Semidefinite programming bounds for spherical three-distance sets

Feng-Yuan Liu[a]   Wei-Hsuan Yu[b]

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Abstract

A spherical three-distance set is a collection X of unit vectors in $\mathbb{R}^n$ such that the set of distances between any two distinct vectors has cardinality three. In this paper, we use the semidefinite programming method to improve the upper bounds for spherical three-distance sets in various dimensions. Specifically, we obtain better bounds in $\mathbb{R}^7$, $\mathbb{R}^{20}$, $\mathbb{R}^{21}$, $\mathbb{R}^{23}$, $\mathbb{R}^{24}$, and $\mathbb{R}^{25}$. Our results show that the maximum size of a spherical three-distance set is 2300 in $\mathbb{R}^{23}$.

Mathematics Subject Classifications: Primary 52C35; Secondary 14N20, 90C22, 90C05

1 Introduction

A spherical s-distance set is a finite collection X of unit vectors in $\mathbb{R}^n$ such that the set of distances between any two distinct vectors has size s. This problem of finding the maximum size of spherical s-distance sets in $\mathbb{R}^n$ has been extensively studied, dating back to the work of Delsarte, Goethals, and Seidel [DGS77]. They proved an upper bound on the cardinality of a spherical s-distance set X, given by the following formula:

$$|X| \leqslant \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}. \tag{1}$$

For instance, if $s = 2$, then $|X| \leqslant \frac{n(n+3)}{2}$, and if $s = 3$, then $|X| \leqslant \frac{n(n+1)(n+5)}{6}$. Spherical designs can be considered as well-distributed sampling points on a sphere, and maximum spherical s-distance sets are closely related to tight spherical t-designs. The notion of spherical designs was introduced by Delsarte, Goethals, and Seidel [DGS77], and the definition is as follows:

[a]Department of Computer Science, National Tsing Hua University, Hsinchu, Taiwan (`smilepa3034@gmail.com`).
[b]Department of Mathematics, National Central University, Taoyuan, Taiwan (`u690604@gmail.com`).

A finite set of points on a sphere X is called a *spherical t-design* in $\mathbb{R}^n$ if it satisfies the following equality for all polynomials f of degree at most t:

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f ds = \frac{1}{|X|}\sum_{y\in X} f(y). \tag{2}$$

Also, there exist lower bounds on the size of spherical t-designs:

$$|X| \geqslant \begin{cases} \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1} & \text{if } t \text{ is even and } t = 2s, \\ \\ 2\binom{n+s-1}{n-1} & \text{if } t \text{ is odd and } t = 2s+1. \end{cases} \tag{3}$$

If a spherical design attains the inequality mentioned above, it is called a *tight* spherical design. Tight spherical designs have only a few distinct distances between their points. A tight spherical $2s$-design is a spherical s-distance set that attains the bound in equation (3) [DGS77]. Similarly, a tight spherical $(2s-1)$-design is an antipodal s-distance set. For example, a tight spherical 6-design is a spherical three-distance set, and a tight spherical 7-design is an antipodal four-distance set. By taking half of the antipodal four-distance set, we obtain a spherical three-distance set. Taking half of an antipodal set means selecting only one point from each pair of antipodal points.

For $s = 2$, which corresponds to the maximum size of spherical two-distance sets, significant progress has been made by Musin [Mus09], Barg-Yu [BY14], Yu [Yu17], and Glazyrin-Yu [GY18]. In fact, the maximum size of spherical two-distance sets in $\mathbb{R}^n$ is almost known for every dimension. It is $\frac{n(n+1)}{2}$ except for some possible exceptions when $n = (2k+1)^2 - 3$, where $k \in \mathbb{N}$ [GY18]. However, for $s = 3$, little is known. The problem has only been solved for dimensions $n = 2$, 3, 4, 8, and 22. In this paper, we solve this problem for $\mathbb{R}^{23}$, where the answer is 2300 points with inner product values $\pm\frac{1}{3}, 0$ (**Table 1**). The configuration is half of a tight spherical 7-design, which is a sharp code (a four-distance set and a 7-design), and also a universal optimal code discussed in Cohn-Kumar [CK07].

n	Size	Structure	Inner product value
2	7	Heptagon (attaining bound (1))	
3	12	Icosahedron [Shi13]	$(-1, -\sqrt{5}/5, \sqrt{5}/5)$
4	13	**Theorem 3.5** of [SÖ20]	
8	120	Subset of E_8 root system [MN11]	$(-1/2, 0, 1/2)$
22	2025	Subset of the minimum vectors in the Leech lattice [MN11]	$(-4/11, -1/44, 7/22)$
23	2300	Half of a tight spherical 7-design [*new result]	$(-1/3, 0, 1/3)$

Table 1: Known results of maximum spherical three-distance sets in $\mathbb{R}^n$.

The tight spherical 7-design in $\mathbb{R}^{23}$ is also known as the kissing configuration, consisting of 4600 points with inner product values $-1, 0, \pm\frac{1}{3}$. Interestingly, we prove that half of this configuration forms the maximum spherical three-distance set in $\mathbb{R}^{23}$. The uniqueness of the tight spherical 7-design has been discussed in previous work [BS81, Cuy05, CK07], and it can be constructed as a subset of the Leech lattice: $\{x \in \text{Leech lattice} : \langle x \cdot e_1 \rangle = \frac{1}{2},\ e_1 = (1, 0, 0, \ldots, 0)\}$. The Leech lattice is an elegant configuration with many interesting properties, such as being the solution to the sphere packing problem [CKM+17] and the kissing number problem [BV08, OS79] in $\mathbb{R}^{24}$. We can also note that a slice of the Leech lattice yields a maximum spherical three-distance set in one lower dimension, which is another interesting property.

Tight spherical 7-designs exist only in very specific dimensions; specifically, when the dimension n is equal to three times the square of an integer minus 4, that is, $n = 3k^2 - 4$, where $k \in \mathbb{N}$ and $k \geqslant 2$ [BB09]. For $k = 2$, i.e. $n = 8$, the tight spherical 7-design in $\mathbb{R}^8$ corresponds to the root system of E_8. Musin and Nozaki [MN11] proved that the maximum spherical three-distance set in $\mathbb{R}^8$ is formed by half of the E_8 root system, which consists of 120 points. Our result extends this to $k = 3$ and $n = 23$. We prove that half of the tight spherical 7-design in $\mathbb{R}^{23}$ is also a maximum spherical three-distance set in $\mathbb{R}^{23}$. Could you consider investigating the existence of tight spherical 7-designs for $k \geqslant 4$? Unfortunately, the existence is not yet clear for $k \geqslant 4$, which means that there are currently no known constructions for tight spherical 7-designs in dimensions $n = 44, 71, 104, \cdots$. Bannai, Munemasa, and Venkov [BMV05] have proven some cases of nonexistence. However, there are still infinitely many cases that remain open. We believe that any tight spherical 7-design will give rise to a maximum spherical three-distance set, but a proof of this assertion is still elusive.

We define $A(\mathbb{S}^{n-1})$ as the maximum size of spherical 3-distance sets in $\mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Our work is motivated by Barg and Yu [BY13], who used the semidefinite programming (SDP) method to improve the upper bounds for the size of spherical two-distance sets. They obtained exact values of 276 for dimension $n = 23$ and $\frac{n(n+1)}{2}$ for $40 \leqslant n \leqslant 93$ except for $n = 46$ and $n = 78$. Musin and Nozaki [MN11] improved the upper bounds for $A(\mathbb{S}^{n-1})$ using Delsarte's linear programming method. They obtained exact answers for $A(\mathbb{S}^7) = 120$ and $A(\mathbb{S}^{21}) = 2025$. Additionally, they improved upper bounds for dimensions from $n = 6$ to 50, e.g., $A(\mathbb{S}^6) \leqslant 91$. Our contribution is to achieve tighter upper bounds using the SDP method. We prove that $A(\mathbb{S}^6) \leqslant 84$, $A(\mathbb{S}^{19}) \leqslant 1540$, $A(\mathbb{S}^{20}) \leqslant 1771$, $A(\mathbb{S}^{23}) \leqslant 2600$, $A(\mathbb{S}^{24}) \leqslant 2925$, and we obtain the exact answer for $\mathbb{R}^{23}$, where $A(\mathbb{S}^{22}) = 2300$. We summarize our results in **Table 4** for readers' convenience.

This paper is organized into the following sections: Section 1 provides a brief introduction to spherical three-distance sets. Section 2 discusses previous methods for obtaining upper bounds on the maximum size of spherical three-distance sets, including absolute harmonic bounds and linear programming bounds. Section 3 introduces the concept of semidefinite programming and its application to improving upper bounds on $A(\mathbb{S}^{n-1})$. Section 4 presents a method that combines Nozaki's theorem (**Theorem 6**) with SDP/LP methods to perform sampling on the $(0, 1)$ interval. Section 5 uses the sum of squares method to rigorously prove the relationship between the sampling points. Section 6 includes discussions and conclusions.

2 Previous methods

In this section, we introduce two previous methods for studying the upper bounds of spherical three-distance sets: the harmonic absolute bounds and the linear programming bounds.

Denote the Gegenbauer polynomials of degree k with dimension parameter n as $G_k^n(x)$. They are defined using the following recurrence relation:

$$G_0^n(x) = 1, G_1^n(x) = x,$$

$$G_k^n(x) = \frac{(2k+n-4)\, x\, G_{k-1}^n(x) - (k-1)G_{k-2}^n(x)}{k+n-3}, k \geqslant 2.$$

For instance, we have:

$$G_2^n(x) = \frac{nx^2-1}{n-1}, \quad G_3^n(x) = \frac{x}{n-1}\left(nx^2+2x^2-3\right).$$

These Gegenbauer polynomials play important roles in the harmonic absolute bounds.

Harmonic absolute bound

Harmonic absolute bound (HB) was originally proven by Delsarte [Del73a, Del73b, Lev92]. A more advanced version is mentioned in Godsil's [God17] book, while Nozaki [Noz09] provides detailed proof in his paper.

Theorem 1. *(Harmonic absolute bound) [God17][Noz09]*

Let X be a three-distance set in $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ with $D(X) = \{d_1, d_2, d_3\}$, where $D(X)$ collects all the inner product values of any two distinct points in X. Consider the polynomial $f(x) = \prod_{i=1}^3 (x-d_i)$ and suppose its expansion in the basis $G_k^n(x)$ has the form $f(x) = \sum_{k=0}^{3} f_k^n G_k^n(x)$ in the basis G_k^n. Then, we have

$$|X| \leqslant \sum_{k: f_k^n > 0} h_k^n,$$

where $h_k^n = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$ which represents the dimension of linear space of all real harmonic homogeneous polynomials of degree k.

If we are given the dimension and three inner product values, we can calculate the harmonic bounds. Let's consider the following example:

Example 2. Considering the half of tight spherical 7-designs in $\mathbb{R}^{23}$, the three inner product values between distinct points are $(\frac{-1}{3}, 0, \frac{1}{3})$. We calculate the harmonic bounds with given three inner product values in the following.

Let $f(x) = \prod_{i=1}^{3}(x - d_i) = \sum_{k=0}^{3} f_k^n G_k^n(x)$, f_k^n can be written as the following term:

$$
\begin{aligned}
f_0^n &= -d_1 d_2 d_3 - \frac{d_1 + d_2 + d_3}{n}, \\
f_1^n &= d_1 d_2 + d_1 d_3 + d_2 d_3 + \frac{3}{n+2}, \\
f_2^n &= \frac{1-n}{n}(d_1 + d_2 + d_3), \\
f_3^n &= \frac{n-1}{n+2}.
\end{aligned}
$$

Substitute f_k^n with the dimension $n = 23$, and $(d_1, d_2, d_3) = (\frac{-1}{3}, 0, \frac{1}{3})$, then $f_1^n, f_3^n > 0$ and $f_0^n, f_2^n \leqslant 0$. Thus, the harmonic absolute bound is

$$|X| \leqslant \sum_{k: f_k^n > 0} h_k^n = h_1^n + h_3^n = 23 + 2277 = 2300.$$

Linear programming bound

Linear programming (LP) is another method to estimate the upper bound of spherical few distance sets. This theorem is established by Delsarte et al. [DGS77]. Musin and Nozaki [MN11] incorporate LP method and Nozaki theorem (**Theorem 6**) to obtain the upper bound of spherical codes.

Theorem 3. *(Delsarte's inequality) [DGS77]*

For any finite set of points $X \subset \mathbb{S}^{n-1}$

$$\sum_{(x,y) \in X^2} G_k^n(\langle x, y \rangle) \geqslant 0, \forall k \in \mathbb{N}.$$

Delsarte proved this inequality by the addition formula for spherical harmonics. With this linear inequality, we can derive the Delsarte linear programming bounds for the spherical three-distance sets.

Theorem 4. *(Delsarte's linear programming bound) [MN11][DGS77]*

Let $X \in \mathbb{S}^{n-1}$ be a finite set and assume that for any $x, y \in X$, $\langle x, y \rangle \in \{d_1, d_2, d_3\}$. Then the cardinality of X is bounded above by the solution of the following linear programming problem:

$$
\begin{aligned}
&\textit{maximize} && 1 + x_1 + x_2 + x_3 && \text{(4a)} \\
&\textit{subject to} && 1 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geqslant 0, \forall k \in \mathbb{N} && \text{(4b)} \\
& && x_j \geqslant 0, j = 1, 2, 3. && \text{(4c)}
\end{aligned}
$$

Therefore, if the dimension n and the three inner product values d_1, d_2, d_3 are given, we can solve the above LP problem to obtain the upper bounds for the size of a spherical three-distance set. For instance, if we set $n = 23, (d_1, d_2, d_3) = (-1/3, 0, 1/3)$ (half of a tight 7-design in $\mathbb{R}^{23}$) and $k \leqslant 18$, we can obtain the upper bound 2300 which is coherent to the harmonic bound Example 2.

3 Semidefinite programming method

3.1 Semidefinite Programming

A semidefinite program (SDP) is an optimization problem of the form [VB96]

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad F(x) \succeq 0, \end{aligned}$$

where

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i.$$

The vector $c \in \mathbb{R}^m$ and $F_0, \cdots, F_m$ are symmetric matrices in $\mathbb{R}^{n \times n}$. The inequality sign in $F(x) \succeq 0$ means that $F(x)$ is positive semidefinite, i.e.,

$$z^T F z \geqslant 0, \forall z \in \mathbb{R}^n.$$

SDP is an extension of linear programming, which has been utilized to bound the size of codes under specific constraints. For instance, Schrijver [Sch05] applied the SDP method to enhance the bounds for the $A(n, d)$ problem, which seeks the maximum size of a binary code of length n with the constraint of a minimum distance at least d. Schrijver's work was grounded in the block diagonalization of the Terwilliger algebra of the Hamming cube.

As another example, the kissing number problem has also been improved by SDP. This problem seeks to determine how many unit spheres can touch the center unit sphere without overlapping. It is equivalent to finding the maximum size of spherical codes such that each pair of points has inner product values in the interval $[-1, 1/2]$.

Bachoc-Vallentin [BV08] applied SDP to enhance the upper bounds for the kissing number problem in various dimensions. We adapted their formula to the case of spherical three-distance sets. The SDP method might yield tighter upper bounds compared to the linear programming method, as we incorporate matrix constraints in addition to the linear constraints.

Following [BV08], we define the matrices $Y_k^n(u, v, t)$ and $S_k^n(u, v, t)$ with dimensions $(p_{\text{SDP}} - k + 1) \times (p_{\text{SDP}} - k + 1)$, where p_{SDP} is the parameter of the SDP matrix constraints.

For all $0 \leqslant i, j \leqslant p_{\text{SDP}} - k$,

$$(Y_k^n(u, v, t))_{ij} = u^i v^j ((1 - u^2)(1 - v^2))^{\frac{k}{2}} G_k^{n-1} \left(\frac{t - uv}{\sqrt{(1 - u^2)(1 - v^2)}} \right), \quad 0 \leqslant k \leqslant p_{\text{SDP}},$$

$$S_k^n(u,v,t) = \frac{1}{6}\sum_{\sigma\in S_3} Y_k^n(\sigma(u,v,t)),$$

$$S_k^n(1,1,1) = \mathbf{0}_M, \quad k \geqslant 1,$$

where $\sigma(u,v,t)$ represents all permutations in the permutation group S_3. $\mathbf{0}_M$ denotes the zero matrix.

Then

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succeq 0.$$

Semidefinite programming bound

Barg and Yu [BY13] utilized SDP to derive upper bounds for spherical two-distance sets. Nozaki proved that d_1 and d_2 can be expressed as functions of d_3 (Theorems 6, 7, 8). We incorporated their approach to implement SDP programming for establishing upper bounds on spherical three-distance sets.

Theorem 5. *Let p_{LP}, p_{SDP} be the parameters of LP constraints and SDP matrix constraints.* [1] *If X is a spherical three-distance set with inner product values d_1, d_2, d_3 in $\mathbb{R}^n$, then the cardinality of X is bounded above by the solution of the following semidefinite programming problem:*

$$\begin{aligned}
&\textit{maximize} && 1+\frac{1}{3}(x_1+x_2+x_3) && \text{(5a)}\\
&\textit{subject to} && \begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0&1\\1&1\end{pmatrix}(x_1+x_2+x_3)+\begin{pmatrix}0&0\\0&1\end{pmatrix}\sum_{i=4}^{13}x_i \succeq 0, && \text{(5b)}\\
& && 3+x_1G_k^n(d_1)+x_2G_k^n(d_2)+x_3G_k^n(d_3)\geqslant 0,\ k=1,2,\cdots,p_{LP}, && \text{(5c)}\\
& && S_k^n(1,1,1)+x_1S_k^n(d_1,d_1,1)+x_2S_k^n(d_2,d_2,1)+x_3S_k^n(d_3,d_3,1) && \\
& && +x_4S_k^n(d_1,d_1,d_1)+x_5S_k^n(d_2,d_2,d_2)+x_6S_k^n(d_3,d_3,d_3) && \\
& && +x_7S_k^n(d_1,d_1,d_2)+x_8S_k^n(d_1,d_1,d_3)+x_9S_k^n(d_2,d_2,d_1) && \text{(5d)}\\
& && +x_{10}S_k^n(d_2,d_2,d_3)+x_{11}S_k^n(d_3,d_3,d_1)+x_{12}S_k^n(d_3,d_3,d_2) && \\
& && +x_{13}S_k^n(d_1,d_2,d_3)\succeq 0,\ k=0,1,2,\cdots,p_{SDP}, && \\
& && x_j\geqslant 0,\ j=1,2,\cdots,13. && \text{(5e)}
\end{aligned}$$

In this theorem, the variables x_i represent the number of triple points in X associated with certain combinations of inner product values d_1, d_2, d_3, and 1. For instance, x_1 is related to the counting of triple points in X where the three inner product values are $(d_1, d_1, 1)$. In a similar setting, x_2 corresponds to $(d_2, d_2, 1)$, and so on up to x_{13}, which is associated with (d_1, d_2, d_3).

[1] In our paper, we set $(p_{LP}, p_{SDP}) = (18, 6)$. Besides, we also tried with $(p_{LP}, p_{SDP}) = (18, 5)$, then our experiment is not able to obtain the upper bound $A(\mathbb{S}^{22}) \leqslant 2300$. We conclude that the matrix condition with $p_{SDP} = 6$ is crucial, though we do not have a theoretical explanation.

4 Discrete sampling points with Nozaki theorem

With given three inner product values d_1, d_2, d_3, we can explore the harmonic absolute bound (**Section 2**), the linear programming bound (**Section 2**), and the semidefinite programming bound (**Section 3.1**). Moreover, when the size of X is sufficiently large, there exist certain relations among d_1, d_2, d_3. In this section, we introduce Nozaki's Theorem. With this result, we can express d_1, d_2 as function of d_3, thereby reducing the three variables d_1, d_2, d_3 to a single variable d_3. Upon reduction to a single variable d_3, we sample points in the interval $d_3 \in (0, 1)$ [2] [3].

First, define

$$(K_1, K_2, K_3) = (\frac{(d_2-1)(d_3-1)}{(d_2-d_1)(d_3-d_1)}, \frac{(d_1-1)(d_3-1)}{(d_1-d_2)(d_3-d_2)}, \frac{(d_1-1)(d_2-1)}{(d_1-d_3)(d_2-d_3)}).$$

Then, if the size of spherical three-distance set is large enough, then K_i will be integers and bounded above.

Theorem 6. *(Nozaki Theorem) [Noz11]*

If X is a spherical three-distance set and the size of X is greater than or equal to $2N(\mathbb{S}^{n-1})$, then (K_1, K_2, K_3) are all integers, and K_i are bounded as follows:

$$|K_i| \leqslant \lfloor 1/2 + \sqrt{N(\mathbb{S}^{n-1})^2/(2N(\mathbb{S}^{n-1})-2)+1/4} \rfloor, \; i = 1, 2, 3.$$

where $N(\mathbb{S}^{n-1}) := h_0^n + h_1^n + h_2^n$.

The numbers (K_1, K_2, K_3) also satisfy the following equation [MN11].

$$\begin{cases} K_1 + K_2 + K_3 = 1 \\ d_1K_1 + d_2K_2 + d_3K_3 = 1 \\ d_1^2K_1 + d_2^2K_2 + d_3^2K_3 = 1. \end{cases} \tag{6}$$

Through observation of (K_1, K_2, K_3) and simple calculation, we can derive more properties involving (K_1, K_2, K_3). (**Prop 7**)

Proposition 7. *K_1, K_2 and K_3 have the following properties when $-1 \leqslant d_1 < d_2 < d_3 < 1$.*

1. $|K_1| < |K_2|$
2. $K_1K_2 < 0$
3. $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0.$

[2] If $d_1 < d_2 < d_3$ and $d_3 < 0$, then the cardinality of the set is at most $2n+1$ by Rankin's bound [Ran47].

[3] For the sampling experiment in the paper, we place 1000 uniform sampling points for the interval $d_3 \in (0, 1)$.

Furthermore, we can solve the system of equations 6. Without loss of generality, let's assume that $d_1 < d_2 < d_3$, and then we can find the roots using the `Matlab Symbolic Toolbox` [TM19]:

$$(d_1, d_2) = (\frac{K_1 - d_3K_1K_3 - (d_3-1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)}, \frac{K_2 - d_3K_2K_3 + (d_3-1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)})$$

or

$$(d_1^*, d_2^*) = (\frac{K_1 - d_3K_1K_3 + (d_3-1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)}, \frac{K_2 - d_3K_2K_3 - (d_3-1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)})$$

However, (d_1^*, d_2^*) is invalid. By proposition 7,

$$d_1^* - d_2^* = \frac{(d_3-1)\sqrt{-K_1K_2K_3}}{K_1K_2} \geqslant 0.$$

This contradicts the initial assumption $d_1^* < d_2^*$. Therefore, the following theorem holds.

Theorem 8. *Suppose that $d_1 < d_2 < d_3$. Using the system of equations 6, we can solve for d_1 and d_2 with the following formula:*

$$d_1 = \frac{K_1 - d_3K_1K_3 - (d_3-1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)}$$
$$d_2 = \frac{K_2 - d_3K_2K_3 + (d_3-1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)}$$

Theorem 9. *(Improved bounds of [MN11] with SDP)*

Let $\mathfrak{D}(\mathbb{S}^{n-1})$ denote the set of all possible spherical three distances $D(X) = \{d_1, d_2, d_3\}$ such that (K_1, K_2, K_3) are integers and satisfy the upper bounds in Theorem 6. For each $D \in \mathfrak{D}(\mathbb{S}^{n-1})$, we have two bounds: the Harmonic absolute bound (HB) [1] and the semidefinite programming bound (SDP) [3.1]. The following result holds:

Let

$$B(D) := \min_{D\in\mathfrak{D}(\mathbb{S}^{n-1})} \{SDP(D), HB(D)\},$$ [4]

then

$$A(\mathbb{S}^{n-1}) \leqslant \max_{D\in\mathfrak{D}(\mathbb{S}^{n-1})} \{B(D), 2N(\mathbb{S}^{n-1}) - 1\}.$$

Notice that the above theorem is not rigorous until we use the sum of squares (SOS) methods to guarantee that there is no significant oscillation between the sampling points. The details can be found in "Section 5."

We illustrate the upper bound in Figure 1.

[4] In Musin and Nozaki's paper [MN11], they use two bounds: harmonic absolute bound HB and Delsarte's linear programming bound LP. Therefore, they define the upper bound $B(D) := \min_{D\in\mathfrak{D}(\mathbb{S}^{n-1})}\{LP(D), HB(D)\}$.

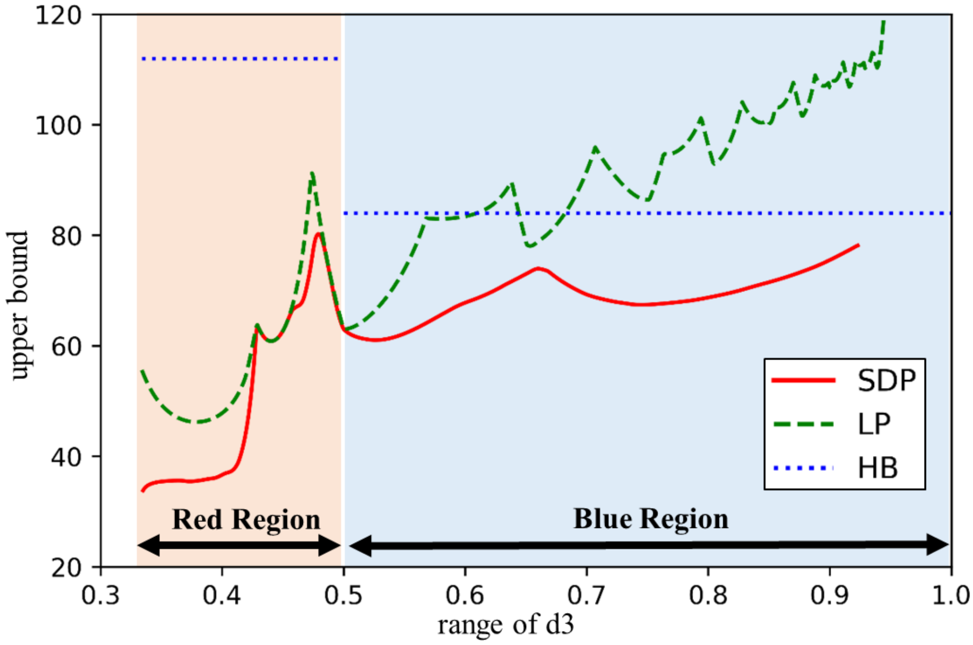

Figure 1: Sampling points figure with $(K_1, K_2, K_3) = (1, -3, 3)$ on $\mathbb{R}^7$

In the blue region of **Figure 1**, LP/SDP bound will become unbounded when d_3 is close to 1. Therefore, we utilize the harmonic bound to control this area. Furthermore, the harmonic bound in this region is always less than or equal to $h_1 + h_3$ (which is 84, if $n = 7$). On the other hand, the performance of LP/SDP bounds is consistently better than the harmonic bound in the red region. However, in certain cases of (K_1, K_2, K_3), the LP bound is worse than the harmonic bound in the blue region. In fact, SDP plays a crucial role in dominating the red region with a much smaller upper bound. We can observe how the SDP bounds outperform the LP bounds in Table 2.

When the dimension n is given, there are many possibilities for (K_1, K_2, K_3) in $\mathbb{R}^n$. Our SDP method only improves the upper bounds of certain (K_1, K_2, K_3), but these improvements do indeed contribute to enhancing the overall upper bounds. For example, for $n = 23$ and $K_i = (1, -3, 3)$ and $(2, -6, 5)$, the LP bounds are 2385 and 2319 respectively. The SDP bounds are 1072 and 1693 respectively, which are crucial improvements that SDP can achieve by proving the upper bounds to 2300, a feat that LP cannot accomplish. For other cases not listed, the LP bounds are already well controlled below 2300. We will include additional figures in the Appendix.

In these dimensions, there are 338 different possible choices of (K_1, K_2, K_3). The number of choices for K_i is determined by Theorem 6. For example, in $\mathbb{R}^7$, there are six choices of K_i: $(1, -4, 4), (1, -3, 3), (1, -2, 2), (2, -4, 3), (2, -3, 2)$, and $(3, -4, 2)$. We have successfully improved 12 cases of (K_1, K_2, K_3) out of all 338 cases. (see **Table 3**)

Dimension	(K_1, K_2, K_3)	Max LP of Red region	Max SDP of Red region	HB of Blue region
7	(1, -3, 3)	(91.22, (d_3 = 0.474))	(80.23, (d_3 = 0.479))	84
20	(1, -4, 4)	(1589.65, (d_3 = 0.543))	(756.18, (d_3 = 0.543))	1540
21	(2, -6, 5)	(1867.02, (d_3 = 0.420))	(1332.83, (d_3 = 0.420))	1771
23	(1, -3, 3)	(2385.60, (d_3 = 0.590))	(1072.29, (d_3 = 0.593))	2300
23	(2, -6, 5)	(2319.82, (d_3 = 0.421))	(1693.01, (d_3 = 0.430))	2300
23	(3, -8, 6)	(2300.03, (d_3 = 0.332))	(2298.12, (d_3 = 0.333))	2300
24	(1, -5, 5)	(2821.84, (d_3 = 0.500))	(1594.81, (d_3 = 0.500))	2600
24	(1, -4, 4)	(2681.29, (d_3 = 0.556))	(1759.73, (d_3 = 0.556))	2600
24	(1, -3, 3)	(2758.20, (d_3 = 0.589))	(1293.60, (d_3 = 0.596))	2600
25	(1, -5, 5)	(4138.41, (d_3 = 0.511))	(2472.46, (d_3 = 0.522))	2925
25	(1, -4, 4)	(3210.08, (d_3 = 0.559))	(2238.27, (d_3 = 0.559))	2925
25	(1, -3, 3)	(3166.53, (d_3 = 0.588))	(1883.75, (d_3 = 0.600))	2925

Table 2: Sampling points table with SDP bounds perform better than LP bounds

Dimension	Improved (K_1, K_2, K_3)	Total (K_1, K_2, K_3)
7	1	6
20	1	55
21	1	55
23	3	66
24	3	78
25	3	78

Table 3: Numbers of improved (K_1, K_2, K_3) with respect to total numbers

By combining the SDP bound of the red region with the harmonic bound of the blue region, we obtain a new upper bound for $\mathbb{R}^7$, $\mathbb{R}^{20}$, $\mathbb{R}^{21}$, $\mathbb{R}^{23}$, $\mathbb{R}^{24}$, and $\mathbb{R}^{25}$. (see **Table 4**)

Dimension	Original bound [MN11]	New bound
7	91	84
20	1541	1540
21	1772	1771
23	2301	2300
24	2601	2600
25	2926	2925

Table 4: Our new results on upper bounds of maximum spherical three-distance sets

5 Rigorous proof with sum-of-squares (SOS) decomposition

We presented the SDP bounds on the sampling points in the previous section. However, there might be significant oscillation in the upper bounds between two adjacent sampling points. We need a rigorous proof to ensure that the upper bound between the sampling points is also well bounded. Theorem 5 provides us with the upper bound for a single sampling point d_3. We introduce another advanced approach, the **sum-of-squares decomposition (SOS)**, to handle the upper bound for the interval $[a_1, a_2]$. Let's begin with a simple example of SOS:

Example 10. (SOS example with YALMIP) [5]

Given the non-negative polynomial constraint $p(x) = x^2+(y+2)x+(3-y)$ for $x \in \mathbb{R}$, it can be expressed as the sum of squares of polynomials [Hil88]. Our objective is to minimize the objective function $y^2 - 3y + 1$. We can perform sum-of-squares decomposition using the **YALMIP** package in the **MATLAB 2019a** environment.

```
function sos_example()
    sdpvar x y
    p = x^2 + (y+2)*x + (3-y);
    F = sos(p);                   % Create sos constraint
    obj = y^2 - 3*y + 1;          % Minimize object function
    solvesos(F, obj);             % Solve sos problem
    disp(value(obj))              % Show min(obj)
    disp(value(y))                % Show corresponding argument y
    sdisplay(sosd(F))             % Show sum-of-squares decomposition
end
```

After performing the calculation, we obtain $min(obj) = -0.8888$, with $y = 0.8990$. The polynomial $p(x) = x^2 + (y + 2)x + (3 - y)$ can be expressed as the sum of $(x + 1.44948974278)^2$ and $(-4.39561910079 \times 10^{-7}x + 3.03252860027 \times 10^{-7})^2$ numerically, indicating a sum-of-squares decomposition.

We can provide further details about **solvesos**. Nesterov [Nes00] proved that a polynomial $p(x)$ can be expressed in sum-of-squares form if and only if there exists a positive semidefinite matrix Q such that $p = XQX^T$, where $X = (1, x, x^2, \cdots, x^m)$ if deg $p = 2m$. In this example, we can express the polynomial $p(x) = x^2 + (y + 2)x + (3 - y)$ as $p(x) = \begin{pmatrix} 1 & x \end{pmatrix} Q \begin{pmatrix} 1 \\ x \end{pmatrix}$, where Q is a 2 by 2 semidefinite matrix. Therefore, this sum-of-squares (SOS) problem is equivalent to the following semidefinite programming (SDP) problem: [6]

[5] YALMIP Sum-of-squares programming tutorial:
https://yalmip.github.io/tutorial/sumofsquaresprogramming/

[6] YALMIP will automatically transform the SOS problem into an SDP problem and use an SDP solver to solve it.

Example 11. (Equivalent format of SOS example 10)

$$\text{minimize} \qquad y^2 - 3y + 1 \tag{7a}$$

$$\text{subject to} \qquad \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \succeq 0 \tag{7b}$$

$$q_{11} = 3 - y \tag{7c}$$

$$q_{12} + q_{21} = y + 2 \tag{7d}$$

$$q_{22} = 1 \tag{7e}$$

Actually, our problem is much more complex than this simple example. Barg and Yu [BY13] employed the SOS method to prove the upper bound of spherical two-distance sets. Nozaki demonstrated that d_1 and d_2 could be expressed as functions of d_3 (Theorem 6, 7, 8). We integrated their approach to perform sum-of-squares programming in order to rigorously establish the upper bound of spherical three-distance sets.

We will proceed with the proof through the following steps:(1) Rewrite the original SDP problem into its dual form and extend the non-negative polynomial constraints, originally applicable only on the finite interval $[a_1, a_2]$, to non-negative polynomials over the entire real line $\mathbb{R}$. (2) Convert the non-negative polynomial constraints over $\mathbb{R}$ into a sum-of-squares decomposition. (3) Transform the constraints of the sum-of-squares decomposition into an SDP problem.

Theorem 12. *(SDP Dual Form of Theorem 5)*

$$\textit{minimize} \qquad 1 + \{\sum_{i=1}^{p_{LP}} \alpha_i + \beta_{11} + \langle F_0, S_0^n(1,1,1)\rangle\} \tag{8a}$$

$$\textit{subject to} \qquad \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \succeq 0 \tag{8b}$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_1)) + 3 \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(d_1, d_1, 1)\rangle \leqslant -1 \tag{8c}$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_2)) + 3 \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(d_2, d_2, 1)\rangle \leqslant -1 \tag{8d}$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_3)) + 3 \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(d_3, d_3, 1)\rangle \leqslant -1 \tag{8e}$$

$$\beta_{22} + \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(y_1, y_2, y_3)\rangle \leqslant 0 \tag{8f}$$

$$\alpha_i \geqslant 0,\ i = 1, 2, \cdots, p_{LP} \tag{8g}$$

$$F_i \succeq 0,\ i = 0, 1, 2, \cdots, p_{SDP} \tag{8h}$$

where

$$(y_1, y_2, y_3) \in \{(d_1, d_1, d_1), (d_2, d_2, d_2), (d_3, d_3, d_3), (d_1, d_1, d_2), (d_1, d_1, d_3), \\ (d_2, d_2, d_1), (d_2, d_2, d_3), (d_3, d_3, d_1), (d_3, d_3, d_2), (d_1, d_2, d_3)\}.$$

First, let's rewrite the original SDP problem (**Theorem 5**) in its dual form. Constraints (8c)-(8f) impose positivity conditions on the univariate polynomials of d_3 (where d_1 and d_2 can be expressed as functions of d_3, according to Nozaki's theorem 6), denoting d_3 as a. To extend these non-negative polynomial constraints from the small interval $[a_1, a_2]$ to the entire real numbers, we can utilize the following theorem:

Theorem 13. *If $f(a)$ is a polynomial of degree m satisfies $f(a) \geqslant 0$ for $a \in [a_1, a_2]$, then*

$$f^+(a) = (1 + a^2)^m f(\frac{a_1 + a_2 a^2}{1 + a^2}) \geqslant 0, \, \forall a \in \mathbb{R}.$$

Proof. Define $g(a) = \frac{a_1+a_2a^2}{1+a^2}$ and consider the values of g on $a \in [0, \infty)$. The function values at the boundary points of the interval $[0, \infty)$ are $g(0) = a_1$, and $\underset{a\to\infty}{lim}\, g(a) = a_2$. The function g is increasing for $a \in [0, \infty)$ since $g'(a) = \frac{2a(a_2-a_1)}{(a^2+1)^2} \geqslant 0$. Consequently, $f(g(a))$ is non-negative for $a \in [0, \infty)$. Moreover, $g(a)$ is an even function and $f(g(a))$ is also non-negative for $a \in (-\infty, 0]$. Therefore, $f^+(a) = (1 + a^2)^m f(g(a))$ is non-negative for all real numbers. □

Secondly, Hilbert proved that a non-negative polynomial over the entire real numbers can be expressed as the sum of squares [Hil88]. Specifically, $f^+(a) = \sum_i r_i^2(a)$, where r_i are polynomials. Finally, according to Nesterov's result [Nes00], a polynomial $f^+(a)$ can be expressed in sum-of-squares form if and only if there exists a positive semidefinite matrix Q such that $f^+ = XQX^T$, where $X = (1, a, a^2, \cdots, a^m)$ if deg $f^+ = 2m$. Therefore, constraints (8c)-(8f) can be reformulated as conditions involving positive semidefinite matrices.

If there are positive semidefinite matrices $M_1, M_2, \ldots, M_k$, then we can deduce that the block matrix $\begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{pmatrix}$ is also positive semidefinite. We employ this technique to handle the constraints involving semidefinite matrices.

Consider $d_3 \in I = [0, 1]$. Partition it into n sub-intervals I_k, where $I_k = [\frac{i-1}{n}, \frac{i}{n}]$. Denote the upper bound as UB, $UB(I) := \underset{k\in[1,n],k\in\mathbb{N}}{max} UB(I_k)$. In our SOS experiment, we set $n = 200$, and each sub-interval has a width of 0.005.[7] The results of the sum-of-squares experiment are presented in Table 5.

The SOS decomposition ensures that there are no issues between the sampling points. All values are smaller than the harmonic bound of the blue region, thus completing the rigorous proof.

[7] Using a higher value of n would provide a tighter upper bound for I, but it would also require more computing resources. To balance the accuracy of the upper bound and computational resources, we choose $n = 200$ and the corresponding sub-interval width of 0.005.

Dimension	(K_1, K_2, K_3)	SOS Decompostion Value (covered Max SDP in Red Region)	Harmonic Bound (Blue Region)
7	(1, -3, 3)	SOS(0.475, 0.480) = 80.29	84
20	(1, -4, 4)	SOS(0.540, 0.545) = 804.06	1540
21	(2, -6, 5)	SOS(0.415, 0.420) = 1343.66	1771
23	(1, -3, 3)	SOS(0.590, 0.595) = 1234.62	2300
23	(2, -6, 5)	SOS(0.425, 0.430) = 1703.71	2300
23	(3, -8, 6)	SOS(0.332, 0.335) = 2300.85 [8]	2300
24	(1, -5, 5)	SOS(0.495, 0.500) = 1594.80	2600
24	(1, -4, 4)	SOS(0.555, 0.560) = 2159.78	2600
24	(1, -3, 3)	SOS(0.595, 0.600) = 1605.49	2600
25	(1, -5, 5)	SOS(0.520, 0.525) = 2495.09	2925
25	(1, -4, 4)	SOS(0.555, 0.560) = 2474.14	2925
25	(1, -3, 3)	SOS(0.595, 0.600) = 2080.54	2925

Table 5: SOS Decomposition Value Covered Max SDP

6 Discussions and conclusions

We improved the bounds up to $n = 25$ since our SDP bounds did not perform well in achieving tighter bounds for larger dimensions. For example, for $n = 26$ and $K_i = (1, -5, 5)$, the SDP bound exceeded the harmonic bound. Consequently, we did not obtain better results than Musin-Nozaki [MN11]. In $\mathbb{R}^{23}$, we found that the SDP bound for spherical three-distance sets is 2300, and we also have knowledge of constructing such sets with 2300 points from tight spherical 7-designs. Therefore, we are curious about the existence of constructions for spherical three-distance sets with 84 points in $\mathbb{R}^7$.

Initially, we considered the orbit of a group. Eiichi Bannai suggested trying the group $\mathrm{PSL}(2, 7)$, which has an order of 168, just double our target of 84 points. We attempted to find a suitable initial vector such that the orbit of the group action by $\mathrm{PSL}(2, 7)$ forms a three-distance set with 84 points. However, the orbit of the irreducible or reducible representation of $\mathrm{PSL}(2, 7)$ in 7 dimensions could not produce such three-distance sets. Next, we performed numerical non-linear optimization to calculate the energy-minimizing configuration of 84 points on $\mathbb{S}^6$.

Unfortunately, neither of these two methods has helped us find the desired construction. We will intensify our efforts to identify a more powerful algorithm capable of discovering this construction. We firmly believe that such a construction exists for the 84 points to form a spherical three-distance set in $\mathbb{R}^7$. If we manage to find such a construction, in conjunction with our new bounds, we can establish that $A(\mathbb{S}^6) = 84$.

[8] Since $SOS(0.330, 0.335) = 2302.64$, it is not tight enough. We cut it into two smaller intervals, and calculate their SOS result separately: $SOS(0.330, 0.332) = 2297.42$ and $SOS(0.332, 0.335) = 2300.85$. This is the tighter upper bound.

Currently, the largest known construction size for spherical three-distance sets on $\mathbb{S}^6$ is 64, which can be derived from a 3-class association scheme with 64 vertices and a Krein array [7, 6, 5; 1, 2, 3]. Additionally, the half of the E_7 root system forms a spherical three-distance set in $\mathbb{R}^7$ with 63 vertices. However, it is evident that this cannot be the maximum size.

We also attempted to estimate the upper bounds without relying on the integer conditions of d_i (Nozaki Theorem 6). In $\mathbb{R}^4$ and $\mathbb{R}^5$, we conducted SDP on sampling points covering all possible inner product values, effectively dividing the intervals of d_1, d_2, and d_3 into numerous segments. Specifically, we subdivided $d_1 \in (-1, 1)$ and $d_2 \in (-1, 1)$ into 100 segments each, and $d_3 \in (0, 1)$ into 50 segments. In $\mathbb{R}^4$, the numerical SDP bound was reduced to 26, while LP yielded an upper bound of 27 [MN11]. For instance, for $(d_1, d_2, d_3) = (-0.76, -0.16, 0.54)$, the LP bound was 27, whereas the SDP bound was 13. In certain instances, the SDP bound appeared large, while at that moment, the harmonic bound was 26. Therefore, the SDP method only yielded the optimal bound of 26 in $\mathbb{R}^4$.

However, [SÖ20] proved the bound to be 13 and provided the construction. In $\mathbb{R}^5$, when we do not use Nozaki's Theorem 6, the SDP bounds on some d_i are larger than the harmonic bound (45 points). Therefore, we do not improve upon the results of Musin and Nozaki [MN11]. For other dimensions, the results from sampling points do not improve the upper bounds, and we opt to save our time by not conducting sum-of-squares experiments.

Furthermore, we attempt to utilize the bounds for s-distance sets proposed by Glazyrin and Yu when $s = 3$. The bound is derived from the polynomial method and certain rank arguments. Readers can refer to Section 2.3 in [GY18] for further details. According to Corollary 3 in [GY18], if X is a spherical three-distance set in $\mathbb{R}^n$ with inner product values (d_1, d_2, d_3), then

$$|X| \leqslant \frac{1 + n(n+3)}{1 - \frac{n-1}{(n+2)(1-d_1)(1-d_2)(1-d_3)}}, \qquad \text{if the right hand side is positive.}$$

While the upper bound may prove efficient for small d_i, this represents a distinct approach to estimating the upper bounds for spherical three-distance sets. It could potentially serve as a valuable tool when LP and SDP methods fail to produce satisfactory results. Nevertheless, its contribution to our current bounds is limited.

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Appendix

Experiments in $\mathbb{R}^7$

There are 6 types of (K_1, K_2, K_3) in $\mathbb{R}^7$ [**Proposition 7**]. The LP, SDP, and HB results are presented in the following table and figures.

K_1	K_2	K_3	d_1	d_2	d_3	Notes
1	-4	4	$\frac{8d_3}{3} - \frac{5}{3}$	$\frac{5d_3}{3} - \frac{2}{3}$	d_3	Already controlled by LP and HB [2]
1	-3	3	$3d_3 - 2$	$2d_3 - 1$	d_3	**Improved bound** by **SDP** and HB [3]
1	-2	2	$4d_3 - 3$	$3d_3 - 2$	d_3	Already controlled by LP and HB [4]
2	-4	3	$\frac{3d_3}{2} + \frac{\sqrt{6}(d_3-1)}{2} - \frac{1}{2}$	$\frac{3d_3}{2} + \frac{\sqrt{6}(d_3-1)}{4} - \frac{1}{2}$	d_3	Already controlled by LP and HB [5]
2	-3	2	$2d_3 + \sqrt{3}\,(d_3 - 1) - 1$	$2d_3 + \frac{2\sqrt{3}(d_3-1)}{3} - 1$	d_3	Already controlled by LP and HB [6]
3	-4	2	$2d_3 + \frac{2\sqrt{6}(d_3-1)}{3} - 1$	$2d_3 + \frac{\sqrt{6}(d_3-1)}{2} - 1$	d_3	Already controlled by LP and HB [7]

Table 6: Six kinds of (K_1, K_2, K_3) in $\mathbb{R}^7$ and corresponding three inner product values

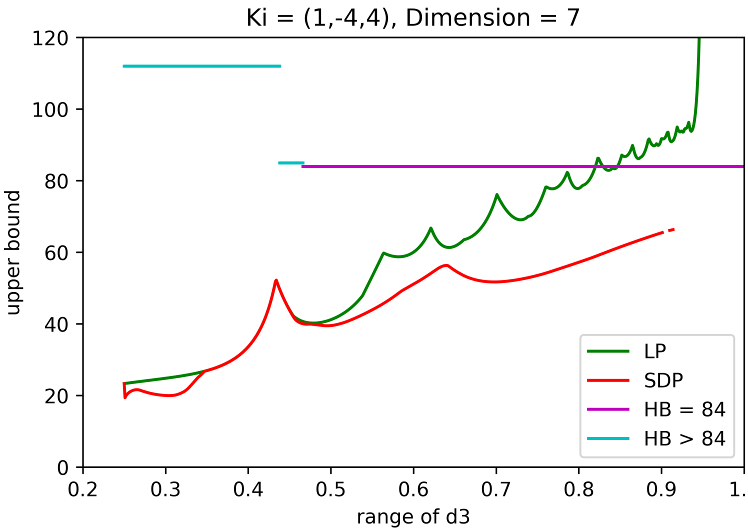

Figure 2: Sampling points with $(K_1, K_2, K_3) = (1, -4, 4)$ in $\mathbb{R}^7$

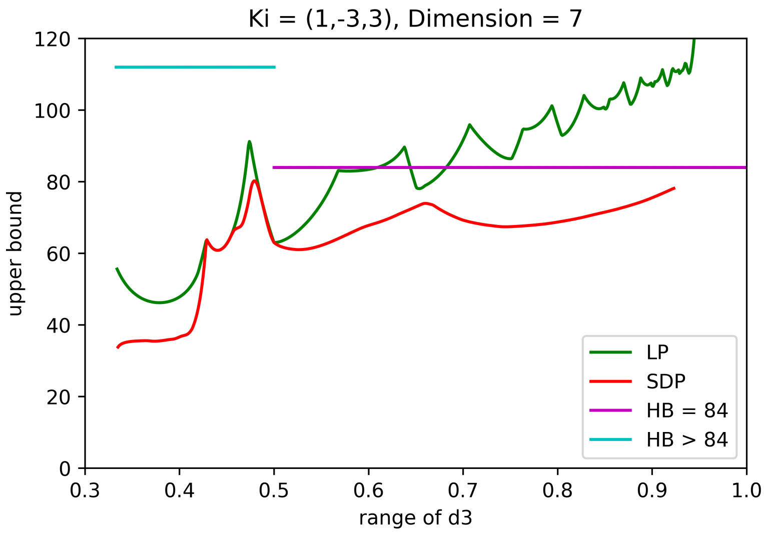

Figure 3: Sampling points with $(K_1, K_2, K_3) = (1, -3, 3)$ in $\mathbb{R}^7$

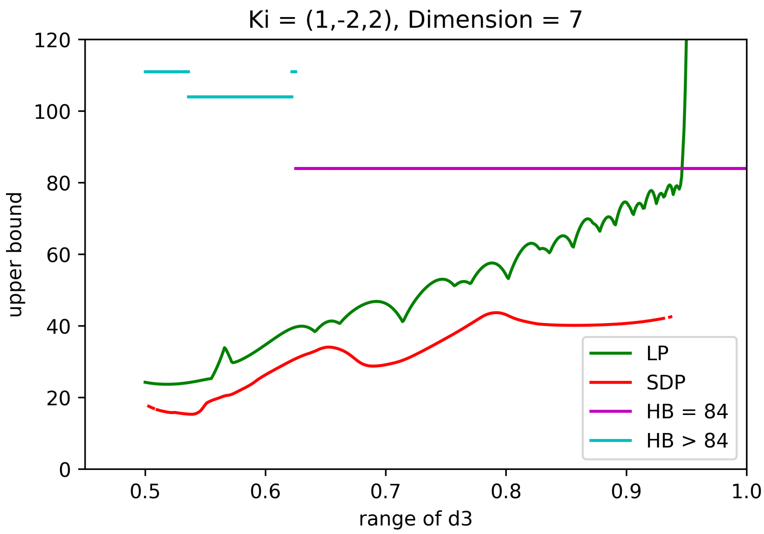

Figure 4: Sampling points with $(K_1, K_2, K_3) = (1, -2, 2)$ in $\mathbb{R}^7$

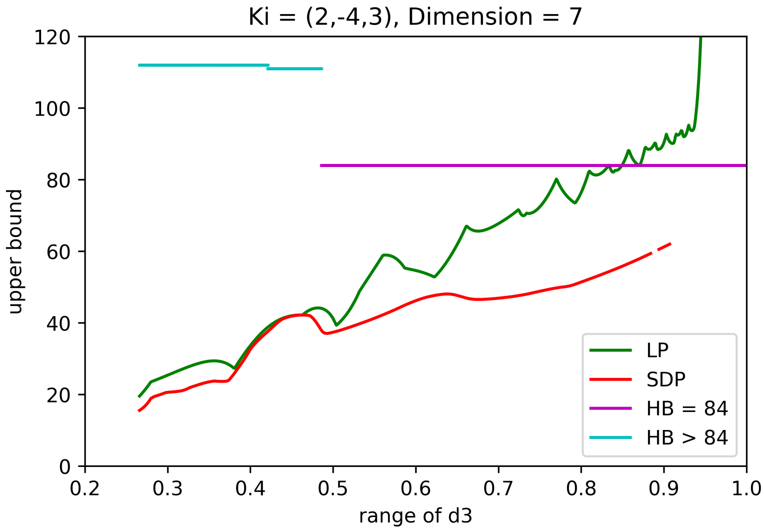

Figure 5: Sampling points with $(K_1, K_2, K_3) = (2, -4, 3)$ in $\mathbb{R}^7$

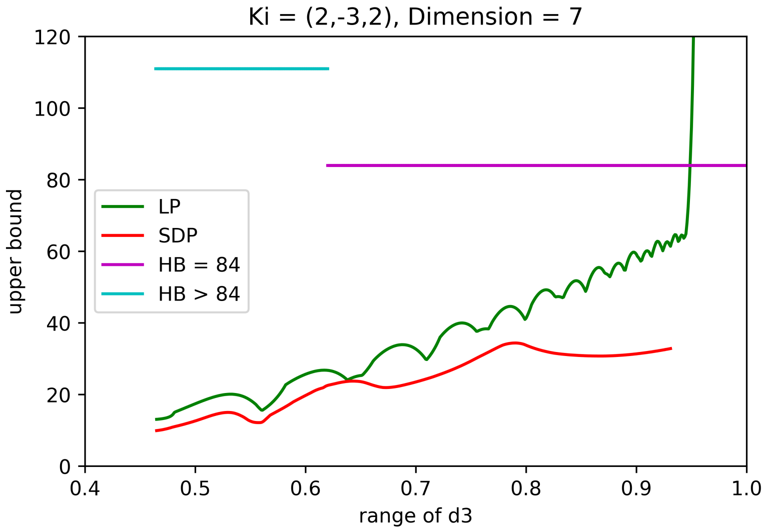

Figure 6: Sampling points with $(K_1, K_2, K_3) = (2, -3, 2)$ in $\mathbb{R}^7$

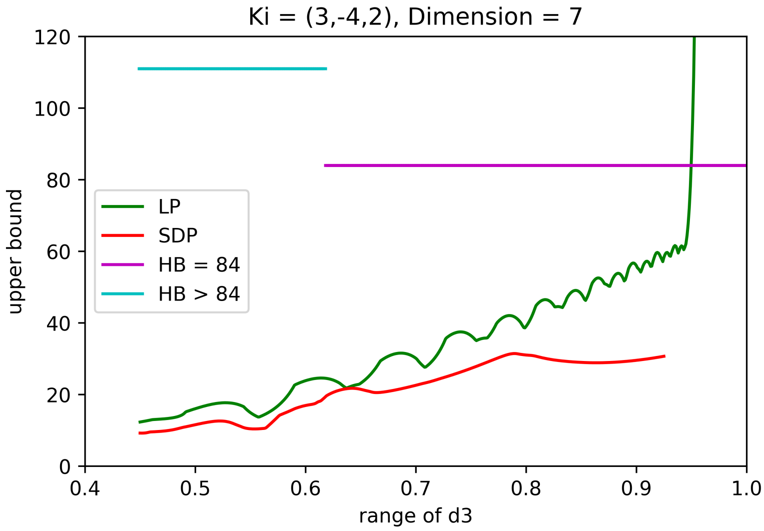

Figure 7: Sampling points with $(K_1, K_2, K_3) = (3, -4, 2)$ in $\mathbb{R}^7$

Experiment in $\mathbb{R}^{23}$

There are 66 types of (K_1, K_2, K_3) in $\mathbb{R}^{23}$ [**Prop 7**]. We only improve two instances of K_i and present the LP, SDP, and HB results in the following table and figures.

K_1	K_2	K_3	d_1	d_2	d_3	Notes
1	-3	3	$3d_3 - 2$	$2d_3 - 1$	d_3	**Improved bound** by **SDP** and HB [8]
2	-6	5	$\frac{5d_3}{4} + \frac{\sqrt{15}(d_3-1)}{4} - \frac{1}{4}$	$\frac{5d_3}{4} + \frac{\sqrt{15}(d_3-1)}{12} - \frac{1}{4}$	d_3	**Improved bound** by **SDP** and HB [9]
			(Others)			Already controlled by LP and HB

Table 7: (K_1, K_2, K_3) in $\mathbb{R}^{23}$ and corresponding three inner product values

In the $(1, -3, 3)$ case, the purple line represents the harmonic bound 2301, while the green line (LP) exceeds the purple line in some regions around $d_3 = 0.59$. Therefore, in those regions, the best bound we can obtain is 2301. However, the red line (SDP) always stays below the purple line in that region. Therefore, SDP indeed plays a crucial role in achieving a tighter upper bound of 2300. A similar phenomenon occurs for the $(2, -6, 5)$ case; without SDP, the best bound is 2301. For other cases of K_i in $\mathbb{R}^{23}$, the LP bounds have already performed well in achieving the bound 2300.

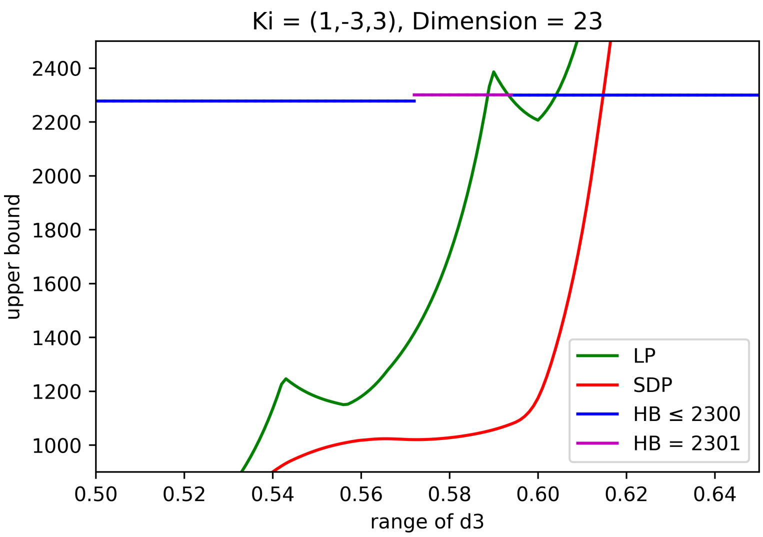

Figure 8: Sampling points with $(K_1, K_2, K_3) = (1, -3, 3)$ in $\mathbb{R}^{23}$

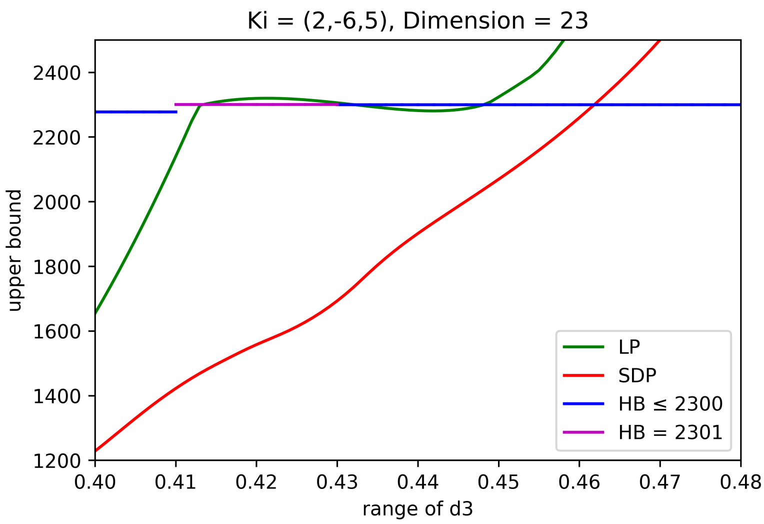

Figure 9: Sampling points with $(K_1, K_2, K_3) = (2, -6, 5)$ in $\mathbb{R}^{23}$

For $\mathbb{R}^{23}$, we also provide the detailed experimental data in the GitHub website: https://github.com/smileyung/SEMIDEFINITE-PROGRAMMING-BOUNDS-FOR-SPHERICAL-THREE-DISTANCE-SETS-Appendix

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