A counterexample to directed-KKL

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Abstract

We show that the natural directed analogues of the KKL theorem [6] and the Eldan–Gross inequality [4] from the analysis of Boolean functions fail to hold. This is in contrast to several other isoperimetric inequalities on the Boolean hypercube (such as the Poincaré inequality, Margulis's inequality [9] and Talagrand's inequality [14]) for which directed strengthenings have recently been established.

Mathematics Subject Classifications: Primary 05D05; Secondary 06E30

In this note, we consider isoperimetric inequalities over the Boolean hypercube $\{0,1\}^n$. Our notation and terminology follow O'Donnell [11]; in particular, we refer the reader to introductory chapters of [11] for further background.

Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and an input $x \in \{0,1\}^n$, we define the *sensitivity of f at x* as

$$\mathrm{sens}_f(x) = \left|\left\{i : f(x) \neq f(x^{\oplus i})\right\}\right|,$$

where $x^{\oplus i} = (x_1, \ldots, 1 - x_i, \ldots, x_n)$. Two closely related isoperimetric quantities are the *influence of a variable $i \in [n]$ on f*, given by

$$\mathrm{Inf}_i[f] = \mathop{\mathbb{P}}_{x\sim\{0,1\}^n}\left[f(x) \neq f(x^{\oplus i})\right],$$

where $x \sim \{0,1\}^n$ represents a uniform sample from $\{0,1\}^n$, and the *total influence of f*, given by

$$\mathbf{I}[f] = \sum_{i=1}^{n} \mathrm{Inf}_i[f].$$

It is easy to check that $\mathbf{I}[f] = \mathbb{E}[\mathrm{sens}_f(x)]$, and so the total influence of a function is sometimes also referred to as its *average sensitivity*.

To set the stage, we recall perhaps the simplest isoperimetric inequality on the Boolean hypercube, the Poincaré inequality, which says that

$$\mathbf{I}[f] \geqslant \mathrm{Var}[f].$$

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The following strengthening of the Poincaré inequality was obtained by Talagrand [14], and is known to imply yet another isoperimetric inequality due to Margulis [9].

Theorem 1. *Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f(x)}\right] = \Omega(\mathrm{Var}[f]).$$

An alternative (and incomparable) strengthening of the Poincaré inequality is given by the celebrated Kahn–Kalai–Linial theorem [6].

Theorem 2. *Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, there exists $i \in [n]$ such that*

$$\mathrm{Inf}_i[f] = \Omega\left(\mathrm{Var}[f] \cdot \frac{\log_2 n}{n}\right).$$

Talagrand [15] conjectured the following common generalization of Theorems 1 and 2, which was proved by Eldan and Gross [4].

Theorem 3. *Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f(x)}\right] = \Omega\left(\mathrm{Var}[f]\sqrt{\log_2\left(2 + \frac{e}{\sum_{i=1}^n \mathrm{Inf}_i[f]^2}\right)}\right).$$

Here, we will be concerned with directed versions of such results in the Boolean hypercube. Recall that a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is said to be *monotone* (respectively *anti-monotone*) if for all $x, y \in \{0,1\}^n$, $x \preccurlyeq y$ implies $f(x) \leqslant f(y)$ (respectively $f(x) \geqslant f(y)$), where we write $x \preccurlyeq y$ to mean $x_i \leqslant y_i$ for all $i \in [n]$. In connection with the problem of monotonicity testing, Khot, Minzer, and Safra [7] obtained a 'directed' analogue of Theorem 1. Writing

$$\mathrm{sens}_f^-(x) = \left|\left\{i : f(x) > f(x^{\oplus i}) \text{ and } x \preccurlyeq x^{\oplus i}\right\}\right|$$

for the *negative sensitivity of f at x*, and

$$\varepsilon(f) = \min_{g \text{ monotone}} \mathrm{dist}(f, g)$$

for the *distance to monotonicity of f*, where $\mathrm{dist}(f,g) = \mathbb{P}_{x\sim\{0,1\}^n}[f(x) \neq g(x)]$, the following result, a slight sharpening of the result of Khot, Minzer, and Safra [7], was established by Pallavoor, Raskhodnikova, and Waingarten [12].

Theorem 4. *Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f^-(x)}\right] = \Omega(\varepsilon(f)).$$

Indeed, prior results on monotonicity testing due to Goldreich, Goldwasser, Lehman, Ron and Samordinsky [5] and Chakrabarty and Seshahdri [3] can be viewed as directed analogues of the Poincaré inequality and Margulis's inequality [9] respectively. Finally, a directed analogue of an inequality due to Pisier [13] was obtained by Canonne, Chen, Kamath, Levi and Waingarten [2].

Although the directed analogues are known to imply their undirected counterparts, see [7], their proofs bear little resemblance to the proofs in the undirected setting (with the exception of the directed Pisier inequality) and are usually much more involved.

These results suggest an informal analogy between the undirected and the directed cube, with isoperimetric quantities being replaced with their directed counterparts and $\mathrm{Var}[f]$ being replaced with $\varepsilon(f)$ in the latter. Writing

$$\mathrm{Inf}_i^-[f] = \left|\left\{x : f(x) > f(x^{\oplus i}) \text{ and } x \preccurlyeq x^{\oplus i}\right\}\right| \cdot \frac{1}{2^{n-1}}$$

for the *negative influence of i on f*, the following directed variant of Theorem 2 would yield a natural directed analogue of the KKL inequality.

Conjecture 5. Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, there exists $i \in [n]$ such that

$$\mathrm{Inf}_i^-[f] \geqslant \Omega\left(\varepsilon(f) \cdot \frac{\log_2 n}{n}\right).$$

Conjecture 5, as well as a Fourier analytic reformulation thereof, appears to have been first raised by Khot [8]. Our aim in this short note is to show that Conjecture 5 fails to hold.

Theorem 6. *There is a function $f : \{0,1\}^{2n} \to \{0,1\}$ with*

1. $\mathrm{Inf}_i^-[f] = 0$ *for all* $i \in [n]$,
2. $\mathrm{Inf}_i^-[f] = O(1/n)$ *for all* $i \in [2n] \setminus [n]$, *and*
3. $\varepsilon(f) = \Omega(1)$,

We note that this further rules out a natural directed analogue of Theorem 3 (which would imply Conjecture 5).

To prove Theorem 6, We view $\{0,1\}^{2n}$ as $\{0,1\}^n \times \{0,1\}^n$ and construct a function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ with

1. $\mathrm{Inf}_i^-[f] = 0$ for all $i \in [n]$,
2. $\mathrm{Inf}_i^-[f] = O(1/n)$ for all $i \in [2n] \setminus [n]$, and
3. $\varepsilon(f) = \Omega(1)$,

thereby refuting Conjecture 5.

Proof of Theorem 6. Let $T_1, \ldots, T_n \in \binom{[n]}{\log_2 n}$ be drawn independently and uniformly at random. Set

$$f(x,y) := \bigvee_{i=1}^{n} \left(\left(\bigwedge_{j \in T_i} x_j \right) \wedge (1 - y_i) \right).$$

We note that this function is closely related to the well-known 'tribes' function due to Ben-Or and Linial [1].

It is clear that f is monotone in the first n coordinates and anti-monotone in the last n coordinates; consequently $\mathrm{Inf}_i^-[f] = 0$ for all $i \in [n]$. A coordinate $i \in [2n] \setminus [n]$ is relevant only on $x \in \{0,1\}^n$ for which $\bigwedge_{j \in T_i} x_j = 1$; as $|T_i| = \log_2 n$, this set has measure at most

$$\frac{2^{n - \log_2 n}}{2^n} = \frac{1}{n}.$$

It follows that $\mathrm{Inf}_i^-[f] = O(1/n)$ for all $i \in [2n]$.

Before turning to the third item above, we recall the following fact from [7] without proof.

Lemma 7. *For $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ such that f is monotone in the first n coordinates and anti-monotone in the last n coordinates, we have*

$$\varepsilon(f) = \Theta\left(\mathop{\mathbb{E}}_{x \sim \{0,1\}^n} \left[\mathop{\mathrm{Var}}_{y \sim \{0,1\}^n} [f(x,y)] \right] \right). \qquad \square$$

Suppose, for convenience, that $x \in \{0,1\}^n$ is such that $\bigwedge_{j \in T_i} x_j = 1$ for exactly one $i \in [n]$. Then the restricted function $f(x, \cdot) : \{0,1\}^n \to \{0,1\}$ is simply the anti-dictatorship $(1 - y_i)$, and has $\mathrm{Var}[f(x, \cdot)] = \Omega(1)$. We will be done if we can show that there exist $T_1, \ldots, T_n$ for which this happens for $\Omega(1)$ fraction of $x \in \{0,1\}^n$. As before, for fixed $i \in [n]$ we have

$$\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\bigwedge_{j \in T_i} x_j \right] = \frac{1}{n},$$

which tells us that

$$\mathop{\mathbb{E}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| \right] = 1.$$

By Markov's inequality, we thus have

$$\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| \geqslant 2 \right] \leqslant \frac{1}{2}.$$

In addition, with expectation over choices of $T_1, \ldots, T_n$,

$$\mathbb{E}\left[\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| = 0 \right] \right] \approx \left(1 - \frac{1}{n}\right)^n \approx \frac{1}{e},$$

since for fixed x, the events $\{\bigwedge_{j\in T_i} x_j = 1\}$ ($i \in [n]$) are independent, and for x with $|\sum x_i| < n^{2/3}$ (where almost all of the measure of $\{0,1\}^n$ lies), $\mathbb{P}(\bigwedge_{j\in T_i} x_j = 1) = 1/n + O(n^{-5/4})$. Therefore there is a choice of $T_1, \ldots, T_n$ with

$$\mathop{\mathbb{P}}_{x\sim\{0,1\}^n}\left[\left|\left\{i : \bigwedge_{j\in T_i} x_j = 1\right\}\right| = 1\right] \gtrapprox \frac{1}{2} - \frac{1}{e} = \Omega(1),$$

and we are done. □

Note added in proof

After a draft of this paper was circulated, it was brought to our attention that Minzer and Khot [10] have independently discovered a similar construction to the one establishing our main result.

References

[1] M. Ben-Or and N. Linial, *Collective coin flipping, robust voting schemes and minima of Banzhaf values*, Proc. 26th Annual Symposium on Foundations of Computer Science (FOCS), 1985, pp. 408–416. doi:10.1109/SFCS.1985.15.

[2] C. L. Canonne, X. Chen, G. Kamath, A. Levi, and E. Waingarten, *Random Restrictions of High Dimensional Distributions and Uniformity Testing with Subcube Conditioning*, Proc. 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), 2021, pp. 321–336. doi:10.1137/1.9781611976465.21.

[3] D. Chakrabarty and C. Seshadhri, *An $o(n)$ Monotonicity Tester for Boolean Functions over the Hypercube*, SIAM J. Comput. **45** (2016), 461–472. doi:10.1137/13092770X.

[4] R. Eldan and R. Gross, *Concentration on the Boolean hypercube via pathwise stochastic analysis*, Inventiones mathematicae 230, no. 3 (2022). doi:10.1007/s00222-022-01135-8.

[5] O. Goldreich, S. Goldwasser, E. Lehman, D. Ron, and A. Samordinsky, *Testing Monotonicity*, Combinatorica **20** (2000), 301–337. doi:10.1007/s004930070011.

[6] J. Kahn, G. Kalai, and N. Linial, *The influence of variables on boolean functions*, Proc. 29th Annual Symposium on Foundations of Computer Science (FOCS), 1988, pp. 68–80. doi:10.1109/SFCS.1988.21923.

[7] S. Khot, D. Minzer, and M. Safra, *On Monotonicity Testing and Boolean Isoperimetric Type Theorems*, Proc. 56th Annual Symposium on Foundations of Computer Science (FOCS), 2015, pp. 52–58. doi:10.1109/FOCS.2015.13.

[8] H. Lee, *Notes on Simons Algorithms and Geometry Meetings*, 2022.

[9] G. Margulis, *Probabilistic characteristics of graphs with large connectivity*, Prob. Peredachi Inform. **10** (1974), 101–108.

[10] D. Minzer, *Personal communication*, 2022.

[11] R. O'Donnell, *Analysis of Boolean Functions*, Cambridge University Press, New York, 2014. doi:10.1017/CBO9781139814782.

[12] R. K. S. Pallavoor, S. Raskhodnikova, and E. Waingarten, *Approximating the distance to monotonicity of boolean functions*, Random Struct. Algorithms **60** (2022), 233–260. doi:10.1002/rsa.21029.

[13] G. Pisier, *Probabilistic methods in the geometry of Banach spaces*, Lecture notes in Math., Springer, 1986, pp. 167–241. doi:10.1007/BFb0076302.

[14] M. Talagrand, *Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem*, Geom. Funct. Anal. **3** (1993), 295–314. doi:10.1007/BF01895691.

[15] M. Talagrand, *On boundaries and influences*, Combinatorica **17** (1997), 275–285. doi:10.1007/BF01200910.