Spectral extremal problem on t copies of ℓ-cycle

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Submitted: Jul 10, 2023; Accepted: Jul 9, 2024; Published: Oct 18, 2024

Abstract

Extremal problem on cycles plays an important role in extremal graph theory. Let $ex(n, F)$ and $spex(n, F)$ be the maximum size and spectral radius over all n-vertex F-free graphs, respectively. In this paper, we shall pay attention to the study of both $ex(n, tC_\ell)$ and $spex(n, tC_\ell)$. On the one hand, we determine $ex(n, tC_{2\ell+1})$ and characterize the extremal graph for any integers t, ℓ and $n \geqslant f(t, \ell)$, where $f(t, \ell) = O(t\ell^2)$. This generalizes the result on $ex(n, tC_3)$ of Erdős [Arch. Math. 13 (1962) 222–227] as well as the research on $ex(n, C_{2\ell+1})$ of Füredi and Gunderson [Combin. Probab. Comput. 24 (2015) 641–645]. On the other hand, motivated by the spectral Turán-type problem proposed by Nikiforov, we obtain the extremal spectral radius $spex(n, tC_\ell)$ for any fixed t, ℓ and large enough n. Our results extend some classic spectral extremal results or conjectures on odd cycles and even cycles. Our results also give some inspirations for general spectral Turán-type problem $spex(n, F)$ on bipartite or non-partite F.

Mathematics Subject Classifications: 05C35; 05C50

1 Introduction

Given a graph F, a graph is said to be *F-free* if it does not contain a subgraph isomorphic to F. The *Turán number* of F, denoted by $ex(n, F)$, is the maximum number of edges in an n-vertex F-free graph. An F-free graph is said to be *extremal* with respect to $ex(n, F)$, if it has n vertices and $ex(n, F)$ edges. Denote by $T_{n,r}$ the complete r-partite graph on n vertices in which all parts are as equal in size as possible. An interesting graph in Turán-type problems is a cycle. In 2015, Füredi and Gunderson [14] determined $ex(n, C_{2\ell+1})$ for all n and ℓ, and specially, $T_{n,2}$ is the unique extremal graph when $n \geqslant 4\ell$. However, up to now the exact value of $ex(n, C_{2\ell})$ is still open. Given a graph F, we denote by tF the disjoint union of t copies of F. The study of the Turán number of tC_ℓ can be dated back to 1962, Erdős [10] determined $ex(n, tC_3)$ for $n > 400(t-1)^2$, and characterized the unique extremal graph $K_{t-1} + T_{n-t+1,2}$, (that is, the join of K_{t-1} and $T_{n-t+1,2}$, which is obtained

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by connecting each vertex of K_{t-1} with all vertices of $T_{n-t+1,2}$). Subsequently, Moon [20] proved that Erdős's result is still valid whenever $n > \frac{9t-11}{2}$. In addition, Erdős and Pósa [12] also showed that $ex(n, t\mathcal{C}) = (2t-1)(n-t)$ for $t \geqslant 2$ and $n \geqslant 24t$, where $t\mathcal{C}$ is the family of graphs consisting of t vertex-disjoint cycles without length restriction. In this paper, we further determine the Turán number $ex(n, tC_{2\ell+1})$ by the following theorem. It should be noted that if n is sufficiently large, our result is a special case of a theorem due to Simonovits [29].

Theorem 1. *Let t, ℓ, n be three integers with $t, \ell \geqslant 2$ and $n \geqslant \left\lfloor \frac{(8t\ell+4\ell+3t-6)^2}{4\lfloor \frac{t}{2} \rfloor} \right\rfloor + 8t\ell + 4t + 4\ell - 5$. Then $K_{t-1} + T_{n-t+1,2}$ is the unique extremal graph with respect to $ex(n, tC_{2\ell+1})$.*

Let $A(G)$ be the adjacency matrix of a graph G, and $\rho(G)$ be its spectral radius. The spectral extremal value of a given graph F, denoted by $spex(n, F)$, is the maximum spectral radius over all n-vertex F-free graphs. An F-free graph on n vertices with maximum spectral radius is called an *extremal graph* with respect to $spex(n, F)$. Note that $\rho(G) \geqslant \frac{2m}{n}$ for each graph G with n vertices and m edges. Thus we always have $ex(n, F) \leqslant \frac{n}{2} spex(n, F)$, which sometimes presents a best upper bound on the Turán number of F (see [27]).

In recent years, the investigation on $spex(n, F)$ has become very popular (see [5, 8, 9, 16, 17, 18, 19, 30, 31, 33, 35]). In this paper, we are interested in studying $spex(n, tF)$ for some given F. Let P_k, C_k, S_k, K_k denote a path, a cycle, a star and a complete graph of order k, respectively. Up to now, $spex(n, tF)$ and its corresponding extremal graphs were studied for some special cases (see $spex(n, tK_2)$ [13], $spex(n, tP_\ell)$ [2], $spex(n, tS_\ell)$ [3], $spex(n, tK_\ell)$ [21]).

In this paper, we consider that F is a cycle of given length. We first investigate the case that F is an odd cycle. Note that Nikiforov [23] determined $spex(n, C_{2\ell+1})$ for sufficiently large n. Using Theorem 1 and Nikiforov's result on $spex(n, C_{2\ell+1})$, we prove the following theorem.

Theorem 2. *For any two given positive integers t, ℓ and sufficiently large n, $K_{t-1} + T_{n-t+1,2}$ is the unique extremal graph with respect to $spex(n, tC_{2\ell+1})$.*

Next, we focus on an even cycle F. When $t = 1$, it can be reduced to a classic spectral Turán-type problem $spex(n, C_{2\ell})$, which was initially investigated by Nikiforov [22, 26]. Denote by $S_{n,\ell}$ the join of an ℓ-clique with an independent set of size $n - \ell$. Furthermore, let $S^+_{n,\ell}$ be the graph obtained from $S_{n,\ell}$ by adding an edge within its independent set, and $S^{++}_{n,\ell}$ be the graph obtained from $S_{n,\ell}$ by embedding a maximum matching within its independent set. Nikiforov [22] and Zhai et al. [34] determined the unique extremal graph $S^{++}_{n,1}$ with respect to $spex(n, C_4)$ for odd and even n respectively. In 2010, Nikiforov [26] gave a *spectral even cycle conjecture* as follows: $S^+_{n,\ell-1}$ is the unique extremal graph with respect to $spex(n, C_{2\ell})$ for $\ell \geqslant 3$ and n large enough. In 2022, Cioabă, Desai and Tait [6, 7] established a new spectral extremal method by which they completely solved the above conjecture and a spectral Erdős-Sós conjecture which was also proposed by Nikiforov [26]. In this paper, we develop Nikiforov's conjecture by the following result.

Theorem 3. *Let t, ℓ be given positive integers and n be sufficiently large. Then*
(i) $S^{++}_{n,2t-1}$ is the unique extremal graph with respect to $spex(n, tC_4)$;
(ii) $S^{+}_{n,\ell t-1}$ is the unique extremal graph with respect to $spex(n, tC_{2\ell})$ for $\ell \geqslant 3$.

In fact, Cioabă, Desai and Tait's method is very powerful for $spex(n, F)$ when $ex(n, F) = o(n^{\frac{3}{2}})$ and the numbers of local edges are $O(n)$ in F-free graphs, more precisely, there are only $O(n)$ edges within $N_1(u)$ as well as between $N_1(u)$ and $N_2(u)$ for every vertex u, where $N_i(u)$ denotes the set of vertices at distance i from u. Unfortunately, $ex(n, tC_4) = \Theta(n^{\frac{3}{2}})$ and the numbers of local edges are $O(n^{1+\frac{1}{\ell}})$ in $tC_{2\ell}$-free graphs with $t \geqslant 2$. To this end, we prove an important structural property on the extremal graph G with respect to $spex(n, tC_{2\ell})$, that is, $G - \{u\}$ always contains exactly $t-1$ vertex-disjoint 2ℓ-cycles for each $u \in V(G)$. Moreover, we show a special property on the maximum degree of the extremal graph with respect to $spex(n, tC_4)$. These give two key approaches to prove Theorem 3.

Theorems 2 and 3 also give some inspirations on studying $spex(n, F)$ for general F. To be precise, if F is non-partite with $\chi(F) = r + 1$, its spectral extremal graph maybe tend to contain a complete r-partite graph or r-partite Turán graph as a spanning subgraph; if F is bipartite with $ex(n, F) = o(n^{\frac{3}{2}})$, its spectral extremal graph maybe tend to contain a complete bipartite graph $K_{k,n-k}$ as a spanning subgraph.

The remainder of this paper is organized as follows. In Section 2, some preliminary lemmas are introduced. In Section 3, we use the Erdős-Moon theorem on $ex(n, tC_3)$ and structural analysis to prove Theorem 1. In Section 4, we use Theorem 1 and a stability method to show Theorem 2. In Section 5, we present the proof of Theorem 3 by a combination of structural analysis, induction and the Cioabă-Desai-Tait method.

2 Preliminaries

Given a simple graph G, we use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\nu(G)$ the matching number, $\Delta(G)$ the maximum degree, $\delta(G)$ the minimum degree, respectively. For a vertex $v \in V(G)$, we denote by $N_G(v)$ its neighborhood and $d_G(v)$ its degree in G. Given two disjoint vertex subsets S and T. Let $G[S]$ be the subgraph induced by S, $G - S$ be the subgraph induced by $V(G) \setminus S$, and $G[S, T]$ be the bipartite subgraph on the vertex set $S \cup T$ which consists of all edges with one endpoint in S and the other in T. For short, we write $e(S) = e(G[S])$ and $e(S, T) = e(G[S, T])$. Let $K_{n_1,\ldots,n_r}$ be the complete r-partite graph with classes of sizes $n_1, \ldots, n_r$. If $\sum_{i=1}^{r} n_i = n$ and $|n_i - n_j| \leqslant 1$ for any two integers $i, j \in \{1, \ldots, r\}$, then $K_{n_1,\ldots,n_r}$ is exactly the n-vertex r-partite Turán graph $T_{n,r}$. Let $F + H$ be the join and $F \cup H$ be the union, of F and H, respectively. Particularly, we denote by tF the disjoint union of t copies of F.

In this section, we introduce some lemmas which will be used in the proofs of Theorems 1, 2 and 3. The first one is due to Erdős [10] and Moon [20].

Lemma 4. ([10, 20]) *Let t, n be two positive integers with $n \geqslant \lfloor \frac{19t-9}{2} \rfloor$. Then*

$$ex(n, tC_3) = \binom{t-1}{2} + (t-1)(n-t+1) + \left\lfloor \frac{(n-t+1)^2}{4} \right\rfloor.$$

Furthermore, $K_{t-1} + T_{n-t+1,2}$ is the unique extremal graph with respect to $ex(n, tC_3)$.

Given two integers ν and Δ, define $f(\nu, \Delta) = \max\{e(G) \mid \nu(G) \leqslant \nu, \Delta(G) \leqslant \Delta\}$. In 1976, Chvátal and Hanson [4] obtained the following result.

Lemma 5. ([4]) *For every two integers $\nu \geqslant 1$ and $\Delta \geqslant 1$, we have*

$$f(\nu, \Delta) = \Delta\nu + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \frac{\Delta}{2} \rceil} \right\rfloor \leqslant \nu(\Delta + 1).$$

The following spectral version of the Erdős-Stone-Simonovits stability theorem was given by Nikiforov [24].

Theorem 6. ([24]) *Let $r \geqslant 2$, $\frac{1}{\ln n} < c < r^{-8(r+21)(r+1)}$, $0 < \varepsilon < 2^{-36} r^{-24}$ and G be an n-vertex graph. If $\rho(G) > (1 - \frac{1}{r} - \varepsilon)n$, then one of the following holds:*
(i) G contains a $K_{r+1}(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$;
(ii) G differs from $T_{n,r}$ in fewer than $(\varepsilon^{\frac{1}{4}} + c^{\frac{1}{8r+8}})n^2$ edges.

From Theorem 6, Desai et al. [9] obtained the following stability result. Theorem 6 and the following lemma present an efficient approach to study spectral extremal problems.

Lemma 7. ([9]) *Let F be a graph with chromatic number $\chi(F) = r + 1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that if G is an F-free graph on $n \geqslant n_0$ vertices with $\rho(G) \geqslant (1 - \frac{1}{r} - \delta)n$, then G can be obtained from $T_{n,r}$ by adding and deleting at most εn^2 edges.*

The following spectral extremal result on odd cycles is due to Nikiforov [23].

Lemma 8. ([23]) *Let ℓ be a given positive integer and n be large enough. Then, $T_{n,2}$ is the unique extremal graph with respect to $spex(n, C_{2\ell+1})$.*

The following result is known as the Erdős-Gallai theorem.

Lemma 9. ([11]) *Let n and ℓ be two integers with $n \geqslant \ell \geqslant 2$. Then $ex(n, P_\ell) \leqslant \frac{(\ell-2)n}{2}$, with equality if and only if $n = t(\ell - 1)$ and $G \cong tK_{\ell-1}$.*

We note that the best current bound for $ex(n, C_{2\ell})$ was given by He [15], who improved on a bound $ex(n, C_{2\ell}) \leqslant \big(80\sqrt{\ell}\log \ell + o(1)\big)n^{1+\frac{1}{\ell}}$ of Bukh and Jiang [1] by reducing a factor of $\sqrt{5 \log \ell}$. However, for our purposes the dependence of the multiplicative constant on ℓ is not important. For convenience, we use the following version, which improves a known bound of Verstraëte [32] by a factor $8 + o(1)$ when $n \gg k$.

Lemma 10. ([28]) *For all $\ell \geqslant 2$ and $n \geqslant 1$, we have*

$$ex(n, C_{2\ell}) \leqslant (\ell - 1)n(n^{\frac{1}{\ell}} + 16).$$

3 Proof of Theorem 1

In this section, we give the proof of Theorem 1. More precisely, we will extend the Turán-type result on vertex-disjoint triangles to the disjoint union of general odd cycles. First of all, we shall prove two structural lemmas.

Lemma 11. *Let t, ℓ, n be three positive integers with $n \geqslant 8t\ell + 4\ell + 4t - 6$. Let G be a graph on n vertices with $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$, and $S \subseteq V(G)$ with $|S| \leqslant (t-1)(2\ell+1)$. If $G - S$ contains a triangle C^*, then $G - S$ also contains a $(2\ell+1)$-cycle.*

Proof. The result holds trivially for $\ell = 1$. Assume now that $\ell \geqslant 2$. Set $G' = G - S$ and $C^* = u_0v_0w_0u_0$. Note that $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$ and $n \geqslant 8t\ell + 4\ell + 4t - 6$. Then,

$$\delta(G') \geqslant \delta(G) - |S| \geqslant \left\lfloor \frac{n}{2} \right\rfloor - (t-1)(2\ell+1) > 3\ell.$$

Hence, there exist three vertices u_1, v_1, w_1 such that $u_1 \in N_{G'}(u_0) \setminus V(C^*)$, $v_1 \in N_{G'}(v_0) \setminus (V(C^*) \cup \{u_1\})$ and $w_1 \in N_{G'}(w_0) \setminus (V(C^*) \cup \{u_1, v_1\})$. Now let $H_0 = C^*$. Moreover, we define a subgraph $H_1 \subseteq G'$ with $V(H_1) = V(H_0) \cup \{u_1, v_1, w_1\}$ and $E(H_1) = E(H_0) \cup \{u_0u_1, v_0v_1, w_0w_1\}$. If $\ell \geqslant 3$, then there exist three vertices u_2, v_2, w_2 such that $u_2 \in N_{G'}(u_1) \setminus V(H_1)$, $v_2 \in N_{G'}(v_1) \setminus (V(H_1) \cup \{u_2\})$ and $w_2 \in N_{G'}(w_1) \setminus (V(H_1) \cup \{u_2, v_2\})$. Repeat the above steps, we can obtain a sequence of subgraphs $H_0, \cdots, H_{\ell-1}$ such that $V(H_i) = V(H_{i-1}) \cup \{u_i, v_i, w_i\}$ and

$$E(H_i) = E(H_{i-1}) \cup \{u_{i-1}u_i, v_{i-1}v_i, w_{i-1}w_i\}$$

for $1 \leqslant i \leqslant \ell - 1$. Clearly, $|H_i| = 3i + 3$ for each $i \in \{0, \ldots, \ell-1\}$. Then we can easily check that $\frac{n+3}{4} \geqslant |S| + |H_{\ell-1}| + \frac{1}{4}$. Furthermore, for each $x \in \{u_{\ell-1}, v_{\ell-1}, w_{\ell-1}\} \subseteq V(H_{\ell-1})$ we can see that

$$\begin{aligned} |N_{G'}(x) \setminus V(H_{\ell-1})| &\geqslant d_{G'}(x) - (|H_{\ell-1}| - 1) \\ &\geqslant \delta(G') - |H_{\ell-1}| + 1 \\ &\geqslant \frac{n-1}{2} - |S| - |H_{\ell-1}| + 1 \\ &= \frac{1}{3}\Big(n - |S| - |H_{\ell-1}|\Big) + \frac{2}{3}\Big(\frac{n+3}{4} - |S| - |H_{\ell-1}|\Big). \end{aligned}$$

Thus we have

$$3|N_{G'}(x) \setminus V(H_{\ell-1})| > n - |S| - |H_{\ell-1}| = |V(G') \setminus V(H_{\ell-1})|.$$

By the pigeonhole principle, there exists some $y \in V(G') \setminus V(H_{\ell-1})$ such that y is adjacent to at least two vertices, say $v_{\ell-1}$ and $w_{\ell-1}$, of $\{u_{\ell-1}, v_{\ell-1}, w_{\ell-1}\}$. Hence, the subgraph $G'[\{y, v_0, \ldots, v_{\ell-1}, w_0, \ldots, w_{\ell-1}\}]$ contains a cycle of length $2\ell+1$, as $v_0w_0 \in E(H_0)$. The result follows. □

Lemma 12. *Let t, k, n be three integers with $t \geqslant 2$, $k \geqslant \lfloor \frac{19t-9}{2} \rfloor$ and $n \geqslant \left\lfloor \frac{(k-t)^2}{4\lfloor \frac{t+1}{2} \rfloor} \right\rfloor + (k+1)$. If G is a graph of order n with $e(G) \geqslant ex(n, tC_3)$ and $\delta(G) \leqslant \lfloor \frac{n}{2} \rfloor - 1$, then there exists an induced subgraph $G' \subseteq G$ on $n' \geqslant k$ vertices with $e(G') \geqslant ex(n', tC_3) + 1$ and $\delta(G') \geqslant \lfloor \frac{n'}{2} \rfloor$.*

Proof. By Lemma 4, for any integer $n^* \geqslant \lfloor \frac{19t-7}{2} \rfloor$ we have

$$ex(n^*, tC_3) - ex(n^* - 1, tC_3) = \left\lfloor \frac{n^* + t - 1}{2} \right\rfloor. \tag{1}$$

Since $\delta(G) \leqslant \lfloor \frac{n}{2} \rfloor - 1$, there is a vertex $u_0 \in V(G)$ such that $d_G(u_0) \leqslant \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-2}{2} \rfloor$. Set $G_0 = G$ and $G_1 = G_0 - \{u_0\}$. Combining $e(G_0) \geqslant ex(n, tC_3)$, $d_{G_0}(u_0) \leqslant \lfloor \frac{n-2}{2} \rfloor$ and (1) gives

$$e(G_1) = e(G_0) - d_{G_0}(u_0) \geqslant ex(n-1, tC_3) + \left\lfloor \frac{t+1}{2} \right\rfloor, \tag{2}$$

as $\lfloor \frac{n+t-1}{2} \rfloor - \lfloor \frac{n-2}{2} \rfloor \geqslant \lfloor \frac{t+1}{2} \rfloor$. Now, if $\delta(G_1) \geqslant \lfloor \frac{n-1}{2} \rfloor$, then we define $G' = G_1$ and we are done. Otherwise, there is a vertex $u_1 \in V(G_1)$ such that $d_{G_1}(u_1) \leqslant \lfloor \frac{n-3}{2} \rfloor$. Then, we set $G_2 = G_1 - \{u_1\}$. By (1) and (2), we obtain

$$e(G_2) = e(G_1) - d_{G_1}(u_1) \geqslant ex(n-2, tC_3) + 2\left\lfloor \frac{t+1}{2} \right\rfloor,$$

as $\lfloor \frac{n+t-2}{2} \rfloor - \lfloor \frac{n-3}{2} \rfloor \geqslant \lfloor \frac{t+1}{2} \rfloor$. Repeating the above steps, we obtain either a G_i for some $i \leqslant n-k-1$ such that it is a desired induced subgraph or a sequence of induced subgraphs $G_0, G_1, \cdots, G_{n-k}$ such that $|G_i| = n - i$ and

$$e(G_i) \geqslant ex(n-i, tC_3) + i\left\lfloor \frac{t+1}{2} \right\rfloor \tag{3}$$

for $1 \leqslant i \leqslant n-k$. Since $n \geqslant \left\lfloor \frac{(k-t)^2}{4\lfloor \frac{t+1}{2} \rfloor} \right\rfloor + (k+1)$, we have

$$(n-k)\left\lfloor \frac{t+1}{2} \right\rfloor > \frac{(k-t)^2}{4} \geqslant \binom{k-t+1}{2} - \left\lfloor \frac{(k-t+1)^2}{4} \right\rfloor. \tag{4}$$

From Lemma 4 we know that

$$ex(k, tC_3) = \binom{t-1}{2} + (t-1)(k-t+1) + \left\lfloor \frac{(k-t+1)^2}{4} \right\rfloor.$$

Combining the above equality with (3) and (4), we obtain

$$\begin{aligned} e(G_{n-k}) &\geqslant ex(k, tC_3) + (n-k)\left\lfloor \frac{t+1}{2} \right\rfloor \\ &> \binom{t-1}{2} + (t-1)(k-t+1) + \binom{k-t+1}{2} = \binom{k}{2}, \end{aligned}$$

contradicting $|G_{n-k}| = k$. Hence, G_i is a desired induced subgraph for some integer $i \leqslant n-k-1$. □

Having Lemmas 11 and 12, we are now ready to give the proof of Theorem 1. Recall that $t \geqslant 2$, $\ell \geqslant 2$ and $n \geqslant \left\lfloor \frac{(8t\ell+4\ell+3t-6)^2}{4\lfloor \frac{t}{2} \rfloor} \right\rfloor + 8t\ell + 4t + 4\ell - 5$. For convenience, we denote $G^* = K_{t-1} + T_{n-t+1,2}$.

Proof. By Lemma 4, we have $e(G^*) = ex(n, tC_3)$ for $t \geqslant 2$ and $n \geqslant \lfloor \frac{19t-9}{2} \rfloor$. Moreover, we can easily check that G^* contains at most $t-1$ vertex-disjoint copies of $C_{2\ell+1}$ for each positive integer ℓ, as every odd cycle in G^* must occupy at least one vertex in the $(t-1)$-clique. Let G be an extremal graph with respect to $ex(n, tC_{2\ell+1})$. Then

$$e(G) = ex(n, tC_{2\ell+1}) \geqslant e(G^*) = ex(n, tC_3).$$

Set $k = 8t\ell + 4\ell + 4t - 6$. Since $\ell \geqslant 2$, we have $k \geqslant \left\lfloor \frac{19t-9}{2} \right\rfloor$. Suppose now that $\delta(G) \leqslant \lfloor \frac{n}{2} \rfloor - 1$. Then by Lemma 12, there exists an induced subgraph $G' \subseteq G$ on $n' \geqslant k$ vertices such that $e(G') \geqslant ex(n', tC_3) + 1$ and $\delta(G') \geqslant \lfloor \frac{n'}{2} \rfloor$. Furthermore, by Lemma 4, G' contains t vertex-disjoint triangles $C^1, C^2, \ldots, C^t$.

Let $S_1 = \cup_{i=2}^t V(C^i)$. Then $|S_1| = 3(t-1) \leqslant (t-1)(2\ell+1)$, and $G' - S_1$ contains a triangle C^1. By Lemma 11, $G' - S_1$ also contains a $(2\ell+1)$-cycle C^{1^*}. Let $S_2 = V(C^{1^*}) \cup (\cup_{i=3}^t V(C^i))$. Then $|S_2| = (2\ell+1) + 3(t-2) \leqslant (t-1)(2\ell+1)$, and $G' - S_2$ contains a triangle C^2. Again by Lemma 11, $G' - S_2$ also contains a $(2\ell+1)$-cycle C^{2^*}.

Repeating the above steps, we obtain a sequence of subsets $S_1, \cdots, S_t$ such that

$$S_j = \left(\cup_{i=1}^{j-1} V(C^{i^*}) \right) \cup \left(\cup_{i=j+1}^{t} V(C^i) \right)$$

and $G' - S_j$ contains a $(2\ell+1)$-cycle C^{j^*} for $2 \leqslant j \leqslant t$. Hence, G' contains t vertex-disjoint $(2\ell+1)$-cycles $C^{1^*}, \ldots, C^{t^*}$, contradicting the fact that G is $tC_{2\ell+1}$-free. Therefore, $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$.

Recall that $e(G) \geqslant ex(n, tC_3)$. Then by Lemma 4, if $G \ncong G^*$, then G contains t vertex-disjoint triangles. Furthermore, by Lemma 11 and a similar way as above to G', we can find t vertex-disjoint $(2\ell+1)$-cycles in G, a contradiction. Therefore, $G \cong G^*$. This completes the proof of Theorem 1. □

4 Proof of Theorem 2

In this section, we give the proof of Theorem 2. By Lemma 8, it holds directly for $t = 1$. In the following, assume that $t \geqslant 2$ and G is an extremal graph with respect to $spex(n, tC_{2\ell+1})$. We first prove that G is connected. Suppose to the contrary, then we can select two distinct components G_1 and G_2 of G with $\rho(G_1) = \rho(G)$. Let G' be a graph obtained from G by adding a new edge between G_1 and G_2. Then G' is $tC_{2\ell+1}$-free and $\rho(G') > \rho(G)$, which contradicts the choice of G. By the Perron-Frobenius theorem, there exists a positive unit eigenvector $X = (x_1, \ldots, x_n)^T$ corresponding to $\rho(G)$. Assume that $u^* \in V(G)$ with $x_{u^*} = \max\{x_i \mid i \in V(G)\}$. We also choose a positive constant $\eta < \frac{1}{75}$, which will be frequently used in the proof. Let $G^* = K_{t-1} + T_{n-t+1,2}$, where $G^* = T_{n,2}$ for $t = 1$. We shall prove $G \cong G^*$ for n sufficiently large.

Lemma 13. $\rho(G) \geqslant \frac{n}{2} + (t-1) - \frac{t^2}{2n}$.

Proof. By Theorem 1, G^* is an extremal graph with respect to $ex(n, tC_{2\ell+1})$. Since $e(T_{n-t+1,2}) = \lfloor \frac{(n-t+1)^2}{4} \rfloor \geqslant \frac{(n-t+1)^2-1}{4}$, we have

$$e(G^*) = e(K_{t-1}) + e(T_{n-t+1,2}) + (t-1)(n-t+1) \geqslant \frac{1}{4}n^2 + \frac{t-1}{2}n - \frac{t^2}{4}.$$

Using the Rayleigh quotient gives

$$\rho(G) \geqslant \rho(G^*) \geqslant \frac{\mathbf{1}^T A(G^*)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2e(G^*)}{n} \geqslant \frac{n}{2} + (t-1) - \frac{t^2}{2n},$$

as desired. □

Lemma 14. *For n sufficiently large, $e(G) \geqslant \left(\frac{1}{4} - \frac{1}{2}\eta^2\right)n^2$. Furthermore, G admits a partition $V(G) = V_1 \cup V_2$ such that $e(V_1, V_2)$ is maximal, $e(V_1) + e(V_2) \leqslant \frac{1}{2}\eta^2 n^2$ and $\left||V_i| - \frac{n}{2}\right| \leqslant \eta n$ for $i \in \{1, 2\}$.*

Proof. Note that $\chi(tC_{2\ell+1}) = 3$ and G is $tC_{2\ell+1}$-free. Moreover, by Lemma 13, $\rho(G) \geqslant \frac{n}{2} + (t-1) - \frac{t^2}{2n}$. Let ε be a positive constant with $\varepsilon < \frac{1}{2}\eta^2$. Then by Lemma 7, $e(G) \geqslant \frac{1}{4}n^2 - \frac{1}{2}\eta^2 n^2$, and there exists a bipartition $V(G) = U_1 \cup U_2$ such that $\lfloor \frac{n}{2} \rfloor \leqslant |U_1| \leqslant |U_2| \leqslant \lceil \frac{n}{2} \rceil$ and $e(U_1) + e(U_2) \leqslant \frac{1}{2}\eta^2 n^2$. We now select a new bipartition $V(G) = V_1 \cup V_2$ such that $e(V_1, V_2)$ is maximal. Then $e(V_1) + e(V_2)$ is minimal, and

$$e(V_1) + e(V_2) \leqslant e(U_1) + e(U_2) \leqslant \frac{1}{2}\eta^2 n^2.$$

On the other hand, assume that $|V_1| = \frac{n}{2} - a$, then $|V_2| = \frac{n}{2} + a$. Thus,

$$e(G) \leqslant |V_1||V_2| + e(V_1) + e(V_2) \leqslant \frac{1}{4}n^2 - a^2 + \frac{1}{2}\eta^2 n^2.$$

Combining $e(G) \geqslant \frac{1}{4}n^2 - \frac{1}{2}\eta^2 n^2$ gives $a^2 \leqslant \eta^2 n^2$, and so $|a| \leqslant \eta n$. □

In the following, we shall define two vertex subsets U and W of G.

Lemma 15. *Let $U = \{v \in V(G) \mid d_G(v) \leqslant \left(\frac{1}{2} - 4\eta\right)n\}$. Then we have $|U| \leqslant \eta n$.*

Proof. Suppose to the contrary that $|U| > \eta n$, then there exists $U' \subseteq U$ with $|U'| = \lfloor \eta n \rfloor$. Moreover, by Lemma 14, we have $e(G) \geqslant \left(\frac{1}{4} - \frac{1}{2}\eta^2\right)n^2$. Now set $n' = |G - U'| = n - \lfloor \eta n \rfloor$. Then $n' - 1 < \left(1 - \eta\right)n$. Thus,

$$\begin{aligned} e(G - U') &\geqslant e(G) - \sum_{v \in U'} d_G(v) \\ &\geqslant \left(\frac{1}{4} - \frac{\eta^2}{2}\right)n^2 - \eta n\left(\frac{1}{2} - 4\eta\right)n \\ &= \frac{1}{4}\left(1 - 2\eta + 14\eta^2\right)n^2 \\ &> \frac{1}{4}\left(n' - 1 + t\right)^2 \end{aligned}$$

for sufficiently large n. We can further check that $\frac{1}{4}(n'+t-1)^2 > e(K_{t-1}+T_{n'-t+1,2})$. Hence, $e(G-U') > e(K_{t-1}+T_{n'-t+1,2})$. By Theorem 1, $G-U'$ contains t vertex-disjoint $(2\ell+1)$-cycles, contradicting the fact that G is $tC_{2\ell+1}$-free. □

Lemma 16. *Let* $W = W_1 \cup W_2$*, where* $W_i = \{v \in V_i \mid d_{V_i}(v) \geqslant 2\eta n\}$ *and* $d_{V_i}(v) = |N_G(v) \cap V_i|$ *for* $i \in \{1,2\}$*. Then we have* $|W| \leqslant \frac{1}{2}\eta n$*.*

Proof. For $i \in \{1,2\}$,

$$2e(V_i) = \sum_{v \in V_i} d_{V_i}(v) \geqslant \sum_{v \in W_i} d_{V_i}(v) \geqslant |W_i| \cdot 2\eta n.$$

Combining Lemma 14, we have

$$\frac{1}{2}\eta^2 n^2 \geqslant e(V_1) + e(V_2) \geqslant \big(|W_1| + |W_2|\big)\eta n = |W|\eta n.$$

Therefore, $|W| \leqslant \frac{1}{2}\eta n$. □

In the following three lemmas, we focus on constructing $(2\ell+1)$-cycles in distinct induced subgraphs of the spectral extremal graph G.

Lemma 17. *For arbitrary* $R \subseteq V(G)$ *with* $|R| \leqslant t(2\ell+1)$*, if there exists an edge within* $V_i \setminus (U \cup W \cup R)$ *for some* $i \in \{1,2\}$*, then* $G - (U \cup W \cup R)$ *contains a* $(2\ell+1)$*-cycle.*

Proof. Let $V' = V_1' \cup V_2'$, where $V_i' = V_i \setminus (U \cup W \cup R)$ for $i \in \{1,2\}$. Moreover, we may assume that $\widehat{i} \in \{1,2\} \setminus \{i\}$. We first claim that for each vertex $u \in V_i'$,

$$|N_{V'}(u)| \geqslant |N_{V_{\widehat{i}}'}(u)| > \frac{2}{5}n, \tag{5}$$

where $N_{V'}(u) = N_G(u) \cap V'$. Since $u \notin U \cup W$, we know that $d_{V_i}(u) < 2\eta n$ and $d_G(u) > (\frac{1}{2} - 4\eta)n$. Recall that $V_1 \cup V_2$ is a bipartition of $V(G)$. Thus $d_{V_{\widehat{i}}}(u) = d_G(u) - d_{V_i}(u) > (\frac{1}{2} - 6\eta)n$. Combining Lemmas 15 and 16 gives

$$|N_{V_{\widehat{i}}'}(u)| \geqslant |N_{V_{\widehat{i}}}(u)| - \big(|U| + |W| + |R|\big) > \Big(\frac{1}{2} - 6\eta\Big)n - \frac{3}{2}\eta n - t(2\ell+1) > \frac{2}{5}n,$$

as the constant $\eta < \frac{1}{75}$ and n is sufficiently large. Thus, (5) follows.

Now let u_0v_0 be an arbitrary edge within V_i'. From (5) we know that both $|N_{V_{\widehat{i}}'}(u_0)| > \frac{2}{5}n$ and $|N_{V_{\widehat{i}}'}(v_0)| > \frac{2}{5}n$. Moreover, by Lemma 14, $|V_{\widehat{i}}'| \leqslant |V_{\widehat{i}}| \leqslant \frac{n}{2} + \eta n$. Thus,

$$\big|N_{V_{\widehat{i}}'}(u_0) \cap N_{V_{\widehat{i}}'}(v_0)\big| \geqslant \big|N_{V_{\widehat{i}}'}(u_0)\big| + \big|N_{V_{\widehat{i}}'}(v_0)\big| - \big|V_{\widehat{i}}'\big| > \frac{3}{10}n - \eta n > 0,$$

and hence there exists a vertex $w_0 \in N_{V_{\widehat{i}}'}(u_0) \cap N_{V_{\widehat{i}}'}(v_0)$. Since $w_0 \in V_{\widehat{i}}'$, it follows from (5) that $|N_{V'}(w_0)| \geqslant |N_{V_i'}(u)| > \frac{2}{5}n$.

Let $H_0 = G[\{u_0, v_0, w_0\}]$. Then $H_0 \cong C_3$ and $H_0 \subseteq G-(U\cup W\cup R)$. If $\ell = 1$, then H_0 is a desired $(2\ell+1)$-cycle. Assume now that $\ell \geqslant 2$. Since $|N_{V'}(u)| > \frac{2}{5}n$ for each $u \in V(H_0)$, there exist $u_1, v_1, w_1 \in V'$ such that $u_1 \in N_{V'}(u_0) \setminus V(H_0)$, $v_1 \in N_{V'}(v_0) \setminus (V(H_0) \cup \{u_1\})$ and $w_1 \in N_{V'}(w_0) \setminus (V(H_0) \cup \{u_1, v_1\})$. Then, we define a subgraph $H_1 \subseteq G$ with $V(H_1) = V(H_0) \cup \{u_1, v_1, w_1\}$ and $E(H_1) = E(H_0) \cup \{u_0u_1, v_0v_1, w_0w_1\}$. If $\ell \geqslant 3$, then there exist u_2, v_2, w_2 such that $u_2 \in N_{V'}(u_1) \setminus V(H_1)$, $v_2 \in N_{V'}(v_1) \setminus (V(H_1) \cup \{u_2\})$ and $w_2 \in N_{V'}(w_1) \setminus (V(H_1) \cup \{u_2, v_2\})$. Repeating the above steps, we obtain a sequence of subgraphs $H_0, H_1, \cdots, H_{\ell-1}$ such that $V(H_j) = V(H_{j-1}) \cup \{u_j, v_j, w_j\}$ and $E(H_j) = E(H_{j-1}) \cup \{u_{j-1}u_j, v_{j-1}v_j, w_{j-1}w_j\}$ for $1 \leqslant j \leqslant \ell - 1$. Then, $|H_{\ell-1}| = 3\ell$ and $H_{\ell-1} \subseteq G - (U \cup W \cup R)$. Set $V'' = V' \setminus V(H_{\ell-1})$. For each $u \in \{u_{\ell-1}, v_{\ell-1}, w_{\ell-1}\}$, we have $|N_{V''}(u)| \geqslant |N_{V'}(u)| - |H_{\ell-1}| + 1 > \frac{2}{5}n - 3\ell + 1$, and thus

$$|N_{V''}(u_{\ell-1})| + |N_{V''}(v_{\ell-1})| + |N_{V''}(w_{\ell-1})| > n > |V''|$$

for n sufficiently large. This implies that there exists $w \in V''$ such that w is adjacent to at least two vertices, say $u_{\ell-1}$ and $v_{\ell-1}$, of $\{u_{\ell-1}, v_{\ell-1}, w_{\ell-1}\}$. Therefore, $G-(U\cup W\cup R)$ contains a $(2\ell + 1)$-cycle $u_0 \dots u_{\ell-1}wv_{\ell-1} \dots v_0u_0$. The proof is completed. ☐

Lemma 18. *For arbitrary $R \subseteq V(G)$ with $|R| \leqslant t(2\ell + 1)$, if there exists a vertex $u_0 \in W \setminus U$, then $G - \big((U \cup W \cup R) \setminus \{u_0\}\big)$ contains a $(2\ell + 1)$-cycle.*

Proof. Since $V(G) = V_1 \cup V_2$, we may assume without loss of generality that $u_0 \in V_1$. Then by the definitions of U and W, we have

$$d_G(u_0) > \Big(\frac{1}{2} - 4\eta\Big)n \quad \text{and} \quad d_{V_1}(u_0) \geqslant 2\eta n.$$

Moreover, by Lemmas 15 and 16, $|U| \leqslant \eta n$ and $|W| \leqslant \frac{1}{2}\eta n$. Thus,

$$\big|N_{V_1\setminus(U\cup W\cup R)}(u_0)\big| \geqslant d_{V_1}(u_0) - (|U| + |W| + |R|) \geqslant \frac{1}{2}\eta n - t(2\ell + 1) > 0.$$

Then, there exists a vertex v_0 in $N_{V_1}(u_0) \setminus (U \cup W \cup R)$. Again by the definitions of U and W, we can see that $d_G(v_0) > (\frac{1}{2} - 4\eta)n$ and $d_{V_1}(v_0) < 2\eta n$. It follows that

$$d_{V_2}(v_0) = d_G(v_0) - d_{V_1}(v_0) > \Big(\frac{1}{2} - 6\eta\Big)n. \tag{6}$$

Recall that $V(G) = V_1\cup V_2$ is a bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximal. Hence, $d_{V_1}(u_0) \leqslant \frac{1}{2}d_G(u_0)$. Since $d_G(u_0) > (\frac{1}{2} - 4\eta)n$, we get that

$$d_{V_2}(u_0) = d_G(u_0) - d_{V_1}(u_0) \geqslant \frac{1}{2}d_G(u_0) > \Big(\frac{1}{4} - 2\eta\Big)n. \tag{7}$$

Furthermore, Lemma 14 gives $|V_2| \leqslant \frac{n}{2} + \eta n$. Combining with (6) and (7), we obtain

$$|N_{V_2}(u_0) \cap N_{V_2}(v_0)| \geqslant |N_{V_2}(u_0)| + |N_{V_2}(v_0)| - |V_2| \geqslant \Big(\frac{1}{4} - 9\eta\Big)n.$$

Note that $\eta < \frac{1}{75}$ and n is sufficiently large. It follows that

$$\Big|\big(N_{V_2}(u_0) \cap N_{V_2}(v_0)\big) \setminus (U \cup W \cup R)\Big| \geqslant \Big(\frac{1}{4} - 9\eta\Big)n - \frac{3}{2}\eta n - t(2\ell+1) > 0.$$

Hence, there exists $w_0 \in \big(N_{V_2}(u_0) \cap N_{V_2}(v_0)\big) \setminus (U \cup W \cup R)$. Let $H_0 = G[\{u_0, v_0, w_0\}]$. Then $H_0 \cong C_3$ and $H_0 \subseteq G - \big((U \cup W \cup R) \setminus \{u_0\}\big)$. For $\ell = 1$, H_0 is a $(2\ell+1)$-cycle. For $\ell \geqslant 2$, using the same method as in the proof of Lemma 17, we can find a $(2\ell+1)$-cycle in $G - \big((U \cup W \cup R) \setminus \{u_0\}\big)$. □

Lemma 19. *Let $\nu = \sum_{i=1}^{2} \nu\big(G[V_i \setminus (U \cup W)]\big)$. Then $\nu \leqslant t-1$. Moreover, $G - (U \cup W)$ contains at least ν vertex-disjoint $(2\ell+1)$-cycles.*

Proof. The case $\nu = 0$ is trivial. Now assume that $\nu \geqslant 1$, and let $u_1u_2, \ldots, u_{2\nu-1}u_{2\nu}$ be ν independent edges in $G[V_1 \setminus (U \cup W)] \cup G[V_2 \setminus (U \cup W)]$. Then, we set $R_0 = \{u_j \mid j = 1, 2, \ldots, 2\lambda\}$ and $R_1 = R_0 \setminus \{u_1, u_2\}$, where $\lambda = \min\{\nu, t\}$. Since u_1u_2 is an edge within $V_i \setminus (U \cup W \cup R_1)$ for some $i \in \{1, 2\}$, Lemma 17 indicates that $G - (U \cup W \cup R_1)$ contains a $(2\ell+1)$-cycle C^1. Let $R_2 = \big(R_1 \setminus \{u_3, u_4\}\big) \cup V(C^1)$. Again by Lemma 17, $G - (U \cup W \cup R_2)$ contains a $(2\ell+1)$-cycle C^2, as u_3u_4 is an edge within $V_i \setminus (U \cup W \cup R_2)$ for some $i \in \{1, 2\}$.

Repeating the above steps, we obtain a sequence of vertex subsets $R_1, \cdots, R_\lambda$ such that $R_j = \big(R_{j-1} \setminus \{u_{2j-1}, u_{2j}\}\big) \cup \big(\cup_{k=1}^{j-1} V(C^k)\big)$ and $G - (U \cup W \cup R_j)$ contains a $(2\ell+1)$-cycle C^j for each $j \in \{2, \ldots, \lambda\}$. Clearly, $|R_j| \leqslant (\lambda-1)(2\ell+1)$ for $1 \leqslant j \leqslant \lambda$; moreover, $C^1, C^2, \ldots, C^\lambda$ are vertex-disjoint cycles in $G - (U \cup W)$. Since G is $tC_{2\ell+1}$-free, we have $\lambda \leqslant t-1$. Combining $\lambda = \min\{\nu, t\}$ gives $\nu = \lambda \leqslant t-1$, and thus $C^1, C^2, \ldots, C^\nu$ are vertex-disjoint $(2\ell+1)$-cycles in $G - (U \cup W)$. □

In the following two lemmas, we shall give two local structural properties of G.

Lemma 20. *For $i \in \{1, 2\}$, we have $\Delta\big(G[V_i \setminus (U \cup W)]\big) < t(2\ell+1)$.*

Proof. Our proof is by contradiction. Without loss of generality, suppose that there exists a vertex $u_0 \in V_1 \setminus (U \cup W)$ such that $d_{V_1 \setminus (U \cup W)}(u_0) \geqslant t(2\ell+1)$. Since $u_0 \notin W$, we get $d_{V_1}(u_0) < 2\eta n$ by the definition of W. On the other hand, by Lemma 14, $|V_1| \geqslant \frac{n}{2} - \eta n$, and so

$$|V_1 \setminus (U \cup W)| \geqslant |V_1| - |U| - |W| \geqslant \Big(\frac{1}{2} - \frac{5}{2}\eta\Big)n.$$

Hence, $|V_1 \setminus (U \cup W)| > d_{V_1}(u_0)$, as $\eta < \frac{1}{75}$. This implies that there exist vertices in $V_1 \setminus (U \cup W)$ which are not adjacent to u_0. Let G' be the graph obtained from G by adding all possible edges from u_0 to $V_1 \setminus (U \cup W)$. Then $\rho(G') > \rho(G)$. Since G is extremal with respect to $spex(n, tC_{2\ell+1})$, G' must contain a subgraph H isomorphic to $tC_{2\ell+1}$. From the construction of G', we can further see that $u_0 \in V(C)$ for some $(2\ell+1)$-cycle C in H. Set $H' = H - V(C)$. Then $H' \subseteq G$. Since $d_{V_1 \setminus (U \cup W)}(u_0) \geqslant t(2\ell+1)$ while $|H'| = (t-1)(2\ell+1)$, there exists a vertex v_0 with $v_0 \in N_{V_1 \setminus (U \cup W)}(u_0)$ and $v_0 \notin V(H')$ in G.

Now setting $R = V(H')$ in Lemma 17, and noticing that u_0v_0 is an edge within $V_1 \setminus (U \cup W \cup R)$, we obtain that $G - (U \cup W \cup R)$ contains a $(2\ell+1)$-cycle C'. Clearly, $V(C') \cap V(H') = \varnothing$. Therefore, $C' \cup H'$ is a copy of $tC_{2\ell+1}$ in G, which contradicts the fact that G is $tC_{2\ell+1}$-free. □

Lemma 21. *For $i \in \{1,2\}$, $G[V_i \setminus (U \cup W)]$ contains an independent set I_i with $|I_i| > |V_i \setminus (U \cup W)| - 2(t-1)t(2\ell+1)$.*

Proof. Assume that $\nu_i = \nu\big(G[V_i \setminus (U \cup W)]\big)$ for $i \in \{1,2\}$. If $\nu_i = 0$, then $V_i \setminus (U \cup W)$ is a desired independent set. Now assume that $\nu_i \geqslant 1$, and let $u_1u_2, \dots, u_{2\nu_i-1}u_{2\nu_i}$ be ν_i independent edges in $G[V_i \setminus (U \cup W)]$. Let

$$I_i = \big(V_i \setminus (U \cup W)\big) \setminus \big(\cup_{j=1}^{2\nu_i} N_{V_i\setminus(U\cup W)}(u_j)\big).$$

Then, every vertex in I_i is not adjacent to any vertex in $\{u_1, u_2, \dots, u_{2\nu_i}\}$. Now, if $G[I_i]$ contains an edge, then $\nu\big(G[V_i \setminus (U \cup W)]\big) \geqslant \nu_i + 1$, a contradiction. Therefore, I_i is an independent set.

From Lemma 20 we know that $\Delta\big(G[V_i\setminus(U\cup W)]\big) < t(2\ell+1)$. Moreover, $\nu_i \leqslant \nu \leqslant t-1$ by Lemma 19. Thus, we can see that

$$|V_i \setminus (U \cup W)| - |I_i| = |\cup_{j=1}^{2\nu_i} N_{V_i\setminus(U\cup W)}(u_j)| \leqslant 2\nu_i\Delta\big(G[V_i \setminus (U \cup W)]\big) < 2(t-1)t(2\ell+1).$$

The result follows. □

In the following three lemmas, we will give exact characterizations of U and W. Recall that $X = (x_1, \dots, x_n)^T$ is a positive unit eigenvector of G, and $x_{u^*} = \max\{x_i \mid i \in V(G)\}$. Since $|W| \leqslant \frac{1}{2}\eta n < n$ by Lemma 16, we may choose a vertex v^* such that $x_{v^*} = \max\{x_v \mid v \in V(G) \setminus W\}$. We will see that $v^* \notin U$. Then

$$\rho(G)x_{u^*} = \sum_{v\in N_W(u^*)} x_v + \sum_{v\in N_{G-W}(u^*)} x_v \leqslant \sum_{v\in W} x_v + \sum_{v\in V(G)\setminus W} x_v \leqslant |W|x_{u^*} + (n-|W|)x_{v^*}.$$

Moreover, $\rho(G) > \frac{n}{2}$ by Lemma 13. It follows that

$$x_{v^*} \geqslant \frac{\rho(G)-|W|}{n-|W|}x_{u^*} \geqslant \frac{\rho(G)-|W|}{n}x_{u^*} > \frac{1}{2}\big(1-\eta\big)x_{u^*}. \tag{8}$$

Since $\eta < \frac{1}{75}$, we have $x_{v^*} > \frac{2}{5}x_{u^*}$. On the other hand,

$$\rho(G)x_{v^*} = \sum_{v\in N_W(v^*)} x_v + \sum_{v\in N_{G-W}(v^*)} x_v \leqslant |W|x_{u^*} + d_G(v^*)x_{v^*}.$$

Combining with $x_{v^*} > \frac{2}{5}x_{u^*}$, $\rho(G) > \frac{n}{2}$ and $|W| \leqslant \frac{1}{2}\eta n$, we obtain

$$d_G(v^*) \geqslant \rho(G) - \frac{x_{u^*}}{x_{v^*}}|W| \geqslant \rho(G) - \frac{5}{2}|W| > \Big(\frac{1}{2} - \frac{5}{4}\eta\Big)n.$$

Recall that $U = \{v \in V(G) \mid d_G(v) \leqslant \left(\frac{1}{2} - 4\eta\right)n\}$. Then $v^* \notin U$, and so $v^* \in V(G) \setminus (U \cup W)$.

Assume now that $v^* \in V_{i^*} \setminus (U \cup W)$ for some $i^* \in \{1, 2\}$, and set $\widehat{i^*} \in \{1, 2\} \setminus \{i^*\}$. Then by Lemma 20, $|N_{V_{i^*}}(v^*) \setminus (U \cup W)| < t(2\ell + 1)$. Thus,

$$\begin{aligned}
\rho(G)x_{v^*} &= \sum_{v \in N_{U \cup W}(v^*)} x_v + \sum_{v \in N_{V_{i^*}}(v^*) \setminus (U \cup W)} x_v + \sum_{v \in N_{V_{\widehat{i^*}}}(v^*) \setminus (U \cup W)} x_v \\
&< \big(|W|x_{u^*} + |U|x_{v^*}\big) + t(2\ell + 1)x_{v^*} + \sum_{v \in V_{\widehat{i^*}} \setminus (U \cup W \cup I_{\widehat{i^*}})} x_v + \sum_{v \in I_{\widehat{i^*}}} x_v \\
&\leqslant \big(|W|x_{u^*} + |U|x_{v^*}\big) + (2t - 1)t(2\ell + 1)x_{v^*} + \sum_{v \in I_{\widehat{i^*}}} x_v,
\end{aligned}$$

where $I_{\widehat{i^*}}$ is an independent set of $G[V_{\widehat{i^*}} \setminus (U \cup W)]$ such that $\left|V_{\widehat{i^*}} \setminus (U \cup W \cup I_{\widehat{i^*}})\right| < 2(t - 1)t(2\ell + 1)$ (see Lemma 21). Subsequently,

$$\sum_{v \in I_{\widehat{i^*}}} x_v > \big(\rho(G) - |U| - (2t - 1)t(2\ell + 1)\big)x_{v^*} - |W|x_{u^*}. \tag{9}$$

Lemma 22. *We have $U = \varnothing$.*

Proof. Suppose to the contrary that there exists $u_0 \in U$. Let G' be the graph obtained from G by deleting edges incident to u_0 and joining all possible edges from $I_{\widehat{i^*}}$ to u_0.

We claim that G' is $tC_{2\ell+1}$-free. Otherwise, G' contains a subgraph H isomorphic to $tC_{2\ell+1}$. From the construction of G', we can see that H must contain a $(2\ell + 1)$-cycle C' with $u_0 \in V(C')$. Set $H' = H - V(C')$. Then $H' \subseteq G$. Assume that $N_{C'}(u_0) = \{u_1, u_2\}$, then $u_1, u_2 \in I_{\widehat{i^*}}$ by the definition of G'. Since $I_{\widehat{i^*}} \subseteq V_{\widehat{i^*}} \setminus (U \cup W)$, we have $u_1, u_2 \notin U \cup W$. By the definitions of U and W, we know that $d_G(u_j) > \left(\frac{1}{2} - 4\eta\right)n$ and $d_{V_{\widehat{i^*}}}(u_j) < 2\eta n$ for $j \in \{1, 2\}$. Hence, $|N_{V_{i^*}}(u_1)| = d_G(u_1) - d_{V_{\widehat{i^*}}}(u_1) > \left(\frac{1}{2} - 6\eta\right)n$. Similarly, $|N_{V_{i^*}}(u_2)| > \left(\frac{1}{2} - 6\eta\right)n$. Moreover, $|V_{i^*}| \leqslant \frac{n}{2} + \eta n$ by Lemma 14. It follows that

$$|N_{V_{i^*}}(u_1) \cap N_{V_{i^*}}(u_2)| \geqslant |N_{V_{i^*}}(u_1)| + |N_{V_{i^*}}(u_2)| - |V_{i^*}| > \left(\frac{1}{2} - 13\eta\right)n.$$

Now, note that $|H| = t(2\ell + 1)$. Then $|N_{V_{i^*}}(u_1) \cap N_{V_{i^*}}(u_2)| > |H|$, and hence we can find a vertex $u \in \big(N_{V_{i^*}}(u_1) \cap N_{V_{i^*}}(u_2)\big) \setminus V(H)$. This implies that $G - V(H')$ contains a $(2\ell + 1)$-cycle C'', which is obtained from C' by replacing $\{u_0u_1, u_0u_2\}$ with $\{uu_1, uu_2\}$. Hence, $C'' \cup H'$ is a copy of $tC_{2\ell+1}$ in G, a contradiction. Therefore, the above claim holds.

Now, $d_G(u_0) \leqslant (\frac{1}{2} - 4\eta)n$ by the definition of U. Recall that $\rho(G) > \frac{n}{2}$ and $|U| \leqslant \eta n$. Then

$$\rho(G) - d_G(u_0) - |U| > 3\eta n. \tag{10}$$

Moreover,

$$\sum_{v \in N_G(u_0)} x_v = \sum_{v \in N_W(u_0)} x_v + \sum_{v \in N_{G-W}(u_0)} x_v \leqslant |W|x_{u^*} + d_G(u_0)x_{v^*}. \tag{11}$$

Recall that $x_{v^*} > \frac{2}{5}x_{u^*}$ and $|W| \leqslant \frac{1}{2}\eta n$. Combining (9), (10) and (11), we get that

$$\begin{aligned}\sum_{v\in I_{\widehat{i^*}}} x_v - \sum_{v\in N_G(u_0)} x_v &\geqslant \sum_{v\in I_{\widehat{i^*}}} x_v - \big(|W|x_{u^*} + d_G(u_0)x_{v^*}\big)\\ &> \big(\rho(G)-d_G(u_0)-|U|-(2t-1)t(2\ell+1)\big)x_{v^*} - 2|W|x_{u^*}\\ &> \big(3\eta n-(2t-1)t(2\ell+1)\big)\frac{2}{5}x_{u^*} - \eta n x_{u^*}\\ &> \frac{1}{10}\eta n x_{u^*}\end{aligned}$$

for n sufficiently large. Thus,

$$\rho(G') - \rho(G) \geqslant X^T\big(A(G') - A(G)\big)X = 2x_{u_0}\Big(\sum_{v\in I_{\widehat{i^*}}} x_v - \sum_{v\in N_G(u_0)} x_v\Big) > 0,$$

contradicting the fact that G is extremal with respect to $spex(n, tC_{2\ell+1})$. □

Lemma 23. *For each $v \in V(G)$, we have $x_v > \frac{2}{5}x_{u^*}$.*

Proof. Recall that $\rho(G) > \frac{n}{2}$ and $|W| \leqslant \frac{1}{2}\eta n$. Then $|W| < \eta\rho(G)$. Moreover, $U = \varnothing$ by Lemma 22. Combining (9), we obtain that

$$\sum_{v\in I_{\widehat{i^*}}} x_v > \big(\rho(G) - (2t-1)t(2\ell+1)\big)x_{v_0} - \eta\rho(G)x_{u^*}.$$

From (8) we know that $x_{v^*} > \frac{1}{2}\big(1-\eta\big)x_{u^*}$. Thus, for n sufficiently large,

$$\sum_{v\in I_{\widehat{i^*}}} x_v > \Big(\frac{1}{2} - 2\eta\Big)\rho(G)x_{u^*}.$$

Now, suppose to the contrary that there exists $u_0 \in V(G)$ such that $x_{u_0} \leqslant \frac{2}{5}x_{u^*}$. Let G' be the graph obtained from G by deleting edges incident to u_0 and joining all edges from $I_{\widehat{i^*}}$ to u_0. By a similar discussion as in the proof of Lemma 22, we claim that G' is $tC_{2\ell+1}$-free. However,

$$\sum_{v\in I_{\widehat{i^*}}} x_v - \sum_{v\in N_G(u_0)} x_v = \sum_{v\in I_{\widehat{i^*}}} x_v - \rho(G)x_{u_0} > \left(\frac{1}{2} - 2\eta - \frac{2}{5}\right)\rho(G)x_{u^*} > 0,$$

which implies that

$$\rho(G') - \rho(G) \geqslant X^T\big(A(G') - A(G)\big)X = 2x_{u_0}\Big(\sum_{v\in I_{\widehat{i^*}}} x_v - \sum_{v\in N_G(u_0)} x_v\Big) > 0,$$

contradicting the fact that G is extremal with respect to $spex(n, tC_{2\ell+1})$. □

Lemma 24. $|W| = t-1$ *and* $\nu = 0$.

Proof. Note that $U = \varnothing$. By Lemma 19, $\nu = \nu(\cup_{i=1}^{2} G[V_i \setminus W]) \leqslant t-1$; and if $\nu \geqslant 1$, then $G - W$ contains ν vertex-disjoint $(2\ell+1)$-cycles $C^1, C^2, \dots, C^\nu$.

We first claim that $|W| \leqslant t-1-\nu$. Otherwise, $|W| \geqslant t-\nu$. Let $R_0 = \{u_1, u_2, \dots, u_{t-\nu}\}$ be a subset of W. Furthermore, we define $R_1 = R_0 \setminus \{u_1\}$ if $\nu = 0$; and $R_1 = (R_0 \setminus \{u_1\}) \cup (\cup_{i=1}^{\nu} V(C^i))$ if $\nu \geqslant 1$. Then $|R_1| \leqslant (t-1)(2\ell+1)$. By Lemma 18, $G - \big((W \cup R_1) \setminus \{u_1\}\big)$ contains a $(2\ell+1)$-cycle $C^{\nu+1}$, where $V(C^{\nu+1}) \cap R_0 \subseteq \{u_1\}$. If $t-\nu \geqslant 2$, then we further define $R_2 = \big(R_1 \setminus \{u_2\}\big) \cup V(C^{\nu+1})$. Clearly, $|R_2| \leqslant (t-1)(2\ell+1)$. Again by Lemma 18, $G - \big((W \cup R_2) \setminus \{u_2\}\big)$ contains a $(2\ell+1)$-cycle $C^{\nu+2}$, where $V(C^{\nu+2}) \cap R_0 \subseteq \{u_2\}$. Repeating the above steps, we obtain a sequence of vertex subsets $R_1, R_2, \dots, R_{t-\nu}$ with $R_j = \big(R_{j-1} \setminus \{u_j\}\big) \cup \big(\cup_{k=1}^{j-1} V(C^{\nu+k})\big)$ and $|R_j| \leqslant (t-1)(2\ell+1)$ such that $G - \big((W \cup R_j) \setminus \{u_j\}\big)$ contains a $(2\ell+1)$-cycle $C^{\nu+j}$ for each $j \in \{1, \dots, t-\nu\}$. Furthermore, $V(C^{\nu+j}) \cap R_0 \subseteq \{u_j\}$ for $1 \leqslant j \leqslant t-\nu$. Thus we can observe that $C^1, C^2, \dots, C^t$ are vertex-disjoint, which contradicts the fact that G is $tC_{2\ell+1}$-free.

Now define $H = \cup_{i=1}^{2} G[V_i \setminus W]$. Then $\nu(H) = \nu$. We further claim that

$$e(H) \leqslant (t-1)(2t\ell + t + 1). \tag{12}$$

The case $\nu = 0$ is trivial. Assume that $\nu \geqslant 1$. By Lemma 20, $\Delta(H) < t(2\ell+1)$. Recall that $f(\nu, \Delta) = \max\{e(G) \mid \nu(G) \leqslant \nu, \Delta(G) \leqslant \Delta\}$, and by Lemma 5 $f(\nu, \Delta) \leqslant \nu(\Delta+1)$. Thus,

$$e(H) \leqslant f\big(\nu(H), \Delta(H)\big) \leqslant f(\nu, t(2\ell+1)) \leqslant \nu \cdot (2t\ell + t + 1).$$

Note that $\nu \leqslant t-1$. Therefore, (12) holds.

Note that $|W| \leqslant t-1-\nu \leqslant t-1$. It suffices to prove $|W| = t-1$, as it implies that $\nu = 0$. Suppose to the contrary that $|W| \leqslant t-2$. Take $S \subseteq V_1 \setminus W$ with $|S| = t-1-|W|$, and let G' be the graph obtained from G by deleting all edges in $E(H)$ and adding all possible edges from S to $V_1 \setminus (W \cup S)$. Clearly, G' is a spanning subgraph of $K_{|W \cup S|} + K_{|V_1 \setminus (W \cup S)|, |V_2 \setminus W|}$. Since $|W \cup S| = t-1$, G' contains at most $t-1$ vertex-disjoint odd cycles, and so G' is $tC_{2\ell+1}$-free.

Recall that $|V_1| \geqslant \frac{1}{2}n - \eta n$, and by Lemma 23, $x_v > \frac{2}{5} x_{u^*}$ for each $v \in V(G)$. Combining (12), we have

$$\begin{aligned} \rho(G') - \rho(G) &\geqslant X^T\big(A(G') - A(G)\big)X \geqslant \sum_{u \in S, v \in V_1 \setminus (W \cup S)} 2x_u x_v - \sum_{uv \in E(H)} 2x_u x_v \\ &\geqslant |S|\Big(\frac{n}{2} - \eta n - t + 1\Big)\frac{8}{25} x_{u^*}^2 - (t-1)(2t\ell + t + 1) 2x_{u^*}^2 \\ &> 0, \end{aligned}$$

contradicting the fact that G is an extremal graph with respect to $spex(n, tC_{2\ell+1})$. $\square$

In the following, we complete the proof of Theorem 2.

Proof. Recall that $G^* = K_{t-1} + T_{n-t+1,2}$ and we shall prove $G \cong G^*$. We first look for a $(t-1)$-clique in which each vertex is adjacent to all other vertices of G. By Lemma 24, we know that $|W| = t-1$. It suffices to show that $d_G(u) = n-1$ for each $u \in W$.

Suppose to the contrary that there exists a vertex $u \in W$ with $d(u) < n-1$. Then we can select a non-neighbor v of u in G. Let $G' = G + \{uv\}$. Then $\rho(G') > \rho(G)$. Since G is extremal with respect to $spex(n, tC_{2\ell+1})$, G' contains a subgraph H isomorphic to $tC_{2\ell+1}$, where $uv \in E(H)$. More precisely, H contains a $(2\ell+1)$-cycle C with $uv \in V(C)$. Set $H' = H - V(C)$. Then $H' \subseteq G$, and by Lemma 18, $G - \big((W \cup V(H')) \setminus \{u\}\big)$ contains a $(2\ell+1)$-cycle C'. Since $u \notin V(H')$, $H' \cup C'$ is a copy of $tC_{2\ell+1}$ in G, a contradiction. Therefore, $d_G(u) = n-1$ for each $u \in W$.

Let $|V_i \setminus W| = n_i$ for $i \in \{1,2\}$. Assume without loss of generality that $n_1 \geqslant n_2$. By Lemma 24, $\nu = \nu(\cup_{i=1}^{2} G[V_i \setminus W]) = 0$, and thus $G - W \subseteq K_{n_1,n_2}$. Since G is extremal, we have $G - W \cong K_{n_1,n_2}$. To show $G \cong G^*$, it suffices to show $G - W \cong T_{n-t+1,2}$, or equivalently, $n_1 - n_2 \leqslant 1$.

Suppose to the contrary that $n_1 \geqslant n_2 + 2$. By symmetry, we may assume $x_u = x_i$ for each $u \in V_i \setminus W$ and $i \in \{1,2\}$. Moreover, let $x_u = x_3$ for each $u \in W$. Thus,

$$\rho(G)x_1 = n_2 x_2 + (t-1)x_3, \;\; \rho(G)x_2 = n_1 x_1 + (t-1)x_3,$$

and $\rho(G)x_3 = n_1 x_1 + n_2 x_2 + (t-2)x_3$. It follows that

$$x_1 = \frac{\rho(G)+1}{\rho(G)+n_1} x_3 \;\; \text{and} \;\; x_2 = \frac{\rho(G)+1}{\rho(G)+n_2} x_3. \tag{13}$$

Select $u_0 \in V_1 \setminus W$. Let G'' be the graph obtained from G by deleting edges from u_0 to $V_2 \setminus W$ and adding all edges from u_0 to $V_1 \setminus (W \cup \{u_0\})$. Then $G'' \cong K_{t-1} + K_{n_1-1,n_2+1}$, and thus G'' is still $tC_{2\ell+1}$-free. Moreover,

$$\rho(G'') - \rho(G) \geqslant \sum_{v \in V_1 \setminus (W \cup \{u_0\})} 2x_{u_0}x_v - \sum_{v \in V_2 \setminus W} 2x_{u_0}x_v = 2x_1\big((n_1-1)x_1 - n_2 x_2\big).$$

In view of (13), we have

$$(n_1-1)x_1 - n_2 x_2 = \frac{(\rho(G)+1)\left((n_1-n_2-1)\rho(G) - n_2\right)}{(\rho(G)+n_1)(\rho(G)+n_2)} x_3 > 0,$$

since $n_1 \geqslant n_2 + 2$ and $\rho(G) > \frac{n}{2} > n_2$. It follows that $\rho(G'') > \rho(G)$, a contradiction. Therefore, $n_1 - n_2 \leqslant 1$ and $G \cong K_{t-1} + T_{n-t+1,2}$. This completes the proof. □

5 Proof of Theorem 3

In this section, we will often assume that n is sufficiently large without saying so explicitly. We first give the lower and upper bounds of $\rho(S^{+}_{n,\ell})$ and $\rho(S^{++}_{n,\ell})$.

Lemma 25. *For fixed ℓ and sufficiently large n, we have*
(i) $\rho(S^{++}_{n,\ell}) \geqslant \rho(S^{+}_{n,\ell}) \geqslant \frac{\ell-1+\sqrt{(\ell-1)^2+4\ell(n-\ell)}}{2} \geqslant \sqrt{\ell n}$ *if* $\ell \geqslant 2$;
(ii) $\rho(S^{++}_{n,\ell}) \leqslant \sqrt{\left(\ell + \frac{1}{4\ell}\right)n}$ *if* $\ell \geqslant 1$.

Proof. (i) From [26, p. 2246] we obtain $\rho(S_{n,\ell}) = \frac{\ell-1+\sqrt{(\ell-1)^2+4\ell(n-\ell)}}{2}$. Since $S_{n,\ell} \subseteq S^{+}_{n,\ell} \subseteq S^{++}_{n,\ell}$, the inequality holds obviously for $\ell \geqslant 2$.

(ii) By the Perron-Frobenius theorem, there exists a positive unit eigenvector $X = (x_1, \ldots, x_n)^T$ corresponding to ρ, where $\rho = \rho(S^{++}_{n,\ell})$. Let W be the set of dominating vertices in $S^{++}_{n,\ell}$, and $\overline{W} = V(S^{++}_{n,\ell}) \backslash W$. Choose $u_0 \in W$ and $v_0 \in \overline{W}$ with $x_{u_0} = \max_{u \in W} x_u$ and $x_{v_0} = \max_{v \in \overline{W}} x_v$. Note that $|W| = \ell$. Then, $\rho x_{u_0} \leqslant (\ell-1)x_{u_0} + (n-\ell)x_{v_0}$ and $\rho x_{v_0} \leqslant \ell x_{u_0} + x_{v_0}$. Combining these two inequalities, we obtain

$$(\rho - \ell + 1)(\rho - 1) \leqslant (n-\ell)\ell.$$

If $\rho > \sqrt{\left(\ell + \frac{1}{4\ell}\right)n}$, then $(\rho-\ell+1)(\rho-1) > (n-\ell)\ell$, a contradiction. Thus, $\rho \leqslant \sqrt{\left(\ell + \frac{1}{4\ell}\right)n}$. □

Recall that $\ell \geqslant 2$ in Theorem 3. We shall proceed the proof by induction on t. When $t = 1$, the result holds immediately by [6, 22, 34]. In the following, we assume that $t \geqslant 2$.

For convenience, set $\lambda = \ell t - 1$, then $\lambda \geqslant 2\ell - 1$. Let G be an extremal graph with respect to $spex(n, tC_{2\ell})$. Clearly, G is connected. By the Perron-Frobenius theorem, there exists a positive unit eigenvector $X = (x_1, \ldots, x_n)^T$ corresponding to $\rho(G)$. Choose $u^* \in V(G)$ with $x_{u^*} = \max\{x_i \mid i = 1, 2, \ldots, n\}$. For a vertex u and a positive integer i, let $N_i(u)$ denote the set of vertices at distance i from u in G. By the induction hypothesis, we obtain that for n' sufficiently large,

$$spex\big(n', (t-1)C_{2\ell}\big) = \begin{cases} \rho(S^{++}_{n',\lambda-\ell}) & \text{if } \ell = 2, \\ \rho(S^{+}_{n',\lambda-\ell}) & \text{if } \ell \geqslant 3. \end{cases} \tag{14}$$

We then show that for each $u \in V(G)$, $G - \{u\}$ contains $t-1$ vertex-disjoint copies of $C_{2\ell}$ through Lemmas 26 and 27. This will be used to bound $\rho(G)$ in Lemma 29, to bound $\sum_{v \in V(G)} d^2_G(v)$ in Lemma 30 and to prove a key property in Lemma 31.

Lemma 26. *Let H be a graph on $n-1$ vertices. Then $\rho(H) \geqslant \rho(K_1 + H) - \frac{n-1}{\rho(K_1+H)}$.*

Proof. Let $V(H) \cup \{\overline{u}\}$ be the vertex set of $K_1 + H$. Set $\overline{\rho} := \rho(K_1 + H)$ and let $Y = (y_u)$ be an eigenvector to $\overline{\rho}$. Using the Rayleigh quotient gives

$$\overline{\rho} = \frac{2\sum_{uv \in E(K_1+H)} y_u y_v}{\sum_{u \in V(K_1+H)} y_u^2} = \frac{2\sum_{uv \in E(H)} y_u y_v + 2y_{\overline{u}} \sum_{u \in V(H)} y_u}{y_{\overline{u}}^2 + \sum_{u \in V(H)} y_u^2}. \tag{15}$$

Since $\overline{\rho} y_{\overline{u}} = \sum_{u \in V(H)} y_u$, we have $y_{\overline{u}} \sum_{u \in V(H)} y_u = \overline{\rho} y_{\overline{u}}^2 = \frac{1}{\overline{\rho}}\big(\sum_{u \in V(H)} y_u\big)^2$. Thus by (15), we obtain

$$2\sum_{uv \in E(H)} y_u y_v = \overline{\rho} \sum_{u \in V(H)} y_u^2 - \overline{\rho} y_{\overline{u}}^2 = \overline{\rho} \sum_{u \in V(H)} y_u^2 - \frac{1}{\overline{\rho}}\Big(\sum_{u \in V(H)} y_u\Big)^2.$$

By the Cauchy-Schwarz inequality we have $\left(\sum_{u\in V(H)} y_u\right)^2 \leqslant (n-1)\sum_{u\in V(H)} y_u^2$. It follows that

$$\rho(H) \geqslant \frac{2\sum_{uv\in E(H)} y_u y_v}{\sum_{u\in V(H)} y_u^2} \geqslant \overline{\rho} - \frac{n-1}{\overline{\rho}},$$

as desired. □

Lemma 27. *For every vertex $u \in V(G)$, $G - \{u\}$ contains $t-1$ vertex-disjoint 2ℓ-cycles.*

Proof. Suppose to the contrary that there exists a vertex u such that $G-\{u\}$ is $(t-1)C_{2\ell}$-free. Then $\rho(G-\{u\}) \leqslant spex\big(n-1,(t-1)C_{2\ell}\big)$. It follows from (14) that

$$\rho\big(G-\{u\}\big) \leqslant \rho\big(S^{++}_{n-1,\lambda-\ell}\big), \tag{16}$$

as $\rho(S^{+}_{n-1,\lambda-\ell}) \leqslant \rho(S^{++}_{n-1,\lambda-\ell})$.

Recall that $t,\ell \geqslant 2$ and $\lambda \geqslant 2\ell-1 \geqslant 3$. We can easily check that $\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}} > \sqrt{\lambda-\ell+\frac{1}{4(\lambda-\ell)}}$. By Lemma 25 (ii), we further have

$$\sqrt{\lambda n} - \frac{n}{\sqrt{\lambda n}} > \sqrt{\Big(\lambda-\ell+\frac{1}{4(\lambda-\ell)}\Big)n} \geqslant \rho(S^{++}_{n,\lambda-\ell}) > \rho(S^{++}_{n-1,\lambda-\ell}). \tag{17}$$

On the one hand, u is a dominating vertex of G. Otherwise, there exists a vertex v not adjacent to u. Let G^* be the graph obtained from G by adding the edge uv. Since $G^*-\{u\} = G-\{u\}$, $G^*-\{u\}$ is also $(t-1)C_{2\ell}$-free, and thus G^* is $tC_{2\ell}$-free. However, $G \subset G^*$ indicates that $\rho(G) < \rho(G^*)$, contradicting the fact that G is extremal with respect to $spex(n,tC_{2\ell})$.

On the other hand, notice that $S^{+}_{n,\lambda}$ is $tC_{2\ell}$-free, then $\rho(G) \geqslant \rho(S^{+}_{n,\lambda})$, and so $\rho(G) \geqslant \sqrt{\lambda n}$ by Lemma 25 (i). Since u is a dominating vertex of G, one can see $G \cong K_1+(G-\{u\})$. Combining $\rho(G) \geqslant \sqrt{\lambda n}$ and (17) with Lemma 26, we have

$$\rho\big(G-\{u\}\big) \geqslant \rho(G) - \frac{n-1}{\rho(G)} \geqslant \sqrt{\lambda n} - \frac{n}{\sqrt{\lambda n}} > \rho\big(S^{++}_{n-1,\lambda-\ell}\big),$$

which contradicts (16). Therefore, the lemma holds. □

Lemma 28. *For every vertex $u \in V(G)$ and every subset $W_0 \subseteq V(G)$, we have $e(N_1(u)) \leqslant (2\lambda - \frac{3}{2})|N_1(u)|$ and $e(N_1(u), N_2(u)\cap W_0) \leqslant (2\lambda-\frac{1}{2})(|N_1(u)| + |N_2(u)\cap W_0|)$.*

Proof. By Lemma 27, $G-\{u\}$ contains $t-1$ vertex-disjoint 2ℓ-cycles, say $C^1,\ldots,C^{t-1}$. Let $V' = \cup_{j=1}^{t-1} V(C^j)$ and $G' = G - V'$. Then G' is $C_{2\ell}$-free. Set $N'_i(u) = N_i(u)\setminus V'$ for $i\in\{1,2\}$. Clearly, $G'[N'_1(u)]$ is $P_{2\ell-1}$-free. By Lemma 9, $e(N'_1(u)) \leqslant (\ell-\frac{3}{2})|N'_1(u)| \leqslant (\ell-\frac{3}{2})|N_1(u)|$ and so

$$\begin{aligned} e\big(N_1(u)\big) &\leqslant e\big(N'_1(u)\big) + |N_1(u)\cap V'||N_1(u)| \leqslant \Big(\ell - \frac{3}{2} + 2\ell(t-1)\Big)|N_1(u)| \\ &\leqslant \Big(2\lambda - \frac{3}{2}\Big)|N_1(u)|, \end{aligned}$$

as $\ell \geqslant 2$ and $\lambda = \ell t - 1$. Clearly, the bipartite subgraph $G'[N_1'(u), N_2'(u) \cap W_0]$ is $P_{2\ell+1}$-free (otherwise, we can find a $P_{2\ell-1}$ with both endpoints in $N_1'(u)$ and thus a $C_{2\ell}$ in G'). By Lemma 9, $e(N_1'(u), N_2'(u) \cap W_0) \leqslant (\ell - \frac{1}{2})(|N_1(u)| + |N_2(u) \cap W_0|)$. Since $N_1(u) \setminus N_1'(u)$ and $(N_2(u) \cap W_0) \setminus (N_2'(u) \cap W_0)$ are two subsets of V', we have

$$\begin{aligned} e\big(N_1(u), N_2(u) \cap W_0\big) &\leqslant e\big(N_1'(u), N_2'(u) \cap W_0\big) + |V'|(|N_1(u)| + |N_2(u) \cap W_0|) \\ &\leqslant \Big(\ell - \frac{1}{2} + 2\ell(t-1)\Big)(|N_1(u)| + |N_2(u) \cap W_0|) \\ &\leqslant \Big(2\lambda - \frac{1}{2}\Big)\big(|N_1(u)| + |N_2(u) \cap W_0|\big), \end{aligned}$$

completing the proof. □

Lemma 29. $\sqrt{\lambda n} \leqslant \rho(G) \leqslant \sqrt{6\lambda n}$.

Proof. Recall that $S^+_{n,\lambda}$ is $tC_{2\ell}$-free and G is a spectral extremal graph. Then $\rho(G) \geqslant \rho(S^+_{n,\lambda})$, and the lower bound follows from Lemma 25 (i). We then prove the upper bound. Note that

$$\rho^2(G)x_{u^*} = \sum_{u \in N_1(u^*)} \sum_{w \in N_1(u)} x_w \leqslant |N_1(u^*)|x_{u^*} + 2e\big(N_1(u^*)\big)x_{u^*} + e\big(N_1(u^*), N_2(u^*)\big)x_{u^*}.$$

Setting $u = u^*$ and $W_0 = N_2(u^*)$ in Lemma 28, we obtain $\rho^2(G) \leqslant \big(6\lambda - \frac{5}{2}\big)n \leqslant 6\lambda n$. □

In [25], Nikiforov studied an extremal problem on degree power, which is an extension of Turán's problem. Nikiforov showed that $\sum_{u \in V(H)} d_H^2(u) \leqslant 2(\ell-1)e(H) + (\ell-1)(|H|-1)|H|$ for every $C_{2\ell}$-free graph H. Inspired by this result, we obtain the following one on $tC_{2\ell}$-free graphs.

Lemma 30. *We have* $e(G) \leqslant \ell n^{1+\frac{1}{\ell}}$ *and* $\sum_{v \in V(G)} d_G^2(v) < 2\lambda n^2$.

Proof. From the above definition of G', we know that $|G'| = n - 2\ell(t-1)$ and G' is $C_{2\ell}$-free. By Lemma 10, we have

$$e(G') \leqslant ex\big(n - 2\ell(t-1), C_{2\ell}\big) \leqslant (\ell-1)(n - 2\ell(t-1))^{1+\frac{1}{\ell}} + 16(\ell-1)n.$$

It follows that

$$e(G) \leqslant e(G') + \sum_{v \in V'} d_G(v) \leqslant e(G') + 2\ell(t-1)n \leqslant \ell n^{1+\frac{1}{\ell}}. \tag{18}$$

Hence, the first statement holds. For an arbitrary vertex $u \in V(G)$,

$$\sum_{v \in N_1(u)} d_G(v) = |N_1(u)| + 2e\big(N_1(u)\big) + e\big(N_1(u), N_2(u)\big).$$

Combining this with Lemma 28, where W_0 is chosen as $N_2(u)$, we get that

$$\sum_{v \in N_1(u)} d_G(v) < (4\lambda - 2)|N_1(u)| + \Big(2\lambda - \frac{1}{2}\Big)n.$$

Summing the above inequality over all vertices $u \in V(G)$ and using (18), we obtain

$$\begin{aligned}\sum_{u\in V(G)}\sum_{v\in N_1(u)} d_G(v) &< (4\lambda-2)\sum_{u\in V(G)} d_G(u) + \Big(2\lambda-\frac{1}{2}\Big)n^2\\ &= (8\lambda-4)e(G) + \Big(2\lambda-\frac{1}{2}\Big)n^2\\ &< 2\lambda n^2.\end{aligned}$$

Observe that $\sum_{v\in V(G)} d_G^2(v) = \sum_{u\in V(G)}\sum_{v\in N_1(u)} d_G(v)$. Hence, the second statement follows. □

Choose a positive constant $\eta < \frac{1}{20000\lambda^5}$, and define $W = \{u \in V(G) \mid x_u \geqslant \eta x_{u^*}\}$. We shall give an upper bound for $|W|$ and a lower bound for degrees of vertices in W (see Lemmas 33 and 34). However, we are in trouble when $\ell = 2$ as $ex(n, tC_4) = \Theta(n^{\frac{3}{2}})$. Hence, we prove a special structural property as follows.

Lemma 31. *For $\ell = 2$, we have $\Delta(G) \geqslant (1 - \frac{\eta}{40\lambda})n$.*

Proof. Set $\alpha = 1 - \frac{\eta}{40\lambda}$ and suppose to the contrary that $\Delta(G) < \alpha n$. Specially, we have $d_G(u^*) < \alpha n$. By Lemma 27, $G - \{u^*\}$ contains $t-1$ vertex-disjoint quadrilaterals $C^1, \dots, C^{t-1}$. Given an arbitrary $i \in \{1, 2, \dots, t-1\}$, we assume that $V(C^i) = \{u_{ij} \mid j = 1, 2, 3, 4\}$. We then define $M_i = \{u_{ij_1}, u_{ij_2}\}$ if there exist two distinct vertices $u_{ij_1}, u_{ij_2} \in V(C^i)$ with $|N_1(u_{ij_1}) \cap N_1(u_{ij_2})| \geqslant (1-\alpha)n$, and $M_i = V(C^i)$ otherwise. Furthermore, set $M := \cup_{i=1}^{t-1} M_i$.

If $M_i = \{u_{ij_1}, u_{ij_2}\}$ for some $u_{ij_1}, u_{ij_2} \in V(C^i)$, then $e\big(M_i, V(G-M)\big) \leqslant d_G(u_{ij_1}) + d_G(u_{ij_2}) < 2\alpha n$. If $M_i = V(C^i)$, then

$$e\big(M_i, V(G-M)\big) \leqslant \Big|\cup_{j=1}^4 N_1(u_{ij})\Big| + \sum_{1\leqslant j_1<j_2\leqslant 4}\Big|N_1(u_{ij_1}) \cap N_1(u_{ij_2})\Big| < n + 6(1-\alpha)n < 2\alpha n,$$

where the last inequality follows from $a = 1 - \frac{\eta}{40\lambda} > \frac{7}{8}$. Hence, we always have

$$e\big(M, V(G-M)\big) = \sum_{i=1}^{t-1} e\big(M_i, V(G-M)\big) < 2(t-1)\alpha n. \tag{19}$$

Now, we will see that $G - M$ is C_4-free. Otherwise, let $\widetilde{C}^t$ be a 4-cycle in $G - M$. If $M_1 = V(C^1)$, then we define a 4-cycle $\widetilde{C}^1 = C^1$, where $V(\widetilde{C}^1) \cap V(\widetilde{C}^t) = \varnothing$ obviously. If $M_1 = \{u_{1j_1}, u_{1j_2}\}$ for some $u_{1j_1}, u_{1j_2} \in V(C^1)$, then $|N_1(u_{1j_1}) \cap N_1(u_{1j_2})| \geqslant (1-\alpha)n$, and thus there exists a 4-cycle $\widetilde{C}^1 = u_{1j_1}v_1u_{1j_2}w_1u_{1j_1}$ such that $v_1, w_1 \notin M \cup V(\widetilde{C}^t)$. Similarly, if $M_2 = V(C^2)$, then we define $\widetilde{C}^2 = C^2$; otherwise, $M_2 = \{u_{2j_1}, u_{2j_2}\}$, then we can find a 4-cycle $\widetilde{C}^2 = u_{2j_1}v_2u_{2j_2}w_2u_{2j_1}$ such that $v_2, w_2 \notin M \cup V(\widetilde{C}^1) \cup V(\widetilde{C}^t)$. Repeating the above steps, we obtain a sequence of vertex-disjoint 4-cycles $\widetilde{C}^1, \cdots, \widetilde{C}^{t-1}$ such that $V(\widetilde{C}^i) \cap V(\widetilde{C}^t) = \varnothing$ for $1 \leqslant i \leqslant t-1$. Consequently, we obtain t vertex-disjoint 4-cycles in G, a contradiction. Therefore, $G - M$ is C_4-free.

We know that

$$\rho^2(G)x_{u^*} = \sum_{v\in N_1(u^*)} \rho(G)x_v = \sum_{v\in N_1(u^*)} \sum_{w\in N_1(v)} x_w. \tag{20}$$

In the following, we have to distinguish (20) into three cases. Let $\widetilde{N}_i(v)$ be the set of vertices at distance i from a vertex v in $G-M$. Then $u^* \in V(G-M)$ and $\widetilde{N}_i(u^*) \subseteq N_i(u^*)$ for $i \in \{1,2\}$.

Case (i) We have $w \in M$. We shall evaluate $\sum_{v\in N_1(u^*)} \sum_{w\in N_1(v)\cap M} x_w$. Note that $N_1(u^*) \setminus M = \widetilde{N}_1(u^*)$. On the one hand, $\sum_{v\in N_1(u^*)\setminus M} \sum_{w\in N_1(v)\cap M} x_w \leqslant e\big(\widetilde{N}_1(u^*), M\big)x_{u^*}$. On the other hand,

$$\sum_{v\in N_1(u^*)\cap M} \sum_{w\in N_1(v)\cap M} x_w \leqslant 2e(M)x_{u^*} \leqslant 2\binom{|M|}{2}x_{u^*} < (1-\alpha)nx_{u^*},$$

as $|M| \leqslant 4(t-1)$. Thus we have

$$\sum_{v\in N_1(u^*)} \sum_{w\in N_1(v)\cap M} x_w < \Big(e\big(\widetilde{N}_1(u^*), M\big) + (1-\alpha)n\Big)x_{u^*}. \tag{21}$$

Case (ii) Both w and v belong to $G-M$. We shall evaluate $\sum_{v\in \widetilde{N}_1(u^*)} \sum_{w\in \widetilde{N}_1(v)} x_w$. Clearly, $\widetilde{N}_1(v) \subseteq \{u^*\} \cup \widetilde{N}_1(u^*) \cup \widetilde{N}_2(u^*)$ for $v \in \widetilde{N}_1(u^*)$. Since $G-M$ is C_4-free, vertices in $\widetilde{N}_1(u^*)$ have no common neighbors in $\widetilde{N}_2(u^*)$, which implies that $e\big(\widetilde{N}_1(u^*), \widetilde{N}_2(u^*)\big) = |\widetilde{N}_2(u^*)|$. Hence,

$$\sum_{v\in \widetilde{N}_1(u^*)} \sum_{w\in \widetilde{N}_1(v)\cap \widetilde{N}_2(u^*)} x_w \leqslant e\big(\widetilde{N}_1(u^*), \widetilde{N}_2(u^*)\big)x_{u^*} = |\widetilde{N}_2(u^*)|x_{u^*}.$$

Since $G-M$ is C_4-free, there also exits no P_3 within $\widetilde{N}_1(u^*)$. Thus,

$$\sum_{v\in \widetilde{N}_1(u^*)} \sum_{w\in \widetilde{N}_1(v)\cap \widetilde{N}_1(u^*)} x_w \leqslant \sum_{v\in \widetilde{N}_1(u^*)} x_v \leqslant \sum_{v\in N_1(u^*)} x_v = \rho(G)x_{u^*}.$$

Observe that $\sum_{w\in \widetilde{N}_1(v)} x_w = x_{u^*} + \sum_{w\in \widetilde{N}_1(v)\cap \widetilde{N}_1(u^*)} x_w + \sum_{w\in \widetilde{N}_1(v)\cap \widetilde{N}_2(u^*)} x_w$ for each $v \in \widetilde{N}_1(u^*)$. Combining above two inequalities, we obtain

$$\sum_{v\in \widetilde{N}_1(u^*)} \sum_{w\in \widetilde{N}_1(v)} x_w \leqslant \big(|\widetilde{N}_1(u^*)| + |\widetilde{N}_2(u^*)| + \rho(G)\big)x_{u^*} < \big(n - |M| + \rho(G)\big)x_{u^*}. \tag{22}$$

Case (iii) w belongs to $G-M$ but $v \in M$. We shall calculate the term $\sum_{v\in N_1(u^*)\cap M} \sum_{w\in N_1(v)\setminus M} x_w$. Now set $\widetilde{N}_{2^+}(u^*) := V(G-M) \setminus \big(\{u^*\} \cup \widetilde{N}_1(u^*)\big)$. We can observe that

$$\begin{aligned}
\sum_{v\in N_1(u^*)\cap M} \sum_{w\in N_1(v)\setminus M} x_w &= \sum_{v\in N_1(u^*)\cap M} \Big(x_{u^*} + \sum_{w\in N_1(v)\cap \widetilde{N}_1(u^*)} x_w + \sum_{w\in N_1(v)\cap \widetilde{N}_{2^+}(u^*)} x_w\Big) \\
&\leqslant \sum_{v\in M} \Big(x_{u^*} + \sum_{w\in N_1(u^*)} x_w + \sum_{w\in \widetilde{N}_{2^+}(u^*)} x_w\Big) \\
&\leqslant |M|\big(1+\rho(G)\big)x_{u^*} + e\big(\widetilde{N}_{2^+}(u^*), M\big)x_{u^*}.
\end{aligned} \tag{23}$$

Summing (21), (22) and (23) into (20), we obtain

$$\rho^2(G) < (|M|+1)\rho(G) + (2-\alpha)n + e\big(V(G-M), M\big).$$

Note that $|M| \leqslant 4(t-1)$, and by Lemma 29, $\rho(G) \leqslant \sqrt{6\lambda n}$. Thus, $(|M|+1)\rho(G) \leqslant (4t-3)\sqrt{6\lambda n} < (1-\alpha)n$. Moreover, $e\big(V(G-M), M\big) < 2(t-1)\alpha n$ by (19). Consequently,

$$\rho^2(G) < \big(3 + (2t-4)\alpha\big)n \leqslant (2t-1)n,$$

as $t \geqslant 2$ and $\alpha = 1 - \frac{\eta}{40\lambda} < 1$. However, by Lemma 29 we have $\rho^2(G) \geqslant \lambda n = (\ell t - 1)n \geqslant (2t-1)n$, a contradiction, Therefore, $\Delta(G) \geqslant \alpha n$, completing the proof. □

Lemma 32. *Let $W' = \{u \in V(G) \mid x_u \geqslant \frac{\eta}{5}x_{u^*}\}$. Then $|W'| \leqslant \frac{\eta}{20\lambda}n$.*

Proof. We first consider the case $\ell \geqslant 3$. By Lemma 29, $\rho(G) \geqslant \sqrt{\lambda n}$. Hence,

$$\sqrt{\lambda n}\frac{\eta}{5}x_{u^*} \leqslant \sqrt{\lambda n}x_u \leqslant \rho(G)x_u = \sum_{v \in N_1(u)} x_v \leqslant d_G(u)x_{u^*}$$

for each $u \in W'$. Summing this inequality over all vertices $u \in W'$, we obtain

$$|W'|\sqrt{\lambda n}\frac{\eta}{5}x_{u^*} \leqslant \sum_{u \in W'} d_G(u)x_{u^*} \leqslant \sum_{u \in V(G)} d_G(u)x_{u^*} \leqslant 2e(G)x_{u^*}. \tag{24}$$

Combining (18) and (24), we get $|W'| \leqslant \frac{10\ell n^{1+\frac{1}{\ell}}}{\sqrt{\lambda n}\eta} \leqslant \frac{\eta}{20\lambda}n$ for n large enough.

Now, there remains the case $\ell = 2$. By Lemma 31, there exists a vertex $v^* \in V(G)$ with $d_G(v^*) \geqslant (1 - \frac{\eta}{40\lambda})n$. Hence,

$$|W' \setminus N_1(v^*)| \leqslant |V(G) \setminus N_1(v^*)| = n - d_G(v^*) \leqslant \frac{\eta}{40\lambda}n.$$

Let $W^* = \{v \in N_1(v^*) \mid x_v \geqslant \sqrt{6\lambda}n^{-0.4}x_{u^*}\}$. Note that $\rho(G) \leqslant \sqrt{6\lambda n}$ by Lemma 29. Thus

$$\big|W^*\big|\sqrt{6\lambda}n^{-0.4}x_{u^*} \leqslant \sum_{v \in N_1(v^*)} x_v = \rho(G)x_{v^*} \leqslant \sqrt{6\lambda n}x_{u^*},$$

yielding $|W^*| \leqslant n^{0.9} \leqslant \frac{\eta}{40\lambda}n$. Since $\frac{\eta}{5}x_{u^*} > \sqrt{6\lambda}n^{-0.4}x_{u^*}$, we have $W' \cap N_1(v^*) \subseteq W^*$, and so $|W' \cap N_1(v^*)| \leqslant |W^*| \leqslant \frac{\eta}{40\lambda}n$. Combining $|W' \setminus N_1(v^*)| \leqslant \frac{\eta}{40\lambda}n$ gives $|W'| \leqslant \frac{\eta}{20\lambda}n$, as claimed. □

Lemma 33. $|W| \leqslant \frac{128\lambda^3}{\eta^2}$.

Proof. We first prove that $d_G(u) > \frac{\eta}{8\lambda}n$ for each $u \in W$. Suppose to the contrary that there exists a vertex $\widetilde{u} \in W$ with $d_G(\widetilde{u}) \leqslant \frac{\eta}{8\lambda}n$. Then $x_{\widetilde{u}} \geqslant \eta x_{u^*}$ as $\widetilde{u} \in W$, and by Lemma 29 $\rho(G) \geqslant \sqrt{\lambda n}$. Thus we have

$$\eta\lambda n x_{u^*} \leqslant \rho^2(G)x_{\widetilde{u}} = |N_1(\widetilde{u})|x_{\widetilde{u}} + \sum_{u \in N_1(\widetilde{u})} d_{N_1(\widetilde{u})}(u)x_u + \sum_{u \in N_2(\widetilde{u})} d_{N_1(\widetilde{u})}(u)x_u. \tag{25}$$

By Lemma 28, we have $e(N_1(\widetilde{u})) \leqslant (2\lambda - \frac{3}{2})|N_1(\widetilde{u})|$. Note that $|N_1(\widetilde{u})| \leqslant \frac{\eta}{8\lambda}n$. Thus,

$$|N_1(\widetilde{u})|x_{\widetilde{u}} + \sum_{u\in N_1(\widetilde{u})} d_{N_1(\widetilde{u})}(u)x_u \leqslant \big(|N_1(\widetilde{u})| + 2e(N_1(\widetilde{u}))\big)x_{u^*} \leqslant (4\lambda-2)|N_1(\widetilde{u})|x_{u^*} \leqslant \frac{1}{2}\eta n x_{u^*}.$$

Combining the above inequality with (25), we obtain

$$\sum_{u\in N_2(\widetilde{u})} d_{N_1(\widetilde{u})}(u)x_u \geqslant (\lambda - \frac{1}{2})\eta n x_{u^*}. \tag{26}$$

Now, setting $u = \widetilde{u}$ and $W_0 = W'$ in Lemma 28, we have

$$e\big(N_1(\widetilde{u}), N_2(\widetilde{u}) \cap W'\big) \leqslant \big(2\lambda - \frac{1}{2}\big)\big(|N_1(\widetilde{u})| + |N_2(\widetilde{u}) \cap W'|\big) \leqslant 2\lambda\big(|N_1(\widetilde{u})| + |W'|\big).$$

Since $|N_1(\widetilde{u})| \leqslant \frac{\eta}{8\lambda}n$, and $|W'| \leqslant \frac{\eta}{20\lambda}n$ by Lemma 32, it follows that

$$\sum_{u\in N_2(\widetilde{u})\cap W'} d_{N_1(\widetilde{u})}(u)x_u \leqslant e\big(N_1(\widetilde{u}), N_2(\widetilde{u}) \cap W'\big)x_{u^*} \leqslant \frac{7}{20}\eta n x_{u^*}. \tag{27}$$

Note that $x_u < \frac{\eta}{5}x_{u^*}$ for each $u \in V(G) \setminus W'$. Setting $u = \widetilde{u}$ and $W_0 = V(G) \setminus W'$ in Lemma 28, we get $e\big(N_1(\widetilde{u}), N_2(\widetilde{u}) \setminus W'\big) \leqslant (2\lambda - \frac{1}{2})n$. Consequently,

$$\sum_{u\in N_2(\widetilde{u})\setminus W'} d_{N_1(\widetilde{u})}(u)x_u \leqslant e\big(N_1(\widetilde{u}), N_2(\widetilde{u}) \setminus W'\big)\frac{\eta}{5}x_{u^*} \leqslant (2\lambda - \frac{1}{2})n\frac{\eta}{5}x_{u^*}.$$

Combining (27) gives

$$\sum_{u\in N_2(\widetilde{u})} d_{N_1(\widetilde{u})}(u)x_u \leqslant \Big(\frac{7}{20} + \frac{4\lambda-1}{10}\Big)\eta n x_{u^*} < (\lambda - \frac{1}{2})\eta n x_{u^*}$$

as $\lambda = \ell t - 1 > \frac{5}{4}$, contradicting (26). Therefore, $d_G(u) > \frac{\eta}{8\lambda}n$ for each $u \in W$. It follows that $\sum_{u\in V(G)} d_G^2(u) \geqslant \sum_{u\in W} d_G^2(u) \geqslant |W|\big(\frac{\eta}{8\lambda}n\big)^2$. Moreover, $\sum_{u\in V(G)} d_G^2(u) < 2\lambda n^2$ by Lemma 30. Thus, $|W| \leqslant \frac{128\lambda^3}{\eta^2}$, as claimed. □

Lemma 34. *For each $u \in W$, we have $d_G(u) \geqslant \big(\frac{x_u}{x_{u^*}} - 20\eta\big)n$.*

Proof. Let u be an arbitrary vertex in W. For convenience, we use W_i and $\overline{W_i}$ instead of $N_i(u)\cap W$ and $N_i(u)\setminus W$, respectively. By Lemma 28, $\max\{e(N_1(u)), e(N_1(u), N_2(u))\} \leqslant 2\lambda n$. Since $W_i \cup \overline{W_i} = N_i(u)$ for $i \in \{1, 2\}$, we can see that

$$\max\{e(\overline{W_1}), e(W_1, \overline{W_1}), e(W_1, \overline{W_2}), e(\overline{W_1}, \overline{W_2})\} \leqslant 2\lambda n. \tag{28}$$

Recall that $\rho(G) \geqslant \sqrt{\lambda n}$. We also have

$$\lambda n x_u \leqslant \rho^2(G)x_u = \sum_{v\in N_1(u)}\sum_{w\in N_1(v)} x_w = |N_1(u)|x_u + \sum_{v\in N_1(u)}\sum_{w\in N_1(v)\setminus\{u\}} x_w. \tag{29}$$

Note that $N_1(u) = W_1 \cup \overline{W_1}$ and for any $v \in N_1(u)$,

$$N_1(v) \setminus \{u\} = N_1(v) \cap \big(N_1(u) \cup N_2(u)\big) = N_1(v) \cap (W_1 \cup \overline{W_1} \cup W_2 \cup \overline{W_2}).$$

We now calculate the term $\sum_{v \in N_1(u)} \sum_{w \in N_1(v)\setminus\{u\}} x_w$ in (29). We first consider the case $v \in W_1$. Note that $x_w \leqslant x_{u^*}$ for $w \in W_1 \cup W_2$ and $x_w \leqslant \eta x_{u^*}$ for $w \in \overline{W_1} \cup \overline{W_2}$. Thus,

$$\sum_{v \in W_1} \sum_{w \in N_1(v)\setminus\{u\}} x_w \leqslant \big(2e(W_1) + e(W_1, W_2)\big)x_{u^*} + \big(e(W_1, \overline{W_1}) + e(W_1, \overline{W_2})\big)\eta x_{u^*}.$$

On the one hand, $|W| < \frac{128\lambda^3}{\eta^2}$ by Lemma 33. Note that $W_1 \cup W_2 \subseteq W$. Thus, $2e(W_1) + e(W_1, W_2) \leqslant 2\binom{|W|}{2} \leqslant \eta\lambda n$. On the other hand, we have $e(W_1, \overline{W_1}) + e(W_1, \overline{W_2}) \leqslant 4\lambda n$ by (28). Therefore,

$$\sum_{v \in W_1} \sum_{w \in N_1(v)\setminus\{u\}} x_w \leqslant 5\lambda\eta n x_{u^*}. \tag{30}$$

Now, we consider the case $v \in \overline{W_1}$. We can see that

$$\begin{aligned} \sum_{v \in \overline{W_1}} \sum_{w \in N_1(v)\setminus\{u\}} x_w &\leqslant \sum_{v \in \overline{W_1}} \sum_{w \in N_1(v)\cap(W_1 \cup W_2)} x_w + \sum_{v \in \overline{W_1}} \sum_{w \in N_1(v)\cap(\overline{W_1} \cup \overline{W_2})} x_w \\ &\leqslant e(\overline{W_1}, W_1 \cup W_2)x_{u^*} + \big(2e(\overline{W_1}) + e(\overline{W_1}, \overline{W_2})\big)\eta x_{u^*} \\ &\leqslant e(\overline{W_1}, W_1 \cup W_2)x_{u^*} + 6\lambda\eta n x_{u^*}, \end{aligned} \tag{31}$$

where the last inequality follows from (28).

In the following, we shall evaluate $e(\overline{W_1}, W_1 \cup W_2)$. Let $\overline{W_1}'$ be the subset of $\overline{W_1}$ in which each vertex has at least λ neighbors in $W_1 \cup W_2$. If $|W_1 \cup W_2| \leqslant \lambda - 1$, then $|\overline{W_1}'| = 0$. If $|W_1 \cup W_2| \geqslant \lambda$, then we claim that $|\overline{W_1}'| < (\lambda+1)\binom{|W_1 \cup W_2|}{\lambda}$. Otherwise, since there are only $\binom{|W_1 \cup W_2|}{\lambda}$ options for all vertices in $\overline{W_1}'$ to choose a set of λ neighbors from $W_1 \cup W_2$, we can find λ vertices in $W_1 \cup W_2$ with at least $|\overline{W_1}'|/\binom{|W_1 \cup W_2|}{\lambda} \geqslant \lambda + 1$ common neighbors in $\overline{W_1}'$. Moreover, note that $u \notin W_1 \cup W_2$ and $\overline{W_1}' \subseteq \overline{W_1} \subseteq N_1(u)$. Hence, G contains a copy of $K_{\lambda+1,\lambda+1}$, and thus t vertex-disjoint 2ℓ-cycles, a contradiction. Therefore, we always have $|\overline{W_1}'| < (\lambda+1)\binom{|W_1 \cup W_2|}{\lambda} \leqslant (\lambda+1)\binom{|W|}{\lambda}$.

By Lemma 33, $|W|$ is constant. Now $|\overline{W_1}'|$ is also constant. Thus, $|\overline{W_1}'||W_1 \cup W_2| \leqslant 9\lambda\eta n$. Moreover, from the definition of $\overline{W_1}'$ we know $e(\overline{W_1} \setminus \overline{W_1}', W_1 \cup W_2) \leqslant (\lambda-1)|\overline{W_1} \setminus \overline{W_1}'|$. Thus

$$e(\overline{W_1}, W_1 \cup W_2) \leqslant e(\overline{W_1}', W_1 \cup W_2) + e(\overline{W_1} \setminus \overline{W_1}', W_1 \cup W_2) \leqslant 9\lambda\eta n + (\lambda-1)|N_1(u)|. \tag{32}$$

Back to (31), we obtain $\sum_{v \in \overline{W_1}} \sum_{w \in N_1(v)\setminus\{u\}} x_w \leqslant \big(15\lambda\eta n + (\lambda-1)|N_1(u)|\big)x_{u^*}$. Combining this with (29) and (30), we get that

$$\lambda n x_u \leqslant |N_1(u)|x_u + 20\lambda\eta n x_{u^*} + (\lambda-1)|N_1(u)|x_{u^*} \leqslant \big(20\lambda\eta n + \lambda|N_1(u)|\big)x_{u^*},$$

which yields $|N_1(u)| \geqslant \big(\frac{x_u}{x_{u^*}} - 20\eta\big)n$, as desired. □

Now, we define $W'' = \{u \in V(G) \mid x_u \geqslant 5000\lambda^4\eta x_{u^*}\}$. Recall that $\eta < \frac{1}{20000\lambda^5}$ and $W = \{u \in V(G) \mid x_u \geqslant \eta x_{u^*}\}$. Clearly, $u^* \in W''$ and $W'' \subseteq W$.

Lemma 35. *For every $v \in W''$, we have $x_v \geqslant (1 - \frac{1}{200\lambda^3})x_{u^*}$ and $d_G(v) \geqslant (1 - \frac{1}{100\lambda^3})n$. Moreover, we have $|W''| = \lambda$.*

Proof. Suppose to the contrary that there exists $v_0 \in W''$ with $x_{v_0} < (1 - \frac{1}{200\lambda^3})x_{u^*}$. We use W_i and $\overline{W_i}$ to denote $N_i(u^*) \cap W$ and $N_i(u^*) \setminus W$, respectively. We first prove that $|\overline{W_1} \cap N_1(v_0)| \geqslant 4000\lambda^4\eta n$. By Lemma 34, we have

$$|N_1(u^*)| \geqslant (1 - 20\eta)n \quad \text{and} \quad |N_1(v_0)| \geqslant (5000\lambda^4\eta - 20\eta)n,$$

as $x_{v_0} \geqslant 5000\lambda^4\eta n x_{u^*}$. Moreover, by Lemma 33, we have $|W| \leqslant \frac{128\lambda^3}{\eta^2} \leqslant 10\eta n$. Hence, $|\overline{W_1}| = |N_1(u^*) \setminus W| \geqslant (1 - 30\eta)n$, and so

$$\left|\overline{W_1} \cap N_1(v_0)\right| \geqslant \left|\overline{W_1}\right| + \left|N_1(v_0)\right| - n \geqslant (5000\lambda^4\eta - 50\eta)n > 4000\lambda^4\eta n. \tag{33}$$

In view of (33), v_0 has neighbors in $\overline{W_1}$. Then v_0 is of distance at most two from u^*, that is, $v_0 \in N_1(u^*) \cup N_2(u^*)$. Note that $v_0 \in W'' \subseteq W$. Thus, $v_0 \in W_1 \cup W_2$. Recall that $x_{v_0} < (1 - \frac{1}{200\lambda^3})x_{u^*}$. Now, setting $u = u^*$ in (29)-(31), we can observe that

$$\begin{aligned} \lambda n x_{u^*} &\leqslant |N_1(u^*)|x_{u^*} + 11\lambda\eta n x_{u^*} + e\big(\overline{W_1}, (W_1 \cup W_2) \setminus \{v_0\}\big)x_{u^*} + e\big(\overline{W_1}, \{v_0\}\big)x_{v_0} \\ &< |N_1(u^*)|x_{u^*} + 11\lambda\eta n x_{u^*} + e\big(\overline{W_1}, (W_1 \cup W_2)\big)x_{u^*} - e\big(\overline{W_1}, \{v_0\}\big)\frac{x_{u^*}}{200\lambda^3}, \end{aligned}$$

where $e(\overline{W_1}, W_1 \cup W_2) \leqslant 9\lambda\eta n + (\lambda - 1)|N_1(u^*)|$ by (32). Thus,

$$\lambda n \leqslant \lambda|N_1(u^*)| + 20\lambda\eta n - \frac{e\big(\overline{W_1}, \{v_0\}\big)}{200\lambda^3} < \lambda n + 20\lambda\eta n - \frac{e\big(\overline{W_1}, \{v_0\}\big)}{200\lambda^3}.$$

Consequently, $e\big(\overline{W_1}, \{v_0\}\big) < 4000\lambda^4\eta n$, contradicting (33). Thus $x_v \geqslant (1 - \frac{1}{200\lambda^3})x_{u^*}$ for $v \in W''$.

Recall that $\eta < \frac{1}{20000\lambda^5}$. Then by Lemma 34, we can see that for each $v \in W''$,

$$d_G(v) \geqslant \big(\frac{x_v}{x_{u^*}} - 20\eta\big)n \geqslant \Big(1 - \frac{1}{200\lambda^3} - 20\eta\Big)n \geqslant \Big(1 - \frac{1}{100\lambda^3}\Big)n.$$

It remains to show $|W''| = \lambda$. We first suppose that $|W''| \geqslant \lambda + 1$. Note that every $v \in W''$ has at most $\frac{n}{100\lambda^3}$ non-neighbors. It follows that any $\lambda + 1$ vertices in W'' have at least $n - \frac{(\lambda+1)n}{100\lambda^3} \geqslant \lambda + 1$ common neighbors. Thus, G contains $K_{\lambda+1,\lambda+1}$ as a subgraph. Recall that $\lambda = \ell t - 1$. Thus G also contains $tC_{2\ell}$, a contradiction. Therefore, $|W''| \leqslant \lambda$.

Next, suppose that $|W''| \leqslant \lambda - 1$. Since $u^* \in W'' \setminus (W_1 \cup W_2)$, we have $|W'' \cap (W_1 \cup W_2)| \leqslant \lambda - 2$, and so $e(\overline{W_1}, (W_1 \cup W_2)) \cap W'' \leqslant (\lambda - 2)n$. On the other hand, setting $u = u^*$ and $W_0 = N_2(u^*)$ in Lemma 28, we get $e(N_1(u^*)) + e(N_1(u^*), N_2(u^*)) \leqslant (4\lambda - 2)n$, and thus

$$\begin{aligned} e\big(\overline{W_1}, (W_1 \cup W_2) \setminus W''\big) &\leqslant e\big(\overline{W_1}, W_1\big) + e\big(\overline{W_1}, W_2\big) \\ &\leqslant e(N_1(u^*)) + e(N_1(u^*), N_2(u^*)) \leqslant (4\lambda - 2)n. \end{aligned}$$

Recall that $x_w \geqslant 5000\lambda^4\eta x_{u^*}$ if and only if $w \in W''$. Now, setting $u = u^*$, $x_w \leqslant x_{u^*}$ for $w \in (W_1 \cup W_2) \cap W''$ and $x_w < 5000\lambda^4\eta x_{u^*}$ for $w \in (W_1 \cup W_2) \setminus W''$ in (29)-(31), we obtain

$$\begin{aligned}\lambda n x_{u^*} &\leqslant \Big(|N_1(u^*)|+11\lambda\eta n+e\big(\overline{W_1},(W_1\cup W_2)\cap W''\big)+e\big(\overline{W_1},(W_1\cup W_2)\setminus W''\big)5000\lambda^4\eta\Big)x_{u^*}\\ &\leqslant \Big(n+11\lambda\eta n+(\lambda-2)n+5000\lambda^4\eta(4\lambda-2)n\Big)x_{u^*}\\ &\leqslant \lambda n x_{u^*},\end{aligned}$$

as $\eta < \frac{1}{20000\lambda^5}$. This gives a contradiction. Therefore, $|W''| = \lambda$. □

In the following, we complete the proof of Theorem 3.

Proof. By Lemma 35, we see that $|W''| = \lambda = \ell t - 1$ and every vertex in W'' has at most $\frac{n}{100\lambda^3}$ non-neighbors. Now, let U be the subset of $V(G)\setminus W''$ in which every vertex is a non-neighbor of some vertex in W'' and $U' = V(G) \setminus (W'' \cup U)$. Then, $G[W'', U'] \cong K_{|W''|,|U'|}$. Note that $|U| \leqslant |W''|\frac{n}{100\lambda^3} = \frac{n}{100\lambda^2}$, and thus $|U'| \geqslant n - \lambda - \frac{n}{100\lambda^2} \geqslant \frac{n}{2}$.

We will see that $U = \varnothing$. Suppose to the contrary that $U \neq \varnothing$. Given $u \in U$ arbitrarily. We first prove that u has at most one neighbor in U'. Otherwise, u has two neighbors $u_1, u_2 \in U'$. Assume that $\{u_1, u_2, \ldots, u_\ell\} \subseteq U'$ and $\{w_1, w_2, \ldots, w_{\ell-1}\} \subseteq W''$. By the definition of U', we can see that

$$C^1 := u_1uu_2w_1u_3\ldots u_{\ell-1}w_{\ell-2}u_\ell w_{\ell-1}u_1$$

is a 2ℓ-cycle in G. Clearly, $|W'' \setminus V(C^1)| = (t-1)\ell$ and $|U' \setminus V(C^1)| = |U'| - \ell \geqslant (t-1)\ell$ as n is sufficiently large. This implies that $G[W'' \setminus V(C^1), U' \setminus V(C^1)]$ contains a copy of $K_{(t-1)\ell,(t-1)\ell}$, and hence contains $t-1$ vertex-disjoint 2ℓ-cycles, say $C^2, \ldots, C^t$. Thus, G contains t vertex-disjoint 2ℓ-cycles $C^1, C^2, \ldots, C^t$, which gives a contradiction. Hence, u has at most one neighbor in U'. Moreover, by the definition of U, $|N_1(u) \cap W''| \leqslant |W''| - 1 = \lambda - 1$. It follows that

$$\sum_{w\in N_1(u)\cap(W''\cup U')} x_w = \sum_{w\in N_1(u)\cap W''} x_w + \sum_{w\in N_1(u)\cap U'} x_w \leqslant (\lambda-1)x_{u^*} + 5000\lambda^4\eta x_{u^*}. \tag{34}$$

We now claim that $\rho(G)x_u \geqslant (\lambda - \frac{1}{200\lambda^2})x_{u^*}$. Otherwise, let G^* be the graph obtained from G by deleting all edges incident to u and joining u to all vertices in W''. Note that $|U'| \geqslant \frac{n}{2}$ and $N_{G^*}(u) \subseteq N_{G^*}(v)$ for any $v \in U'$. Then G^* is $tC_{2\ell}$-free (otherwise, $G^* - \{u\}$ contains $tC_{2\ell}$, and thus $G - \{u\}$ too, a contradiction). Moreover,

$$\rho(G^*) - \rho(G) \geqslant X^T\big(A(G^*) - A(G)\big)X = 2x_u\Big(\sum_{w\in W''} x_w - \sum_{w\in N_1(u)} x_w\Big).$$

Note that $\sum_{w\in W''} x_w \geqslant |W''|(1 - \frac{1}{200\lambda^3})x_{u^*} = (\lambda - \frac{1}{200\lambda^2})x_{u^*}$ by Lemma 35, but

$$\sum_{w\in N_1(u)} x_w = \rho(G)x_u < (\lambda - \frac{1}{200\lambda^2})x_{u^*}$$

by assumption. Thus, $\rho(G^*) > \rho(G)$, a contradiction.

Now we have

$$\Big(\lambda - \frac{1}{200\lambda^2}\Big)x_{u^*} \leqslant \rho(G)x_u = \sum_{w\in N_1(u)\cap(W''\cup U')} x_w + \sum_{w\in N_1(u)\cap U} x_w.$$

Combining (34) gives

$$\frac{\sum_{w\in N_1(u)\cap U} x_w}{\rho(G)x_u} \geqslant \frac{(\lambda - \frac{1}{200\lambda^2})x_{u^*} - (\lambda - 1 + 5000\lambda^4\eta)x_{u^*}}{(\lambda - \frac{1}{200\lambda^2})x_{u^*}} \geqslant \frac{4}{5\lambda},$$

as $\eta < \frac{1}{20000\lambda^5} < \frac{1}{5000\lambda^4}(\frac{1}{5} - \frac{1}{200\lambda^2} + \frac{1}{250\lambda^3})$. Thus, $\sum_{w\in N_1(u)\cap U} x_w \geqslant \frac{4}{5\lambda}\rho(G)x_u$.

Now consider the matrix $A' = A(G[U])$ and the vector $X' = X|_U$ (the restriction of X to U). We can observe that

$$(A'X')_u = \sum_{w\in N_1(u)\cap U} x_w \geqslant \frac{4}{5\lambda}\rho(G)x_u$$

for each $u \in U$. Since X is a positive unit eigenvector of G, X' is a positive vector and thus $A'X' \geqslant \frac{4}{5\lambda}\rho(G)X'$ entrywise. Moreover, $\rho(G) \geqslant \sqrt{\lambda n}$ by Lemma 29. Hence,

$$\rho(G[U]) \geqslant \frac{X'^T A' X'}{X'^T X'} \geqslant \frac{4}{5\lambda}\rho(G) \geqslant \frac{4}{5}\sqrt{\frac{n}{\lambda}},$$

which also implies that $|U| = \Omega(\sqrt{n})$. Since $G[U]$ is $tC_{2\ell}$-free, we have $\rho(G[U]) \leqslant \sqrt{6\lambda|U|}$ by Lemma 29. Recall that $|U| \leqslant \frac{n}{100\lambda^2}$. It follows that

$$\rho(G[U]) \leqslant \sqrt{\frac{6\lambda n}{100\lambda^2}} < \frac{4}{5}\sqrt{\frac{n}{\lambda}},$$

a contradiction. Therefore, $U = \varnothing$.

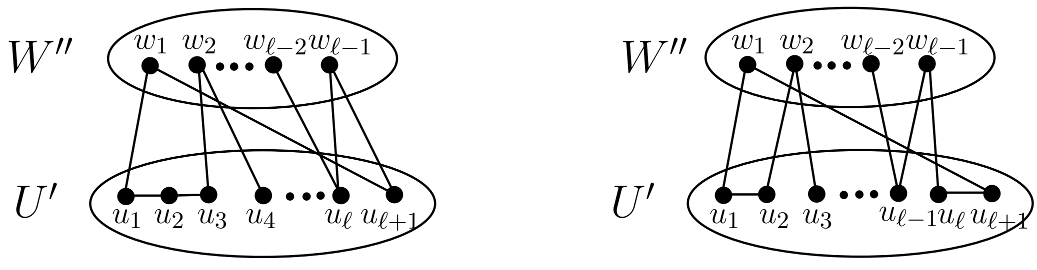

Figure 1: A special 2ℓ-cycle in G.

Now we have $V(G) = W'' \cup U'$ and $G[W'', U'] \cong K_{\lambda,n-\lambda}$. In the following, we consider two cases of Theorem 3.

(i) $\ell = 2$. Recall that P_k denotes a path of order k. Since $|W''| = \lambda = \ell t - 1 = 2t - 1$, we can see that $G[U']$ is P_3-free (otherwise, we can find tC_4 in G). Thus, $G[U']$ consists of independent edges and isolated vertices. Since G is extremal with respect to $spex(n, tC_4)$, we know that G is edge-maximal, which implies that W'' is a $(2t-1)$-clique and $G \cong S^{++}_{n,2t-1}$.

(ii) $\ell \geqslant 3$. Since $|W''| = \lambda = \ell t - 1$, we will see that $e(U') \leqslant 1$. Otherwise, whether $G[U']$ contains a P_3 or two independent edges, we can always find t vertex-disjoint copies of $C_{2\ell}$, which consist of $t-1$ 2ℓ-cycles in $G[W'', U']$, and a special 2ℓ-cycle (see Figure 1). Since G is edge-maximal, we similarly have $G \cong S^{+}_{n,2t-1}$.

This completes the proof. □

Acknowledgements

The authors would like to show great gratitude to anonymous referees for their helpful comments which largely improve the quality of this paper.

Mingqing Zhai was supported by the National Natural Science Foundation of China (No. 12171066) and Major Natural Science Research Project of Universities in Anhui Province (No. 2022AH040151). Huiqiu Lin was supported by the National Natural Science Foundation of China (Nos. 12271162, 12326372), Natural Science Foundation of Shanghai (Nos. 22ZR1416300 and 23JC1401500) and The Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning (No. TP2022031).

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