Two classes of posets with real-rooted chain polynomials

Christos A. Athanasiadis[a] Theo Douvropoulos[b]
Katerina Kalampogia-Evangelinou[a]

Submitted: Jul 16, 2023; Accepted: Jun 10, 2024; Published: Oct 18, 2024

Abstract

The coefficients of the chain polynomial of a finite poset enumerate chains in the poset by their number of elements. It has been a challenging open problem to determine which posets have real-rooted chain polynomials. Two new classes of posets, namely those of all rank-selected subposets of Cohen-Macaulay simplicial posets and all noncrossing partition lattices associated to finite Coxeter groups, are shown to have this property. The first result generalizes one of Brenti and Welker. As a special case, the descent enumerator of permutations of the set $\{1, 2, \ldots, n\}$ which have ascents at specified positions is shown to be real-rooted, hence log-concave and unimodal, and a good estimate for the location of the peak is deduced.

Mathematics Subject Classifications: 05A15, 05E45, 06A07, 26C10

1 Introduction

The coefficients of the chain polynomial $f_P(x)$ of a finite poset (partially ordered set) P enumerate chains in P by their number of elements. From a face enumeration point of view, $f_P(x)$ is the f-polynomial of a flag simplicial complex of special type, namely the order complex $\Delta(P)$ (see [34] for basic definitions and terminology on simplicial complexes and their face enumeration). Thus,

$$f_P(x) = f(\Delta(P), x) = \sum_{i=0}^{n} f_{i-1}(\Delta(P))x^i, \tag{1}$$

where $f_{i-1}(\Delta(P))$ is the number of i-element chains in P (which are precisely the $(i-1)$-dimensional faces of $\Delta(P)$) and n is the largest size of such a chain (which is one more

[a]Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, 15784 Athens, Greece (caath@math.uoa.gr, kalampogia@math.uoa.gr).
[b]Department of Mathematics, Brandeis University, Waltham, MA 02453, USA (tdouvropoulos@brandeis.edu).

than the dimension of $\Delta(P)$). For some purposes, one may focus on the corresponding h-polynomial

$$h(\Delta(P), x) = \sum_{i=0}^{n} f_{i-1}(\Delta(P)) x^i (1-x)^{n-i} \tag{2}$$

instead. For instance, $f_P(x)$ has only real roots if and only if so does $h(\Delta(P), x)$. The following general question was posed in [5].

Question 1. ([5, Question 1.1]) For which finite posets P does the chain polynomial $f_P(x)$ have only real roots?

There are several motivations behind this question. For finite distributive lattices, it is known to be equivalent to the poset conjecture for natural labelings, posed in the seventies by Neggers [27] (see also [32, Conjecture 1]) and finally disproved by Stembridge [40], after counterexamples to a more general conjecture were found by Brändén [12]. It is open for face posets of convex polytopes [15, Question 1] and more general regular cell complexes [5, Section 5], in which case the chain polynomial coincides with the f-polynomial of the barycentric subdivision of the complex, and is conjectured to hold for all geometric lattices [5, Conjecture 1.2]. Among other positive results, an affirmative answer to Question 1 has been given for simplicial posets with nonnegative h-vector [15] and cubical posets with nonnegative cubical h-vector [3] (in particular, for face lattices of simplicial and cubical convex polytopes), for partition and subspace lattices [5] and for posets which do not contain the disjoint union of a three-element chain and a one-element chain as an induced subposet [36]. An overview of positive results will appear in [24].

This paper contributes an affirmative answer to Question 1 for two other broad classes of posets, namely those of all rank-selected subposets of Cohen–Macaulay (over some field) simplicial posets (more generally, of simplicial posets with nonnegative h-vector) and all noncrossing partition lattices associated to finite Coxeter groups. A special case of the first class of independent interest comes from considering rank-selected subposets of Boolean lattices. Given $T \subseteq [n-1] := \{1, 2, \ldots, n-1\}$ we set

$$A_n^T(x) = \sum_{w \in \mathfrak{S}_n \colon \mathrm{Des}(w) \subseteq T} x^{\mathrm{des}(w)}, \tag{3}$$

where $\mathfrak{S}_n$ is the symmetric group of permutations of the set $[n]$ and for $w \in \mathfrak{S}_n$, $\mathrm{Des}(w) = \{i \in [n-1] : w(i) > w(i+1)\}$ and $\mathrm{des}(w)$ is the set and the number of descents of w, respectively. Thus, $A_n^T(x)$ is the descent enumerator for permutations in $\mathfrak{S}_n$ which have ascents at specified positions (those in $[n-1] \smallsetminus T$); it reduces to the classical Eulerian polynomial $A_n(x)$ [37, Section 1.4] for $T = [n-1]$. A q-analogue of $A_n^T(x)$ was studied in [17] in the special case $T = [r] \subseteq [n-1]$ (see also Example 11). To the best of our knowledge, except for the important special case of Eulerian polynomials, the unimodality, log-concavity and real-rootedness of $A_n^T(x)$ have not been considered before.

The following statement is the first main result of this paper (basic definitions and terminology on posets can be found in Sections 2, 4, 5). Part (b) generalizes the main result of [15], which corresponds to the special case $T = [n]$. Throughout this paper, we denote by $\hat{P}$ the poset obtained from P by adding a maximum element and set $A_n^T(x) := A_n^{T\cap[n-1]}(x)$ for $T \subseteq \mathbb{N}$.

Theorem 2. *Let n be a positive integer.*

(a) The polynomial $A_n^T(x)$ has only real roots for every $T \subseteq [n-1]$.

(b) Let P be a Cohen–Macaulay simplicial poset of rank n. Then, every rank-selected subposet $\hat{P}_T$ of $\hat{P}$ has a real-rooted chain polynomial. Moreover, $h(\Delta(\hat{P}_T), x)$ is interlaced by $A_n^T(x)$ for every $T \subseteq [n]$.

Noncrossing partition lattices associated to Coxeter groups are central objects of study in Coxeter-Catalan combinatorics; see [1, Chapter 2] for an overview. The enumeration of chains in these posets has been a very popular topic [6, 16, 19, 23, 26, 29, 30]. Let us denote by NC_W the noncrossing partition lattice associated to W. The second main result of this paper is as follows.

Theorem 3. *Let W be a finite Coxeter group.*

(a) The noncrossing partition lattice NC_W has a real-rooted chain polynomial.

(b) The polynomial $h(\Delta(\mathrm{NC}_W), x)$ has a nonnegative real-rooted symmetric decomposition with respect to $r_W - 1$ for every irreducible finite Coxeter group W, where r_W is the rank of W. In particular, $h(\Delta(\mathrm{NC}_W), x)$ is unimodal, with a peak at position $\lfloor r_W/2 \rfloor$.

Question 1 cannot have an affirmative answer for all Cohen–Macaulay posets since, as already explained, it fails for finite distributive lattices. However, and since the proper parts of face lattices of polytopes, geometric lattices, noncrossing partition lattices of types A and B [25] and rank-selected subposets of Boolean lattices are doubly Cohen–Macaulay (see [34, Section III.3] for information about doubly Cohen–Macaulay posets), it seems reasonable to pose the following question.

Question 4. Does the chain polynomial of any doubly Cohen–Macaulay lattice (or even doubly Cohen–Macaulay poset) have only real roots?

This paper is organized as follows. Section 2 reviews definitions and tools from the theory of real-rooted polynomials (and especially the theory of interlacing) and the enumerative combinatorics of posets which are essential in understanding the main results and their proofs. The proof of Theorem 2 splits in two sections. Section 3 proves that $A_n^T(x)$ is real-rooted (see Theorem 8), hence unimodal, gives a good estimate for the location of the peak and discusses some interesting special cases. Section 4 proves part (b) of the theorem by combining Theorem 8 with an exercise from [34] (see Lemma 14) and, as an application, generalizes part (a) in the setting of colored permutations. Part (a) of Theorem 3 is proven in Section 5 in two different ways. The first proof does not assume the classification of finite Coxeter groups. The second proof is based on explicit combinatorial interpretations (which are of independent interest), as descent enumerators of certain families of words, of the h-polynomials of the order complexes $\Delta(\mathrm{NC}_W)$ for the irreducible finite Coxeter groups W of classical types (see Proposition 17) and on computer computations for the exceptional groups. These combinatorial interpretations are extracted from the known explicit formulas for the entries of the flag f-vectors of noncrossing partition lattices [6, 19, 30], the case of groups of type D being the trickiest. Part (b) of Theorem 3 follows from Proposition 17 by an application of a result of Jochemko [22] about Veronese operators on formal power series.

2 Preliminaries

This section reviews basic concepts and tools from the theory of real-rooted polynomials and the enumerative combinatorics of posets (the theory of rank selection, in particular) which will be essential in the following three sections. Standard references for these topics are [13, 20, 32, 34, 37].

2.1 Polynomials

A polynomial $p(x) = h_0 + h_1x + \cdots + h_nx^n \in \mathbb{R}[x]$ is called

- *symmetric*, with center of symmetry $n/2$, if $h_i = h_{n-i}$ for all $0 \leqslant i \leqslant n$,
- *unimodal*, with a peak at position k, if $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_k \geqslant h_{k+1} \geqslant \cdots \geqslant h_n$,
- *log-concave*, if $h_i^2 \geqslant h_{i-1}h_{i+1}$ for $1 \leqslant i \leqslant n-1$,
- *real-rooted*, if every root of $p(x)$ is real, or $p(x) \equiv 0$.

Every real-rooted polynomial with nonnegative coefficients is log-concave and unimodal; see [13, 32] for more information about these concepts.

A real-rooted polynomial $p(x)$, with roots $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$, is said to *interlace* a real-rooted polynomial $q(x)$, with roots $\beta_1 \geqslant \beta_2 \geqslant \cdots$, if

$$\cdots \leqslant \alpha_2 \leqslant \beta_2 \leqslant \alpha_1 \leqslant \beta_1.$$

We then write $p(x) \preceq q(x)$. By convention, the zero polynomial interlaces and is interlaced by every real-rooted polynomial and nonzero constant polynomials interlace all polynomials of degree at most one. A sequence $(p_0(x), p_1(x), \dots, p_m(x))$ of real-rooted polynomials is called *interlacing* if $p_i(x) \preceq p_j(x)$ for $0 \leqslant i < j \leqslant m$. The following statement lists well known properties of interlacing sequences; see, for instance, [13, Section 7.8] [20, Chapter 3].

Lemma 5. *Let $(p_0(x), p_1(x), \dots, p_m(x))$ be an interlacing sequence of real-rooted polynomials with positive leading coefficients.*

(a) Every nonnegative linear combination $p(x)$ of $p_0(x), p_1(x), \dots, p_m(x)$ is real-rooted. Moreover, $p_0(x) \preceq p(x) \preceq p_m(x)$.

(b) The sequence $(q_0(x), q_1(x), \ldots, q_{m+1}(x))$ *of partial sums*

$$q_k(x) = \sum_{i=k}^{m} p_i(x)$$

for $k \in \{0, 1, \ldots, m+1\}$ *is also interlacing.*

(c) The sequence $(t_0(x), t_1(x), \ldots, t_{m+1}(x))$ *defined by*

$$t_k(x) = x \sum_{i=0}^{k-1} p_i(x) + \sum_{i=k}^{m} p_i(x)$$

for $k \in \{0, 1, \ldots, m+1\}$ *is also interlacing.*

Given a polynomial $p(x) \in \mathbb{R}[x]$ of degree at most n, there exist unique symmetric polynomials $a(x), b(x) \in \mathbb{R}[x]$ with centers of symmetry $n/2$ and $(n-1)/2$, respectively, such that $p(x) = a(x) + xb(x)$. This expression is known as the *symmetric decomposition* (or *Stapledon decomposition*) of $p(x)$ with respect to n. Then, $p(x)$ is said to have a *nonnegative* (respectively, *unimodal* or *real-rooted*) *symmetric decomposition* with respect to n if $a(x)$ and $b(x)$ have nonnegative coefficients (respectively, are unimodal or real-rooted); see [7, 14] for more information about these concepts. Every polynomial which has a nonnegative unimodal symmetric decomposition with respect to n is unimodal, with a peak at position $\lceil n/2 \rceil$.

2.2 Poset combinatorics

Our notation and terminology generally follows that of [37, Chapter 3]. Let P be a finite graded poset of rank n, having a minimum element $\hat{0}$ and rank function $\rho : P \to \{0, 1, \ldots, n\}$, and let $\hat{P}$ be the poset obtained from P by adding a maximum element $\hat{1}$. Given $T \subseteq [n]$, the T-rank-selected subposet of $\hat{P}$ is defined as

$$\hat{P}_T = \{y \in P : \rho(y) \in T\} \cup \{\hat{0}, \hat{1}\}.$$

We denote by $\alpha_{\hat{P}}(T)$ the number of maximal chains of $\hat{P}_T$ and set

$$\beta_{\hat{P}}(T) = \sum_{S \subseteq T} (-1)^{|T \smallsetminus S|} \alpha_{\hat{P}}(S) \tag{4}$$

for $T \subseteq [n]$. Equivalently, we have

$$\alpha_{\hat{P}}(T) = \sum_{S \subseteq T} \beta_{\hat{P}}(S) \tag{5}$$

for $T \subseteq [n]$. The collections of numbers $(\alpha_{\hat{P}}(T))_{T \subseteq [n]}$ and $(\beta_{\hat{P}}(T))_{T \subseteq [n]}$ are the *flag f-vector* and the *flag h-vector* of $\hat{P}$, respectively.

The *order complex* of a finite poset Q is defined as the simplicial complex $\Delta(Q)$ which consists of all chains in Q. The f-polynomial and the h-polynomial of $\Delta(Q)$ are defined by Equations (1) and (2), respectively, when P is replaced by Q. Since the h-polynomial is unaffected when maximum or minimum elements are removed from Q, we have the equivalent expressions

$$f(\Delta(\hat{P}_T \smallsetminus \{\hat{0}, \hat{1}\}), x) = \sum_{S \subseteq T} \alpha_{\hat{P}}(S) x^{|S|} = \sum_{S \subseteq T} \beta_{\hat{P}}(S) x^{|S|} (1+x)^{|T \smallsetminus S|} \tag{6}$$

and

$$h(\Delta(\hat{P}_T), x) = \sum_{S \subseteq T} \alpha_{\hat{P}}(S) x^{|S|} (1-x)^{|T \smallsetminus S|} = \sum_{S \subseteq T} \beta_{\hat{P}}(S) x^{|S|} \tag{7}$$

for every $T \subseteq [n]$, where the second equality in each case is a consequence of Equation (5).

Example 6. Let $\hat{P}$ be the Boolean lattice B_n of subsets of $[n]$, partially ordered by inclusion. Then, $\beta_{\hat{P}}(S)$ is equal to the number of permutations $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w) = S$ for every $S \subseteq [n-1]$ [37, Corollary 3.13.2] and Equation (7) yields that

$$h(\Delta((B_n)_T), x) = \sum_{w \in \mathfrak{S}_n \colon \mathrm{Des}(w) \subseteq T} x^{\mathrm{des}(w)} = A_n^T(x).$$

We note that, by definition of $A_n^T(x)$ and a standard argument, we have $A_n^T(x) = A_n^{n-T}(x)$ for every $T \subseteq [n-1]$, where $n - T := \{n - a : a \in T\}$. $\square$

The *zeta polynomial* $\mathcal{Z}(P, x)$ is another important enumerative invariant of a finite poset P [37, Section 3.12]. For the ease of notation, we define it here by letting $\mathcal{Z}(P, k)$ be the number of multichains $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ of length $k-1$ (rather than $k-2$) of elements of P, where $\mathcal{Z}(P, 0) := 1$. A comparison of [37, Proposition 3.12.1 (a)] with [34, Theorem II.1.4] then shows that

$$\sum_{k \geqslant 0} \mathcal{Z}(P, k) x^k = \frac{h(\Delta(P), x)}{(1-x)^n}, \tag{8}$$

where n is the largest cardinality of a chain in P. The following lemma states that the product $P \times Q$ of posets has a real-rooted chain polynomial, provided that so do P and Q and that $\Delta(P)$ and $\Delta(Q)$ have nonnegative h-vectors; it complements some of the results of [5, Section 5].

Lemma 7. *Let P, Q be finite posets. If $h(\Delta(P), x)$ and $h(\Delta(Q), x)$ have nonnegative coefficients and only real roots, then so does $h(\Delta(P \times Q), x)$.*

Proof. Let m and n be the largest cardinality of a chain in P and Q, respectively. Clearly, we have $\mathcal{Z}(P \times Q, k) = \mathcal{Z}(P, k)\mathcal{Z}(Q, k)$ for every k. Hence, by Equation (8),

$$\sum_{k \geqslant 0} \mathcal{Z}(P, k)\mathcal{Z}(Q, k) x^k = \frac{h(\Delta(P \times Q), x)}{(1-x)^{m+n-1}}$$

and the proof follows by an application of [43, Theorem 0.2]. $\square$

3 Permutations with restricted descent set

This section proves that the polynomials $A_n^T(x)$ are real-rooted, as claimed in part (a) of Theorem 2, and in particular unimodal, locates their peak and discusses some interesting special cases and formulas. The applications of the real-rootedness of $A_n^T(x)$ discussed here have a probabilistic flavor; see [13, Section 7.2] [28] for overviews of this topic. For a probabilistic approach to the theory of descents in permutations, we recommend [11, Section 5].

Crucial to the proof will be the polynomials

$$p_{n,k}^T(x) = \sum_{w \in \mathfrak{S}_{n+1,k+1} :\, \mathrm{Des}(w) \subseteq T} x^{\mathrm{des}(w)}, \tag{9}$$

where $T \subseteq [n]$, $k \in \{0, 1, \dots, n\}$ and $\mathfrak{S}_{n+1,k+1}$ is the set of permutations $w \in \mathfrak{S}_{n+1}$ such that $w(1) = k+1$. We note that $p_{n,0}^T(x) = A_n^{T-1}(x)$ and

$$p_{n,n}^T(x) = \begin{cases} xA_n^{T-1}(x), & \text{if } 1 \in T \\ 0, & \text{if } 1 \notin T \end{cases}$$

for $T \subseteq [n]$, where $T - 1 := \{a - 1 : a \in T\}$ and, as mentioned in Section 1, $A_n^T(x) := A_n^{T \cap [n-1]}(x)$ for $T \subseteq \mathbb{N}$. We set $p_{n,k}^T(x) = p_{n,k}(x)$ when $T = [n]$; these polynomials appeared in [15] [17, Section 2.2] and have been studied intensely since then; see, for instance, [4, Section 2] [8, Section 3] [13, Example 7.8.8] and the references given there. They can also be defined by the formula [15, Equation (4)]

$$\sum_{m \geqslant 0} m^k (1+m)^{n-k} x^m = \frac{p_{n,k}(x)}{(1-x)^{n+1}}. \tag{10}$$

The polynomials $p_{n,k}(x)$ are real-rooted and $(p_{n,k}(x))_{0 \leqslant k \leqslant n}$ is an interlacing sequence for every $n \in \mathbb{N}$; see, for instance, [13, Example 7.8.8]. This fact is generalized by the main result of this section.

Theorem 8. *For all $n \in \mathbb{N}$ and $T \subseteq [n]$,*

$$(p_{n,0}^T(x), p_{n,1}^T(x), \dots, p_{n,n}^T(x)) \tag{11}$$

is an interlacing sequence of real-rooted polynomials.

In particular, $A_n^{T-1}(x)$ is real-rooted and it interlaces $A_{n+1}^T(x)$ for all positive integers n and $T \subseteq [n]$.

Proof. We proceed by induction on n, the result being trivial for $n = 0$. Suppose that $n \geqslant 1$ and that (11) is an interlacing sequence of real-rooted polynomials when n is replaced by $n - 1$. It is straightforward to verify from the defining equation (9) that

$$p_{n,k}^T(x) = \sum_{i=k}^{n-1} p_{n-1,i}^{T-1}(x)$$

for $k \in \{0, 1, \ldots, n\}$, if $1 \notin T$, and

$$p_{n,k}^{T}(x) = x \sum_{i=0}^{k-1} p_{n-1,i}^{T-1}(x) + \sum_{i=k}^{n-1} p_{n-1,i}^{T-1}(x)$$

for $k \in \{0, 1, \ldots, n\}$, if $1 \in T$. This recurrence generalizes that of the special case $T = [n]$; see [13, Example 7.8.8]. An application of Lemma 5 shows that, in either case, (11) is an interlacing sequence of real-rooted polynomials as well. This completes the inductive step. The last statement follows from part (a) of Lemma 5 since $p_{n,0}^{T}(x) = A_n^{T-1}(x)$ and $\sum_{k=0}^{n} p_{n,k}^{T}(x) = A_{n+1}^{T}(x)$. □

We recall that a polynomial $p(x) = \sum_{k \geqslant 0} h_k x^k \in \mathbb{R}[x]$ with nonnegative and unimodal coefficients is said to have a *mode* m if there exists a unique $m \in \frac{1}{2}\mathbb{Z}$ such that either $h_m = \max_k h_k$, or $h_{m \pm 1/2} = \max_k h_k$.

Corollary 9. *The polynomial $A_n^T(x)$ is unimodal and log-concave for every $T \subseteq [n-1]$. Moreover, $A_n^T(x)$ has a mode $m_n(T)$ such that $\lfloor \mu_n(T) \rfloor \leqslant m_n(T) \leqslant \lceil \mu_n(T) \rceil$ for*

$$\mu_n(T) := r - \sum_{i=1}^{r} \binom{c_i + c_{i+1}}{c_i}^{-1}, \tag{12}$$

where $T = \{a_1, a_2, \ldots, a_r\}$ with $1 \leqslant a_1 < \cdots < a_r < n$ and $c_i = a_i - a_{i-1}$ for $i \in [r+1]$, with $a_0 := 0$ and $a_{r+1} := n$. In particular,

$$h_0(T) \leqslant h_1(T) \leqslant \cdots \leqslant h_{\lfloor r/2 \rfloor}(T), \tag{13}$$

if $A_n^T(x) = \sum_{i=0}^{r} h_i(T) x^i$.

Proof. Since $A_n^T(x)$ is real-rooted and has nonnegative coefficients, by a result of Darroch [18, Theorem 4] (see also [13, Theorem 2.2] [28, p. 284]) we only need to verify that the right-hand side of Equation (12) is equal to the expected number of descents when a permutation $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w) \subseteq T$ is selected uniformly at random. This holds because the probability that $a_i \in \mathrm{Des}(w)$ for such w is easily computed to be $1 - \binom{c_i + c_{i+1}}{c_i}^{-1}$. The last statement follows since $\mu_n(T) \geqslant r/2$. □

Remark 10. The fact (see Example 6) that $A_n^T(x) = h(\Delta((B_n)_T), x)$, combined with Equation (7), yields the explicit formula

$$A_n^T(x) = \sum_{S \subseteq T} \alpha_{B_n}(S)\, x^{|S|} (1-x)^{|T \smallsetminus S|},$$

where $\alpha_{B_n}(S)$ is a multinomial coefficient. We thank Ira Gessel [21] for pointing out that this is equivalent to the determinantal formula $x^r A_n^T(1/x) = n! \det(\theta_{ij}(x))_{0 \leqslant i,j \leqslant r}$, where

$$\theta_{ij}(x) = \begin{cases} 0, & \text{if } i > j+1 \\ 1, & \text{if } i = j+1 \\ \dfrac{(1-x)^{j-i}}{(a_{j+1} - a_i)!}, & \text{if } i \leqslant j \end{cases}$$

and $T = \{a_1, a_2, \ldots, a_r\}$ is as in Corollary 9, and for suggesting a direct combinatorial proof. $\square$

Example 11. (a) For $T = [r] \subseteq [n-1]$, the polynomial $A_n^T(x)$ is the descent enumerator for permutations $w \in \mathfrak{S}_n$ which have ascents in the last $n-r-1$ positions. A q-analogue of $A_n^T(x)$ in this case was studied in [17] (although the unimodality of $A_n^T(x)$ was not addressed there). Theorem 8 and Corollary 9 imply that $A_n^T(x)$ is a real-rooted, hence unimodal, polynomial of degree r and that it has a mode m such that $\lfloor r/2 \rfloor \leqslant m \leqslant \lceil (r+1)/2 \rceil$. Since $(B_n)_T$, with its maximum element removed, is a simplicial poset in this case, the real-rootedness of $A_n^T(x)$ already follows from the main result of [15]. Setting $q = 1$ in the formula of [17, Theorem 2.10] gives that

$$\sum_{m \geqslant 0} \sum_{i=0}^{r} \binom{n-r+i-1}{i} m^i (m+1)^{r-i} = \frac{A_n^T(x)}{(1-x)^{r+1}}$$

or equivalently, by Equation (10), that

$$A_n^T(x) = \sum_{i=0}^{r} \binom{n-r+i-1}{i} p_{r,i}(x).$$

In particular, $A_n^T(x)$ is interlaced by the Eulerian polynomial $p_{r,0}(x) = A_r(x)$.

(b) More generally, for $T = \{s+1, s+2, \ldots, s+r\} \subseteq [n-1]$, the polynomial $A_n^T(x)$ is the descent enumerator for permutations $w \in \mathfrak{S}_n$ which have ascents in the first s and the last $n-r-s-1$ positions. According to Theorem 8 and Corollary 9, $A_n^T(x)$ is a real-rooted, hence unimodal, polynomial of degree r which has a mode m such that $\lfloor (r-1)/2 \rfloor \leqslant m \leqslant \lceil (r+1)/2 \rceil$. $\square$

Example 12. Let $T = \{2, 4, \ldots, 2n-2\}$, so that $A_{2n}^T(x)$ is the descent enumerator for permutations $w \in \mathfrak{S}_{2n}$ which have an ascent in every odd position. By Theorem 8 and Corollary 9, $A_{2n}^T(x)$ is a real-rooted, hence unimodal, polynomial of degree $n-1$ which has a mode m_n such that $\lfloor 5(n-1)/6 \rfloor \leqslant m_n \leqslant \lceil 5(n-1)/6 \rceil$.

Let us choose a permutation $w \in \mathfrak{S}_{2n}$ with $\mathrm{Des}(w) \subseteq T = \{2, 4, \ldots, 2n-2\}$ uniformly at random and let $X_n(w) = \mathrm{des}(w)$ for such $w \in \mathfrak{S}_{2n}$. One may compute the variance of the random variable X_n as $\sigma_n^2 = (19n-13)/180$ for $n \geqslant 2$. As a consequence of Corollary 9, X_n has mean $\mu_n = 5(n-1)/6$. Given the real-rootedness of $A_{2n}^T(x)$, a theorem of Bender [9] (see also [13, Theorem 2.1] [28, p. 286]) implies that $(X_n - \mu_n)/\sigma_n$ converges to the standard normal distribution as $n \to \infty$. $\square$

We conclude this section with the following question. Part (b) provided a lot of the motivation behind this paper; it is an open problem [7, Question 7.2] to decide whether the inequalities which appear there hold for the h-vectors of all $(r-1)$-dimensional doubly Cohen–Macaulay simplicial complexes. An affirmative answer to part (a) would imply the (weaker) top-heavy inequalities $h_i(T) \leqslant h_{r-i}(T)$ for $0 \leqslant i \leqslant \lfloor r/2 \rfloor$; we refer the reader to [41] for this implication and for the concept of a convex ear decomposition.

Question 13. Let $A_n^T(x) = \sum_{i=0}^r h_i(T)x^i$, where $T \subseteq [n-1]$ has size r.

(a) Does the order complex of the rank-selected subposet $(B_n)_T$ of the Boolean lattice B_n (with its minimum and maximum elements removed) have a convex ear decomposition?

(b) Do the inequalities

$$\frac{h_0(T)}{h_r(T)} \leqslant \frac{h_1(T)}{h_{r-1}(T)} \leqslant \cdots \leqslant \frac{h_r(T)}{h_0(T)}$$

hold?

4 Rank-selected subposets of simplicial posets

This section proves part (b) of Theorem 2 and gives an application. We recall that a finite poset P with a minimum element $\hat{0}$ is said to be *simplicial* [33] [34, Section III.6] if the interval $[\hat{0}, y]$ is isomorphic to a Boolean lattice for every $y \in P$. The enumerative invariant of a graded simplicial poset P of rank n which will be essential to the proof is the *h-polynomial* of P. This was defined by Stanley [33] as

$$h(P, x) = \sum_{i=0}^{n} f_{i-1}(P)\, x^i (1-x)^{n-i},$$

where $f_{i-1}(P)$ is the number of elements of P of rank i. We then have

$$f_{j-1}(P) = \sum_{i=0}^{j} \binom{n-i}{j-i} h_i(P) \tag{14}$$

for $j \in \{0, 1, \ldots, n\}$ and $h(P, x) = h(\Delta, x)$, if P is the face poset of an $(n-1)$-dimensional simplicial complex Δ. Stanley [33] (see also [34, Section III.6]) showed that $h(P, x)$ has nonnegative coefficients for every Cohen–Macaulay simplicial poset P.

Another essential ingredient for the proof of Theorem 2 is the following statement (an exercise from [34]), which expresses the flag h-vector of a graded simplicial poset in terms of its h-vector. We provide a proof for the convenience of the reader.

Lemma 14. ([34, Exercise III.15]) *Let P be a graded simplicial poset of rank n. Then,*

$$\beta_{\hat{P}}(S) = \sum_{k=0}^{n} h_k(P)\ \#\{w \in \mathfrak{S}_{n+1} : w(n+1) = k+1, \mathrm{Des}(w) = [n+1] \smallsetminus S\}$$

for every $S \subseteq [n]$.

Proof. Let $\mathrm{Asc}(w) := [n] \smallsetminus \mathrm{Des}(w)$ be the set of ascents of a permutation $w \in \mathfrak{S}_{n+1}$. We need to show that

$$\beta_{\hat{P}}(S) = \sum_{k=0}^{n} h_k(P)\ \#\{w \in \mathfrak{S}_{n+1} : w(n+1) = k+1, \mathrm{Asc}(w) = S\}$$

for every $S \subseteq [n]$ or, equivalently, that

$$\alpha_{\hat{P}}(T) = \sum_{k=0}^{n} h_k(P)\ \#\{w \in \mathfrak{S}_{n+1} :\ w(n+1) = k+1, \mathrm{Asc}(w) \subseteq T\}$$

for every $T \subseteq [n]$. Let us write $T = \{a_1, a_2, \ldots, a_r\} \subseteq [n]$, with $1 \leqslant a_1 < \cdots < a_r \leqslant n$. There are $f_{a_r-1}(P)$ elements of rank a_r in P and $\binom{a_r}{a_1, a_2-a_1, \ldots, a_r-a_{r-1}}$ chains of elements of ranks $a_1, a_2, \ldots, a_{r-1}$ in any Boolean lattice of rank a_r. Given this and Equation (14), we find that

$$\begin{aligned}\alpha_{\hat{P}}(T) &= f_{a_r-1}(P)\binom{a_r}{a_1, a_2-a_1, \ldots, a_r-a_{r-1}} \\ &= \sum_{k=0}^{n} \binom{n-k}{a_r-k} h_k(P) \binom{a_r}{a_1, a_2-a_1, \ldots, a_r-a_{r-1}}.\end{aligned}$$

Thus, it suffices to verify that $\binom{n-k}{a_r-k}\binom{a_r}{a_1, a_2-a_1, \ldots, a_r-a_{r-1}}$ is equal to the number of permutations $w \in \mathfrak{S}_{n+1}$ such that $w(n+1) = k+1$ and $\mathrm{Asc}(w) \subseteq T$, a task which can safely be left to the reader. ☐

Proof of Theorem 2. Given Theorem 8, we only need to show part (b). By Lemma 14 we have

$$\begin{aligned}\beta_{\hat{P}}(S) &= \sum_{k=0}^{n} h_k(P)\ \#\{w \in \mathfrak{S}_{n+1} :\ w(n+1) = k+1, \mathrm{Asc}(w) = S\} \\ &= \sum_{k=0}^{n} h_k(P)\ \#\{w \in \mathfrak{S}_{n+1} :\ w(1) = k+1, \mathrm{Des}(w) = n+1-S\}.\end{aligned}$$

Therefore, by Equation (7),

$$\begin{aligned}h(\Delta(\hat{P}_T), x) &= \sum_{S \subseteq T} \beta_{\hat{P}}(S) x^{|S|} \\ &= \sum_{k=0}^{n} h_k(P) \sum_{S \subseteq T} \#\{w \in \mathfrak{S}_{n+1} :\ w(1) = k+1, \mathrm{Des}(w) = n+1-S\}\, x^{|n+1-S|} \\ &= \sum_{k=0}^{n} h_k(P) \sum_{w \in \mathfrak{S}_{n+1} :\, w(1) = k+1, \mathrm{Des}(w) \subseteq n+1-T} x^{\mathrm{des}(w)} \\ &= \sum_{k=0}^{n} h_k(P) p_{n,k}^{n+1-T}(x)\end{aligned}$$

and the proof follows from Theorem 8, Lemma 5 (a) and the fact that $p_{n,0}^{n+1-T}(x) = A_n^{n-T}(x) = A_n^T(x)$. ☐

Colored permutations. As an application, let us generalize part (a) of Theorem 2 to r-colored permutations. An *r-colored permutation* of the set $[n]$ is defined as a pair $w \times \mathbf{z}$, where $w = (w(1), w(2), \dots, w(n)) \in \mathfrak{S}_n$ and $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \{0, 1, \dots, r-1\}^n$. The number z_i is thought of as the color assigned to $w(i)$. The set of all r-colored permutations of $[n]$ is denoted by $\mathfrak{S}_n[\mathbb{Z}_r]$.

Let $u = w \times \mathbf{z} \in \mathfrak{S}_n[\mathbb{Z}_r]$ be an r-colored permutation, as before, and set $w(n+1) = n+1$ and $z_{n+1} = 0$. A *descent* of u is any index $i \in [n]$ such that either $z_i > z_{i+1}$, or $z_i = z_{i+1}$ and $w(i) > w(i+1)$. Thus, n is a descent of u if and only if $w(n)$ has nonzero color. As usual, we denote by $\mathrm{Des}(u)$ and $\mathrm{des}(u)$ the set and the number of descents of $u \in \mathfrak{S}_n[\mathbb{Z}_r]$, respectively. The polynomial

$$A^T_{n,r}(x) = \sum_{u \in \mathfrak{S}_n[\mathbb{Z}_r]:\, \mathrm{Des}(u) \subseteq T} x^{\mathrm{des}(u)}, \tag{15}$$

defined for $T \subseteq [n]$, provides a common generalization of $A^T_n(x)$ (the special case $r = 1$) and the r-colored Eulerian polynomial $A_{n,r}(x)$ (the special case $T = [n]$), introduced and studied by Steingrímsson [38, 39]. The latter was shown to be real-rooted in [38, Theorem 3.19] [39, Theorem 19].

Theorem 15. *The polynomial $A^T_{n,r}(x)$ is real-rooted and interlaced by $A^T_n(x)$ for all positive integers n, r and every $T \subseteq [n]$.*

Proof. We will apply Theorem 2 to the poset of r-colored subsets of the set $[n]$, defined as follows. We consider the subsets Ω of $[n] \times \{0, 1, \dots, r-1\}$ for which for every $i \in [n]$ there is at most one $j \in \{0, 1, \dots, r-1\}$ such that $(i, j) \in \Omega$ and let P be the set of all such subsets, partially ordered by inclusion. Thus, P is a graded simplicial poset of rank n which is isomorphic to the Boolean lattice B_n for $r = 1$. It was shown in the proof of [2, Theorem 1.3] that P is shellable, hence Cohen–Macaulay, and that $\beta_{\hat{P}}(S)$ is equal to the number of r-colored permutations $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ with descent set equal to S, for every $S \subseteq [n]$. As a result, in view of Equation (7),

$$h(\Delta(\hat{P}_T), x) = \sum_{S \subseteq T} \beta_{\hat{P}}(S) x^{|S|} = \sum_{u \in \mathfrak{S}_n[\mathbb{Z}_r]:\, \mathrm{Des}(u) \subseteq T} x^{\mathrm{des}(u)} = A^T_{n,r}(x)$$

for every $T \subseteq [n]$ and the proof follows from Theorem 2. $\square$

5 Noncrossing partition lattices

This section proves Theorem 3. We first recall the definition of NC_W. Let W be a finite Coxeter group with rank r_W and set of reflections $\mathcal{T}$. For $\alpha \in W$ we denote by $\ell_{\mathcal{T}}(\alpha)$ the smallest k such that α can be written as a product of k reflections in $\mathcal{T}$. We define the partial order $\preceq$ on W by letting $\alpha \preceq \beta$ if $\ell_{\mathcal{T}}(\alpha) + \ell_{\mathcal{T}}(\alpha^{-1}\beta) = \ell_{\mathcal{T}}(\beta)$, in other words if there exists a shortest factorization of α into reflections which is a prefix of such a shortest factorization of β. Then, NC_W is defined as the closed interval $[e, \gamma]$ in $(W, \preceq)$, where $e \in W$ is the identity element and γ is any Coxeter element of W. The noncrossing partition poset NC_W is a rank-symmetric, graded lattice with rank function $\ell_{\mathcal{T}}$ and rank r_W; its combinatorial type is independent of the choice of γ. A detailed exposition of noncrossing partition lattices can be found in [1, Chapter 2].

An important role in our first proof of Theorem 3 (a) will be played by the polynomials

$$p_{n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} \colon w(1)=k+1} x^{\mathrm{des}(w)},$$

where $k \in \{0, 1, \ldots, n\}$, discussed in Section 3.

First proof of Theorem 3 (a). Let $r_W = n$ be the rank of W. We first assume that W is irreducible. Then, the zeta polynomial of NC_W is given by the formula (see, for instance, [1, Theorem 3.5.2])

$$\begin{aligned}\mathcal{Z}(\mathrm{NC}_W, m) &= \frac{1}{|W|} \prod_{i=1}^{n} (mh + d_i) \\ &= \frac{1}{|W|} \prod_{i=1}^{n} \left((h - d_i)m + d_i(m+1) \right),\end{aligned}$$

where h is the Coxeter number of W and $d_1, d_2, \ldots, d_n$ are its degrees. Since $d_i \leqslant h$ for every i, the second expression shows that $\mathcal{Z}(\mathrm{NC}_W, m)$ can be written as a nonnegative linear combination of the polynomials $m^k(1+m)^{n-k}$ for $k \in \{0, 1, \ldots, n\}$. Moreover, this must be the case for every W, since NC_W is isomorphic to the product of posets $\prod_{i=1}^{\ell} \mathrm{NC}_{W_i}$, where $W_1, W_2, \ldots, W_\ell$ are the irreducible components of W, and hence $\mathcal{Z}(\mathrm{NC}_W, m) = \prod_{i=1}^{\ell} \mathcal{Z}(\mathrm{NC}_{W_i}, m)$.

In view of Equations (8) and (10), we conclude that $h(\Delta(\mathrm{NC}_W), x)$ can be written as a nonnegative linear combination of the polynomials $p_{n,k}(x)$ for $k \in \{0, 1, \ldots, n\}$. Since $(p_{n,k}(x))_{0 \leqslant k \leqslant n}$ is an interlacing sequence, this and Lemma 5 imply that $h(\Delta(\mathrm{NC}_W), x)$ is real-rooted and is interlaced by the Eulerian polynomial $A_n(x)$. □

Remark 16. An explicit expression for $\mathcal{Z}(\mathrm{NC}_W, m)$ as a nonnegative linear combination of the polynomials $m^k(1+m)^{n-k}$ for $k \in \{0, 1, \ldots, n\}$ can be deduced from results of [10, 23] (see [10, Section 4.6]). Specifically, assuming that W is irreducible of rank n, we have

$$\mathcal{Z}(\mathrm{NC}_W, m) = \sum_{k=0}^{n-1} \frac{JV(W;k)}{n!} \cdot m^k (1+m)^{n-k},$$

where $JV(W;k)$ is equal to the number of shortest factorizations $\gamma = \tau_1 \tau_2 \cdots \tau_n$ of the Coxeter element γ into reflections such that there are exactly k indices $i \in [n-1]$ for which $\tau_1 \tau_2 \cdots \tau_i$ is greater than $\tau_1 \tau_2 \cdots \tau_{i+1}$ in the Bruhat order on W. □

The second proof of part (a) and the proof of part (b) of Theorem 3 are based on explicit combinatorial interpretations of the polynomial $h(\Delta(\mathrm{NC}_W), x)$ for the irreducible finite Coxeter groups of classical types. Before stating them, we need to introduce some definitions and notation. A *descent* of a word $w \in [r]^n$ is any index $i \in [n-1]$

such that $w(i) \geqslant w(i+1)$. We denote by $\mathcal{D}_n$ the set of words $w \in \mathbb{Z}^n$ such that $(|w(1)|, w(2), \ldots, w(n)) \in [n-1]^n$. A *descent* of such a word $w \in \mathcal{D}_n$ is defined as any index $i \in [n-1]$ such that

- $|w(i)| > w(i+1)$, or
- $w(i) = w(i+1) > 0$.

As usual, we denote by $\mathrm{Des}(w)$ and $\mathrm{des}(w)$ the set and the number of descents, respectively, of a word w.

Proposition 17. *Let W be an irreducible finite Coxeter group of Coxeter type $\mathcal{X}$. Then,*

$$h(\Delta(\mathrm{NC}_W), x) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{w \in [n]^{n-1}} x^{\mathrm{des}(w)}, & \textit{if } \mathcal{X} = A_{n-1} \\[2ex] \displaystyle\sum_{w \in [n]^n} x^{\mathrm{des}(w)}, & \textit{if } \mathcal{X} = B_n \\[2ex] \displaystyle\sum_{w \in \mathcal{D}_n} x^{\mathrm{des}(w)}, & \textit{if } \mathcal{X} = D_n. \end{cases}$$

Moreover,

$$h(\Delta(\mathrm{NC}_W), x) = \begin{cases} 1 + (m-1)x, & \textit{if } \mathcal{X} = I_2(m) \\ 1 + 28x + 21x^2, & \textit{if } \mathcal{X} = H_3 \\ 1 + 275x + 842x^2 + 232x^3, & \textit{if } \mathcal{X} = H_4 \\ 1 + 100x + 265x^2 + 66x^3, & \textit{if } \mathcal{X} = F_4 \\ 1 + 826x + 10778x^2 + 21308x^3 + 8141x^4 + 418x^5, & \textit{if } \mathcal{X} = E_6 \\ 1 + 4152x + 110958x^2 + 446776x^3 + 412764x^4 & \\ \quad + 85800x^5 + 2431x^6, & \textit{if } \mathcal{X} = E_7 \\ 1 + 25071x + 1295238x^2 + 9523785x^3 + 17304775x^4 & \\ \quad +8733249x^5 + 1069289x^6 + 17342x^7, & \textit{if } \mathcal{X} = E_8. \end{cases}$$

Proof. Let us write $P = \mathrm{NC}_W$ and first consider the case $\mathcal{X} = A_{n-1}$. The explicit formula of [19, Theorem 3.2] (see also [30, p. 196]) for the entries of the flag f-vector of P can be rewritten as

$$\alpha_P(T) = \frac{1}{n} \#\{w \in [n]^{n-1} : \mathrm{Des}(w) \subseteq T\}$$

for $T \subseteq [n-2]$. From Equation (5) it readily follows that

$$\beta_P(S) = \frac{1}{n} \#\{w \in [n]^{n-1} : \mathrm{Des}(w) = S\}$$

for $S \subseteq [n-2]$ and hence that

$$h(\Delta(P), x) = \sum_{S \subseteq [n-2]} \beta_P(S) x^{|S|} = \frac{1}{n} \sum_{w \in [n]^{n-1}} x^{\mathrm{des}(w)}.$$

One can reach the same conclusion by using the combinatorial interpretation of $\beta_P(S)$ in terms of parking functions, given in [35, Proposition 3.2]. The proof of the formula for $\mathcal{X} = B_n$ is entirely similar, once one rewrites the formula of [35, Proposition 7] for the flag f-vector of P as

$$\alpha_P(T) = \#\{w \in [n]^n : \mathrm{Des}(w) \subseteq T\}$$

for $T \subseteq [n-1]$.

Let us now consider the case $\mathcal{X} = D_n$, which is more involved: it is not true any more that $\beta_P(S)$ is equal to the number of words $w \in \mathcal{D}_n$ with descent set equal to S. Let us write $\bar{P} = P \smallsetminus \{\hat{0}, \hat{1}\}$. The formula of [6, Theorem 1.2] for the flag f-vector of P shows that

$$\begin{aligned} f_{k-1}(\Delta(\bar{P})) = 2 &\sum_{(a_1, a_2, \dots, a_{k+1}) \vDash n} \binom{n-1}{a_1}\binom{n-1}{a_2} \cdots \binom{n-1}{a_{k+1}} + \\ &\sum_{(a_1, a_2, \dots, a_{k+1}) \vDash n} \sum_{i=1}^{k+1} \binom{n-1}{a_1} \cdots \binom{n-2}{a_i - 2} \cdots \binom{n-1}{a_{k+1}}, \end{aligned}$$

where the first two sums run through all compositions $(a_1, a_2 \dots, a_{k+1})$ of n with $k+1$ parts. Using the fact that $\binom{n-2}{a_i-2} = \frac{a_i-1}{n-1}\binom{n-1}{a_i-1}$, changing the order of summation in the double sum and replacing a_i with $a_i + 1$ yields that

$$\begin{aligned} f_{k-1}(\Delta(\bar{P})) = 2 &\sum_{(a_1, a_2 \dots, a_{k+1}) \vDash n} \binom{n-1}{a_1}\binom{n-1}{a_2} \cdots \binom{n-1}{a_{k+1}} + \\ &\sum_{i=1}^{k+1} \sum_{(a_1, a_2, \dots, a_{k+1}) \vDash n-1} \frac{a_i}{n-1} \binom{n-1}{a_1} \cdots \binom{n-1}{a_2} \cdots \binom{n-1}{a_{k+1}}. \end{aligned}$$

Changing again the order of summation in the double sum, since $\sum_{i=1}^{k+1} a_i = n-1$, we find that

$$\begin{aligned} f_{k-1}(\Delta(\bar{P})) = 2 &\sum_{(a_1, a_2, \dots, a_{k+1}) \vDash n} \binom{n-1}{a_1}\binom{n-1}{a_2} \cdots \binom{n-1}{a_{k+1}} + \\ &\sum_{(a_1, a_2, \dots, a_{k+1}) \vDash n-1} \binom{n-1}{a_1} \cdots \binom{n-1}{a_2} \cdots \binom{n-1}{a_{k+1}}. \end{aligned}$$

We may rewrite this formula as

$$f_{k-1}(\Delta(\bar{P})) = 2 \sum_{T\subseteq[n-1],\,|T|=k} \#\{w \in [n-1]^n : \mathrm{Des}(w) \subseteq T\} + \sum_{T\subseteq[n-2],\,|T|=k} \#\{w \in [n-1]^{n-1} : \mathrm{Des}(w) \subseteq T\},$$

whence

$$\begin{aligned}
h(\Delta(P), x) = h(\Delta(\bar{P}), x) &= \sum_{k=0}^{n-1} f_{k-1}(\Delta(\bar{P}))\, x^k (1-x)^{n-1-k} \\
&= 2 \sum_{k=0}^{n-1} \sum_{T\subseteq[n-1],\,|T|=k} \#\{w \in [n-1]^n : \mathrm{Des}(w) \subseteq T\} x^k (1-x)^{n-1-k} \\
&+ \sum_{k=0}^{n-1} \sum_{T\subseteq[n-2],\,|T|=k} \#\{w \in [n-1]^{n-1} : \mathrm{Des}(w) \subseteq T\} x^k (1-x)^{n-1-k} \\
&= 2 \sum_{T\subseteq[n-1]} \#\{w \in [n-1]^n : \mathrm{Des}(w) \subseteq T\} x^{|T|} (1-x)^{n-1-|T|} \\
&+ \sum_{T\subseteq[n-2]} \#\{w \in [n-1]^{n-1} : \mathrm{Des}(w) \subseteq T\} x^{|T|} (1-x)^{n-1-|T|}.
\end{aligned}$$

Setting $\mathrm{Des}(w) = S$ in each sum, summing over all $S \subseteq T$ and changing the order of summation yields that

$$\begin{aligned}
h(\Delta(P), x) = 2 &\sum_{S\subseteq[n-1]} \#\{w \in [n-1]^n : \mathrm{Des}(w) = S\} \sum_{S\subseteq T\subseteq[n-1]} x^{|T|} (1-x)^{n-1-|T|} \\
+ &\sum_{S\subseteq[n-2]} \#\{w \in [n-1]^{n-1} : \mathrm{Des}(w) = S\} \sum_{S\subseteq T\subseteq[n-2]} x^{|T|} (1-x)^{n-1-|T|}
\end{aligned}$$

and hence that

$$\begin{aligned}
h(\Delta(P), x) &= 2 \sum_{S\subseteq[n-1]} \#\{w \in [n-1]^n : \mathrm{Des}(w) = S\} x^{|S|} \\
&+ (1-x) \sum_{S\subseteq[n-2]} \#\{w \in [n-1]^{n-1} : \mathrm{Des}(w) = S\} x^{|S|} \\
&= 2 \sum_{w\in[n-1]^n} x^{\mathrm{des}(w)} + (1-x) \sum_{w\in[n-1]^{n-1}} x^{\mathrm{des}(w)}. \qquad (16)
\end{aligned}$$

Considering the cases $|w(1)| \neq w(2)$ and $|w(1)| = w(2)$ for a word $w \in \mathcal{D}_n$ shows that the number of words $w \in \mathcal{D}_n$ with $\mathrm{des}(w) = k$ is equal to the coefficient of x^k in the expression (16) and the proof follows.

The exceptional types are handled with straightforward computations via Sage [42]. $\square$

Second proof of Theorem 3 (a). We recall that NC_W is isomorphic to the product of posets $\prod_{i=1}^{\ell} \mathrm{NC}_{W_i}$, where $W_1, W_2, \ldots, W_\ell$ are the irreducible components of W. This fact and Lemma 7 show that we may assume that W is irreducible.

Let us first consider the case of groups of type D. By Proposition 17, it suffices to show that $h_n(x) := \sum_{w \in \mathcal{D}_n} x^{\mathrm{des}(w)}$ is real-rooted for every $n \geqslant 2$. For $k \geqslant 2$, we denote by $\mathcal{D}_{n,k}$ the set of words $w \in \mathbb{Z}^k$ such that $(|w(1)|, w(2), \ldots, w(k)) \in [n-1]^k$ and note that $\mathcal{D}_{n,n} = \mathcal{D}_n$. We define the notion of descent for words $w \in \mathcal{D}_{n,k}$ just as in the special case $k = n$ and set

$$h_{n,k,j}(x) = \sum_{w \in \mathcal{D}_{n,k}\colon w(k)=j} x^{\mathrm{des}(w)}$$

for $j \in [n-1]$. We will prove that $(h_{n,k,n-1}(x), \ldots, h_{n,k,2}(x), h_{n,k,1}(x))$ is an interlacing sequence of real-rooted polynomials for all $n, k \geqslant 2$ by induction on k. This holds for $k = 2$ since then $h_{n,k,j}(x) = (2j-1) + (2n-2j-1)x$ for every $j \in [n-1]$. The inductive step follows by an application of part (c) of Lemma 5, since

$$h_{n,k+1,j}(x) = \sum_{i=1}^{j-1} h_{n,k,i}(x) \, + \, x \sum_{i=j}^{n-1} h_{n,k,i}(x)$$

for $j \in [n-1]$. In particular, $h_{n,n+1,1}(x) = x h_n(x)$ is real-rooted for every $n \geqslant 2$ and hence so is $h_n(x)$.

A similar (and even simpler) argument shows that $\sum_{w \in [r]^n} x^{\mathrm{des}(w)}$ is real-rooted for all $n, r \geqslant 1$. This covers the cases of groups of types A and B. The exceptional groups can be treated with a case by case verification. $\square$

Symmetric decompositions. The second part of Theorem 3 will be proven by an application a result of Jochemko [22], after the expressions of Proposition 17 for $h(\Delta(\mathrm{NC}_W), x)$ are suitably rewritten. For a polynomial or formal power series $H(x) = \sum_{n \geqslant 0} h_n x^n \in \mathbb{C}[[x]]$ we use the notation $\mathcal{S}_r(H(x)) = \sum_{n \geqslant 0} h_{rn} x^n$.

Lemma 18. *Let $E_{n,r}(x) = \sum_{w \in [r]^n} x^{\mathrm{des}(w)}$. Then,*

$$x^n E_{n,r}(1/x) = \mathcal{S}_r \left(x(1 + x + x^2 + \cdots + x^{r-1})^{n+1} \right)$$

for all $n, r \geqslant 1$.

Proof. First we relate the polynomials $E_{n,r}(x)$ to the

$$\tilde{E}_{n,r}(x) := \sum_{w \in \mathcal{W}_{n,r}} x^{\mathrm{asc}^*(w)},$$

where $\mathcal{W}_{n,r}$ is the set of words $w : \{0, 1, \ldots, n\} \to [r]$ with $w(0) = 1$ and $\mathrm{asc}^*(w)$ is the number of indices $i \in [n]$ such that $w(i-1) < w(i)$. We note that

$$x^{n-1} E_{n,r}(1/x) = \sum_{w \in [r]^n} x^{n-1-\mathrm{des}(w)} = \sum_{w \in [r]^n} x^{\mathrm{asc}^*(w)},$$

where $\mathrm{asc}^*(w) = n-1-\mathrm{des}(w)$ is the number of strict ascents of $w \in [r]^n$. Distinguishing the cases $w(1) = 1$ and $w(1) \geqslant 2$ for such a word and for a word $w \in \mathcal{W}_{n,r}$ we get

$$\begin{aligned} x^{n-1} E_{n,r}(1/x) &= \tilde{E}_{n-1,r}(x) + \sum_{w \in [r]^n :\, w(1) \geqslant 2} x^{\mathrm{asc}^*(w)}, \\ \tilde{E}_{n,r}(x) &= \tilde{E}_{n-1,r}(x) + x \sum_{w \in [r]^n :\, w(1) \geqslant 2} x^{\mathrm{asc}^*(w)}. \end{aligned}$$

These equalities imply that

$$x^n E_{n,r}(1/x) = \tilde{E}_{n,r}(x) + (x-1)\tilde{E}_{n-1,r}(x). \tag{17}$$

We now recall that

$$\tilde{E}_{n,r}(x) = \mathcal{S}_r\left((1+x+x^2+\cdots+x^{r-1})^{n+1}\right). \tag{18}$$

This formula follows from the identity

$$\sum_{m \geqslant 0} \binom{n+rm}{n} x^m = \frac{\tilde{E}_{n,r}(x)}{(1-x)^{n+1}}, \tag{19}$$

which can be proved by a standard 'placing balls into boxes' argument (see [31, Corollary 8] for a q-analogue) and the computation

$$\begin{aligned} \sum_{m \geqslant 0} \binom{n+rm}{n} x^m &= \mathcal{S}_r\left(\frac{1}{(1-x)^{n+1}}\right) = \mathcal{S}_r\left(\frac{(1+x+x^2+\cdots+x^{r-1})^{n+1}}{(1-x^r)^{n+1}}\right) \\ &= \frac{\mathcal{S}_r\left((1+x+x^2+\cdots+x^{r-1})^{n+1}\right)}{(1-x)^{n+1}}. \end{aligned}$$

Combining Equations (17) and (18) we get

$$\begin{aligned} x^n E_{n,r}(1/x) &= \mathcal{S}_r\left((1+x+x^2+\cdots+x^{r-1})^{n+1}\right) + (x-1)\mathcal{S}_r\left((1+x+x^2+\cdots+x^{r-1})^{n}\right) \\ &= \mathcal{S}_r\left((1+x+x^2+\cdots+x^{r-1})^{n+1} + (x^r-1)(1+x+x^2+\cdots+x^{r-1})^{n}\right) \\ &= \mathcal{S}_r\left(x(1+x+x^2+\cdots+x^{r-1})^{n+1}\right) \end{aligned}$$

and the proof follows. □

The following result of Jochemko [22] will be applied in the proof of Theorem 3 (b).

Theorem 19. ([22, Theorem 1.1]) *Let $h(x) = h_0 + h_1 x + \cdots + h_d x^d$ be a polynomial of degree $s \leqslant d$ with nonnegative coefficients such that*

- $h_0 + h_1 + \cdots + h_i \geqslant h_d + h_{d-1} + \cdots + h_{d-i+1}$, *and*

- $h_0 + h_1 + \cdots + h_i \leqslant h_s + h_{s-1} + \cdots + h_{s-i}$

for all i*. Then,* $\mathcal{S}_r\left(h(x)(1+x+x^2+\cdots+x^{r-1})^{d+1}\right)$ *has a nonnegative real-rooted symmetric decomposition with respect to* d *whenever* $r \geqslant \max\{s, d+1-s\}$*.*

Proof of Theorem 3 (b). Using the notation of Lemma 18, by Proposition 17 and its proof we have

$$h(\Delta(\mathrm{NC}_W), x) = \begin{cases} (1/n)E_{n-1,n}(x), & \text{if } \mathcal{X} = A_{n-1} \\ E_{n,n}(x), & \text{if } \mathcal{X} = B_n \\ 2E_{n,n-1}(x) + (1-x)E_{n-1,n-1}(x), & \text{if } \mathcal{X} = D_n. \end{cases}$$

In view of Lemma 18, these formulas may be rewritten as

$$x^{r_W} h(\Delta(\mathrm{NC}_W), 1/x) = \begin{cases} (1/n)\mathcal{S}_n\left(x(1+x+x^2+\cdots+x^{n-1})^n\right), & \text{if } \mathcal{X} = A_{n-1} \\ \mathcal{S}_n\left(x(1+x+x^2+\cdots+x^{n-1})^{n+1}\right), & \text{if } \mathcal{X} = B_n \\ \mathcal{S}_{n-1}\left((x+x^2)(1+x+x^2+\cdots+x^{n-2})^{n+1}\right), & \text{if } \mathcal{X} = D_n. \end{cases}$$

These expressions and Theorem 19 imply in each case that $x^{r_W} h(\Delta(\mathrm{NC}_W), 1/x)$ has a nonnegative real-rooted symmetric decomposition with respect to r_W. Since $h(\Delta(\mathrm{NC}_W), x)$ has degree $r_W - 1$, it has a nonnegative real-rooted symmetric decomposition with respect to $r_W - 1$. The exceptional groups are again handled by a routine case by case verification. ☐

We close this section with the analogue of Question 13.

Question 20. Let $h(\Delta(\mathrm{NC}_W), x) = \sum_{i=0}^{r} h_i(W)x^i$, where $r = r_W - 1$.

(a) Does the order complex $\Delta(\overline{\mathrm{NC}}_W)$ of the noncrossing partition lattice NC_W (with its minimum and maximum elements removed) have a convex ear decomposition?

(b) Do the inequalities
$$\frac{h_0(W)}{h_r(W)} \leqslant \frac{h_1(W)}{h_{r-1}(W)} \leqslant \cdots \leqslant \frac{h_r(W)}{h_0(W)}$$
hold?

Acknowledgements

First and third authors supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the '2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers' (Project Number: HFRI-FM20-04537).

Part of the motivation behind Theorem 2 was developed during the workshop 'Interactions between Topological Combinatorics and Combinatorial Commutative Algebra', held at BIRS (Banff, Canada) in April 2023. The first named author wishes to thank the organizers Mina Bigdeli, Sara Faridi, Satoshi Murai and Adam Van Tuyl for the invitation and the participants for useful discussions. The authors also wish to thank Christian Stump for help with the computation of the chain polynomial of the noncrossing partition lattice of type E_8.

References

[1] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc, **202** (2009), no. 949, pp. x+159.

[2] C.A. Athanasiadis, *Edgewise subdivisions, local h-polynomials and excedances in the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$*, SIAM J. Discrete Math. **28** (2014), 1479–1492.

[3] C.A. Athanasiadis, *Face numbers of barycentric subdivisions of cubical complexes*, Israel J. Math. **246** (2021), 423–439.

[4] C.A. Athanasiadis, *Triangulations of simplicial complexes and theta polynomials*, arXiv:2209.01674.

[5] C.A. Athanasiadis and K. Kalampogia-Evangelinou, *Chain enumeration, partition lattices and polynomials with only real roots*, Combinatorial Theory **3** (2023), Article # 12, 21pp.

[6] C.A. Athanasiadis and V. Reiner, *Noncrossing partitions for the group D_n*, SIAM J. Discrete Math. **18** (2004), 397–417.

[7] C.A. Athanasiadis and E. Tzanaki, *Symmetric decompositions, triangulations and real-rootedness*, Mathematika **67** (2021), 840–859.

[8] M. Beck, K. Jochemko and E. McCullough, *h^*-polynomials of zonotopes*, Trans. Amer. Math. Soc. **371** (2019), 2021–2042.

[9] E.A. Bender, *Central and local limit theorems applied to asymptotic enumeration*, J. Combin. Theory Series A **15** (1973), 91–111.

[10] P. Biane and M. Josuat-Vergés, *Noncrossing partitions, Bruhal order and the cluster complex*, Ann. Inst. Fourier **69** (2019), 2241–2289.

[11] A. Borodin, P. Diaconis and J. Fulman, *On adding a list of numbers (and other one-dependent determinantal point processes)*, Bull. Amer. Math. Soc. **47** (2010), 639–670.

[12] P. Brändén, *Counterexamples to the Neggers–Stanley conjecture*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 155–158 (electronic).

[13] P. Brändén, *Unimodality, log-concavity, real-rootedness and beyond*, in *Handbook of Combinatorics* (M. Bona, ed.), CRC Press, 2015, pp. 437–483.

[14] P. Brändén and L. Solus, *Symmetric decompositions and real-rootedness*, Int. Math. Res. Not. **2021** (2021), 7764–7798.

[15] F. Brenti and V. Welker, *f-vectors of barycentric subdivisions*, Math. Z. **259** (2008), 849–865.

[16] G. Chapuy and T. Douvropoulos, *Counting chains in the noncrossing partition lattice via the W-Laplacian*, J. Algebra **602** (2022), 381–404.

[17] S. Corteel, I.M. Gessel, C.D. Savage and H.S. Wilf, *The joint distribution of descent and major index over restricted sets of permutations*, Ann. Comb. **11** (2007), 375–386.

[18] J.N. Darroch, *On the distribution of the number of successes in independent trials*, Ann. Math. Statist. **35** (1964), 1317–1321.

[19] P.H. Edelman, *Chain enumeration and noncrossing partitions*, Discrete Math. **31** (1980), 171–180.

[20] S. Fisk, *Polynomials, roots, and interlacing*, arXiv:0612833.

[21] I.M. Gessel, personal communication, April 15, 2023.

[22] K. Jochemko, *Symmetric decompositions and the Veronese construction*, Int. Math. Res. Not. **2021** (2021), 11427–11447.

[23] M. Josuat-Vergés, *Refined enumeration of noncrossing chains and hook formulas*, Ann. Comb. **19** (2015), 443–460.

[24] K. Kalampogia-Evangelinou, *Some real-rooted polynomials in algebraic combinatorics*, Doctoral Dissertation, University of Athens, in preparation.

[25] M. Kallipoliti and M. Kubitzke, *A poset fiber theorem for doubly Cohen-Macaulay posets and its applications*, Ann. Comb. **17** (2013), 711–731.

[26] J.S. Kim, *Chain enumeration of k-divisible noncrossing partitions of classical types*, J. Combin. Theory Series A **118** (2011), 879–898.

[27] J. Neggers, *Representations of finite partially ordered sets*, J. Comb. Inform. Syst. Sci. **3** (1978), 113–133.

[28] J. Pitman, *Probabilistic bounds on the coefficients of polynomials with only real zeros*, J. Combin. Theory Series A **77** (1997), 279–303.

[29] N. Reading, *Chains in the noncrossing partition lattice*, SIAM J. Discrete Math. **22** (2008), 875–886.

[30] V. Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Math. **177** (1997), 195–222.

[31] C.D. Savage and M.J. Schuster, *Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences*, J. Combin. Theory Series A **119** (2012), 850–870.

[32] R.P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, in *Graph Theory and its Applications: East and West*, Annals of the New York Academy of Sciences **576**, New York Acad. Sci., New York, 1989, pp. 500–535.

[33] R.P. Stanley, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), 319–331.

[34] R.P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhäuser, Basel, 1996.

[35] R.P Stanley, *Parking functions and noncrossing partitions*, Electron. J. Combin. **4** (1997), #R20, 14 pp (electronic).

[36] R.P. Stanley, *Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications & notable open problems*, Discrete Math. **193** (1998), 267–286.

[37] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, second edition, Cambridge, 2011.

[38] E. Steingrímsson, *Permutation statistics of indexed and poset permutations*, Ph.D. thesis, MIT, 1992.

[39] E. Steingrímsson, *Permutation statistics of indexed permutations*, European J. Combin. **15** (1994), 187–205.

[40] J.R. Stembridge, *Counterexamples to the poset conjectures of Neggers, Stanley, and Stembridge*, Trans. Amer. Math. Soc. **359** (2007), 1115–1128.

[41] E. Swartz, *g-Elements, finite buildings and higher Cohen–Macaulay connectivity*, J. Combin. Theory Series A **113** (2006), 1305–1320.

[42] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 8.4)*, 2018, `https://www.sagemath.org`.

[43] D.G. Wagner, *Total positivity of Hadamard products*, J. Math. Anal. Appl. **163** (1992), 459–483.