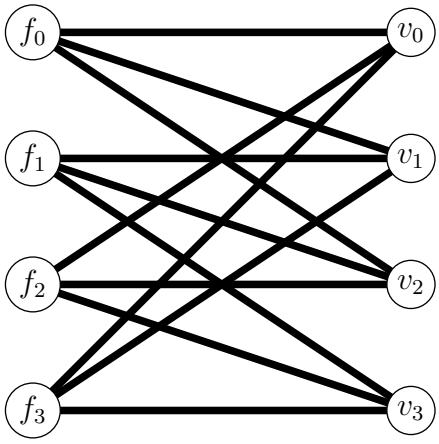
$$\left(2\widehat{P}_0\widehat{P}_0^T - I\right)\left(2\widehat{P}_1\widehat{P}_1^T - I\right) = \left(2\widehat{M}\widehat{M}^T - I\right)\left(2\widehat{N}\widehat{N}^T - I\right),$$

i.e., the transition matrix of the bipartite walk on the incidence graph of the embedding $\mathcal{M}$ equals the transition matrix of the vertex-face walk on $\mathcal{M}$.



(a) The circular embedding of K_4.
The facial walks on K_4 embedding above:

$$f_0 = \{(0,1),(1,2),(2,0)\}$$
$$f_1 = \{(1,3),(3,2),(2,1)\}$$
$$f_2 = \{(0,2),(2,3),(3,0)\}$$
$$f_3 = \{(0,3),(3,1),(1,0)\}$$



(b) The face-vertex incidence graph of the planar embedding of K_4.

Figure 2: The circular embedding of K_4 and its corresponding vertex-face incidence graph.

In [11], Zhan focuses on the circular orientable embedding of graph G such that both

G and its dual graph are regular. The embedding $\mathcal{M}$ has type (k,l) if each vertex has degree l and each faces uses k vertices. Note that a vertex-face walk over a (k,l)-type embedding $\mathcal{M}$ corresponds to a bipartite walk on a (k,l)-regular bipartite graph that is the vertex-face incidence graph of $\mathcal{M}$.

Theorem 7 (Theorem **6.5.4** in [5])**.** *Let G be a semi-regular bipartite graph with degree (k,l) and P_0, P_1 denote its two parts. Let π_0, π_1 denote the partitions of edges of G according to P_0, P_1 respectively. Let U be the bipartite walk transition matrix for G. Then*

$$U^2 = \exp\left(\gamma(U - U^T)\right)$$

for some real number γ if and only if G has four or five distinct eigenvalues. Moreover,

$$S = \frac{kl}{4}(U^T - U)$$

is the skew-adjacency matrix of some oriented graph on the edges of G.

Let $c_{0,k}$ denote the cell of partition π_0 containing edge e_k and similarly, $c_{1,k}$ denote the cell of partition π_1 containing edge e_k. Then we have

$$S_{i,j} = \begin{cases} 1, & \text{if } |c_{0,i} \cap c_{1,j}| = 1 \text{ and } |c_{1,i} \cap c_{0,j}| = 0, \\ -1, & \text{if } |c_{0,i} \cap c_{1,j}| = 0 \text{ and } |c_{1,i} \cap c_{0,j}| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A partial geometric design with parameters (d,k,t,c) is a point-d-regular and block-k-regular design, where for each point-block pair (p,B), the number of incident point-block pairs

$$|\{(p',B') : p' \neq p, B' \neq B, p' \in B, p \in B'\}|$$

equals c or t, depending on whether p is in B or not. In Section 6.5 [5], Godsil and Zhan have showed that when G is an incidence graph of a partial geometric design, then we have that

$$U^2 = \exp\left(\gamma(U - U^T)\right)$$

for some real number γ.

7 Vertex-Face walks on complete graphs

In [2], Biggs states that K_n has a regular embedding if and only if n is a prime power and every regular embedding of K_n must arise from the rotation system stated in [11].

Lemma 8 (Theorem **5.6.2** in [11])**.** *Let $n = p^k$ for some prime p. Let g be a primitive generator of the finite field $\mathbb{F}$ of order n. For each element u in $\mathbb{F}$, define the cyclic permutation*

$$\pi_u = \{v + g^0, v + g^1, \ldots, v + g^{n-2}\}.$$

The rotation system $\{\pi_u : u \in V(K_m)\}$ gives a circular embedding of K_n.

In the case of H-digraphs arised from the vertex-face walk on K_n, we know that the skew-adjacency matrix of H-digraph $A(\overrightarrow{H})$ is indexed by arcs of K_n. Let f_{ab} denote the unique face that contains arc (a, b). From the proof of Theorem 6.5.4 in [5], we have that

$$A(\overrightarrow{H})_{(a,b),(c,d)} = \begin{cases} 1, & \text{if } c \in f_{ab} \text{ and } a \notin f_{cd}, \\ -1, & \text{if } a \in f_{cd} \text{ and } c \notin f_{ab}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that in a self-dual circular embedding of K_n, each face consists of $n-1$ distinct vertices, which implies that each face misses a unique vertex of K_n.

We use $LD(K_n)$ to denote the line digraph of K_n.

Theorem 9. *The H-digraphs Z_n obtained from the vertex-face walks of a self-dual embedding of K_n is the line digraphs of K_n.*

Proof. We construct an isomorphism from Z_n to $LD(K_n)$. Define a map $f : V(Z_n) \to V(LD(K_n))$ as

$$(a, b) \mapsto (u, a),$$

where u is the unique vertex missed by f_{ab}. First we show that f is a homomorphism. Say

$$f(a, b) = (u, a), \quad f(c, d) = (v, c),$$

which implies that u is the unique vertex missed by f_{ab} and v is the unique vertex missed by f_{cd}. We know that there is an arc from (a, b) to (c, d) in Z_n if and only if

$$c \in f_{ab} \text{ and } a \notin f_{cd}.$$

Since each face miss a unique vertex in the circular embedding of K_n, we must have that

$$a = v,$$

which means that there is an arc from $f(a, b)$ to $f(c, d)$ in $LD(K_n)$. Thus, the map f is indeed a homomorphism.

Now we prove that f is a bijection and since $LD(K_n)$ is finite, it suffices to prove that f is an injection. Assume towards contradictions that two distinct arcs (a, b) and (a', b') get mapped to (x, y) by the map f. Then by how we define the map f, we know that

$$a = a' = y.$$

The vertex x is missed by f_{ab} and $f_{a'b'} = f_{ab'}$. Since the faces here arised from facial walks on the circular embedding of K_n, we must have that

$$(a, b) = (a', b').$$

This means that f has to be an injection and hence, a bijection. Therefore, we can conclude that the map f gives an isomorphism from Z_n to $LD(K_n)$. □

Theorem 10 (Theorem **5.6.3** in [11])**.** *Let n be a prime power. Let U be the transition matrix of the vertex-face walk for a regular embedding of K_n. Then there is a $\gamma \in \mathbb{R}$ such that*

$$U = \exp\left(\gamma(U^T - U)\right).$$

Further $U^T - U$ is a scalar multiple of the skew-adjacency matrix of an oriented graph, which

(i) has $n(n-1)$ vertices,

(ii) is $(n-2)$-regular, and

(iii) has exactly three eigenvalues: 0 and $\pm i\sqrt{n(n-2)}$

We rephrase Theorem 9 in terms of bipartite walk and we get the following theorem.

Theorem 11. *Let G_n be a $(n-1)$-regular bipartite graph with each part of size n. Then the H-digraph obtained from the bipartite walk on G_n is the line digraph of K_n.*

Proof. Since there is every cell of π_1 miss a unique vertex in C_0 and every cell of π_0 misses a unique vertex in C_1, the proof of Theorem 9 applies here. □

8 Paths and even cycles

The vertex-face incidence graph of a cellular embedding of a graph must have degree at least three for each vertex. So neither a path nor a cycle can be a bipartite graph raised from the vertex-face incidence relation of an circular embedding. In this section, we discuss the bipartite walk defined on paths and even cycles.

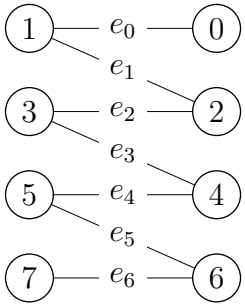


Figure 3: P_8.

We label the vertices of P_n as $v_0, v_1, \ldots, v_{n-1}$ accordingly from the leftmost vertices to the rightmost vertices of P_n. Note that v_0, v_{n-1} are the only two vertices of degree 1 with all the others of degree 2. Partition π_0 is the partition of edges such that edges with the same end at a vertex in $\{v_1, v_3, \ldots, v_{n-1}\}$ are in the same cell of π_0. Partition π_1 is the partition of edges such that edges with the same end at a vertex in $\{v_0, v_2, \ldots, v_{n-2}\}$ are in the same cell of π_1. Edge e_i is the edge between v_i, v_{i+1} for all integer $0 \leqslant i \leqslant n-2$.

Recall that P, Q are the projections onto the vectors that is constant on the cells of π_0, π_1 respectively. Let c_i denote the characteristic vector of the edges adjacent to vertex i. The column space of Q is

$$\mathrm{Col}(Q) = \mathrm{span}\{c_0, c_2, \ldots, c_{n-2}\},$$

The matrix $2Q - I$ is a reflection about the column space of Q, which is the span of cells of π_1. If two edges belong to the same cell, then they are the "cellmate" of each other.

Note that every vertex of a path has degree $\leqslant 2$, which means that each edge has at most one cellmate in the partitions. For each $0 \leqslant i \leqslant n-2$, let e_j be the cellmate of e_i in π_1. Using that each cell in π_0, π_1 has size $\leqslant 2$, we have that

$$(2Q - I)e_i = e_j.$$

Similarly, if e_i, e_j are cellmates in π_0, then we have that

$$(2P - I)e_i = e_j.$$

Here both reflections $2P - I$ and $2Q - I$ is permutation matrices. Thus, the transition matrix $U = (2P - I)(2Q - I)$ of bipartite walk on P_n is a permutation matrix such that for each integer $0 \leqslant i \leqslant n-2$,

$$Ue_i = \begin{cases} e_{i+2}, & \text{if } i \text{ is odd and } i \neq n-3; \\ e_{i-2}, & \text{if } i \text{ is even and } i \neq 0; \\ e_1, & \text{if } i = 0; \\ e_{n-2}, & \text{if } i = n-3. \end{cases} \tag{8.1}$$

Theorem 12. *The transition matrix of the bipartite walk on P_n corresponds to a $(n-1)$-cycle permutation whose cycle form is*

$$(e_0, e_1, e_3, \ldots, e_{n-3}, e_{n-2}, e_{n-4}, \ldots, e_2)\,.$$

Proof. It follows from the discussion above. □

For example, the transition matrix of the bipartite walk on P_8 is

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This correspond to the permutation (0135642) in S_7 and we have that

$$U^7 = I.$$

Since $U(P_8)$ is a permutation matrix of order 7, it is easy to see that every edge of P_8 can be mapped to any other edges within 7 steps in the bipartite walk. This is an interesting phenomenon called *universal perfect state transfer*. Note that if U is the transition matrix of bipartite walk on P_n, then

$$U^{n-1} = I,$$

which implies that for every n, the bipartite walk on P_n has the universal perfect state transfer. We will discuss this property further in the next section.

Cyclic permutation matrix U is of order $n-1$, then it has eigenvalue

$$\lambda_k = \left(e^{\frac{2\pi i}{n-1}}\right)^k$$

with eigenvector

$$f_k = \begin{pmatrix} 1 & \lambda_k^{-1} & \lambda_k & \lambda_k^{-2} & \lambda_k^2 & \cdots & \lambda_k^{-(n-2)/2} & \lambda_k^{(n-2)/2} \end{pmatrix}^T, \tag{8.2}$$

for $k = 0, \ldots, n-2$. The λ_k-eigenspace of U is

$$E_{\lambda_k} = \frac{1}{n-1} f f^*.$$

Note that $E_1 = \frac{1}{n-1} J$.

From the eigenvectors (8.2) of U, we know that if s, t are integers in $\{1, \ldots, n-2\}$, we have that

$$(E_{\lambda_r})_{s,t} = \begin{cases} \frac{1}{n-1}(\lambda_r)^{-\frac{s+1}{2}}(\lambda_r)^{\frac{t+1}{2}} & \text{if both } s,t \text{ are odd;} \\ \frac{1}{n-1}(\lambda_r)^{\frac{s}{2}}(\lambda_r)^{\frac{t+1}{2}} & \text{if } s \text{ is even and } t \text{ is odd;} \\ \frac{1}{n-1}(\lambda_r)^{-\frac{s+1}{2}}(\lambda_r)^{-\frac{t}{2}} & \text{if } s \text{ is odd and } t \text{ is even;} \\ \frac{1}{n-1}(\lambda_r)^{\frac{s}{2}}(\lambda_r)^{-\frac{t}{2}} & \text{if both } s,t \text{ are even.} \end{cases} \tag{8.3}$$

Theorem 13. *For an even $n \geqslant 4$, the H-digraph obtained from the bipartite walk on P_n is an oriented K_{n-1}.*

Proof. As the discussion above, the transition matrix of bipartite walk on P_n has spectral decomposition

$$U = \sum_{k=0}^{n-2} \lambda_k E_{\lambda_k},$$

where

$$\lambda_k = \left(e^{\frac{2\pi i}{n-1}}\right)^k.$$

When n is even, the Hamiltonian of U is

$$H = \sum_{k=0}^{(n-2)/2} \frac{2k\pi}{n-1} \left(E_{\lambda_k} - \overline{E_{\lambda_k}}\right).$$

To prove that the H-digraph is an oriented complete graph, we show that the Hamiltonian H has non-zero off-diagonal entries. As shown above that the eigenvector of U with eigenvalue λ_k is of the form 8.2, each row of E_{λ_k} is a permutation of its first row, which implies that each row of H is a permutation of its first row. So in order to prove that all the off-diagonal entries of H are non-zero, it is sufficient to prove that

$$H_{0,t} \neq 0$$

for all $t \neq 0$.

Based on the formula of the (s,t)-th entry of E_{λ_r} shown in 8.3 we have that for $r \in \{0,1,2,\ldots,n-2\}$ and, $s,t \in \{0,1,\ldots,n-2\}$, we have that

$$\left(E_{\lambda_r} - \overline{E_{\lambda_r}}\right)_{s,t} = \begin{cases} \frac{2}{n-1}\sin\left(\frac{2\pi r}{n-1}\cdot\frac{t-s}{2}\right)i, & \text{if both } s,t \text{ are odd;} \\ \frac{2}{n-1}\sin\left(\frac{2\pi r}{n-1}\cdot\frac{s+t+1}{2}\right)i & \text{if } s \text{ is even and } t \text{ is odd;} \\ \frac{2}{n-1}\sin\left(\frac{2\pi r}{n-1}\cdot\frac{-t-s-1}{2}\right)i, & \text{if } s \text{ is odd and } t \text{ is even;} \\ \frac{2}{n-1}\sin\left(\frac{2\pi r}{n-1}\cdot\frac{s-t}{2}\right)i, & \text{if both } s,t \text{ are even.} \\ 0 \quad \text{if } s=t. \end{cases}$$

Then entries of the first row of H are

$$(H)_{0,t} = \sum_{k=0}^{(n-2)/2} \frac{2k\pi}{n-1}\left(E_{\lambda_k}\right)_{0,t} = \begin{cases} \sum_{k=0}^{(n-2)/2} \frac{4k\pi}{(n-1)^2}\sin\left(\frac{2k\pi}{n-1}\cdot\frac{t+1}{2}\right) & \text{if } t \text{ is odd;} \\ \sum_{k=0}^{(n-2)/2} \frac{4k\pi}{(n-1)^2}\sin\left(\frac{2k\pi}{n-1}\cdot\frac{-t}{2}\right) & \text{if } t \text{ is even;} \\ 0 \quad \text{if } t=0 \end{cases}$$

When $n = 2a+2$ for some integer $a \geqslant 1$, then for each positive odd integer b, we have that

$$\sum_{k=0}^{(n-2)/2} \frac{2k\pi}{n-1}\sin\left(\frac{2k\pi}{n-1}\cdot b\right) = \frac{\pi\csc\left(\frac{b\pi}{2a+1}\right)\left(2(a+1)+\sin\left(\frac{2b\pi(a+1)}{2a+1}\right)\csc\left(\frac{b\cdot\pi}{2a+1}\right)\right)}{4a+2} \tag{8.4}$$

and for each positive even integer b, we have that

$$\sum_{k=0}^{(n-2)/2} \frac{2k\pi}{n-1}\sin\left(\frac{2k\pi}{n-1}\cdot b\right) = \frac{\pi\csc\left(\frac{b\pi}{2a+1}\right)\left(-2(a+1)+\sin\left(\frac{2b\pi(a+1)}{2a+1}\right)\csc\left(\frac{b\cdot\pi}{2a+1}\right)\right)}{4a+2}. \tag{8.5}$$

Since the sine function is an odd function, we only need to show that $H_{0,t} \neq 0$ for all odd $1 \leqslant t \leqslant \frac{n}{2}$. Since $\csc(x) \neq 0$ over all its domain and when $1 \leqslant b \leqslant a+1$,

$$\sin\left(\frac{2b\pi(a+1)}{2a+1}\right)\csc\left(\frac{b\cdot\pi}{2a+1}\right) \pm 2(a+1) \neq 0.$$

The sum shown in 8.5 and 8.4 are non-zero for all $1 \leqslant b \leqslant a+1$. Thus, we have that

$$(H)_{0,t} \neq 0$$

for all $t \neq 0$. Therefore, we can conclude that the H-digraph is an oriented K_{n-1}. □

Note that when n is odd, the adjacency matrix of P_n is not invertible and so we consider the Hamiltonian of U^2. When $n = 3$, the Hamiltonian of U^2 is zero matrix. When $n \equiv 1 \pmod 4$, the square of its transition matrix U^2 still has -1 as an eigenvalue, which implies that there is no real skew-symmetric S such that Hamiltonian of U^2 is of the form iS. So here, we omit the case when $n \equiv 1 \pmod 4$.

Corollary 14. *When $n \equiv 3 \pmod 4$, let*

$$U^2 = \exp(iH),$$

then H is the weighted skew adjacency matrix of two copies of oriented $K_{\frac{n-1}{2}}$.

Proof. By Theorem 12, we know that U^2 corresponds to two $\left(\frac{n-1}{2}\right)$-cycles. Each $\left(\frac{n-1}{2}\right)$-cycle is equivalent to the permutation associated with the transition matrix of $P_{\frac{n+1}{2}}$. The result follows from Theorem 13. □

Even cycles are another class of bipartite graphs that cannot be raised from the vertex-face incidence relation of a circular embedding.

For an even integer n, consider a path P_n with the same labelling as before and add an edge e_{n-1} between v_0, v_{n-1}, which gives us a even cycle C_n. Partition π_0 are the partition of edges based on vertices $\{v_1, v_3, \ldots, v_{n-1}\}$ and partition π_1 are the partition of edges based on vertices $\{v_0, v_2, \ldots, v_{n-2}\}$.

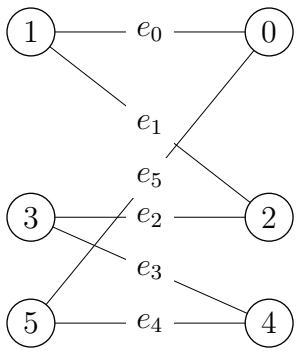


Figure 4: C_6.

When n is even and U is the transition matrix of bipartite walk on C_n, using the same argument as we do when we discuss the transition matrix of bipartite walk on paths, we have that

$$Ue_i = \begin{cases} e_{i+2 \pmod n} & \text{if } i \text{ is odd;} \\ e_{i-2 \pmod n} & \text{if } i \text{ is even.} \end{cases} \tag{8.6}$$

Theorem 15. *When n is even, the transition matrix U of the bipartite walk on C_n is a cyclic permutation matrix of order $n/2$.*

Proof. The mapping relation 8.6 implies that U is a cyclic permutation whose cycle form is

$$(e_0, e_{n-2}, \cdots, e_2)(e_1, e_3, \ldots, e_{n-1}). \qquad \square$$

Note that eigenvalues of C_n are

$$\left\{ 2\cos\left(\frac{2\pi k}{n}\right) : k \in \{0, 1, \ldots, n-1\} \right\}.$$

So when $n \equiv 0 \pmod 4$, the adjacency matrix of C_n is not invertible and we consider the Hamiltonian of U^2 instead.

Corollary 16. *Let U be the transition matrix of bipartite walk on C_n for some even n. When $n \equiv 2 \pmod 4$, let H be the Hamiltonian of U, then the corresponding H-digraph is two copies of a weighted oriented $K_{\frac{n}{2}}$. When $n \equiv 0 \pmod 4$ and $n \geqslant 12$, let H be the Hamiltonian of U^2, then the corresponding H-digraph is three copies of a weighted oriented $K_{\frac{n}{4}}$.*

Proof. From Theorem 15, the transition matrix of U is two $\frac{n}{2}$-cycles and each cycle is the permutation associated with the transition matrix of bipartite walk on $P_{\frac{n}{2}+1}$. Results follow from Theorem 13 and Corollary 14. $\square$

Note that when $n = 4$, the Hamiltonian of U is zero matrix. When $n = 8$, the transition matrix U and U^2 both have -1 as eigenvalues. There is no real skew-symmetric S such that the Hamiltonian of U or the Hamiltonian of U^2 is of the form iS and so, we omit the case when $n = 8$.

9 Universal PST

Let U be the transition matrix of the continuous walk defined over graph G, then we say there is perfect state transfer from state a to state b if

$$|U(t)_{a,b}| = 1.$$

A graph G has universal perfect state transfer if it has perfect state transfer between every pair of its vertices. According to Cameron et al. in [3], the only known graphs that have universal perfect state transfer are oriented K_2, C_3 with constant weight i assigned on each arc.

In this section, we show that bipartite walk can help us to construct weighted oriented graphs where the continuous quantum walk has universal perfect state transfer. Note that

when we talk about continuous walks on weighted graph, the Hamiltonian is the weighted adjacency matrix A of the graph, i.e., the transition matrix is of the form

$$\exp(iA).$$

If the transition matrix U of a bipartite walk is a permutation matrix with finite order, then its H-digraph has universal perfect state transfer.

Lemma 17. *Let G be a connected bipartite walk. The transition matrix of the bipartite walk on G is a permutation matrix if and only if every vertex of G has degree either 1 or 2.*

Proof. Here, we use the same notations as defined in Section 2. If every vertex of G has degree either 1 or 2, using the same notations as before, then both $2P - I$ and $2Q - I$ are permutation matrices. Hence, the transition matrix U is also a permutation matrix.

For the other direction, note that $2P - I, 2Q - I$ are reflections about the spaces spanned by characteristic vectors of cells of π_0, π_1 respectively and cells in one partition are disjoint. Then in order for U to map an edge e_i to another edge e_j, the size of each cell of both partitions π_1, π_2 cannot be greater than two. □

We have shown in Theorem 12 that the transition matrix of the bipartite walk over P_n for some even n is a permutation matrix with finite order. We can use this to produce weighted graphs over which continuous walks have universal perfect state transfer.

The following theorem follows directly from the fact that $U^{n-1} = I$ and Theorem 13.

Corollary 18. *Let n be an even integer. Let s, t be distinct integer in $\{0, \ldots, n-2\}$. we define*

$$\alpha = \begin{cases} \frac{t-s}{2}, & \text{if both } s,t \text{ are odd;} \\ \frac{s+t+1}{2} & \text{if } s \text{ is even and } t \text{ is odd;} \\ \frac{-t-s-1}{2}, & \text{if } s \text{ is odd and } t \text{ is even;} \\ \frac{s-t}{2}, & \text{if both } s,t \text{ are even.} \end{cases}.$$

When n is even, the edge (s, t) of K_{n-1} is assigned with weight

$$\frac{2}{n-1} \sum_{r=1}^{\frac{n}{2}-1} \frac{2\pi r}{(n-1)} \sin\left(\frac{2\pi r}{n-1}\alpha\right)$$

for all distinct $s, t \in \{0, \ldots, n-2\}$. Let A be the weighted adjacency matrix of the resulting weighted K_{n-1}. Then the continuous walk with transition matrix $\exp(iA)$ has universal perfect state transfer and every state will get transferred perfectly to any other state within time $t \leqslant n - 1$.

10 Open questions

Since continuous quantum walks whose Hamiltonians are symmetric, perfect state transfer is symmetric. That is, in continuous walks, there exists time t when there is perfect state transfer from state a to b and from state b to a. However, perfect state transfer in the discrete quantum walk is not necessarily symmetric. Because the transition matrices of discrete quantum walks are not symmetric in general, there is no guarantee that there exists a positive integer k such that at k-th step there is perfect state transfer between two states. In fact, there may be cases where there is perfect state transfer from state a to state b while there is no perfect state transfer from state b to state a.

Recall that the transition matrix of the bipartite walk defined on the graph in Figure 1 is

$$U = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

State e_i is the characteristic vector of i. It is easy to see that there is perfect state transfer from state e_1 to e_6 at step $k = 1$. But up to $k = 300000$ steps, there is no perfect state transfer observed from e_6 to e_1. We suspect that there is no perfect state transfer from e_6 to e_1. We would like to find a condition on graph G that determines whether or not perfect state transfer is symmetric.

So far, the graphs we observed, over which bipartite walks defined has perfect state transfer, all have minimum degree at most two. We would like to know if there is any graph G with minimum degree at least three that has perfect state transfer in the bipartite walk defined on G.

We would like to know how the structure of the graph G affects behaviors of state transfer in the bipartite walk and if there is any feature of bipartite walk that can be determined by the combinatorial or algebraic properties of the graph it is defined on. This will be the future direction of our studies.

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