

# Some locally Kneser graphs

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*To Jon Hall, on the occasion of his 75th birthday*

## Abstract

The Kneser graph  $K(n, d)$  is the graph on the  $d$ -subsets of an  $n$ -set, adjacent when disjoint. Clearly,  $K(n + d, d)$  is locally  $K(n, d)$ . Hall showed for  $n \geq 3d + 1$  that there are no further examples. Here we give other examples of locally  $K(n, d)$  graphs for  $n = 3d$ , and some further sporadic examples. It follows that Hall's bound is best possible.

**Mathematics Subject Classifications:** 05C75, 05C99, 05C10, 20Exx

## 1 Locally something graphs

A graph  $\Gamma$  is called *locally*  $\Delta$  when for each vertex of  $\Gamma$  the subgraph induced on the set of its neighbors is isomorphic to  $\Delta$ . The Trahtenbrot-Zykov problem [31] asks whether given a finite graph  $\Delta$  there exists a graph  $\Gamma$  that is locally  $\Delta$ . In general, this question is undecidable (Bulitko [14]). It is unknown whether the problem restricted to finite  $\Gamma$  is also undecidable.

More generally, one wants to classify all such graphs  $\Gamma$ . Hall [19] determines the possible  $\Gamma$  for all graphs  $\Delta$  on at most 6 vertices. For some  $\Delta$  a graph  $\Gamma$  that is locally  $\Delta$  is necessarily infinite.

Weetman [28, 29] constructs infinite locally  $\Delta$  graphs for  $\Delta$  of girth at least 6, and proves a diameter bound in certain other cases.

There is a large literature, see e.g. [1–15, 18–30].

## 2 Locally Kneser graphs

The *Kneser graph*  $K(n, d)$  (where  $0 \leq d \leq n$ ) is the graph on the  $d$ -subsets of an  $n$ -set, adjacent when disjoint.

Hall [20] shows that for  $n \geq 3d + 1$  any connected locally  $K(n, d)$  graph is isomorphic to  $K(n + d, d)$ , and wonders whether this bound can be improved.

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In fact there are further examples for  $n = 3d$ : The graph on the  $2^{n-1}$  even weight binary vectors of length  $n$ , adjacent when their difference has weight  $2d$  is locally  $K(n, d)$  and different from  $K(n + d, d)$  (for  $d > 1$ ). It follows that Hall's bound is best possible.

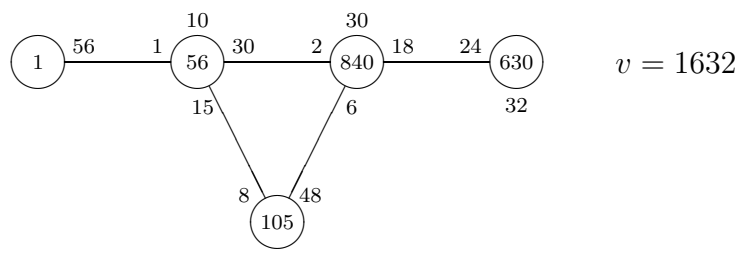
For  $n = 2d + 1$ ,  $d \geq 3$ , the graph  $K(n, d)$  has girth 6, so that there exist infinite locally  $K(n, d)$  graphs by Weetman [28].

There are three locally  $K(5, 2)$  graphs, on 21, 63, and 65 vertices (Hall [18]). Note that  $K(5, 2)$  is the Petersen graph.

There are three locally  $K(6, 2)$  graphs, on 28, 32, and 36 vertices (Buekenhout & Hubaut [10]). Note that  $K(6, 2)$  is the collinearity graph of the generalized quadrangle of order 2.

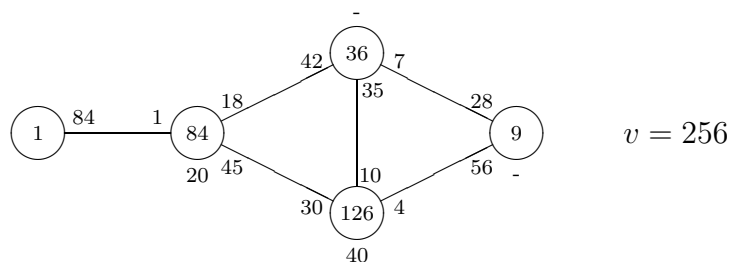
As we saw, there are infinite locally  $K(7, 3)$  graphs. (Question: Are there also finite examples?)

The graph on the elliptic lines in the  $O_8^-(2)$  geometry, adjacent when orthogonal, is locally  $K(8, 3)$ . The automorphism group is  $O_8^-(2):2$ , with point stabilizer  $S_3 \times S_8$ . Diagram:



See also [16].

As we saw, the graph on the 256 binary even weight vectors of length 9, adjacent when they have distance 6, is locally  $K(9, 3)$ . Diagram:



It can be shown using the arguments from [29] that a locally  $K(9, 3)$  graph is necessarily finite. (Question: Is the same true for all locally  $K(3d, d)$  graphs?)

The bimonster  $G = M \text{ wr } 2$  (where  $M$  is the monster) contains a  $S_5$ -subgroup  $S$  whose centralizer is a subgroup  $S_{12}$  in which a 7-point stabilizer is conjugate to  $S$ , see [17]. Let  $\Gamma$  be the graph on the  $S_5$ -subgroups of  $G$  conjugate to  $S$ , adjacent when they commute. Then  $\Gamma$  is locally  $K(12, 5)$ .

### 3 Locally $\lambda = 1$ graphs

Let  $\Delta$  be a graph in which every edge is in a unique triangle (so that  $\Delta$  is the collinearity graph of a partial linear space with lines of size 3). The Kneser graphs  $K(3d, d)$  are examples of such graphs. We study locally  $\Delta$  graphs. The special case where  $\Delta$  is the line graph of the Petersen graph was studied in [6].

Given a partial linear space  $(X, L)$  with lines of size 3, let  $G$  be the group

$$G = \langle X \mid x^2 = 1 = xyz \text{ for all } x \in X \text{ and } \{x, y, z\} \in L \rangle.$$

Suppose  $A, B$  are two disjoint hyperplane complements in  $(X, L)$ , so that each line meets  $A$  and  $B$  in 0 or 2 points. Then the map that sends the elements of  $A, B, X \setminus (A \cup B)$  to  $a, b$ , and 1, respectively, is a map from  $G$  onto the infinite group  $\langle a, b \mid a^2 = b^2 = 1 \rangle$ , so that  $G$  is infinite.

Given a subgroup  $H$  of  $G$ , let  $\Gamma = \Gamma(G, H, X)$  be the graph that has as vertices the cosets  $gH$  for  $g \in G$ , and adjacencies  $g_1H \sim g_2H$  when  $g_2^{-1}g_1 \in HXH$ . Assume that  $H$  is normal in  $G$ . Now the neighbours of  $gH$  are the vertices  $gxH$  for  $x \in X$ . The group  $G$  acts vertex transitively on  $\Gamma$ . The local graph induced on the set of neighbours of the vertex  $H$  of  $\Gamma$  has vertex set  $\{xH \mid x \in X\}$ , and if  $\{x, y, z\}$  is a line, then  $xy = z$  in  $G$ , so that  $xH \sim yH$ . It follows that  $x \mapsto xH$  is a homomorphism from the collinearity graph  $\Delta$  of  $(X, L)$  onto the  $\Gamma$ -neighbourhood of  $H$ .

Is this map injective? Suppose  $H$  is contained in the commutator subgroup  $G'$  of  $G$ . Then  $\Gamma(G, H, X)$  has quotient  $\Gamma(G, G', X)$  and the latter can be identified with the Cayley graph with difference set  $X$  in the  $\mathbb{F}_2$ -vector space  $\langle X \mid x + y + z = 0 \text{ for } \{x, y, z\} \in L \rangle$ . If  $N$  is the point-line incidence matrix of  $(X, L)$ , then this is  $\langle X \rangle / N \langle L \rangle$ , where cosets at Hamming distance 1 are adjacent. Two points remain distinct in the quotient if the column space of  $N$  does not contain vectors of weight 2. No additional adjacencies are introduced if the columns of  $N$  are the only vectors of weight 3 in the column space of  $N$ .

For these latter two conditions to hold, it suffices that for any two distinct points, and for any three pairwise noncollinear points,  $(X, L)$  has a geometric hyperplane missing precisely one of these points.

If these conditions hold,  $\Gamma$  is locally  $\Delta$ .

#### Examples

We construct  $\Gamma = \Gamma(G, G', X)$  for a number of spaces  $(X, L)$  with lines of size 3. Note that  $V = G/G'$  is a binary vector space. In all cases except d), the graph  $\Gamma$  is locally  $\Delta$ , where  $\Delta$  is the collinearity graph of  $(X, L)$ .



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