The algebraic multiplicity of the spectral radius of a uniform hypertree

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Submitted: Jul 20, 2023; Accepted: Sep 8, 2024; Published: Oct 18, 2024

Abstract

It is well-known that the spectral radius of a connected uniform hypergraph is an eigenvalue of the hypergraph. However, its algebraic multiplicity remains unknown. In this paper, we use the Poisson Formula and the matching polynomials to give the algebraic multiplicity of the spectral radius of a uniform hypertree.

Mathematics Subject Classifications: 05C50, 05C65

1 Introduction

From the Perron-Frobenius Theorem (for matrices), it is known that the spectral radius of a connected graph is an eigenvalue of the graph with the algebraic multiplicity 1. Part of the Perron-Frobenius Theorem has been generalized to tensors, in particular, it is known that the spectral radius of a connected uniform hypergraph is one of its eigenvalues [2]. However, it is unknown what its algebraic multiplicity is. In this paper, we aim to give the algebraic multiplicity of the spectral radius of a uniform hypertree.

The characteristic polynomial of a hypergraph is the characteristic polynomial of its adjacency tensor. The Poisson Formula, given in [8, Chapter 3, Theorem 3.4], is a useful method for computing the characteristic polynomials of hypergraphs, particularly hypertrees [1, 3, 7]. Cooper and Dutle [7] gave the characteristic polynomial of the (so-called) "all-one" tensors and the 3-uniform hyperstar. Bao et al. [1] provided a method for computing the characteristic polynomials of hypergraphs with cut vertices, and gave the characteristic polynomial of the k-uniform hyperstar. The authors gave a reduction formula for the characteristic polynomial of a uniform hypergraphs with pendant edges in [3]. And they used the reduction formula iteratively to derive the characteristic polynomial of uniform loose hyperpaths [3].

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The algebraic multiplicity of an eigenvalue refers to the number of times it appears as a root of the characteristic polynomial. The k-power hypergraph is the k-uniform hypergraph that is obtained by adding $k-2$ new vertices to each edge of a graph for $k \geqslant 3$. The algebraic multiplicity of the spectral radius of a power hypergraph was given by the spectral moments in [4]. This paper employs the matching polynomial to give the algebraic multiplicity of the spectral radius of a uniform hypertree.

The matching polynomial of a tree coincides with its characteristic polynomial, as shown in Corollary 2.1 of [10]. However, this correlation does not directly apply to k-uniform hypertrees with $k \geqslant 3$. Zhang et al. showed that the set of roots of the matching polynomial of a k-uniform hypertree is a sub-set of its spectrum [17]. Based on Zhang et al.'s results, Clark and Cooper determined all eigenvalues (without multiplicity) of a k-uniform hypertree T by roots of the matching polynomials of all sub-hypertrees of T [5]. Su et al. used the matching polynomials to investigate a perturbation on the spectral radius of k-uniform hypertrees [16]. Li et al. presented a complete demonstration of the relationship between the characteristic polynomials and matching polynomials of a k-uniform hypertree [13].

The rest of this paper is organized as follows: In Section 2, we present some notation and lemmas about the Poisson Formula for resultants (Section 2.1), the characteristic polynomial of a hypergraph (Section 2.2), and the matching polynomial of a hypergraph (Section 2.3). In Section 3, we apply the Poisson Formula to give the algebraic multiplicity of the spectral radius of a uniform hypertree.

2 Preliminaries

In this section, we present some basic notation and auxiliary lemmas regarding the Poisson Formula for resultants, the characteristic polynomial and matching polynomial of a hypergraph.

2.1 Resultants

For a positive integer n, let $[n] = \{1, \ldots, n\}$. Let $F_1, F_2, \ldots, F_n$ be homogeneous polynomials over an algebraically closed field $\mathbb{K}$ in variables $x_1, \ldots, x_n$, where the degree of F_i is d_i for $i \in [n]$. Denote

$$\begin{aligned} \overline{F_i} &= \overline{F_i}(x_1, x_2, \ldots, x_{n-1}) = F_i(x_1, x_2, \ldots, x_{n-1}, 0), \\ f_i &= f_i(x_1, x_2, \ldots, x_{n-1}) = F_i(x_1, x_2, \ldots, x_{n-1}, 1). \end{aligned}$$

Observe that $\overline{F_i}$'s are still homogeneous, but f_i's are not homogeneous in general for $i \in [n]$.

Let $I = \langle f_1, \ldots, f_{n-1} \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$ be the ideal generated by f_i for $i = 1, \ldots, n-1$. It implies that the set of solutions of the system $f_1 = f_2 = \cdots = f_{n-1} = 0$ is the variety $\mathcal{V}(I)$. Given a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$, define a linear map m_f from $\mathbb{K}[x_1, \ldots, x_n]/I$ to itself using the multiplication. More precisely, the polynomial f gives the coset $[f] \in$

$\mathbb{K}[x_1,\ldots,x_n]/I$, and the linear map m_f is defined by the rules: if $[g]\in\mathbb{K}[x_1,\ldots,x_n]/I$, then

$$m_f([g]) = [f]\cdot[g] = [fg] \in \mathbb{K}[x_1,\ldots,x_n]/I.$$

The ensuing statement of the Poisson Formula for resultants follows from [8, Chapter 3, Theorem 3.4], which is different from the original one in [11, Proposition 2.7].

Lemma 1. *[8, Poisson Formula for resultants] If* $\mathrm{Res}(\overline{F_1},\ldots,\overline{F_{n-1}})\neq 0$, *then the quotient ring* $A=\mathbb{K}[x_1,\ldots,x_n]/\langle f_1,\ldots,f_{n-1}\rangle$ *has dimension* $d_1\cdots d_{n-1}$ *as a vector space over* $\mathbb{K}$, *and*

$$\mathrm{Res}(F_1,\ldots,F_n) = \mathrm{Res}(\overline{F_1},\ldots,\overline{F_{n-1}})^{d_n}\det(m_{f_n}: A\to A), \tag{1}$$

where $m_{f_n}: A\to A$ *is the linear map given by multiplication by* f_n.

The characteristic polynomial of the linear map m_f can be expressed in terms of the points of $\mathcal{V}(I)$ as follows.

Proposition 2. *[8, Chapter 4, Proposition 2.7] Let* $\mathbb{K}$ *be an algebraically closed field and let* I *be a zero-dimensional ideal in* $\mathbb{K}[x_1,\ldots,x_n]$. *If* $f\in\mathbb{K}[x_1,\ldots,x_n]$, *then*

$$\det(\lambda I - m_f) = \prod_{\mathbf{p}\in\mathcal{V}(I)} (\lambda - f(\mathbf{p}))^{m(\mathbf{p})},$$

where $m(\mathbf{p})$ *is the multiplicity* [1] *of the point* $\mathbf{p}\in\mathcal{V}(I)$.

Proposition 2 implies that (1) can be rewritten as

$$\mathrm{Res}(F_1,\ldots,F_n) = \mathrm{Res}(\overline{F_1},\ldots,\overline{F_{n-1}})^{d_n}\prod_{\mathbf{p}\in\mathcal{V}} f_n(\mathbf{p})^{m(\mathbf{p})}, \tag{2}$$

where $\mathcal{V}=\mathcal{V}(f_1,\ldots,f_{n-1})$ is the affine variety defined by the polynomials f_i for all $i\in[n-1]$. The formula (2) is also named Poisson Formula for resultants in the monograph [9, Chapter 13, Theorem 1.3].

2.2 The characteristic polynomial of a hypergraph

A hypergraph $H=(V,E)$ is called *k-uniform* if each edge of H contains exactly k vertices. Similar to the relation between graphs and matrices, there is a natural correspondence between uniform hypergraphs and tensors. For a k-uniform hypergraph H with n vertices, its *(normalized) adjacency tensor* $A_H=(a_{i_1i_2\ldots i_k})$ is a k-order n-dimensional tensor [6], where

$$a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1,i_2,\ldots,i_k\}\in E,\\ 0, & \text{otherwise.}\end{cases}$$

When $k=2$, A_H is the usual adjacency matrix of the graph H.

[1] The multiplicity is sometimes called the local intersection multiplicity, and its definition can be found in [8, Chapter 4, §2].

Let $H=(V,E)$ be a k-uniform hypergraph with $V=[n]$. Given a hyperedge $e \in E$, and a vector $\mathbf{x}=(x_1,x_2,\dots,x_n)^\top \in \mathbb{C}^n$, let $\mathbf{x}_e=\prod_{v\in e} x_v$. Let $E_v=\{e\in E : v\in e\}$ denote the set of hyperedges containing the vertex v. If there exists a nonzero vector $\mathbf{x}$ such that for each $i\in[n]$,

$$\lambda x_i^{k-1}=\sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2}\cdots x_{i_k},$$

or equivalently, for each $v\in V$,

$$\lambda x_v^{k-1}=\sum_{e\in E_v} \mathbf{x}_{e\setminus\{v\}},$$

then λ is called an *eigenvalue* of H and $\mathbf{x}$ is an *eigenvector* of H corresponding to λ ([14, 15]).

For each $v\in V$, define

$$F_v=F_v(x_1,x_2,\dots,x_n)=\lambda x_v^{k-1}-\sum_{e\in E_v}\mathbf{x}_{e\setminus\{v\}}.$$

The polynomial

$$\phi_H(\lambda)\equiv \mathrm{Res}(F_v : v\in V)$$

in the indeterminant λ is called the *characteristic polynomial* of H. For a fixed vertex $u\in V$, let

$$f_v=F_v|_{x_u=1}=F_v(x_w : w\in V, x_u=1).$$

Let $\mathcal{V}=\mathcal{V}(f_v : v\in V\setminus\{u\})$ be the affine variety defined by the polynomials f_v for all $v\in V\setminus\{u\}$. Let $H-u$ denote the hypergraph obtained from H by removing the vertex u and all hyperedges containing u. Applying the Poisson Formula to the characteristic polynomial $\phi_H(\lambda)$, a reduction formula for $\phi_H(\lambda)$ is derived as follows.

Lemma 3. *[3, Formula (1)] Let $H=(V,E)$ be a k-uniform hypergraph with the vertex u. Then the characteristic polynomial*

$$\phi_H(\lambda)=\phi_{H-u}(\lambda)^{k-1}\prod_{\mathbf{p}\in\mathcal{V}}(\lambda-\sum_{e\in E_u}\mathbf{p}_{e\setminus\{u\}})^{m(\mathbf{p})},$$

where $m(\mathbf{p})$ is the multiplicity of $\mathbf{p}$ in $\mathcal{V}=\mathcal{V}(f_v : v\in V\setminus\{u\})$.

The following is the definition of cut vertices of hypergraphs from [1].

Definition 4. [1] Let $k\geqslant 3$, and let $H=(V,E)$ be a k-uniform connected hypergraph and $u\in V$. Denote $E_{\widetilde{u}}=\{e\setminus\{u\} : e\in E_u\}$, and note that the hyperedge $\widetilde{e}=e\setminus\{u\}\in E_{\widetilde{u}}$ has $k-1$ vertices. Deleting the vertex u and changing e to $\widetilde{e}$ for every $e\in E_u$, it can get a non-uniform hypergraph $\widetilde{H}=(\widetilde{V},\widetilde{E})$ with $\widetilde{V}=V\setminus u$ and $\widetilde{E}=(E\setminus E_u)\cup E_{\widetilde{u}}$. The vertex u is called a cut vertex if $\widetilde{H}$ is not connected.

Even if $H-u$ is not connected, the vertex u may not necessarily be a cut vertex of a connected hypergraph H. For instance, when $k \geqslant 3$, the vertex u with degree one is not a cut vertex in a k-uniform hypertree T, even if $T-u$ is not connected. Suppose that $\widetilde{H}_1 = (\widetilde{V}_1, \widetilde{E}_1), \ldots, \widetilde{H}_n = (\widetilde{V}_n, \widetilde{E}_n)$ are connected components of $\widetilde{H}$ defined in Definition 4. For each $i \in [n]$, denote the induced sub-hypergraph of H on $\widetilde{V}_i \cup \{u\}$ by $\widehat{H}_i$, then we call $\widehat{H}_i$ a *branch* of H associated with u. It implies that H can be obtained by coalescing $\widehat{H}_1, \ldots, \widehat{H}_n$ to the vertex u. Recall that the variety $\mathcal{V}(H) = \mathcal{V}(f_v : v \in V \setminus \{u\})$ is defined by the polynomials

$$f_v = \lambda x_v^{k-1} - \sum_{e \in E_v(H)} \mathbf{x}_{e\setminus\{v\}}|_{x_u=1}$$

for all $v \in V \setminus \{u\}$. It is shown that if u is a cut vertex of H, then

$$\begin{aligned}\mathcal{V}(H) &= \mathcal{V}(f_v : v \in V \setminus \{u\}) \\ &= \bigoplus_{i=1}^{n} \mathcal{V}(f_v : v \in V(\widehat{H}_i) \setminus \{u\}) \\ &= \bigoplus_{i=1}^{n} \mathcal{V}(\widehat{H}_i)\end{aligned}$$

in [1]. Bao et al. subsequently gave a reduction formula for the characteristic polynomials of hypergraphs with cut vertices in terms of the linear map [1, Corollary 3.2]. For the convenience of use in Section 3, we restate their formula following a similar approach as the reformulation of (1) to (2).

Lemma 5. *[1, Corollary 3.2] Let H be a k-uniform hypergraph with a cut vertex u and branches $\widehat{H}_1, \cdots, \widehat{H}_n$. Denote $\mathcal{V}^{(i)} = \mathcal{V}(\widehat{H}_i) = \mathcal{V}(f_v : v \in V \setminus \{u\})$ and $E_u^{(i)} = E_u(\widehat{H}_i) \bigcap E_u(H)$. Then*

$$\phi_H(\lambda) = \phi_{H-u}(\lambda)^{k-1} \prod_{\substack{i\in[n] \\ \mathbf{q}^{(i)} \in \mathcal{V}^{(i)}}} (\lambda - \sum_{i=1}^{n} \sum_{e \in E_u^{(i)}} \mathbf{q}^{(i)}_{e\setminus\{u\}})^{\prod_{i=1}^{n} m(\mathbf{q}^{(i)})},$$

where $m(\mathbf{q}^{(i)})$ is the multiplicity of $\mathbf{q}^{(i)}$ in $\mathcal{V}^{(i)}$ for each $i \in [n]$.

When one of the branches is the one-edge hypergraph, it implies that H has a pendant edge incident to u. A more explicit reduction formula for hypergraphs with pendant edges is shown as follows.

Lemma 6. *[3, Theorem 3.2] Let H be a k-uniform hypergraph with a pendant edge incident to the non-pendent vertex u, and we define $\widehat{H}$ as the k-uniform hypergraph obtained by removing the pendant edge and pendent vertices on it from H. Then*

$$\begin{aligned}&\phi_H(\lambda) = \\ &\phi_{H-u}(\lambda)^{k-1} \prod_{\mathbf{p}\in\mathcal{V}(\widehat{H})} (\lambda - \sum_{e\in E_u(\widehat{H})} \mathbf{p}_{e\setminus\{u\}})^{m(\mathbf{p})K_1} \prod_{\mathbf{p}\in\mathcal{V}(\widehat{H})} (\lambda - \frac{1}{\lambda^{k-1}} - \sum_{e\in E_u(\widehat{H})} \mathbf{p}_{e\setminus\{u\}})^{m(\mathbf{p})K_2},\end{aligned}$$

where $K_1 = (k-1)^{k-1} - k^{k-2}$ and $K_2 = k^{k-2}$.

2.3 The matching polynomial of a hypergraph

The *matching* of a k-uniform hypergraph $H=(V,E)$ is a set of the pairwise non-adjacent edges in E. The t-matching is a matching consisting of t edges and the number of t-matching of H is denoted by $m_t(H)$. Set $m_0(H)=1$. The *matching polynomial* of H is defined in [16] as

$$\varphi_H(\lambda)=\sum_{t\geqslant 0}(-1)^t m_t(H)x^{|V|-tk}.$$

Some classical results on the matching polynomials of a graph are extended to the hypergraph case as follows.

Lemma 7. *[16, Theorem 7] Let $H=(V,E)$ and G be two k-uniform hypergraphs. We use $H+G$ to denote the disjoint union of H and G, and use $H-e$ to denote the induced sub-hypergraph of H on the $V\setminus e$ for $e\in E$. Then*

1. $\varphi_{H+G}(\lambda)=\varphi_H(\lambda)\varphi_G(\lambda)$.
2. $\varphi_H(\lambda)=\lambda\varphi_{H-u}(\lambda)-\sum_{e\in E_u}\varphi_{H-e}(\lambda)$.

Zhang et al. [17] showed that the set of roots of the matching polynomial of a k-uniform hypertree is a sub-set of its spectrum. And this result was extended by Clark and Cooper [5] as follows.

Lemma 8. [2]*[5, Theorem 2] Let T be a k-uniform hypertree for $k\geqslant 3$. Then λ is an eigenvalue of T if and only if there exists a sub-tree $\widehat{T}$ of T such that λ is a root of the matching polynomial $\varphi_{\widehat{T}}(\lambda)$.*

Lemma 9. *[16, Proposition 8] Let $k\geqslant 3$, and let T be a k-uniform hypertree. Then the spectral radius $\rho(T)$ of T is a simple root of the matching polynomial $\varphi_T(\lambda)$.*

3 Main results

The algebraic multiplicity of the spectral radius of a k-uniform hypertree T is determined in this section.

For a hypergraph $H=(V,E)$ with the vertex u, recall that $F_v=\lambda x_v^{k-1}-\sum_{e\in E_v}\mathbf{x}_{e\setminus\{v\}}$ and $f_v=F_v|_{x_u=1}$ for all $v\in V$. Let $\mathcal{V}_u(H)=\mathcal{V}(f_v:v\in V\setminus\{u\})$ be the affine variety defined by the polynomials f_v for all $v\in V\setminus\{u\}$.

Lemma 10. *Let $T=(V,E)$ be a uniform hypertree with the vertex u. If $\mathbf{p}$ is a point in $\mathcal{V}_u(T)=\mathcal{V}(f_v:v\in V\setminus\{u\})$ with all coordinates nonzero, then*

$$\mathbf{p}_{e\setminus\{u\}}=\frac{\varphi_{T-e}(\lambda)}{\varphi_{T-u}(\lambda)}$$

[2]The definition of the matching polynomial of a hypergraph varies between [5] and [16]. However, as shown in [16], Lemma 8 is also applicable to the definition in [16].

for each $e \in E_u$. Moreover, we have

$$\lambda - \sum_{e \in E_u} \mathbf{p}_{e \setminus \{u\}} = \frac{\varphi_T(\lambda)}{\varphi_{T-u}(\lambda)}.$$

Proof. We prove the result by the induction on $|E|$.

When $|E| = 1$, it is shown that $\mathbf{p}_{e\setminus\{u\}} = \frac{1}{\lambda^{k-1}}$ in the [3, Equation (5)], which implies that

$$\mathbf{p}_{e\setminus\{u\}} = \frac{\varphi_{T-e}(\lambda)}{\varphi_{T-u}(\lambda)},$$

so the assertion holds.

Assuming the statement holds for any $|E| \leqslant m$, we consider the case $|E| = m + 1$.

When u is a cut vertex of T. Note that there are d_u $(= |E_u|)$ branches of the hypertree T associated with u, and each $e \in E_u$ belongs to a distinct branch. Suppose that $\widehat{T}_i$ are branches of T with $e_i \in E_u$ for all $i \in [d_u]$. Let $\mathbf{q}^{(i)}$ be a point in $\mathcal{V}_u(\widehat{T}_i) = \mathcal{V}(f_v : v \in V(\widehat{T}_i)\setminus\{u\})$ with all coordinates nonzero. Since $|E(\widehat{T}_i)| \leqslant m$, by the induction hypothesis, we have

$$\begin{aligned} \mathbf{q}^{(i)}_{e_i\setminus\{u\}} &= \frac{\varphi_{\widehat{T}_i-e_i}(\lambda)}{\varphi_{\widehat{T}_i-u}(\lambda)} \\ &= \frac{\varphi_{\widehat{T}_i-e_i}(\lambda)\prod_{\substack{j\in[d_u]\\ i\neq j}}\varphi_{\widehat{T}_j-u}(\lambda)}{\prod_{j\in[d_u]}\varphi_{\widehat{T}_j-u}(\lambda)}. \end{aligned} \tag{3}$$

Observe that $T - e_i$ is the disjoint union of $\widehat{T}_i - e_i$ and $\widehat{T}_j - u$ for all $i \neq j$, and $T - u$ is the disjoint union of $\widehat{T}_j - u$ for all $j \in [d_u]$. From Lemma 7 (1), we have

$$\varphi_{T-e_i}(\lambda) = \varphi_{\widehat{T}_i-e_i}(\lambda) \prod_{\substack{j\in[d_u]\\ i\neq j}} \varphi_{\widehat{T}_j-u}(\lambda)$$

and

$$\varphi_{T-u}(\lambda) = \prod_{j\in[d_u]} \varphi_{\widehat{T}_j-u}(\lambda),$$

which implies that $\mathbf{q}^{(i)}_{e_i\setminus\{u\}} = \frac{\varphi_{T-e_i}(\lambda)}{\varphi_{T-u}(\lambda)}$. For $\mathbf{p} \in \mathcal{V}_u(T)$, note that $\mathbf{p}_{e_i\setminus\{u\}} = \mathbf{q}^{(i)}_{e_i\setminus\{u\}}$, then we get

$$\mathbf{p}_{e_i\setminus\{u\}} = \frac{\varphi_{T-e_i}(\lambda)}{\varphi_{T-u}(\lambda)}.$$

When u is not a cut vertex of T. The degree of u is one and we set the hyperedge containing u as $e_0 = \{v_1, v_2, \ldots, v_{k-1}, v_k = u\}$. Let $T \setminus e_0$ denote the hypergraph with $V(T \setminus e_0) = V$ and $E(T \setminus e_0) = E \setminus \{e_0\}$. We observe that $T \setminus e_0$ has k connected components and we use $\widehat{T}^*_t = (\widehat{V}^*_t, \widehat{E}^*_t)$ to denote the connected component containing v_t for each $t \in [k]$.

Recall that $F_v = F_v(x_w : w \in V) = \lambda x_v^{k-1} - \sum_{e\in E_v(T)} \mathbf{x}_{e\setminus\{v\}}$ and $f_v = F_v|_{x_u=1}$ for $v \in V$. For all $t \in [k-1]$ and any $v \in \widehat{V}_t^* \setminus \{v_t\}$, note that $f_v = f_v(x_w : w \in \widehat{V}_t^*)$ is homogeneous. If $\mathbf{p} = (\mathbf{p}_i) \in \mathcal{V}_u(T)$ has all coordinates nonzero, we have

$$f_v(\mathbf{p}) = f_v(\mathbf{p}_w : w \in \widehat{V}_t^*) = f_v(\mathbf{p}_w/\mathbf{p}_{v_t} : w \in \widehat{V}_t^*) = 0. \tag{4}$$

Fix a $t \in [k-1]$. Let $\widehat{F}_v^* = \widehat{F}_v^*(x_w : w \in \widehat{V}_t^*) = \lambda x_v^{k-1} - \sum_{e\in E_v(\widehat{T}_t^*)} \mathbf{x}_{e\setminus\{v\}}$ and $\widehat{f}_v^* = \widehat{F}_v^*|_{x_{v_t}=1}$. For all $v \in \widehat{V}_t^* \setminus \{v_t\}$, note that $\widehat{F}_v^* = f_v$. Then we have $\widehat{f}_v^* = \widehat{F}_v^*|_{x_{v_t}=1} = f_v|_{x_{v_t}=1}$. Set $\mathbf{q}_w = \mathbf{p}_w/\mathbf{p}_{v_t}$ for $w \in \widehat{V}_t^*$ and note that $\mathbf{q}_{v_t} = 1$. By (4), it follows that

$$\widehat{f}_v^*(\mathbf{q}_w : w \in \widehat{V}_t^*) = \widehat{F}_v^*(\mathbf{q}_w : w \in \widehat{V}_t^*)|_{\mathbf{q}_{v_t}=1} = f_v(\mathbf{q}_w : w \in \widehat{V}_t^*)|_{\mathbf{q}_{v_t}=1} = 0$$

for all $v \in \widehat{V}_t^* \setminus \{v_t\}$. Let the vector $\mathbf{q} = (\mathbf{q}_w)$ for $w \in \widehat{V}_t^* \setminus \{v_t\}$. Indeed, $\mathbf{q}$ is a point in the variety $\mathcal{V}_{v_t}(\widehat{T}_t^*)$ defined by the polynomials $\widehat{f}_v^*$ for all $v \in \widehat{V}_t^* \setminus \{v_t\}$. From the induction hypothesis, we have

$$\mathbf{q}_{e\setminus\{v_t\}} = \frac{\mathbf{p}_{e\setminus\{v_t\}}}{\mathbf{p}_{v_t}^{k-1}} = \frac{\varphi_{\widehat{T}_t^*-e}(\lambda)}{\varphi_{\widehat{T}_t^*-v_t}(\lambda)} \tag{5}$$

for each $e \in E_{v_t}(\widehat{T}_t^*)$. By (5), it follows that

$$\mathbf{p}_{e\setminus\{v_t\}} = \frac{\varphi_{\widehat{T}_t^*-e}(\lambda)}{\varphi_{\widehat{T}_t^*-v_t}(\lambda)} \mathbf{p}_{v_t}^{k-1}$$

for $\mathbf{p} \in \mathcal{V}_u(T)$ which only has non-zero coordinates. Since

$$f_{v_t}(\mathbf{p}) = \lambda \mathbf{p}_{v_t}^{k-1} - \sum_{e\in E_{v_t}(\widehat{T}_t^*)} \mathbf{p}_{e\setminus\{v_t\}} - \mathbf{p}_{e_0\setminus\{v_t,u\}} = 0,$$

we have

$$\begin{aligned}\mathbf{p}_{e_0\setminus\{v_t,u\}} &= \left(\lambda - \sum_{e\in E_{v_t}(\widehat{T}_t^*)} \frac{\varphi_{\widehat{T}_t^*-e}(\lambda)}{\varphi_{\widehat{T}_t^*-v_t}(\lambda)}\right) \mathbf{p}_{v_t}^{k-1} \\ &= \frac{\varphi_{\widehat{T}_t^*}(\lambda)}{\varphi_{\widehat{T}_t^*-v_t}(\lambda)} \mathbf{p}_{v_t}^{k-1}\end{aligned}$$

from Lemma 7 (2).

Combining these equations for all $t \in [k-1]$, we get

$$\prod_{t=1}^{k-1} \mathbf{p}_{e_0\setminus\{v_t,u\}} = \prod_{t=1}^{k-1} \frac{\varphi_{\widehat{T}_t^*}(\lambda)}{\varphi_{\widehat{T}_t^*-v_t}(\lambda)} \mathbf{p}_{v_t}^{k-1},$$

which implies that

$$\mathbf{p}_{e_0\setminus\{u\}} = \prod_{t=1}^{k-1} \frac{\varphi_{\widehat{T}_t^*-v_t}(\lambda)}{\varphi_{\widehat{T}_t^*}(\lambda)}.$$

Note that $T-e_0$ is the disjoint union of $\widehat{T}_t^* - v_t$ for all $t \in [k-1]$, and $T-u$ is the disjoint union of $\widehat{T}_t^*$ for all $t \in [k-1]$. From Lemma 7 (1), we have $\mathbf{p}_{e_0\setminus\{u\}} = \frac{\varphi_{T-e_0}(\lambda)}{\varphi_{T-u}(\lambda)}$.

By the induction, we have

$$\mathbf{p}_{e\setminus\{u\}} = \frac{\varphi_{T-e}(\lambda)}{\varphi_{T-u}(\lambda)}$$

for $e \in E_u$. Hence, we get

$$\lambda - \sum_{e \in E_u(T)} \mathbf{p}_{e\setminus\{u\}} = \frac{\varphi_T(\lambda)}{\varphi_{T-u}(\lambda)}$$

from Lemma 7 (2). □

The support of a vector $\mathbf{p} \in \mathcal{V}_u(T)$, denoted by $\mathrm{supp}(\mathbf{p})$, is the set of all indices for nonzero components of $\mathbf{p}$. Let the $|\mathrm{supp}(\mathbf{p})|$-dimensional vector $\mathbf{p}^*$ denote the non-zero projection of $\mathbf{p}$, i.e., $\mathbf{p}^*$ is the vector constructed from all the nonzero components of $\mathbf{p}$. We use $H[U]$ to denote the induced sub-hypergraph of a hypergraph $H = (V, E)$ on $U \subseteq V$. It is seen that $\mathbf{p}^* \in \mathcal{V}(H[\mathrm{supp}(\mathbf{p}) \cup \{u\}])$ only has nonzero components. Applying Lemma 10 to $\mathbf{p}^*$, we can directly extend Lemma 10 to the following general case.

Corollary 11. *Let $T = (V, E)$ be a hypertree with the vertex u. For $\mathbf{p} \in \mathcal{V}_u(T)$, let $T_{\mathbf{p}} = T[\mathrm{supp}(\mathbf{p}) \cup \{u\}]$. For $e \in E_u(T_{\mathbf{p}})$, let $\widehat{T}$ be the connected component of $T_{\mathbf{p}}$ containing e. Then we have*

$$\mathbf{p}_{e\setminus\{u\}} = \frac{\varphi_{\widehat{T}-e}(\lambda)}{\varphi_{\widehat{T}-u}(\lambda)} = \frac{\varphi_{T_{\mathbf{p}}-e}(\lambda)}{\varphi_{T_{\mathbf{p}}-u}(\lambda)}$$

for each $e \in E_u(T_{\mathbf{p}})$. Moreover, we have

$$\lambda - \sum_{e \in E_u} \mathbf{p}_{e\setminus\{u\}} = \frac{\varphi_{T_{\mathbf{p}}}(\lambda)}{\varphi_{T_{\mathbf{p}}-u}(\lambda)}.$$

We are now ready to give the algebraic multiplicity of the spectral radius of a uniform hypertree.

Theorem 12. *The algebraic multiplicity of the spectral radius of a k-uniform hypertree with m edges is $k^{m(k-2)}$.*

Proof. From the Poisson Formula for hypergraphs and Corollary 11, we have

$$\phi_T(\lambda) = \phi_{T-u}(\lambda)^{k-1} \prod_{\mathbf{p} \in \mathcal{V}} \left(\frac{\varphi_{T_{\mathbf{p}}}(\lambda)}{\varphi_{T_{\mathbf{p}}-u}(\lambda)} \right)^{m(\mathbf{p})} \tag{6}$$

for a hypertree $T = (V, E)$ and any $u \in V$.

Lemma 8 shows that the roots of the matching polynomials of sub-hypertrees of a hypertree T are eigenvalues of T. For a connected hypergraph G and its proper subgraph G', it is known that $\rho(G') < \rho(G)$ [12, Corollary 3.5]. It tells that $\rho(G)$ is not an eigenvalue of any proper sub-graph of G. It implies that $\rho(T)$ is not a root of the matching polynomials of any proper sub-graph of T. Then we know that the degree of the factor $\lambda - \rho(T)$ in (6) is solely determined by $\varphi_T(\lambda)$, and its degree in $\varphi_T(\lambda)$ is one, as confirmed by Lemma 9. Let $\mathrm{am}(T)$ denote the algebraic multiplicity of the spectral radius $\rho(T)$, and let $\mathcal{V}^* = \{\mathbf{p} \in \mathcal{V} : T_{\mathbf{p}} = T\}$. By (6), we get

$$\mathrm{am}(T) = \sum_{\mathbf{p} \in \mathcal{V}^*} m(\mathbf{p}). \tag{7}$$

Let $\widetilde{T}$ denote the hypertree obtained from T adding a pendant edge at the vertex u. From Lemma 6, we get

$$\phi_{\widetilde{T}}(\lambda) = \phi_{\widetilde{T}-u}(\lambda)^{k-1} \prod_{\mathbf{p} \in \mathcal{V}} \left(\frac{\varphi_{T_{\mathbf{p}}}(\lambda)}{\varphi_{T_{\mathbf{p}}-u}(\lambda)} \right)^{m(\mathbf{p})(k-1)^{k-1} - k^{k-2}} \prod_{\mathbf{p} \in \mathcal{V}} \left(\frac{\varphi_{\widetilde{T}_{\mathbf{p}}}(\lambda)}{\varphi_{\widetilde{T}_{\mathbf{p}}-u}(\lambda)} \right)^{m(\mathbf{p})k^{k-2}}, \tag{8}$$

where $\widetilde{T}_{\mathbf{p}}$ denotes the hypertree obtained from $T_{\mathbf{p}}$ by adding a pendant edge at the vertex u. Employing a similar trick as used in obtaining (7) from (6), we derive the following equation from (8):

$$\mathrm{am}(\widetilde{T}) = k^{k-2} \sum_{\mathbf{p} \in \mathcal{V}^*} m(\mathbf{p}) = k^{k-2} \mathrm{am}(T).$$

It implies that the algebraic multiplicity of the spectral radius increases k^{k-2}-fold when a pendant edge is added to a hypertree. By starting with a hypertree with one edge, we obtain that the algebraic multiplicity of the spectral radius of a k-uniform hypertree with m edges is $k^{m(k-2)}$. □

Acknowledgement

This work is supported by the National Natural Science Foundation of China (No.12071097, 12371344), the Natural Science Foundation for The Excellent Youth Scholars of the Heilongjiang Province (No. YQ2022A002), the Fundamental Research Funds for the Central Universities.

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