Word-Representable Graphs: Orientations, Posets, and Bounds

Zion Hefty[a] Paul Horn[a] Colby Muir[b] Andrew Owens[c]

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Abstract

Word-representable graphs were originally introduced by Kitaev and Pyatkin, motivated by work of Kitaev and Seif in algebra. Since their introduction, however, there has been a great deal of work in understanding their graph theoretical properties. In this paper, we introduce tools from partially ordered sets, Ramsey theory, finite geometry, as well as probabilistic methods to study them. Through these, we settle a number of open problems in the field, regarding both the existence and length of word-representations for various classes of graphs.

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1 Introduction

Motivated by the work of Kitaev and Seif from [24] studying algebraic problems arising from the Perkins semigroup, Kitaev and Pyatkin introduced the notion of word-representable graphs in [22]. A (simple) graph G on vertex set $V = \{v_1, \dots, v_n\}$ is said to be **word-representable** if there exists a string S in characters $v_1, \dots, v_n$ (with repetitions allowed) so that $v_i \sim v_j$ in G if and only if the characters alternate in S.

While the motivation for introducing these objects arose in algebra, determining the possible properties of word-representable graphs turns out to be a fascinating combinatorial problem. As a window to their properties, in [16] Halldórson, Kitaev and Pyatkin proved that a graph G being word-representable is equivalent to G admitting a certain type of orientation, known as a semi-transitive orientation. An acyclic orientation of a graph is **semi-transitive** if for any directed path $u_1 \to u_2 \to \cdots \to u_t$ with $t \geqslant 2$, either there is no edge between u_1 and u_t, or all edges $u_i \to u_j$ exist for $1 \leqslant i < j \leqslant t$. In

[a]Department of Mathematics, University of Denver, Denver, Colorado, U.S.A. (`zion.hefty@du.edu`, `paul.horn@du.edu`).
[b]Department of Mathematics and Statistics, Auburn University, Auburn, Alabama, U.S.A. (`cmm0085@auburn.edu`).
[c]Department of Mathematics, Widener University, Chester, Pennsylvania, U.S.A. (`adowens@widener.edu`).

other words, if the edge $u_1 \to u_t$ exists, then the orientation on the subgraph induced by $u_1, u_2, \ldots, u_t$ is transitive.

Beyond the combinatorial properties of word-representable graphs, a natural direction is to also study the minimum length of a word-representation of a graph. The **representation number** of a word-representable graph G is the smallest integer k so that there is a k-uniform word (i.e. a word with each character appearing k times) representing G (which is well defined by a result in [22] stating that any word-representable graph can be represented by a uniform word). Beyond a characterization of graphs with representation number 2, and a few specific examples (see, e.g. [21, 15]) relatively little is known about what combinatorial properties force the representation number to be large.

In this paper we answer a number of open questions raised previously by Kitaev, Lozin, Pyatkin and others. In order to do this, we highlight connections between word-representability and other areas of combinatorics; notably, we apply ideas arising in the study of partially ordered sets, Ramsey theory, probabilistic combinatorics, and finite geometry. We use these connections to study both the (non-)word-representability of graphs, and the representation number of graphs.

Some of our main results include the following:

- Using results from Ramsey theory, we construct graphs of arbitrarily high girth that are not word-representable (cf. Theorem 6). We also construct, using a recent result of Suk and Tomon, graphs of high girth and high chromatic number that are not word-representable (cf. Theorem 32). These answer questions raised by Kitaev and Pyatkin [23], that have been publicized in recent talks of Kitaev (e.g. [20]).

- We prove (cf. Theorem 4) that the random graph $G(n, p)$ is not word-representable when $p = n^{-\kappa}$, for any $0 < \kappa < 1$ asymptotically almost surely (i.e. with probability $\to 1$ as $n \to \infty$). This, in a sense, strengthens the already known fact that $G(n, p)$ for any constant $p \in (0, 1)$ is asymptotically almost surely not word-representable (which, itself, is a slight strengthening of the fact that almost every graph contains an induced wheel W_5, which is not word-representable).

- Exploiting connections between partially ordered sets and word-representability, we give an explicit way of building words representing the cover graph (Hasse diagram) of graded posets (cf. Theorem 15) and general posets (cf. Theorem 17). This bounds the representation number, depending on the dimension, and the combination of width and dimension respectively.

- Utilizing these ideas, along with results of Chandran et al. [9] on the boxicity of the hypercube and of Adiga, Bhowmick and Chandran [1] on the relation of boxicity and dimension we prove (cf. Theorem 28) that the representation number of the n-dimensional hypercube is $O(\log n/ \log \log n)$, answering negatively a question arising from [7] asking if the representation number of the n-cube was n.

- Beyond the n-cube, we give explicit bounds on the representation number for several classes of graphs including showing

- bipartite graphs of maximum degree Δ have representation number $O(\Delta \log^{(1+o(1))} \Delta)$ (cf. Theorem 31), and
- triangle-free graphs of large chromatic number exist with representation number $O(\sqrt{n})$ (cf. Theorem 33).

- Finally, we prove two new lower bounds for the representation number of a graph; first using a widely applicable (but weaker) counting argument (cf. Corollary 20), and then a stronger (but more specialized) bound using the structure of orientations (cf. Theorem 25).

- The **crown graph** $H_{n,n}$ is $K_{n,n}$ with a perfect matching removed. As a consequence of our lower bound method, we also prove (cf. Theorem 26) a tight lower bound on the order dimension of a poset having the crown graph $H_{n,n}$ as a cover graph. Note that $H_{n,n}$ is well-known in poset theory as the comparability graph of the 'standard example,' the first example of a poset whose dimension is n. This is the first result we know of this type, and may be of independent interest. A number of recent results on partially ordered sets have considered the related problem: how *large* can the dimension of a poset be with a fixed (simple) cover graph?

As a reminder of asymptotic notation, for functions $f(n)$ and $g(n)$, we say $f(n) = O(g(n))$ if there exists a constant $C > 0$ so that $|f(n)| \leqslant C|g(n)|$ for n sufficiently large. Meanwhile, $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$ as $n \to \infty$; in particular if $f(n) = o(1)$, then $f(n) \to 0$. All logarithms in the paper are taken to be the base-2 logarithm, unless otherwise specified. (Typically, because logarithms are in $O(\cdot)$ notation, the base of the logarithm seldom matters as changing the base only changes the implied constant.)

The remainder of the paper is organized as follows. In Section 2, we introduce tools from Ramsey Theory and prove results about graphs which are not word-representable. In Section 3, we introduce results from the theory of partially ordered sets and give a method of generating relatively short words representing graphs arising from posets. Then, in Section 4 we develop lower bounds on the representation number for graphs. Finally in Section 5, we give a number of applications of our results including to the representation number of the hypercube.

2 Oriented Ramsey Theory and Semi-Transitive Orientations

Given a graph H, recall that a graph G is a **Ramsey host** for H (written $G \to H$) if whenever the edges of G are 2-colored there is a monochromatic copy of H in G. Standard Ramsey theory considers the case where $G = K_n$ and attempts to find the minimum n so that $K_n \to H$. However there has been a long-standing interest in finding hosts with special properties; for instant, hosts that are relatively sparse, have no large cliques, or have large girth.

For finding graphs without semi-transitive orientations, however, a stronger property than the original Ramsey property is needed. Two related (but different) variants of standard Ramsey theory – namely **ordered Ramsey theory** and **oriented Ramsey theory** – provide a window to finding graphs without semi-transitive orientations. We briefly outline the two definitions and their relation to semi-transitive orientations of graphs.

Oriented Ramsey theory is much what it sounds like – a graph G is an **orientation Ramsey host** for a digraph $\vec{H}$ (written $G \to \vec{H}$) if every orientation of G contains a copy of $\vec{H}$ as a sub-digraph. Here, the connection to graphs with semi-transitive orientations is almost immediate, as observed in the following. Let $\vec{C}_t$ be the orientation of a cycle on the vertex set $u_1, \dots, u_t$ with the oriented path $u_1 \to u_2 \to \cdots \to u_t$ and edge $u_1 \to u_t$. We also highlight that the typical way to show that an orientation is not semi-transitive is to find a *shortcutting edge* – an edge from $u_1 \to u_t$ with an oriented path $u_1 \to u_2 \to \cdots \to u_t$ where some edge $u_i \to u_j$ with $1 \leqslant i < j \leqslant t$ is not present.

Proposition 1. *Suppose G is a graph of girth at least 4 so that $G \to \vec{C}_t$ for some $t \geqslant 4$. Then G has no semi-transitive orientation.*

Proof. Fix an orientation of G. By the Ramsey property, that orientation contains a $\vec{C}_t$. But since the graph has girth 4 this cycle includes a shortcutting edge. Since the orientation is arbitrary, the result follows. □

Thus it suffices, for the point of view of finding graphs of large girth without semi-transitive orientations, to find Orientation Ramsey hosts for some $\vec{C}_t$ of arbitrarily large girth. Fortunately, a recent result of Barros, Calavar, Kohayakawa and Naia from [4], along with an observation dating back to the classic paper [27] of Rödl and Rucinski, will provide exactly this.

We also mention ordered Ramsey theory – as studied in [11], for instance – as this provides another potential source of other non semi-transitively orientable graphs. Now consider a graph G on vertex set $[n]$ (where $[n] = \{1, 2, \dots, n\}$), and a graph H on vertex set $[t]$. An ordered copy of H in G is an isomorphic copy of H in G where the inclusion $\varphi : [t] \to [n]$ is ordered: that is, $\varphi(i) < \varphi(j)$ for all $i < j$. Then G is an ordered Ramsey host for H (written $G \to_< H$) if for every red-blue coloring of the edges of G, G contains a monochromatic ordered copy of H.

Proposition 2. *Suppose G is a graph of girth at least 4 so that $G \to_< C_t$ for some $t \geqslant 4$. Then G has no semi-transitive orientation.*

Proof. Fix an orientation of G. We construct a red-blue coloring of the edges of G as follows: if the edge is ordered in an increasing way (i.e. from a lower indexed vertex to a higher) color it red, otherwise color it blue. By the Ramsey property, this coloring contains a monochromatic ordered C_t. But then this easily seen to be a cycle with a shortcutting edge, regardless of whether it is red or blue. □

The key to showing existence of non-semi transitively oriented graphs of large girth is the result of Barros, Cavalar, Kohayakawa, and Naia we now present.

We recall the Erdős-Renyi $G(n, p)$ model of random graphs: $G(n, p)$ is a probability distribution on n vertex labeled graphs, where edges are present independently with probability p (and missing with probability $1-p$. Following [4], we recall that $p_{\vec{H}}$ is a **threshold** for $G(n,p) \to \vec{H}$ if for a G in $G(n,p)$,

$$\mathbb{P}(G \to \vec{H}) \to \begin{cases} 0 & p = p(n) \ll p_{\vec{H}} \\ 1 & p = p(n) \gg p_{\vec{H}} \end{cases}$$

where here $a \ll b$ (resp. $\gg$) means $\lim_{n\to\infty} a_n/b_n = 0$ (resp. $\lim_{n\to\infty} a_n/b_n = \infty$).

The key proposition is the following, specialized to the cases we are concerned with (cf. Theorem 2 from [4])

Proposition 3 (Special case of Theorem 2 from [4])**.** *For any $t \geqslant 4$,*

$$p_{\vec{C}_t}(n) = n^{-\frac{t-2}{t-1}}$$

is the threshold for $G \to \vec{C}_t$.

We remark that the upper bound for the threshold – that is, the fact that there exists a constant $C = C(t)$ so that if $p = p(n) \geqslant Cn^{-\frac{t-2}{t-1}}$ then $\mathbb{P}(G \to \vec{C}_t) \to 1$ – appears in the masters thesis of Cavalar [8]; this is actually the most important direction for our purposes. As mentioned in [4] there are some other, more classical, ways of deriving this threshold via graph regularity – Cavalar uses more modern machinery, namely graph containers, to achieve the result. We also note, for the convenience of the reader referring to the statement in [4], that the quantity in the exponent, $-\frac{t-2}{t-1}$, is the 2-density of C_t; where the 2-density of a graph H is

$$m_2(H) = \max_{\substack{F\subseteq H \\ v(F)\geqslant 3}} \frac{e(F)-1}{v(F)-2}.$$

This quantity is well-known to occur as a threshold function for Ramsey properties dating back to the work of Rödl and Ruciński in [27].

An almost immediate consequence of Proposition 3 is the following:

Theorem 4. *Suppose $0 < \kappa < 1$, and G is a $G(n, n^{-\kappa})$ random graph. Then*

$$\mathbb{P}(G \textit{ is word-representable}) \to 0.$$

Proof. Fix a $t = t(\kappa) \geqslant 4$ large enough so that

$$\kappa < \frac{t-2}{t-1} \qquad \textit{and} \qquad \kappa > \frac{2}{t-1}.$$

As $\kappa < \frac{t-2}{t-1}$, Proposition 3 implies that $G(n, n^{-\kappa}) \to \vec{C}_t$ with probability tending to one. On the other hand, the expected number of K_ts in $G(n, n^{-\kappa})$ is at most

$$\binom{n}{t} n^{-\kappa\binom{t}{2}} < n^{t-\kappa\frac{t(t-1)}{2}}.$$

As $\kappa > \frac{2}{t-1}$, this exponent is negative, and hence the expected number of K_ts is $o(1)$. These combine to imply that a random graph $G \in G(n, n^{-\kappa})$ has no semi-transitive orientation with probability tending to 1: With probability tending to 1, any orientation of G must contain $\vec{C}_t$, but since (again, with probability tending to 1) G does not contain a K_t this subgraph induced by these t vertices cannot be complete, so the orientation is not semi-transitive. □

Remark 5. So long as $p \gg n^{-3/5}$ and $1-p \gg n^{-1}$ then $G(n,p)$ will have (with probability tending to 1) induced copies of the (non semi-transitively orientable) wheel W_5, and hence is non semi-transitively orientable. Hence this is of most interest for small κ. Also, taking $p = n^{-\kappa}$ is really for simplicity here; so long as $p = p(n)$ with

$$\limsup \quad -\frac{\log(p)}{\log n} < 1 \qquad\qquad \textit{and} \qquad\qquad \liminf \quad -\frac{\log(p)}{\log n} > 0,$$

the argument works with minor modification.

Finally we prove the main theorem of the section.

Theorem 6. *For any k, there exist graphs of girth k that have no semi-transitive orientation, and hence are not word-representable.*

Per Proposition 1 it suffices to find an orientation Ramsey host for $\vec{C}_k$ of girth k. Proposition 3 implies that a random graph for an appropriate edge density will be such an orientation Ramsey host for $\vec{C}_k$, but a typical graph of that edge density will have girth smaller than k. That said, already dating back to the seminal work of Rödl and Rucinski on sparse Ramsey hosts in [27] it has been known how to take appropriate threshold results (like Proposition 3) and turn them into existence results for Ramsey hosts of large girth.

Though this is not explicitly done by Barros et al. in [4], the method to do so is well-known. Indeed, the existence of the necessary Ramsey hosts is implicit in the thesis of Cavalar, [8]. The explicit probabilities needed to prove Theorem 6 are computed, although the theorem itself does not seem to be stated explicitly. Cavalar instead proves the existence of Ramsey hosts of girth k for a slightly different 'isometric oriented' Ramsey property. Due to Cavalar's work, [4] is more interested in finding the other end of the threshold. The methods are slightly different than those of Rödl and Rucinski [27], though a proof in the vein of Rödl and Rucinski should be possible (see discussion in [8, 4]). We collect, for convenience, the necessary ingredients. We note that key to the proof is the FKG inequality from [13]. In the specialization we need, cf. Theorem 6.3.2 of The Probabilstic Method of Alon and Spencer [2], it states

Lemma 7. *Suppose $\mathcal{Q}_1$, $\mathcal{Q}_2$ are monotonically decreasing graph properties – that is if G has property $\mathcal{Q}_i$ then every subgraph of G has property $\mathcal{Q}_i$. Then if G is a $G(n,p)$ random graph.*

$$\mathbb{P}(G \textit{ has property } \mathcal{Q}_1 \cap \mathcal{Q}_2) \geqslant \mathbb{P}(G \textit{ has property } \mathcal{Q}_1)\mathbb{P}(G \textit{ has property } \mathcal{Q}_2).$$

Proof of Theorem 6. For a fixed k and $p = p(n)$, let $\mathcal{A}_{k,p}$ be the event that a graph in $G(n,p)$ is an orientation Ramsey host for $\vec{C}_k$ and $\mathcal{B}_{k,p}$ be the event that a graph in $G(n,p)$ has girth at least k. It suffices to prove that

$$\mathbb{P}(\mathcal{A}_{k,p} \cap \mathcal{B}_{k,p}) > 0.$$

As $\mathbb{P}(\mathcal{A}_{k,p} \cap \mathcal{B}_{k,p}) \geqslant \mathbb{P}(\mathcal{B}_{k,p}) - \mathbb{P}(\mathcal{A}^c_{k,p})$, it suffices to prove that

$$\mathbb{P}(\mathcal{A}^c_{k,p}) < \mathbb{P}(\mathcal{B}_{k,p})$$

for appropriate choice of p.

Both of these estimates are in [8] – one of which is a completely standard estimate on the probability that a random graph has large girth.

Claim 1: *There exist positive constants C, C', C'' so that if*

$$p \geqslant Cn^{-\frac{k-2}{k-1}},$$

then

$$\mathbb{P}(\mathcal{A}^c) \leqslant \exp\left(-C'n^2p\right) = \exp\left(-C''n^{1+\frac{1}{k-1}}\right). \tag{1}$$

Claim 1 is Theorem 4.1 in [8], specialized to the cycle $\vec{C}_k$ with the estimate (1) from the proof.

Claim 2: If $p = Cn^{-\frac{k-2}{k-1}}$, then

$$p(\mathcal{B}) \geqslant \exp(-O(n)).$$

Claim 2 is a rather standard application of the FKG inequality from [13]; we present a slightly streamlined version of the argument from [8] without trying to optimize any constants. Let Z be the random variable counting the number of cycles of length less than k in a random G from $G(n,p)$. Let $\mathcal{C}$ consist of the collection of potential cycles in G of length less than k. Here, $\mathcal{C}$ can be thought of as cyclically ordered tuples of t distinct vertices. Note that there are fewer than n^t potential cycles of length t in $\mathcal{C}$.

$$\mathbb{E}[Z] = \sum_{C\in\mathcal{C}} p^{|C|} < \sum_{t=3}^{k-1} p^t n^t < k(pn)^{k-1} = O((pn)^{k-1}) = O(n).$$

As containment of different cycles is positively correlated in G – if C_1 is a cycle and C_2 is a different cycle then containing C_1 only can make C_2 more likely; strictly if they share edges – and since a single given cycle is present the FKG inequality implies that

$$\mathbb{P}(\mathcal{B}_{k,c}) = \mathbb{P}(Z=0) \geqslant \prod_{C\in\mathcal{C}}(1-p^{|C|}) \geqslant \exp\left(-\sum_{C\in\mathcal{C}}\left(\frac{p^{|C|}}{1-p^{|C|}}\right)\right)$$
$$= \exp(-O(\mathbb{E}[Z])) = \exp(-O(n)).$$

Here, we note that the first inequality is the FKG inequality specialized as in Lemma 7, applied to the $\mathcal{Q}_i$ being non-containment of various cycles indexed by $\mathcal{C}$. The second follows from the real number inequality $1 - x \geqslant e^{-x/(1-x)}$. Finally we note that the $(1-p^{|C|})^{-1}$ is at most $\frac{1}{2}$ for n sufficiently large (as $p < \frac{1}{2}$) which is absorbed in the implied constant in the $O(\mathbb{E}[Z])$ term.

Thus setting $p = Cn^{-\frac{k-2}{k-1}}$ for an appropriately large C and n, $\mathbb{P}(\mathcal{A}_{k,p}^c) < \mathbb{P}(\mathcal{B}_{k,p})$ so that $\mathbb{P}(\mathcal{A}_{k,p} \cap \mathcal{B}_{k,p}) > 0$ completing the proof as the Ramsey host of large girth is exactly the graph needed. □

3 Partial Orders and Semi-Transitive Orientations

Recall that a (strict) partial order is a set X along with a relation $\prec$ on X that is irreflexive, transitive and antisymmetric. The **comparability graph** of a partially ordered set $(X, \prec)$ is a graph on vertex set X, so that $a, b \in X$ are connected if either $a \prec b$ or $b \prec a$ – that is, if a and b are **comparable**.

Acyclic orientations, including semi-transitive orientations, of a graph naturally give rise to a partial order on a graph, so that if $x, y \in V(G)$, $x \prec y$ if there is a directed path from x to y. Acyclicity, then, precludes the possibility that $x \prec y$ and $y \prec x$ and hence ensures anti-symmetry of the relation.

Thus every semi-transitive orientaton of a graph gives rise to a partial order. In this section we are interested in the reverse direction: in what ways can partial orders give rise to word-representable graphs.

In a partial order, a is said to **cover** b if $b \prec a$ but there does not exist a c so that $b \prec c \prec a$. The **cover graph** of $(X, \prec)$ is a graph on X where a is connected to b if a covers b, or b covers a. A particular representation of the cover graph, known as a Hasse diagram, is an arrangement of the vertices of the cover graph so that elements are arranged on the plane so that lower elements of the poset appear below elements that cover them.

There are close relations between these graphs associated with posets, and word-representable graphs. The starting point of this section is the following simple observation:

Proposition 8. *If $P = (X, \prec)$ is a partial order on a set X then both the comparability graph and cover graph of P are semi-transitive.*

On the other hand, it is shown in [21] that

Proposition 9. *If G is semi-transitive and triangle-free, then it is the Hasse diagram of some poset.*

While a simple observation, this already gives a source of word-representable graphs with interesting properties. For instance, one of the motivating questions of this paper was the existence of graphs of large girth and even larger chromatic number that are word-representable. The chromatic number of Hasse diagrams of girth k can actually be as large as $n^{\frac{1}{2k-3}-o(1)}$, by a recent result of Suk and Tomon [30]. These graphs, combined with our observations imply

Theorem 10. *There exist word-representable graphs of girth k with chromatic number $n^{\frac{1}{2k-3}-o(1)}$.*

Later (cf. Theorem 32) we show that these graphs are not only word-representable, but provide a bound on the length of the word-representations of these graphs.

3.1 Poset Dimension, Realizers, and Shuffling

As partially ordered sets are a natural source of word-representable graphs, it is natural to study the words themselves.

Recall that a **linear extension** of a poset $P = (X, \prec)$ is a total ordering of the elements of X that respects the partial order (that is, if $a \prec b$ in P then $a < b$ in the total ordering.) A linear extension can be thought of as a string of characters, representing the elements in their order. A **realizer** is a set of linear extensions $\{r_1, r_2, \ldots, r_t\}$ with the property that whenever a and b are incomparable in P, then $a < b$ in some extension r_i and $b < a$ in another extension r_j. The **dimension** of a poset is the smallest such t.

A **chain** in a poset is a subset C of P where any two elements of C are comparable and an **antichain** is a subset A of P where no two elements of A are comparable. The **height** of a poset is the size of the largest chain and the **width** of the poset is the size of the largest antichain.

We begin with the following simple observation, which is essentially Theorem 3.4.3 of [21]; for completeness, we provide a short proof as we build on the main idea in Lemma 12 below.

Lemma 11. *Let $P = (X, \prec)$ be a poset. For any realizer $R = \{r_1, \ldots, r_t\}$ of P the string that is the concatenation of $r_1, \ldots, r_t$ is a word-representation of the comparability graph of P.*

Proof. Consider the graph G represented by the concatenation $r_1 r_2 \ldots r_t$; the vertex set of G is X. Fix $a, b \in G$. If a and b are comparable in P – without loss of generality, assume $a \prec b$ so in each r_i, a occurs before b – then a and b alternate, with the ith occurrence of each character coming from r_i. If a and b are incomparable, then a occurs before b in one element of the realizer, and b before a in another – that is, a and b do not alternate in the string. Thus G is precisely the comparability graph. □

A word is a **uniform word** if each vertex appears the same number of times. In particular, a word is ***k*-uniform** if each vertex in the graph appears exactly k times in the word. A graph is called ***k*-representable** if there is a k-uniform word that represents that graph. The smallest k such that G is k-representable is called the representation number of G, denoted $\mathcal{R}(G)$.

Slightly more information is actually gained from the proof of Lemma 11. Given a uniform word w representing a graph G, it naturally induces a semi-transitive orientation of G which we call the **canonical orientation**, where the edge ab is oriented from a to b if a appears first in w. An orientation of a graph is k-representable if there is a k-uniform word which gives the same orientation. One can thus also define the representation number of an orientation of a graph as the smallest k so that a k-uniform word yields that orientation. The representation number of the graph is then the minimum representation number of a semi-transitive orientation of the graph.

Lemma 12. *Let w_1 and w_2 be uniform words representing G_1 and G_2, respectively, that share a vertex set, and G be the graph represented by the concatenation w_1w_2. Then an edge ab is in G if and only if ab is in both G_1 and G_2 and the canonical orientation of ab is the same in both.*

We remark that the representation given by Lemma 11 need not be the *shortest* representation of a comparability graph. That is, the representation number of a comparability graph need not be its dimension. For instance, the crown graph $H_{4,4}$ is the comparability graph of a dimension 4 poset while, as a graph, it is a 3-dimensional cube which has representation number 3 [21]. Indeed, there is even a 3-uniform word representing $H_{4,4}$ whose canonical orientation is the standard orientation of $H_{4,4}$ as a dimension 4 poset. If the vertices of $H_{4,4}$ are labeled with $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ so that the edges are exactly between x_i and y_j when $i \neq j$, then a 3-uniform word representing $H_{4,4}$ with the desired canonical orientation is $x_1x_2x_3y_4x_4y_1y_2x_3y_3x_4x_2y_1x_1y_3y_4x_2y_2x_4x_3x_1y_3y_2y_1y_4$.

This subtlety has actually led to a small amount of confusion in the literature. For instance, the statement of Theorem 5.4.7 in Kitaev and Lozin's book [21] has a minor inaccuracy due to this issue. They define a subset X of the set of vertices V of a graph G to be a **module** if all members of X have the same set of neighbors among vertices not in X. Then the accurate statement of their Theorem 5.4.7 is

Proposition 13. *Suppose that G is a word-representable graph and $x \in V(G)$. Let G' be obtained from G by replacing x with a module M, where M is any comparability graph (in particular, any clique). Then G' is also word-representable. Moreover, if $\mathcal{R}(G) = k_1$ and k_2 is the order dimension of a poset for which M is a comparability graph then $\mathcal{R}(G') = k$, where $k = \max\{k_1, k_2\}$.*

It should be noted that this gives a well defined value for k_2 by a result [31] of Trotter, Moore, and Sumner that says that any two partial orders with the same comparability graph have the same order dimension. In the original statement, k_2 was defined to be the representation number of M. The reason for this modification to the statement is that in the proof, the facts that M can be represented by a concatenation of k_2 linear extensions, and that the concatenation of no fewer than k_2 linear extensions represents M are both reliant on properties of the dimension of a poset rather than just the representation number of M. A small example highlighting the need for order dimension to be considered is that replacing one of the vertices of K_2 with the crown graph $H_{4,4}$ yields a graph that has representation number 4 (by Theorem 4.2.6 of [21]) yet the representation number of K_2 is 1 and the representation number of $H_{4,4}$ is 3.

A **graded poset** is a partially ordered set P paired with a rank function $\varphi : P \to \mathbf{N}$ in which, for $x, y \in P$, if $x \prec y$ in the ordering of P then $\varphi(x) < \varphi(y)$ and if y covers x then $\varphi(y) = \varphi(x) + 1$.

For two linear extensions t_1 and t_2 of a poset P, we define the string $\texttt{shuffle}(t_1, t_2)$ as follows:

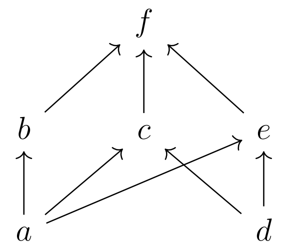
Algorithm: A letter t in t_2 is said to be a legal letter if it has not been printed and:

1. All $x \prec_{t_2} t$ have been printed from t_2.
2. t and all elements that cover t in P have been printed from t_1.

$\texttt{shuffle}(t_1, t_2)$ is generated as follows: Letters are printed iteratively from t_1 and t_2 from left to right. In each iteration, if there is a legal letter from t_2, it is printed. Otherwise, the next unprinted letter from t_1 is printed.

To clarify, we consider a small example.

Example: Consider the following Hasse diagram on elements a, b, c, d, e, f.



We consider $\texttt{shuffle}(t_1, t_2)$ on the linear extensions of P given by $t_1 = dabecf$ and $t_2 = adbcef$. The first letter of t_2, a, does not become legal in t_2 until those elements covering it have been printed. Thus $\texttt{shuffle}(t_1, t_2)$ begins as $dabec$. At this point, a is legal in t_2. So a is printed from t_2. Then d is also legal, so d is printed. Since the next letter in t_2 is b, which is covered by f in P, there is no legal letter in t_2. So f is printed from t_1. At this point, no letters from t_1 remain, and thus the rest of t_2 is printed in order. This yields the string

$$\texttt{shuffle}(t_1, t_2) = dabecadfbcef.$$

Lemma 14. *Suppose C is the cover graph of a poset P, and t_1 and t_2 are linear extensions of P. Then C is a subgraph of the graph represented by the word $\texttt{shuffle}(t_1, t_2)$.*

Proof. Let ab be a directed edge in C. By definition, a must come before b in both t_1 and t_2. By definition, a must be printed before b in both linear extensions. The only way edge ab will be deleted from the resulting word is if vertex a from t_2 is printed before vertex b from t_1. However, the second criteria of the $\texttt{shuffle}$ algorithm states that before vertex a from t_2 can be printed, all vertices dominating a, particularly vertex b, must have been printed from t_1. Therefore, the edge ab is in the graph represented. □

Theorem 15. *Suppose P is a graded poset, and $R = \{r_1, \ldots, r_n\}$ is a realizer of P. Let t be a linear extension of P defined by enumerating vertices in non-decreasing order by their rank. Then the word obtained by concatenating* $\texttt{shuffle}(t,t), r_1, \ldots, r_n$ *is a word-representation of the cover graph, G, of P.*

Proof. By Lemma 11, we know the concatenation of $r_1 \ldots r_n$ eliminates all edges between non-comparable elements of P in G. By Lemma 14, $\texttt{shuffle}(t,t)$ maintains all edges present in G.

Applications of Lemma 12 imply the concatenation of $\texttt{shuffle}(t,t)$ and $r_1 \ldots r_n$ will not delete edges from G.

The only edges remaining that need to be addressed are those between comparable elements with a difference in rank larger than 1, which we show are all deleted. Let $x < z$ be in P where $\varphi(x) + 1 < \varphi(z)$. Since edges are only between elements whose ranks differ by 1, there must exist $y \in P$ such that $\varphi(y) = \varphi(x) + 1$ and $x < y < z$. Then x must come before y which must come before z in t, and $\varphi(x) < \varphi(y) < \varphi(z)$. Applying $\texttt{shuffle}(t,t)$ would ensure x from the second copy of t is printed before z from the first copy of t because all elements of rank $\varphi(x)+1$ would have been printed before z from the first copy of t making all elements of rank $\varphi(x)$ legal to be printed from the second copy of t, deleting the directed edge xz. □

Lemma 16. *Suppose P is a poset, and $X = (x_1, \ldots, x_k)$ is a chain in P. Then there exists a 2-uniform word representing a graph G with the properties that*

1. *G contains the cover graph of P as a subgraph.*
2. *For all $1 \leqslant i \leqslant k$: if $x \prec x_i \prec y$ in P, then G does not contain the edge xy.*
3. *The canonical orientation of G agrees with P: if $x \prec y$ in P and there is an edge from x to y, then the canonical orientation goes from x to y.*

Proof. For $1 \leqslant i \leqslant k$ define the set $B_i = \{y \in P \mid x_i \not\prec y\}$ and define $B_{k+1} = V(G)$. Define $D_i = \{y \in P \mid y \prec x_i\}$ and define $D_{k+1} = V(G)$. By definition we have that $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k \subseteq B_{k+1}$ and $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k \subseteq D_{k+1}$. Note that no element of B_i can be preceded by an element not in B_i and no element of D_i can be preceded by an element not in D_i. Thus we may order the elements of $B_1, B_2 \setminus B_1, \ldots, B_{k+1} \setminus B_k$ such that the concatenation of the orders of $B_1, B_2 \setminus B_1, \ldots, B_{k+1} \setminus B_k$ give us a linear extension of P that we call b. This can be done by taking a linear extension on the poset induced by $B_{i+1} \setminus B_i$. In a similar way, we can make a linear extension d using the D_i. We consider the word $W = \texttt{shuffle}(b,d)$. Note that this word is 2-uniform and by Lemma 14 we have that the graph based on this word contains all the edges in the cover graph of P.

For $1 \leqslant i \leqslant k$, when the shuffle algorithm is about to print an element z of $B_{i+1} \setminus B_i$ from b, all the previous elements in b, which come from either B_1 or $B_{j+1} \setminus B_j$ where $1 \leqslant j \leqslant i-1$, have been printed. This means that all of B_i has been printed. z is greater than x_i. Since all elements from D_i are less than x_i, z does not cover them. Thus every element of D_i is legal to print in d, so these must have been printed before z can.

Thus every element less than x_i is printed twice before any element greater than x_i is printed for the first time so W has no edges from a vertex that precedes x_i to a vertex that is preceded by x_i for all $1 \leqslant x \leqslant k$. □

Theorem 17. *Suppose G is the cover graph of a poset P. Let l be the width of P, and d be the dimension of P. Then G is word-representable by a $(2l + d)$-uniform word.*

Proof. Per Lemma 11, there is a d-uniform word w representing the comparability graph of P. By construction, the canonical orientation of this graph is determined by P; that is, the edge goes from a to b if $a \preceq b$ in P.

By Dilworth's theorem [12], there is a chain decomposition of P consisting of l chains $C_1, C_2 \ldots, C_l$. For each of these chains, C_i, Lemma 16 provides a 2-uniform word w_i which represents a graph including the cover graph, but not including edges from below some element $x \in C_i$ to above that x in the partial order P. The canonical orientation of all these words also agrees with P.

Concatenating these words $ww_1w_2 \ldots w_l$ yields the desired $2l + d$ uniform word by Lemma 12. Indeed, the cover graph is represented by all (and canonically oriented according to P), but all other edges of the comparability graph are killed by one of the words w_i. This is as any non-cover edge xy has some $x \prec z \prec y$, and z lies in some chain C_i, so that xy does not occur in the graph represented by w_i. □

As the dimension is at most the width for cover graphs (see [17]), we obtain

Corollary 18. *Suppose G is the cover graph of a poset P. Let w be the width of P. Then G has representation number at most $3w$.*

4 Lower Bounds for Representation Numbers

In this section, we illustrate two methods for getting lower bounds on representation numbers. The first is an extremely general, but simple, counting argument. Its main benefit is that it makes essentially no assumptions on the graph where it is applied. It yields, however, comparatively weaker lower bounds than the second method, which exploits the structure of possible semi-transitive orientations to prove strong lower bounds when it applies. It applies, for instance, to the crown graph and yields an independent proof of the (sharp) lower bound for these graphs in [15], but also applies to a larger class of graphs and yields a linear lower bound on the representation number in these cases.

4.1 General Lower Bound of Finite Graphs

We begin by giving a general lower bound on the representation number of a finite graph.

To introduce the lemma, we first require a brief definition. Given a graph G, and $X \subset V(G)$, let

$$N_X = \{S \subseteq X : \exists v \in (V \setminus X) \ s.t. \ S = N(v) \cap X\}$$

denote the subsets of X that are the neighborhood of some vertex v restricted to X.

A priori, N_X can be as large as $2^{|X|}$ – this would happen if there were vertices with every possible neighborhood within X. However, we observe the following

Theorem 19. *Suppose G is a graph with a k-uniform word-representation. Then*

$$|N_X| = O(|X|^k).$$

Proof. Suppose G is a graph with a k-uniform word-representation w. Fix $X \subseteq V(G)$ with $X = \{v_1, \ldots, v_t\}$. Then the occurrences of the symbols $v_1, v_2, \ldots, v_t$ in w divide w into $tk+1$ intervals. Note that for any $v \in V \setminus X$ the adjacencies between v and the vertices in X are determined by the intervals that the k repetitions of v fall in. Moreover, if any two v's fall into the same interval, then v is adjacent to none of the vertices in X. Thus

$$|N_X| \leqslant \binom{tk+1}{k} + 1 \leqslant \left(\frac{e(tk+1)}{k}\right)^k + 1 = O(t^k)$$

as desired. □

An immediate consequence, since $|N_X| \leqslant (2et)^k$, is:

Corollary 20. *Suppose G is a word-representable graph, and there exists a set $X \subseteq V$ with $|V| = t$, and $|N_X| = s$. Then*

$$\mathcal{R}(G) \geqslant \frac{\log s}{\log(2et)}.$$

In Kitaev and Lozin's book, [21], the question of whether all bipartite graphs have representation number 3 is raised. The example of the crown graphs $H_{n,n}$, which were shown in [15] to have representation number $\lceil n/2 \rceil$ for $n \geqslant 5$, show this is not true. We remark, however, that Corollary 20 gives, in some sense, a stronger negative answer to this question.

Indeed, it is easy to see that almost every (sufficiently large) bipartite graph gives a negative answer to the question.

Theorem 21. *Suppose G is a uniformly random chosen bipartite graph on (X, Y) where $|X| = |Y| = n$. Then*

$$\mathbb{P}\left(G \text{ is } k\text{-representable for some } k < \tfrac{\log n}{\log(6e \log n)}\right) \to 0$$

as $n \to \infty$.

That is, the smallest k such that G is k representable is at least $(1 - o(1))\frac{\log n}{\log \log n}$ for almost every bipartite graph where both parts have the same size.

Proof. A uniformly random bipartite graph amounts to taking each potential edge between $x \in X$ and $y \in Y$ independently with probability $\frac{1}{2}$. Fix $S \subseteq X$ with $|S| = t = \lfloor 3 \log n \rfloor$.

For $x, y \in Y$,

$$\mathbb{P}(N(x) \cap S = N(y) \cap S) = 2^{-t} \leqslant 2n^{-3}.$$

Thus if Z is the number of such pairs,

$$\mathbb{E}[Z] \leqslant \binom{n}{2}(2n^{-3}) \leqslant n^{-1}.$$

By Markov's inequality $\mathbb{P}(Z > 0) \leqslant \mathbb{E}[Z] \to 0$, so that $\mathbb{P}(Z = 0) \to 1$. But if $Z = 0$ then $|N_S| = n$, while $|S| = t \leqslant 3\log(n)$. The theorem then follows immediately from Corollary 20. □

4.2 Lower Bounds from Orientations

In this section we illustrate how semi-transitive orientations can sometimes be characterized, and this characterization can be used to lower bound the representation number. In particular, we give a lower bound on the representation number for a particular class of bipartite graphs. This class includes crown graphs, and specializing the method gives an alternate proof of the representation numbers of crown graphs, as found in [15].

We remark this method also yields, as a bonus, the minimum order dimension of a poset whose cover graph is the crown graph $H_{n,n}$. This ties to a large stream of fairly recent results in partially ordered sets, which seek to maximize the dimension of posets whose cover graphs are reasonably simple. See, e.g. [29, 5, 18, 19]. This, however, is the first theorem we know of that looks at the opposite question: how small can the dimension of a fairly dense and complicated cover graph be?

We define the **bipartite complement** G^* of a bipartite graph G with parts U, V as the bipartite graph with parts U, V where $u_i v_j \in E(G^*)$ if and only if $u_i v_j \notin E(G)$ for each $u_i \in U$ and $v_j \in V$. A bipartition is **balanced** if the sizes of the two parts differ by at most 1.

We begin by making some observations on the structure of semi-transitive orientations for a class of bipartite graphs, which we can then leverage into a lower bound for the representation number (and order dimension).

Proposition 22. *In a semi-transitive orientation of a C_4 on vertex set $\{a, b, c, d\}$, if ab and bc are oriented edges, then the remaining edges are oriented as ad and dc.*

Proof. If *cd* is an oriented edge, then either orientation of the edge between a and d results in either an oriented cycle or shortcutting edge. The case for *da* follows similarly. □

Lemma 23. *Let G be a balanced $(n - k$-regular bipartite graph on $2n$ vertices.*

Then for any semi-transitive orientation of G there exists an induced subgraph on vertex sets (X, Y, Z) satisfying the following:

- $|X| + |Z| \geqslant n - k$

- $|Y| \geqslant n - 2k$
- *Every vertex in X is a source.*
- *Every vertex in Z is a sink.*

Proof. Let A and B denote the bipartition of G. Fix a semi-transitive orientation of G.

If every vertex in A were either a source or a sink, then we let X consist of the sources in A, Z consist of the sinks, and $Y = B$.

Hence we can assume there is a vertex $y \in A$ such that y is neither a source nor a sink. Let X denote the set of in-neighbors of y, and Z consist of the out neighbors. Then $|X| + |Z| = n - k$.

Since y is neither a source nor sink, X and Z are both non-empty. Fix $x \in X$ and $z \in Z$, and let $Y \subseteq A$ be the common neighborhood of x and z; note that $|Y| \geqslant n - 2k$.

We claim that (X, Y, Z) satisfies the conclusion of the lemma. The fact that every vertex in X is a source is observed as follows. Fix adjacent $x' \in X$, $y' \in Y$. Note that, by construction $x'y$ and yz are oriented edges. Furthermore y' and z are connected by an edge. But then, by Lemma 22, the edges are oriented as $x'y'$ and $y'z$. Since this is true for every $y' \in Y$, every vertex $x' \in X$ is a source to Y. That every vertex in Z is a sink follows similarly, interchanging the role of x and z. □

Given a r-uniform word w, we define how to **unshuffle** w into r 1-uniform words (i.e. permutations) $w_1, w_2, \ldots, w_r$: the i-th word, w_i in the unshuffle of w is the substring consisting of the i-th occurrence of each letter. We denote the i-th occurrence of x in w by x^i, and say that $x^i < y^j$ if x^i occurs before y^j in w.

Lemma 24. *Suppose G is a graph represented by a r-uniform word w, oriented according to its canonical orientation. Let $(P, \preceq)$ be the partial order on $V(G)$ defined so that $v \preceq u$ if there is a directed path from v to u. Then the k words obtained from unshuffling w are linear extensions of $(P, \preceq)$.*

Proof. It suffices to show that whenever u is adjacent to v in $(P, \preceq)$ with $v \preceq u$ that $v^i < u^i$ for all i. This is immediate from the fact that u and v alternate in w. Now if $v \preceq u$ (with u not necessarily adjacent to v) the fact that $v^i < u^i$ follows by considering the cover relations along a directed path from v^i to u^i. □

Theorem 25. *Let G be a balanced bipartite graph on $2n$ vertices whose bipartite complement is both k-regular and does not contain C_4 as a subgraph. Then*

$$\mathcal{R}(G) \geqslant \left\lceil \frac{k(n-3k)}{4k-2} \right\rceil = n\left(\frac{k}{4k-2}\right) + O(k) = |V(G)|\left(\frac{k}{8k-4}\right) + O(k).$$

Proof. We denote the bipartite complement of G by G^*.

Let w be a r-uniform word representing G. We restrict ourselves to a subgraph H as guaranteed by Lemma 23. The restriction of w to the subword whose letters are the vertices in H is a r-uniform word representing H. Let w' be this restriction of w; we consider its unshuffling $w_1, \ldots, w_r$ as defined above.

Let X be the set of sources and let Z the set of sinks as defined in Lemma 23. We let H^* to be the induced subgraph of G^* restricted to the vertices of H.

For a $y \in Y$ and $i \in [r]$, let

$$P(y,i) = \{x \in X : x^{i+1} < y^i\} \cup \{z \in Z : z^i < y^i\}$$
$$S(y,i) = \{x \in X : x^i > y^i\} \cup \{z \in Z : z^{i-1} > y^i\}.$$

We proceed with a sequence of four simple claims.

Claim 1: *The edge $\{u, y\}$ is in H^* if and only if $u \in P(y,i)$ or $u \in S(y,i)$ for some i.*

If $u \notin P(y,i)$ or $S(y,i)$ for any i then either $u^i < y^i < u^{i+1}$ for all i or $y^{i+1} > u^i > y^i$ for all i depending on if $u \in X$ or $u \in Z$, respectively. In other words, u and y alternate and so that u and y are adjacent in H, and hence $\{u, y\}$ is not in H^*. This proves the forward direction.

Conversely, assume $\{u, y\}$ is an edge in H, and hence not in H^*. Let $u = x \in X$. Then $x^i < y^i$ by the definition and thus $u \notin S(y,i)$. Fix the minimal i for which $u \in P(y,i)$. Then w contains a subword $x^i x^{i+1} y^i$ – a contradiction. The case $u = z \in Z$ is similar. □

Let y_j^i denote the j-th vertex of Y to appear in word w_i, and let $t = |Y|$, so that y_t^i is the last element of Y to appear in word w_i.

Claim 2: *For all $1 \leqslant i \leqslant r$, $|S(y_1^i, i)| \leqslant k$ and $|P(y_t^i, i)| \leqslant k$.* □

By Claim 1, if $u \in S(y_1^i, i)$ then u and y_1^i are not adjacent in H. But y is not adjacent to at most k vertices in $X \cup Y$ as G^* is k-regular. The bound on $|P(y_t^i, i)|$ proceeds similarly. □

Claim 3: *For all $1 \leqslant i \leqslant r$ and $j \geqslant 2$, $|S(y_j^i, i)| \leqslant 1$ and $|P(y_{t-j+1}^i, i)| \leqslant 1$.*

Note that by construction $S(y_j^i, i) \subseteq S(y_1^i, i)$. If $u, u' \in S(y_j, i)$, then neither of u or u' would be adjacent to y_j^i or y_1^i in H by Claim 1. But this would be a C_4 in G^*, which is C_4-free. The bound for the $|P(y_{t-j+1}^i, i)|$ follows similarly. □

Claim 4: *For all $1 \leqslant i \leqslant r$ and $j \geqslant k+1$, $|S(y_j^i, i)| = |P(y_{t+1-j}^i, i)| = 0$.*

As $S(y_j^i, i) \subseteq S(y_{j'}^i, i)$ if $j' < j$, then $u \in S(y_j^i, i)$ implies that u is not adjacent to $s_1^i, s_2^i, \ldots, s_j^i$ in H. But as G^* is k-regular, u can be non-adjacent to at most k vertices in H. The bound for the $|P(y_{t-j+1}^i, i)|$ follows similarly. □

Now we combine the claims to yield the result. Claim 1 immediately implies that

$$\sum_{y \in Y} \sum_{i}^{r} (|P(y,i)| + |S(y,i)|) \geqslant |E(H^*)| \tag{2}$$

On the other hand for a fixed $1 \leqslant i \leqslant r$, Claims 2, 3 and 4 combine to show that

$$\sum_{y \in Y} (|P(y,i)| + |S(y,i)|) \leqslant 4k - 2.$$

Finally, we note that $|E(H^*)| \geqslant k(|X|+|Z|) - k(n-|Y|) \geqslant k(n-3k)$. Inserting into (2), we obtain that

$$r(4k-2) \geqslant k(n-3k),$$

and the result follows. □

Corollary 26. *Let G be a bipartite graph on $2n$ vertices whose bipartite complement is k-regular and does not contain C_4 as a subgraph. Then the minimum order dimension among posets for which G is a cover graph is bounded below by $\left\lceil \frac{k(n-3k)}{4k-2} \right\rceil$.*

Proof. Since G is bipartite, it is triangle-free, and so any poset for which G is a cover graph gives a semi-transitive orientation of G. Every semi-transitive orientation of such a G is considered in the previous proof, and when considering a set of linear extensions rather than an unshuffling of a word, the $P(y,i)$ and $S(y,i)$ sets for order dimension are subsets of those for representation number. □

Recall, the crown graph on $2n$ vertices - denoted $H_{n,n}$ - is a complete bipartite graph with a perfect matching deleted. In other words, it could be thought of as two independent sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ where the edge a_ib_i does not exist for each pair but all other edges between the sets do.

The same strategy as in the proof of Theorem 25 can be used as an alternate way to prove a 2016 result of Glen, Kitaev, and Pyatkin from [15] that the representation number of $H_{n,n}$ is $\lceil \frac{n}{2} \rceil$ for $n \geqslant 5$. The details can be inferred from the slight strengthening of a result analogous to Corollary 26:

Theorem 27. *Let $n \geqslant 4$. The minimum order dimension among posets for which $H_{n,n}$ is a cover graph is $\lceil \frac{n}{2} \rceil$.*

Proof. Given a poset, orienting every edge of its cover graph from covered element to covering element yields a semi-transitive orientation, so we can consider semi-transitive orientations of $H_{n,n}$.

Fix an arbitrary semi-transitive orientation. To prove the theorem, we slightly strengthen Lemma 23 in the case where the balanced bipartite graph in question is $H_{n,n}$. Let the partite sets of $H_{n,n}$ be denoted by A and B.

If each element of A is a source (or each a sink), then the poset is oriented as a standard example and has dimension n.

As in the proof of Lemma 23 we fix a $y \in A$ which is neither a source nor sink, let X denote its in-neighbors, and Z denote the out neighbors, so that $|X| + |Z| = n - 1$. If $|X| = 1$ (or $|Z| = 1$) we proceed precisely as in Lemma 23 − let Y denote the common neighborhood of the $x \in X$ and $z \in Z$. Assuming $|X| = 1$, then $|Z| = n - 2$ and after applying Lemma 22 we note that (Y, Z) is a standard example $H_{n-2,n-2}$.

Otherwise, if $|X| \geqslant 2$ and $|Z| \geqslant 2$, we observe that we can take Y to be all of A: Indeed, suppose $w \in A$, and suppose $x \in X$ with w adjacent to x. Then w is also adjacent to some $z \in Z$ (as w is adjacent to all but at most one element of Z and $|Z| \geqslant 2$), and applying Lemma 22 to $\{x, y, z, w\}$ implies that xw is oriented from x to w. Similarly, all edges from w are oriented to Z.

Finally, we show that y' – the vertex in B not adjacent to y – is either a source or a sink. Otherwise, suppose u is an out-neighbor of y' and v is an in-neighbor. Let t be a common neighbor of u and v that is not y' – such exists as $n \geqslant 4$, and u and v only have one non-neighbor in B. Then Lemma 22 applied to $\{v, y', u, t\}$ would imply that t is neither a source nor sink – contradicting what we have already observed. We hence can add y' to either X or Z.

Thus we have shown that the conclusion of Lemma 23 holds with $|X| + |Z| = n$ and $|Y| = n$. One of $|X|, |Z| \geqslant \lceil \frac{n}{2} \rceil$. Taking the larger of these sets (say X) and their partners in Y gives a standard example with dimension $|X|$.

We now construct a poset that meets this bound. Assume that n is even since the odd case follows from the next largest even case. Let $X = \{x_1, \ldots, x_{n/2}\}$, $Y = \{y_1, \ldots, y_n\}$, and $Z = \{z_{n/2+1}, \ldots, z_n\}$ and P be the poset where y_i covers x_j if $j \neq i$ and z_i covers y_j if $j \neq i$. It can be seen that a realizer for this poset can be constructed by having $\frac{n}{2}$ linear extensions that each list both some y_i before x_i and $z_{n/2+1+i}$ before $y_{n/2+1+i}$. The incomparabilities within X, Y, and Z are then simple to eliminate with these linear extensions, and an explicit construction is more tedious than enlightening. □

5 Applications and Examples

In this section we show a number of applications of the work above to study the word-representations of graphs.

5.1 The n-Cube

The n-dimensional hypercube, denoted Q_n, is obtained by taking the Cartesian product of K_2 with itself n-times. This graph can be viewed as the graph on the 2^n length n words in 0/1, with adjacencies between words differing in exactly one position. The *weight* of a word is the number of ones in the word.

In [7] it was proved that Q_n is n-representable. Based on this, and the fact the representation number of Q_n was n for $n = 1, 2, 3$, it was conjectured that the representation number of Q_n was n for all n. This, however, turns out not to be the case.

Theorem 28. $\mathcal{R}(Q_n) = O\left(\frac{\log(n)}{\log\log(n)}\right)$.

In order to prove Theorem 28, we introduce the **boxicity** of a graph. Let $a_1 \cdots a_n$ and $b_1 \cdots b_n$ be elements of $\mathbb{R}$. A k-box is defined as $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$. A k-box representation of a graph G is a mapping of G into k-dimensional Euclidean space such that two vertices are adjacent in G if and only if their corresponding k-boxes have a non-empty intersection. The boxicity of G, denoted $box(G)$, is the minimal d such that G has a d-box representation [1].

Adiga et al. show in [1] that if $P = (V, P)$ is a height 2 poset and G_P is its underlying comparability graph then $box(G_P) \leqslant dim(P) \leqslant 2 \cdot box(G_P)$. In [9], Chandran et al. show that the boxicity of the nth Cartesian product of any finite graph is $O(\log(n)/\log\log(n)))$.

Proof of Theorem 28. We combine the previous two results. Consider the height 2 partial order and semi-transitive orientation constructed by viewing the hypercube as a bipartite graph between the words of even weight and those of odd weight, and directing all edges towards (say) the words of odd weight. (Note that this is *not* the standard order on the boolean lattice/hypercube.)

The hypercube is the comparability graph of this order. Now we combine the above results: This partial order has boxicity $O(\frac{\log n}{\log\log n})$, and hence dimension $O(\frac{\log n}{\log\log n})$. Finally, as Q_n is the comparability graph of this poset, Lemma 11 implies that the representation number is also $O(\frac{\log n}{\log\log n})$. □

Interestingly, a similar $O((\log n)/(\log\log n))$ bound holds for the nth power of *any* bipartite graph by the same argument, as $n \to \infty$. For instance, although the crown graph $H_{k,k}$ is known to have representation number $\lceil k/2 \rceil$, an n-fold Cartesian product of H_k still has representation number $O((\log n)/(\log\log n))$ as $n \to \infty$.

Remark 29. While Theorem 28 implies that for n sufficiently large, $\mathcal{R}(Q_n)$ is *much* smaller than n, it is still interesting to find the lowest such n. In particular, is $\mathcal{R}(Q_4) = 4$?

5.2 General Bipartite Graphs

Building on the observations made above about the n-cube, we note the following result of Scott and Wood from [28].

Proposition 30. *Every poset whose comparibility graph has maximum degree at most Δ has dimension at most*

$$\Delta \log^{1+o(1)} \Delta.$$

Applying this to bipartite graphs, viewed as comparability graphs of height two posets, we observe that

Theorem 31. *If G is bipartite with maximum degree Δ, then $\mathcal{R}(G) \leqslant \Delta \log^{1+o(1))} \Delta$.*

5.3 Highly Chromatic Graphs of Large Girth

In [16], a general upper bound is shown for the representation number of graphs. It is shown that every graph with clique number k has representation number at most $2(n-k)$. For triangle-free graphs, $k = 2$ and this is a rather large upper bound. One, then, wonders, how hard is it to represent fairly complicated triangle-free graphs.

To this end, we observe that graphs of high girth and high chromatic number – which are well-known to be complicated to construct – can be represented fairly efficiently.

Theorem 32. *There are graphs G of girth k and chromatic number $n^{\frac{1}{2k-3}+o(1)}$ with $\mathcal{R}(G) \leqslant n^{\frac{2k-4}{2k-3}+o(1)}$.*

Proof. This follows from Corollary 18, via the construction of Suk and Tomon [30]. Indeed, in Suk and Tomon's construction, the chromatic number is bounded by bounding the size of an independent set in the Hasse diagram. Since the independence number in the Hasse diagram is an upper bound for the width, the result follows. □

In the case of triangle-free graphs, where the girth is 4, we can do slightly better. In [10], Chen, Pach, Szegedy and Tardos show that the Hasse diagram, H, of n random points in $[0,1]^2$ under the dominance order – that is, $(a,b) \preceq (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$ – has independence number

$$\alpha(H) = O\left(\frac{n(\log\log n)^2}{\log n}\right) = o(n),$$

with high probability. As explained in [10], this is well-known to be a partial order of dimension 2.

On the other hand, antichains in the dominance order correspond to decreasing subsequences in the random permutation represented by the points – the permutation from the x order to the y order of the points. The longest decreasing subsequence – that is, the width of the partial order – was famously shown to have a scaled Tracy–Widom distribution by Baik, Deift and Johansson in [3] with mean $2\sqrt{n}$. From this, the width is $O(\sqrt{n})$ with high probability. Thus, with probability tending to one, the obtained random Hasse diagram has width $O(\sqrt{n})$, and independence number $o(n)$ – so that $\chi(H) = \omega(1)$. In all we have obtained

Theorem 33. *For any $k \in \mathbb{N}$, there exist triangle-free word-representable graphs G on n vertices, whose chromatic number is at least k, with $\mathcal{R}(G) = O(\sqrt{n})$.*

5.4 Point-Line (Non-)Incidence Graphs of Finite Projective Planes

The finite projective planes $PG(2,q)$ are linear hypergraphs $H = (\mathcal{P}, \mathcal{L})$ on q^2+q+1 points, with the property that every point is contained in $q+1$ hyperedges (lines) and every line contains in $q+1$ points; hence there are q^2+q+1 lines as well.

The point-line incidence graph, G_q^*, of $PG(2,q)$ is a bipartite graph on $(\mathcal{P}, \mathcal{L})$ where $p \sim \ell$ if and only if $p \in \ell$. These graphs have many nice properties; for the point of view of this example we highlight that G_q^* is $(q+1)$-regular, on $2(q^2+q+1)$ vertices, and has girth 6. G_q^* is known to be an extremal C_4-free bipartite graph (see [26]). Thus if G_q is the bipartite complement of G_q^* (i.e. a bipartite graph on $(\mathcal{P}, \mathcal{L})$ where $p \sim \ell$ if and only if $p \notin \ell$) then, Theorem 25 implies the first part of

Theorem 34. *If G_q is the point-line non-incidence graph of $PG(2,q)$,*

$$\mathcal{R}(G_q) \geqslant (1-o(1))\frac{q^2+q+1}{4} = (1-o(1))\frac{|V(G_q)|}{8},$$

while the largest induced crown $H_{t,t}$ in G_q has

$$t = O(q^{3/2}) = O(|V(G_q)|^{3/4}).$$

To bound the size of the largest crown in G_q, we observe that a crown in G_q corresponds to an induced matching in G_q^*. For this, it suffices to (lower) bound the number of point-line incidences in $PG(2,q)$ for a set of points $P \subseteq \mathcal{P}$ and set of lines $L \subseteq \mathcal{L}$ where $|P| = |L|$. If the number of incidences $I(P,L) > |L|$, then necessarily there can't be an induced matching between P and L, as this is the number of edges between the sets in G_q^*.

To this end, we apply an estimate proved by Vinh in [32] and made explicit – Vinh technically stated only an upper bound for incidences – by Lund and Saraf in [25]. Lund and Saraf (see Theorem 1 of [25]; we specialize to the case where $r = q+1,\ \lambda = 1,\ |B| = q^2+q+1$) prove that

$$\left| I(P,L) - \frac{(q+1)|P||L|}{q^2+q+1} \right| < \sqrt{q|P||L|}.$$

Now a simple computation shows that if $|P| = |L| = 2q^{3/2}$, then $|I(P,L)| > |P|$ for q sufficiently large, and hence no induced matching exists of this size exists. This bound on the largest induced matching in G_q^*, which likewise bounds the size of the largest crown in G_q, completes the proof of Theorem 34.

5.5 Dimension and Representation Numbers

Following Theorem 17, and the discussion above, two natural questions stand out. First, is it possible to improve the bound from Theorem 17, perhaps even proving that the representation number of a Hasse diagram is at most the dimension? Second – and perhaps more temptingly – is the dimension of a poset a *lower* bound on the representation number of a graph.

We now give examples ruling out these possibilities, at least in their strongest sense.

To make the discussion more precise, we give one more definition. Consider a word-representable graph equipped with a semi-transitive orientation. We define the **representation number of the orientation** to be the minimum k so that there is a k-uniform word *yielding that orientation.* The representation number, then, is the minimum over all different semi-transitive orientations.

Proposition 35. *There are graphs with semi-transitive orientations, so that the representation number of the orientation is strictly smaller than the order dimension of the underlying poset.*

Proof. As remarked above, the crown graph $H_{n,n}$ has this property. Indeed, there are $\lceil \frac{n+1}{2} \rceil$-representations of $H_{n,n}$ whose canonical orientation agrees with that of the standard example which has dimension n. For $n = 4$, the word

$$x_1x_2x_3y_4x_4y_1y_2x_3y_3x_4x_2y_1x_1y_3y_4x_2y_2x_4x_3x_1y_3y_2y_1y_4$$

yields a 3-representation of $H_{4,4}$ whose orientation is the standard example.

To construct these words for $n \geqslant 5$, we construct $t = \lceil (n+1)/2 \rceil$ permutations of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ that can be combined to yield the desired word. These permutations have the following properties:

- First,

$$\pi_1 = x_1 x_2 \cdots x_n \qquad \sigma_1 = y_n y_{n-2} y_{n-3} \cdots y_1 y_{n-1}$$
$$\pi_2 = x_{n-1} x_n x_{n-2} x_{n-3} \cdots x_1 \qquad \sigma_2 = y_1 y_{n-1} y_2 \cdots y_{n-3} y_n y_{n-2}$$

  These permutations appear in the words and have the effect of ensuring that the only possible edges between the x's is between x_{n-1} and x_n and in the y's between y_1 and y_{n-1} and y_n and y_{n-2}.

- π_3 must start with x_{n-2}, contain $x_n x_{n-1}$ in the middle in that order, and end with x_2.

  Since π_3 ends with x_2, σ_3 starts with y_2, contains $y_{n-1} y_1 y_{n-2} y_n$ in the middle in that order, and ends with a y_i not previously appearing at the start or end of a word. If $n = 5$, there is no such vertex but by construction $\sigma_3 = y_2 y_4 y_1 y_3 y_5$.

- For $4 \leqslant s \leqslant t$: If y_j ends σ_{i-1}, then x_j starts π_s, and the final vertex of π_s should not have been the start or end of any previous π_i. (This is always possible – $2(s-1) < n$ vertices previously started or ended words as x_1 appeared twice.) The order of the rest does not matter.

  If x_j ends π_s, then σ_s should start with y_j. The last vertex of σ_s should also be distinct from vertices starting/ending previous σ_i – if n is odd, however, this will not be possible when $s = t$. Once $s = t$ all letters have appeared as the start/end of a permutation.

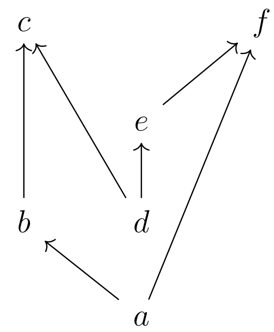
From these permutations the word is constructed by considering the permutations $\pi_1, \sigma_1, \pi_2, \sigma_2, \ldots, \pi_s, \sigma_s$ and interlacing the last vertex of π_i with the first vertex of σ_i, and the last vertex of σ_i with the first vertex of π_{i+1}. Edges between x_i and x_j and y_i and y_j are guaranteed to not exist by the construction of π_1, π_2, π_3 and σ_1, σ_2 and σ_3. The missing edge between y_j and x_j is achieved by the fact that either x_j will end one π_i and start σ_i or end a σ_i and start π_{i+1} for some i by construction. □

Remark 36. As observed, the difference between the order dimension of the underlying poset, and the representation number can be arbitrarily far apart. Still, the following is of interest: For an integer t let $f(t)$, if it exists, denote the largest dimension of a semi-transitive orientation of a graph which has representation number t. What is the behavior of $f(t)$? The standard example shows that $f(t) \geqslant 2t$ if it exists. It would be interesting to get an upper bound on $f(t)$. It would be even more interesting to show that $f(t)$ does not exist for some t – that there is a family of (semi-transitive orientations of) graphs so that the representation number is bounded, but the poset dimension unbounded.

Remark 37. As noted above, the graph $H_{n,n}$ was shown in [15] to have representation number $\lceil \frac{n}{2} \rceil$ for $n \geqslant 5$, while here we've shown that the orientation as a standard example has a representation number of $\lceil \frac{n+1}{2} \rceil$. The difference between these bounds can be seen in the proof of Theorem 25: for the standard example orientation, one edge at a time can be removed either as a transposition in a linear extension or as a 'shuffle'. With t linear extensions, however, there are only $t-1$ potential shuffles. Other orientations allow one to remove two edges using transposition in a single linear extension and this can make up for the missing 'shuffle.'

Proposition 38. *There are semi-transitively oriented graphs so that the representation number of the orientation is strictly greater than the poset dimension.*

Proof. The following orientated graph has representation number (for the given orientation) three, while the order dimension of the underlying poset is 2.



If it were possible to represent this orientation with a 2-uniform word, we may assume without loss of generality that such a word begins with a. Then there are two ways for ad to be a non-edge:

- If the pattern of a and d is $aadd$, then the pattern of a, d, and c must be $aadcdc$ and so f cannot be placed into this pattern to respect its edges in the graph (since the first e must appear after the first d and the first f must appear after the first e).
- If the pattern of a and d is $adda$, then the pattern of a, d, and c must be $adcdca$ and so b cannot be placed into this pattern to respect its edges in the graph.

Thus this orientation is not 2-representable. But it is represented by the 3-uniform word
$adbcedfeafbacbdcef$, and it is the intersection of the linear orders $a < b < d < c < e < f$ and $d < e < a < f < b < c$. ☐

Remark 39. Here, the underlying graph is a 6-cycle, which – as a graph – has representation number two. It is an interesting problem to construct a graph – not just an orientation – of representation number three, where there is an underlying poset of dimension two. It also would be of interest to construct examples where the difference between the parameters is arbitrarily large.

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