The enriched q-monomial basis of the quasisymmetric functions

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Abstract

We construct a new family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ of quasisymmetric functions for each element q of the base ring. We call them the "enriched q-monomial quasisymmetric functions". When $r := q+1$ is invertible, this family is a basis of QSym. It generalizes Hoffman's "essential quasi-symmetric functions" (obtained for $q = 0$) and Hsiao's "monomial peak functions" (obtained for $q = 1$), but also includes the monomial quasisymmetric functions as a limiting case.

We describe these functions $\eta_\alpha^{(q)}$ by several formulas, and compute their products, coproducts and antipodes. The product expansion is given by an exotic variant of the shuffle product which we call the "stufufuffle product" due to its ability to pick several consecutive entries from each composition. This "stufufuffle product" has previously appeared in recent work by Bouillot, Novelli and Thibon, generalizing the "block shuffle product" from the theory of multizeta values.

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Contents

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1 Introduction

Among the combinatorial Hopf algebras that consist of power series in commuting indeterminates, one of the largest and most all-embracing is that of the *quasisymmetric functions*, called QSym. Originally introduced by Gessel in 1984 [Gessel84], it has since found applications (e.g.) to enumerative combinatorics ([Sagan20, Chapter 8], [Stanle24, §7.19–7.23], [GesZhu18]), multizeta values (e.g., [Hoffma15]), algebraic geometry ([Oesing18]) and the representation theory of 0-Hecke algebras ([Meliot17, §6.2]).

It was observed by Ehrenborg ([Ehrenb96, Lemma 4.2]; see [Biller10, §3.3] for a survey) that quasisymmetric functions can also be used to encode the "flag f-vector" of a finite graded poset – i.e., essentially, the number of chains over a given sequence of ranks, for each possible sequence of ranks. Soon after, work of Bergeron, Mykytiuk, Sottile and van Willigenburg ([BMSW00, Example 5.3], but see [Biller10, §3.4] for an explicit statement) showed that if the graded poset is Eulerian (a property shared by face posets of polytopes and simplicial spheres), then the resulting quasisymmetric function is not arbitrary but rather belongs to a certain subalgebra of QSym called *Stembridge's Hopf algebra* or the *peak algebra* or the *odd subalgebra* Π_- *of* QSym. It was initially defined by Stembridge [Stembr97, §3] in order to find a fundamental expansion of the Schur P- and Q-functions, and has since been studied by others for related and unrelated reasons ([AgBeSo06, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5], [Hsiao07] etc.); among other properties, it is a Hopf subalgebra of QSym.

Almost all bases of QSym constructed so far are indexed by *compositions* (i.e., tuples of positive integers), and their structure constants are often governed by versions of shuffle products and deconcatenation coproducts. The peak algebra is smaller, and its bases are often indexed by *odd compositions*, i.e., compositions whose entries are all odd. One of its simplest bases is defined as follows (for the sake of simplicity, we use $\mathbb{Q}$ as a base ring here): If $n \in \mathbb{N}$ and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n (that is, a tuple of positive integers with $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$), then we define the formal power series

$$\eta_\alpha = \sum_{\substack{1 \leqslant g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i = g_{i+1} \text{ for each } i \in E(\alpha)}} 2^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \qquad (1)$$
$$\in \mathbb{Q}\left[\left[x_1, x_2, x_3, \ldots\right]\right],$$

where $E(\alpha)$ denotes the set $\{1, 2, \ldots, n-1\} \setminus \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid 0 < i < \ell\}$. This η_α belongs to the $\mathbb{Q}$-algebra QSym of quasisymmetric functions over $\mathbb{Q}$. If we let α range over all *odd compositions* (i.e., compositions $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ whose entries α_i are all odd), then the η_α form a basis of the peak algebra over $\mathbb{Q}$.

In this form, the power series η_α have been introduced by Hsiao ([Hsiao07, Proposition 2.1], although his η_α differ from ours by a sign), who called them the *monomial peak functions*. Hsiao computed their products, coproducts (in the sense of Hopf algebra) and antipodes, and obtained some structural results for the peak algebra.

In this paper, we generalize the η_α by replacing the power of 2 in (1) by a power of an arbitrary element r of the base ring. We furthermore study the resulting quasisymmetric functions for all compositions α (not only for the odd ones). Thus we obtain a new family $\left(\eta_\alpha^{(q)}\right)_{\alpha \text{ is a composition}}$ of quasisymmetric functions for each element q of the base ring. When $r := q+1$ is invertible, this family is a basis of QSym. It generalizes Hoffman's "essential quasi-symmetric functions" (obtained for $q = 0$) and Hsiao's monomial peak functions (obtained for $q = 1$), but also includes the monomial quasisymmetric functions as a limiting case.

We call our functions $\eta_\alpha^{(q)}$ the *enriched q-monomial quasisymmetric functions*. We describe them by several formulas, and compute their products, coproducts and antipodes (generalizing Hsiao's results). The product expansion is the most interesting one, as it is given by an exotic variant of the shuffle product which we call the "stufufuffle product" due to its ability to pick several consecutive entries from each composition. This "stufufuffle product" has previously appeared in recent work by Bouillot, Novelli and Thibon [BoNoTh22, (1)], where it was proposed as a generalization of the "block shuffle product" from the theory of multizeta values ([HirSat22]). While the authors of [BoNoTh22] have already found a basis of QSym that multiplies according to this product, ours is simpler and more natural. The coproduct and antipode formulas for $\eta_\alpha^{(q)}$ are fairly simple (the coproduct is given by deconcatenation, whereas the antipode involves the parameter q being replaced by its reciprocal $1/q$ and the composition α being reversed). We also express the functions $\eta_\alpha^{(q)}$ in terms of the monomial and fundamental bases of QSym and vice versa. Finally, we discuss how Hopf subalgebras of QSym can be constructed by picking a subset of the set of all compositions. (This generalizes the peak subalgebra.)

This paper is the first of (at least) two. The next shall extend the theory of extended P-partitions to incorporate the parameter q, which will shed a new light onto the enriched q-monomial quasisymmetric functions $\eta_\alpha^{(q)}$ while also leading to a new basis of QSym.

Several results in this paper have appeared (without proof) in the extended abstracts [GriVas21] and [GriVas22].

1.1 Structure of the paper

This paper is organized as follows:

We begin by recalling the definition of quasisymmetric functions (and some concomitant notions) in Section 2.

Then, in Section 3, we define the quasisymmetric functions $\eta_\alpha^{(q)}$ and prove their simplest properties (conversion formulas to the M- and L-bases, formulas for antipode and coproduct). In particular, we show that the family of these functions $\eta_\alpha^{(q)}$ (where α ranges over all compositions) forms a basis of QSym if and only if $r := q + 1$ is invertible in the base ring.

Consequently, in Section 4, we introduce and study the basis of NSym dual to this basis of QSym.

In Section 5, we use this to express the product $\eta_\delta^{(q)}\eta_\varepsilon^{(q)}$ in three equivalent ways.

Finally, we discuss some applications in Section 6, and establish one last formula for $\eta_\alpha^{(q)}$ in Section A.

Remark 1. This paper (in its present version) is written with an expert reader in mind; folklore results and well-known shortcuts are used without proof. Some simple proofs are omitted; others are sketched and/or relegated to an appendix. A more elementary version, which includes such arguments in detail, can be found at [GriVas23a] (and an even more detailed version as an ancillary file to [GriVas23a]). The sheer number of different formulas surrounding the functions $\eta_\alpha^{(q)}$ renders even the present version fairly long; we hope that the structure can serve as a guide.

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2 Quasisymmetric functions

2.1 Formal power series and quasisymmetry

We will use some of the standard notations from [GriRei20, Chapter 5]. Namely:

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- We fix a commutative ring $\mathbf{k}$.
- We consider the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many commuting variables $x_1, x_2, x_3, \ldots$. A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ is a sequence of nonnegative integers such that only finitely many α_i are positive. Formal power series are formal infinite $\mathbf{k}$-linear combinations of such monomials.
- Each monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ has degree $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$.
- A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *of bounded degree* if there exists some $d \in \mathbb{N}$ such that each monomial in f has degree $\leqslant d$ (that is, each monomial of degree $> d$ has coefficient 0 in f).

We now recall the definition of the quasisymmetric functions:

Definition 2. **(a)** Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent* if they can be written in the forms

$$\mathfrak{m} = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell} \qquad \text{and} \qquad \mathfrak{n} = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$$

for some $\ell \in \mathbb{N}$, some positive integers $a_1, a_2, \ldots, a_\ell$ and two strictly increasing ℓ-tuples $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers. (For example, the monomials $x_1^4 x_3^7 x_4 x_9^2$ and $x_3^4 x_4^7 x_{10} x_{16}^2$ are pack-equivalent.)

(b) A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if it has the property that any two pack-equivalent monomials have the same coefficient in f (that is: if $\mathfrak{m}$ and $\mathfrak{n}$ are two pack-equivalent monomials, then the coefficient of $\mathfrak{m}$ in f equals the coefficient of $\mathfrak{n}$ in f).

(c) A *quasisymmetric function* means a formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that is quasisymmetric and of bounded degree.

Quasisymmetric functions have been introduced by Gessel in [Gessel84] (for $\mathbf{k} = \mathbb{Z}$ at least, but the general case is not much different). Introductions to their theory can be found in [GriRei20, Chapters 5–6], [Stanle24, §7.19], [Sagan20, Chapter 8], [Malven93, §4] and various other texts.

It is known (see [Malven93, Corollaire 4.7] or [GriRei20, Proposition 5.1.3]) that the set of all quasisymmetric functions is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. It is denoted by QSym and called the *ring of quasisymmetric functions*. It has several bases (as a $\mathbf{k}$-module), most of which are indexed by compositions.

2.2 Compositions

A *composition* means a finite list $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers. The set of all compositions is denoted by Comp. The *empty composition* $\varnothing$ is the composition $()$, which is a 0-tuple.

The *length* $\ell(\alpha)$ of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is defined to be the number k.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α and is denoted by $|\alpha|$. For any $n \in \mathbb{N}$, we define a *composition of* n to be a composition that has size n. We let Comp_n be the set of all compositions of n (for given $n \in \mathbb{N}$). For example, $(1, 5, 2, 1)$ is a composition with size 9 (since $|(1, 5, 2, 1)| = 1 + 5 + 2 + 1 = 9$), so that $(1, 5, 2, 1) \in \mathrm{Comp}_9$.

For any $n \in \mathbb{Z}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. This set is empty whenever $n \leqslant 0$, and otherwise has size n.

It is well-known that any positive integer n has exactly 2^{n-1} compositions. This has a standard bijective proof ("stars and bars") which is worth recalling, as the bijection itself will be used a lot:

Definition 3. Let $n \in \mathbb{N}$. Let $\mathcal{P}([n-1])$ be the powerset of $[n-1]$ (that is, the set of all subsets of $[n-1]$).

We define a map $D : \mathrm{Comp}_n \to \mathcal{P}([n-1])$ by

$$\begin{aligned} D(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [k-1]\} \\ &= \{\alpha_1,\ \ \alpha_1 + \alpha_2,\ \ \alpha_1 + \alpha_2 + \alpha_3,\ \ \ldots,\ \ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}. \end{aligned}$$

This map D is well-known to be a bijection. (See, e.g., [Grinbe15, detailed version, Proposition 10.17] for a detailed proof of this.)

For example, for $n = 8$, we have $D(2, 1, 3, 2) = \{2,\ 2+1,\ 2+1+3\} = \{2, 3, 6\}$. The notation D presumably originates in the word "**d**escent", but the connection between D and actual descents is indirect and rather misleading. We prefer to call D the "partial sum map" (as $D(\alpha)$ consists of the partial sums of the composition α). Note that Stanley, in [Stanle24, §7.19], writes S_α for $D(\alpha)$ when $\alpha \in \mathrm{Comp}_n$, and writes $\mathrm{co}(I)$ for $D^{-1}(I)$ when $I \in \mathcal{P}([n-1])$.

Note that every composition α of size $|\alpha| > 0$ satisfies $|D(\alpha)| = \ell(\alpha) - 1$, so that $|D(\alpha)| + 1 = \ell(\alpha)$. But this fails if α is the empty composition $\varnothing = ()$ (since $D() = \varnothing$ and $\ell() = 0$).

2.3 The monomial and fundamental bases of QSym

We will only need two bases of QSym: the monomial basis and the fundamental basis.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition, then we define the *monomial quasisymmetric function* $M_\alpha \in \mathrm{QSym}$ by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}. \tag{2}$$

For example,

$$M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \cdots .$$

The family $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, and is known as the *monomial basis* of QSym.

For any composition α, we define the *fundamental quasisymmetric function* $L_\alpha \in \mathrm{QSym}$ by

$$L_\alpha = \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ D(\beta) \supseteq D(\alpha)}} M_\beta, \tag{3}$$

where $n = |\alpha|$ (so that $\alpha \in \mathrm{Comp}_n$). It is not hard to rewrite this as

$$L_\alpha = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (\text{again with } n = |\alpha|) .$$

This quasisymmetric function L_α was originally called F_α in Gessel's paper [Gessel84] (and in some later work such as [Malven93]), but the notation L_α has since spread more widely.

The family $(L_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, and is known as the *fundamental basis* of QSym.

3 The enriched q-monomial functions

3.1 Definition and restatements

Convention 1. From now on, we fix an element q of the base ring $\mathbf{k}$. We set

$$r := q + 1.$$

We shall now introduce a new family of quasisymmetric functions depending on q:

Definition 4. For any $n \in \mathbb{N}$ and any composition $\alpha \in \operatorname{Comp}_n$, we define a quasisymmetric function $\eta_\alpha^{(q)} \in \operatorname{QSym}$ by

$$\eta_\alpha^{(q)} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta. \tag{4}$$

We shall refer to $\eta_\alpha^{(q)}$ as the *enriched q-monomial function* corresponding to α.

Example 5.

(a) Setting $n = 5$ and $\alpha = (1, 3, 1)$ in (4), we obtain

$$\eta_{(1,3,1)}^{(q)} = \sum_{\substack{\beta \in \operatorname{Comp}_5; \\ D(\beta) \subseteq \{1,4\}}} r^{\ell(\beta)} M_\beta \qquad (\text{since } D(1,3,1) = \{1,4\})$$
$$= r^{\ell(5)} M_{(5)} + r^{\ell(1,4)} M_{(1,4)} + r^{\ell(4,1)} M_{(4,1)} + r^{\ell(1,3,1)} M_{(1,3,1)}$$

(since the compositions $\beta \in \operatorname{Comp}_5$ satisfying $D(\beta) \subseteq \{1,4\}$ are (5), $(1,4)$, $(4,1)$ and $(1,3,1)$). This simplifies to

$$\eta_{(1,3,1)}^{(q)} = r M_{(5)} + r^2 M_{(1,4)} + r^2 M_{(4,1)} + r^3 M_{(1,3,1)}.$$

(b) For any positive integer n, we have

$$\eta_{(n)}^{(q)} = r M_{(n)},$$

because the only composition $\beta \in \operatorname{Comp}_n$ satisfying $D(\beta) \subseteq D(n)$ is the composition (n) itself (since $D(n)$ is the empty set $\varnothing$) and has length $\ell(n) = 1$. Likewise, the empty composition $\varnothing = ()$ satisfies

$$\eta_\varnothing^{(q)} = M_\varnothing = 1.$$

The quasisymmetric function $\eta_\alpha^{(q)}$ generalizes several known power series. For $q = 0$, the series $\eta_\alpha^{(q)} = \eta_\alpha^{(0)}$ is the "essential quasi-symmetric function" E_I (for $I = D(\alpha)$) defined in [Hoffma15, (8)]. When α is an odd composition (i.e., all entries of α are odd) and $q = 1$, the series $\eta_\alpha^{(q)} = \eta_\alpha^{(1)}$ is precisely the η_α defined in [AgBeSo06, (6.1)], and differs only in sign from the η_α given in [Hsiao07, (2.1)] (because of [Hsiao07, Proposition 2.1]). (This is the reason for the notation $\eta_\alpha^{(q)}$.) Finally, in an appropriate sense, we can view M_α as the "$q \to \infty$ limit" of $\eta_\alpha^{(q)}$; to be precise, this is saying that when $\eta_\alpha^{(q)}$ is considered as a polynomial in q (over QSym), its leading term is $q^{\ell(\alpha)} M_\alpha$ (which is obvious from (4) and $r = q + 1$).

The following two propositions are essentially restatements of (4) (see the Appendix for proofs):

Proposition 6. *Let $n \in \mathbb{N}$ and $\alpha \in \operatorname{Comp}_n$. Then,*

$$\eta_\alpha^{(q)} = \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha)}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}, \tag{5}$$

where the sum is over all weakly increasing n-tuples $(g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n)$ of positive integers that satisfy $(g_i = g_{i+1}$ for each $i \in [n-1] \setminus D(\alpha))$.

Proposition 7. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}$. Then,*

$$\eta_\alpha^{(q)} = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_\ell} r^{|\{i_1, i_2, \ldots, i_\ell\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \tag{6}$$

where the sum is over all weakly increasing ℓ-tuples $(i_1 \leqslant i_2 \leqslant \cdots \leqslant i_\ell)$ of positive integers.

3.2 The $\eta_\alpha^{(q)}$ as a basis

The equality (4) writes each enriched q-monomial function $\eta_\alpha^{(q)}$ as a $\mathbf{k}$-linear combination of M_β's. Conversely, we can expand each monomial quasisymmetric function M_β as a $\mathbf{k}$-linear combination of $\eta_\alpha^{(q)}$'s, at least after multiplying it by $r^{\ell(\beta)}$:

Proposition 8. *Let $n \in \mathbb{N}$. Let $\beta \in \operatorname{Comp}_n$ be a composition. Then,*

$$r^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \operatorname{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha^{(q)}.$$

For the proof of this proposition (and some later ones as well), we will need the *Iverson bracket notation*:

Convention 2. If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$ (that is, the number 1 if $\mathcal{A}$ is true, and the number 0 if $\mathcal{A}$ is false).

For example, $[2+2=4] = 1$ and $[2+2=5] = 0$.

The following lemma is a classical elementary property of finite sets:

Lemma 9. *Let S and T be two finite sets. Then,*

$$\sum_{\substack{I \subseteq S; \\ T \subseteq I}} (-1)^{|S| - |I|} = [S = T].$$

Proof. See [GriVas23a]. (In a nutshell: If $S = T$ or $T \not\subseteq S$, then this is obvious. Otherwise, fix some element $g \in S \setminus T$ and pair up the addends on the left hand side so that any pair cancels each other out. Specifically, pair the addend corresponding to a subset $I = J$ with the addend corresponding to the subset $I = J \triangle \{g\}$, where the symbol $\triangle$ means "symmetric difference".) ☐

We will also use the following near-trivial property of compositions ([GriVas23b, Corollaries 2.6 and 2.7]):

Lemma 10. *Let $n \in \mathbb{N}$. Then:*

(a) *We have $\ell(\delta) = |D(\delta)| + [n \neq 0]$ for each $\delta \in \operatorname{Comp}_n$.*

(b) *We have $\ell(\beta) - \ell(\alpha) = |D(\beta)| - |D(\alpha)|$ for any $\alpha \in \operatorname{Comp}_n$ and $\beta \in \operatorname{Comp}_n$.*

Proof of Proposition 8. We have

$$\begin{aligned}
&\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}\,\eta_\alpha^{(q)} \\
&= \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} r^{\ell(\gamma)} M_\gamma \qquad \text{(by (4), with } \beta \text{ renamed as } \gamma) \\
&= \sum_{\gamma\in\operatorname{Comp}_n} r^{\ell(\gamma)} M_\gamma \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}. \qquad (7)
\end{aligned}$$

However, for each $\gamma \in \operatorname{Comp}_n$, we have

$$\begin{aligned}
&\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} \underbrace{(-1)^{\ell(\beta)-\ell(\alpha)}}_{\substack{=(-1)^{|D(\beta)|-|D(\alpha)|}\\ \text{(by Lemma 10 (b))}}} \\
&= \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} (-1)^{|D(\beta)|-|D(\alpha)|} = \sum_{\substack{I\subseteq[n-1];\\ D(\gamma)\subseteq I\subseteq D(\beta)}} (-1)^{|D(\beta)|-|I|} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } I \text{ for } D(\alpha) \text{ in the sum,}\\ \text{since the map } D : \operatorname{Comp}_n \to \mathcal{P}([n-1]) \text{ is a bijection}\end{array}\right) \\
&= \sum_{\substack{I\subseteq D(\beta);\\ D(\gamma)\subseteq I}} (-1)^{|D(\beta)|-|I|} \qquad \left(\begin{array}{c}\text{since } D(\beta) \subseteq [n-1]\text{, so that each subset } I\\ \text{of } D(\beta) \text{ is also a subset of } [n-1]\end{array}\right) \\
&= [D(\beta) = D(\gamma)] \qquad \text{(by Lemma 9, applied to } S = D(\beta) \text{ and } T = D(\gamma)) \\
&= [\beta = \gamma] \qquad \text{(since the map } D \text{ is a bijection)}.
\end{aligned}$$

Plugging this into (7), we find

$$\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}\,\eta_\alpha^{(q)} = \sum_{\gamma\in\operatorname{Comp}_n} r^{\ell(\gamma)} M_\gamma\,[\beta = \gamma] = r^{\ell(\beta)} M_\beta$$

(since the factor $[\beta = \gamma]$ in the sum ensures that the only nonzero addend in the sum is the addend for $\gamma = \beta$). This proves Proposition 8. ☐

Proposition 8 shows that the quasisymmetric functions $r^{\ell(\beta)} M_\beta$ for all $\beta \in \text{Comp}$ are $\mathbf{k}$-linear combinations of the enriched q-monomial quasisymmetric functions $\eta_\alpha^{(q)}$. If r is invertible in $\mathbf{k}$, then it follows that the monomial quasisymmetric functions M_β are such combinations as well, and thus the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \text{Comp}}$ spans the $\mathbf{k}$-module QSym in this case. But we can actually say more:

Theorem 11. *Assume that r is invertible in $\mathbf{k}$. Then:*

(a) *The family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \text{Comp}}$ is a basis of the $\mathbf{k}$-module QSym.*

(b) *Let $n \in \mathbb{N}$. Consider the n-th graded component QSym_n of the graded $\mathbf{k}$-module QSym. Then, the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \text{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n.*

Proof. **(b)** Order all compositions of n by increasing length (breaking up ties arbitrarily). Then, (4) shows that each $\eta_\alpha^{(q)}$ (for $\alpha \in \text{Comp}_n$) can be expanded as a linear combination of the M_β (with $\beta \in \text{Comp}_n$). Moreover, all the compositions β that appear in this expansion with nonzero coefficient satisfy $D(\beta) \subseteq D(\alpha)$, and therefore have smaller length than α unless they are equal to α (since $\beta \neq \alpha$ entails $D(\beta) \neq D(\alpha)$, so that $D(\beta)$ is a proper subset of $D(\alpha)$, and thus Lemma 10 **(b)** yields $\ell(\beta) - \ell(\alpha) = |D(\beta)| - |D(\alpha)| < 0$). Thus, if we collect the coefficients of these expansions in a matrix, then this matrix is lower-triangular. Moreover, the diagonal entries of this matrix are $r^{\ell(\alpha)}$, thus invertible (since r is invertible). Hence, the matrix itself is invertible.

So we have shown that the $\eta_\alpha^{(q)}$ can be expanded as linear combinations of the M_β, and the coefficients of these expansions form an invertible matrix. Consequently, the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \text{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n. (since the family $(M_\alpha)_{\alpha \in \text{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n). This proves Theorem 11 **(b)**.

(a) Part **(a)** follows from part **(b)**, since $\bigsqcup_{n \in \mathbb{N}} \text{Comp}_n = \text{Comp}$ and $\bigoplus_{n \in \mathbb{N}} \text{QSym}_n = \text{QSym}$. ☐

Theorem 11 **(a)** has a converse: If the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \text{Comp}}$ is a basis of QSym, then r is invertible. (This is already clear from considering its unique degree-1 entry $\eta_{(1)}^{(q)} = rM_{(1)}$.)

3.3 Relation to the fundamental basis

We can also expand the $\eta_\alpha^{(q)}$ in the fundamental basis and vice versa:

Proposition 12. *Let n be a positive integer. Let $\alpha \in \text{Comp}_n$. Then,*

$$\eta_\alpha^{(q)} = r \sum_{\gamma \in \text{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma.$$

Proposition 13. *Let n be a positive integer. Let $\gamma \in \operatorname{Comp}_n$. Then,*

$$r^n L_\gamma = \sum_{\alpha \in \operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\alpha)|} q^{|[n-1]\setminus(D(\gamma)\cup D(\alpha))|} \eta_\alpha^{(q)}.$$

Note that Proposition 12 generalizes [Hsiao07, Proposition 2.2].

Both propositions can be proved by the help of a rather simple identity:[1]

Lemma 14. *Let S and T be two finite sets. Then,*

$$\sum_{I \subseteq S} (-1)^{|I\setminus T|} q^{|I\cap T|} = [S \subseteq T] \cdot r^{|S|}.$$

Proof of Lemma 14. This is again an exercise in elementary combinatorics, so we refer to [GriVas23a] for a detailed proof. (The two cases to be considered are $S \subseteq T$ and $S \not\subseteq T$. In the former case, the left hand side simplifies to $\sum_{I \subseteq S} q^{|I|}$, which by a simple argument equals $(q+1)^{|S|} = r^{|S|}$. In the latter case, pick an $s \in S \setminus T$ and break up the left hand side into pairs as follows: For each subset $J \subseteq S$, the addend for $I = J$ is paired up with the addend for $I = J \triangle \{s\}$, where the symbol $\triangle$ means "symmetric difference". In each pair, the two partners cancel each other out, and thus the sum is 0.) $\square$

Proof of Proposition 12. We begin by observing that

$$|D(\beta)| + 1 = \ell(\beta) \tag{8}$$

for every $\beta \in \operatorname{Comp}_n$ (an easy consequence of Lemma 10 **(a)**, since $n > 0$).

Let $T := D(\alpha)$. Thus, $D(\alpha) = T$, so that

$$\begin{aligned}
& r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\alpha)|} q^{|D(\gamma)\cap D(\alpha)|} L_\gamma \\
&= r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus T|} q^{|D(\gamma)\cap T|} \underbrace{L_\gamma}_{\substack{= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta)\supseteq D(\gamma)}} M_\beta \\ \text{(by the definition of } L_\gamma)}} \\
&= r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus T|} q^{|D(\gamma)\cap T|} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta)\supseteq D(\gamma)}} M_\beta \\
&= r \sum_{\beta \in \operatorname{Comp}_n} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta)\supseteq D(\gamma)}} (-1)^{|D(\gamma)\setminus T|} q^{|D(\gamma)\cap T|} M_\beta.
\end{aligned}$$

However, every $\beta \in \operatorname{Comp}_n$ satisfies

$$\sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta)\supseteq D(\gamma)}} (-1)^{|D(\gamma)\setminus T|} q^{|D(\gamma)\cap T|}$$

[1] We will use Convention 2.

$$= \sum_{\substack{I \subseteq [n-1]; \\ D(\beta) \supseteq I}} (-1)^{|I \setminus T|} q^{|I \cap T|}$$

$$\left(\begin{array}{c} \text{here, we have substituted } I \text{ for } D(\gamma) \text{ in the sum,} \\ \text{since the map } D : \operatorname{Comp}_n \to \mathcal{P}([n-1]) \\ \text{is a bijection} \end{array} \right)$$

$$= \sum_{I \subseteq D(\beta)} (-1)^{|I \setminus T|} q^{|I \cap T|} \qquad (\text{since } D(\beta) \subseteq [n-1])$$

$$= [D(\beta) \subseteq T] \cdot r^{|D(\beta)|}$$

(by Lemma 14, applied to $S = D(\beta)$). Hence, this becomes

$$r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma$$

$$= r \sum_{\beta \in \operatorname{Comp}_n} \underbrace{\sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|}}_{= [D(\beta) \subseteq T] \cdot r^{|D(\beta)|}} M_\beta$$

$$= r \sum_{\beta \in \operatorname{Comp}_n} [D(\beta) \subseteq T] \cdot r^{|D(\beta)|} M_\beta = r \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} r^{|D(\beta)|} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} \underbrace{r^{|D(\beta)|+1}}_{\substack{= r^{\ell(\beta)} \\ (\text{by } (8))}} M_\beta$$

$$= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} r^{\ell(\beta)} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \qquad (\text{since } T = D(\alpha))$$

$$= \eta_\alpha^{(q)} \qquad \left(\text{by the definition of } \eta_\alpha^{(q)}\right).$$

This proves Proposition 12. ☐

Proof of Proposition 13. For each subset J of $[n-1]$, we let $\overline{J}$ denote its complement $[n-1] \setminus J$. Its size is clearly $\left|\overline{J}\right| = (n-1) - |J|$ (since $n-1 \in \mathbb{N}$ entails that $[n-1]$ has size $n-1$). Thus, every $\beta \in \operatorname{Comp}_n$ satisfies

$$\left|\overline{D(\beta)}\right| = (n-1) - |D(\beta)| = n - \underbrace{(|D(\beta)| + 1)}_{\substack{= \ell(\beta) \\ (\text{by } (8))}} = n - \ell(\beta)$$

and therefore

$$r^{\left|\overline{D(\beta)}\right|} r^{\ell(\beta)} = r^{n-\ell(\beta)} r^{\ell(\beta)} = r^n. \tag{9}$$

Let $T := \overline{D(\gamma)}$. Thus, $D(\gamma) = \overline{T}$, so that

$$\sum_{\alpha \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|[n-1] \setminus (D(\gamma) \cup D(\alpha))|} \eta_\alpha^{(q)}$$

$$= \sum_{\alpha \in \operatorname{Comp}_n} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} \eta_\alpha^{(q)}$$

$$= \sum_{\alpha \in \mathrm{Comp}_n} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \qquad \text{(by (4))}$$

$$= \sum_{\beta \in \mathrm{Comp}_n} \sum_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} r^{\ell(\beta)} M_\beta.$$

However, every $\beta \in \mathrm{Comp}_n$ satisfies

$$\sum_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|}$$

$$= \sum_{\substack{K \subseteq [n-1]; \\ D(\beta) \subseteq K}} (-1)^{\left|\overline{T} \setminus K\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup K\right)\right|}$$

$$\left(\begin{array}{c}\text{here, we have substituted } K \text{ for } D(\alpha) \text{ in the sum,} \\ \text{since the map } D : \mathrm{Comp}_n \to \mathcal{P}([n-1]) \text{ is a bijection}\end{array}\right)$$

$$= \underbrace{\sum_{\substack{I \subseteq [n-1]; \\ D(\beta) \subseteq \overline{I}}}}_{\substack{= \sum_{I \subseteq \overline{D(\beta)}} \\ \text{(since the subsets } I \text{ of } [n-1] \\ \text{satisfying } D(\beta) \subseteq \overline{I} \text{ are precisely} \\ \text{the subsets of } \overline{D(\beta)})}} \underbrace{(-1)^{\left|\overline{T} \setminus \overline{I}\right|}}_{\substack{= (-1)^{|I \setminus T|} \\ \text{(since } \overline{T} \setminus \overline{I} = I \setminus T)}} \underbrace{q^{\left|[n-1] \setminus \left(\overline{T} \cup \overline{I}\right)\right|}}_{\substack{= q^{|I \cap T|} \\ \text{(since we have} \\ [n-1] \setminus \left(\overline{T} \cup \overline{I}\right) = \overline{\overline{T} \cup \overline{I}} = \overline{\overline{I}} \cup \overline{\overline{T}} = I \cap T \\ \text{by de Morgan's laws)}}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } \overline{I} \text{ for } K \text{ in the sum, since} \\ \text{the map } \mathcal{P}([n-1]) \to \mathcal{P}([n-1]) \text{ that sends each} \\ \text{subset } I \text{ to its complement } \overline{I} \text{ is a bijection}\end{array}\right)$$

$$= \sum_{I \subseteq \overline{D(\beta)}} (-1)^{|I \setminus T|} q^{|I \cap T|} = \left[\overline{D(\beta)} \subseteq T\right] \cdot r^{\left|\overline{D(\beta)}\right|}$$

(by Lemma 14, applied to $S = \overline{D(\beta)}$). Hence, this becomes

$$\sum_{\alpha \in \mathrm{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|[n-1] \setminus (D(\gamma) \cup D(\alpha))|} \eta_\alpha^{(q)}$$

$$= \sum_{\beta \in \mathrm{Comp}_n} \underbrace{\sum_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|}}_{= \left[\overline{D(\beta)} \subseteq T\right] \cdot r^{\left|\overline{D(\beta)}\right|}} r^{\ell(\beta)} M_\beta$$

$$= \sum_{\beta \in \mathrm{Comp}_n} \left[\overline{D(\beta)} \subseteq T\right] \cdot r^{\left|\overline{D(\beta)}\right|} r^{\ell(\beta)} M_\beta = \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ \overline{D(\beta)} \subseteq T}} \underbrace{r^{\left|\overline{D(\beta)}\right|} r^{\ell(\beta)}}_{\substack{= r^n \\ \text{(by (9))}}} M_\beta$$

$$= r^n \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ \overline{D(\beta)} \subseteq T}} M_\beta = r^n \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ \overline{D(\beta)} \subseteq \overline{D(\gamma)}}} M_\beta \qquad \left(\text{since } T = \overline{D(\gamma)}\right)$$

$$= r^n \underbrace{\sum_{\substack{\beta \in \mathrm{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} M_\beta}_{\substack{= L_\gamma \\ \text{(by (3), applied to } \alpha = \gamma)}} \qquad \left(\begin{array}{c}\text{since the condition "}\overline{D(\beta)} \subseteq \overline{D(\gamma)}\text{" on a} \\ \text{composition } \beta \in \mathrm{Comp}_n \text{ is equivalent} \\ \text{to the condition "}D(\beta) \supseteq D(\gamma)\text{"}\end{array}\right)$$

$$= r^n L_\gamma .$$

This proves Proposition 13. □

3.4 The antipode of $\eta_\alpha^{(q)}$

The *antipode* of QSym is a certain $\mathbf{k}$-linear map $S : \mathrm{QSym} \to \mathrm{QSym}$ that can be defined in terms of the Hopf algebra structure of QSym, which we have not defined so far. But there are various formulas for its values on certain quasisymmetric functions that can be used as alternative definitions. For example, for any $n \in \mathbb{N}$ and any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathrm{Comp}_n$, we have

$$S(M_\alpha) = (-1)^\ell \sum_{\substack{\gamma \in \mathrm{Comp}_n; \\ D(\gamma) \subseteq D(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)}} M_\gamma . \tag{10}$$

This formula (which appears, e.g., in [Malven93, (4.26)][2] and in [GriRei20, Theorem 5.1.11][3] or in [Grinbe15, detailed version, Proposition 10.70]) can be used to define S (since S is to be $\mathbf{k}$-linear). Also, for each composition α, we have $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}$, where $\omega(\alpha)$ is a certain composition known as the *conjugate* of α. See [Malven93, Corollaire 4.20] or [GriRei20, Theorem 5.1.11 and Proposition 5.2.15] for details and proofs. It is well-known (see, e.g., [GriRei20, Proposition 1.4.10 and Corollary 1.4.12]) that S is a $\mathbf{k}$-algebra homomorphism and an involution (that is, $S^2 = \mathrm{id}$).

We will prove two formulas for the antipode of $\eta_\alpha^{(q)}$. Both rely on the following notation:

Definition 15. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition, then the *reversal* of α is defined to be the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$. It is denoted by $\mathrm{rev}\,\alpha$.

We are now ready to state our first formula for the antipode of $\eta_\alpha^{(q)}$ in the case when q is invertible:

Theorem 16. *Let $p \in \mathbf{k}$ be such that $pq = 1$. Let $\alpha \in \mathrm{Comp}$, and let $n = |\alpha|$. Then, the antipode S of* QSym *satisfies*

$$S\left(\eta_\alpha^{(q)}\right) = (-q)^{\ell(\alpha)} \eta_{\mathrm{rev}\,\alpha}^{(p)} .$$

[2] The proof given in [Malven93] requires $\mathbf{k}$ to be a $\mathbb{Q}$-algebra, but it is easy to see that the truth of (10) for $\mathbf{k} = \mathbb{Q}$ implies the truth of (10) for every commutative ring $\mathbf{k}$.

[3] Note that [GriRei20, Theorem 5.1.11] uses the notation $\mathrm{rev}\,\alpha$ for the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$, and writes "$\gamma$ coarsens $\mathrm{rev}\,\alpha$" for what we call "$\gamma \in \mathrm{Comp}_n$ and $D(\gamma) \subseteq D(\mathrm{rev}\,\alpha)$".

Proof. From $pq = 1$, we obtain $p = q^{-1}$ and

$$\underbrace{r}_{\substack{=q+1 \\ =1+q}} p = (1+q)\, p = p + \underbrace{qp}_{=pq=1} = p + 1. \tag{11}$$

We shall need a few more features of compositions. For any composition $\gamma \in \operatorname{Comp}_n$, we let $\omega(\gamma)$ denote the unique composition of n satisfying

$$D(\omega(\gamma)) = [n-1] \setminus D(\operatorname{rev} \gamma). \tag{12}$$

(This $\omega(\gamma)$ is indeed unique, since the map D is a bijection.) Then, a classical formula ([Malven93, (4.27)] or [GriRei20, (5.2.7)]) says that each $\gamma \in \operatorname{Comp}_n$ satisfies

$$S(L_\gamma) = (-1)^n L_{\omega(\gamma)}. \tag{13}$$

It is also easy to prove (see, e.g., [GriVas23b, Proposition 4.3 **(d)**]) that

$$\omega(\omega(\gamma)) = \gamma \qquad \text{for any } \gamma \in \operatorname{Comp}_n. \tag{14}$$

Thus, the map $\omega : \operatorname{Comp}_n \to \operatorname{Comp}_n$ (which sends each $\gamma \in \operatorname{Comp}_n$ to $\omega(\gamma)$) is a bijection.

We WLOG assume that $n \neq 0$ (since the claim of Theorem 16 is easily checked by hand in the case when $n = 0$).

From $n = |\alpha|$, we obtain $\alpha \in \operatorname{Comp}_n$.

Now, we make the following combinatorial observation:

Observation 1: Let $\gamma \in \operatorname{Comp}_n$. Then,

$$|D(\omega(\gamma)) \cap D(\alpha)| = \ell(\alpha) - 1 - |D(\gamma) \cap D(\operatorname{rev} \alpha)| \tag{15}$$

and

$$|D(\omega(\gamma)) \setminus D(\alpha)| = n - \ell(\alpha) - |D(\gamma) \setminus D(\operatorname{rev} \alpha)|. \tag{16}$$

The proof of Observation 1 is laborious but fairly straightforward, and can be found in [GriVas23b, Proposition 4.4].

Now, Proposition 12 (applied to $\operatorname{rev} \alpha$, p and $p+1$ instead of α, q and r) yields

$$\eta^{(p)}_{\operatorname{rev} \alpha} = (p+1) \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\operatorname{rev} \alpha)|}\, p^{|D(\gamma) \cap D(\operatorname{rev} \alpha)|} L_\gamma. \tag{17}$$

On the other hand, Proposition 12 yields

$$\eta^{(q)}_\alpha = r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|}\, q^{|D(\gamma) \cap D(\alpha)|} L_\gamma.$$

Applying the $\mathbf{k}$-linear map S to both sides of this equality, we obtain

$$S\left(\eta^{(q)}_\alpha\right) = r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|}\, q^{|D(\gamma) \cap D(\alpha)|} \underbrace{S(L_\gamma)}_{\substack{=(-1)^n L_{\omega(\gamma)} \\ \text{(by (13))}}}$$

$$\begin{aligned}
&= r \sum_{\gamma\in\operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\alpha)|} q^{|D(\gamma)\cap D(\alpha)|} (-1)^n L_{\omega(\gamma)} \\
&= r \sum_{\gamma\in\operatorname{Comp}_n} \underbrace{(-1)^{|D(\omega(\gamma))\setminus D(\alpha)|}}_{\substack{=(-1)^{n-\ell(\alpha)-|D(\gamma)\setminus D(\operatorname{rev}\alpha)|} \\ \text{(by (16))}}} \underbrace{q^{|D(\omega(\gamma))\cap D(\alpha)|}}_{\substack{=q^{\ell(\alpha)-1-|D(\gamma)\cap D(\operatorname{rev}\alpha)|} \\ \text{(by (15))}}} (-1)^n \underbrace{L_{\omega(\omega(\gamma))}}_{\substack{=L_\gamma \\ \text{(by (14))}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \omega(\gamma) \text{ for } \gamma \text{ in the sum,} \\ \text{since the map } \omega : \operatorname{Comp}_n \to \operatorname{Comp}_n \text{ is a bijection}\end{array}\right) \\
&= r \sum_{\gamma\in\operatorname{Comp}_n} \underbrace{(-1)^{n-\ell(\alpha)-|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}}_{=(-1)^n(-1)^{\ell(\alpha)}(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}} \underbrace{q^{\ell(\alpha)-1-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}}_{\substack{=q^{\ell(\alpha)}q^{-1}\left(q^{-1}\right)^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|} \\ =q^{\ell(\alpha)}pp^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|} \\ \text{(since } q^{-1}=p)}} (-1)^n L_\gamma \\
&= \underbrace{rp}_{\substack{=p+1 \\ \text{(by (11))}}} \underbrace{(-1)^n (-1)^n}_{=1} \underbrace{(-1)^{\ell(\alpha)} q^{\ell(\alpha)}}_{=(-q)^{\ell(\alpha)}} \sum_{\gamma\in\operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|} p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|} L_\gamma \\
&= (p+1)(-q)^{\ell(\alpha)} \sum_{\gamma\in\operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|} p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|} L_\gamma \\
&= (-q)^{\ell(\alpha)} \underbrace{(p+1) \sum_{\gamma\in\operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|} p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|} L_\gamma}_{\substack{=\eta^{(p)}_{\operatorname{rev}\alpha} \\ \text{(by (17))}}} \\
&= (-q)^{\ell(\alpha)} \eta^{(p)}_{\operatorname{rev}\alpha}.
\end{aligned}$$

This proves Theorem 16. □

Theorem 16 generalizes [Hsiao07, Proposition 2.9].

Our second formula for the antipode of $\eta^{(q)}_\alpha$ needs no requirement on q, but involves a sum:

Theorem 17. *Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$. Then, the antipode S of* QSym *satisfies*

$$S\left(\eta^{(q)}_\alpha\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta\in\operatorname{Comp}_n; \\ D(\beta)\subseteq D(\operatorname{rev}\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta^{(q)}_\beta.$$

We will prove this theorem using the following summation lemma:

Lemma 18. *Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$ and $\gamma \in \operatorname{Comp}_n$ be such that $D(\gamma) \subseteq D(\alpha)$. Then:*

- ***(a)*** *For any $u, v \in \mathbf{k}$, we have*

$$\sum_{\substack{\beta\in\operatorname{Comp}_n; \\ D(\gamma)\subseteq D(\beta)\subseteq D(\alpha)}} u^{\ell(\beta)-\ell(\gamma)} v^{\ell(\alpha)-\ell(\beta)} = (u+v)^{\ell(\alpha)-\ell(\gamma)}.$$

(b) *For any $u \in \mathbf{k}$, we have*

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta) \subseteq D(\alpha)}} u^{\ell(\beta)} = (u+1)^{\ell(\alpha)-\ell(\gamma)} u^{\ell(\gamma)}.$$

(c) *For any $v \in \mathbf{k}$, we have*

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta) \subseteq D(\alpha)}} v^{\ell(\alpha)-\ell(\beta)} = (1+v)^{\ell(\alpha)-\ell(\gamma)}.$$

Proof of Lemma 18. **(a)** This is an easy exercise in combinatorial sums, so we shall be brief; details can be found in [GriVas23a].

Let $u, v \in \mathbf{k}$. Set $A := D(\alpha)$ and $C := D(\gamma)$, so that $C \subseteq A \subseteq [n-1]$.

Recall that the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection. Using this bijection (and Lemma 10 **(b)**), we can translate the claim of Lemma 18 **(a)** from the language of compositions into the language of sets. In the latter language, it says that

$$\sum_{\substack{B \subseteq [n-1]; \\ C \subseteq B \subseteq A}} u^{|B|-|C|} v^{|A|-|B|} = (u+v)^{|A|-|C|}.$$

But this can be shown easily either using the binomial formula (rewriting the left hand side as a sum over all subsets of $A \setminus C$, then filtering it according to the size of the subset) or using a bijection (the choice of a subset $B \subseteq [n-1]$ satisfying $C \subseteq B \subseteq A$ can be reframed as a sequence of choices, one for each element $a \in A \setminus C$, whether the subset should contain a or not contain a). In either case, Lemma 18 **(a)** follows.

(b) This follows by applying part **(a)** to $v = 1$ and then multiplying both sides by $u^{\ell(\gamma)}$.

(c) This follows by applying part **(a)** to $u = 1$. □

Proof of Theorem 17. We replace α by $\operatorname{rev}\alpha$. Thus, α and $\operatorname{rev}\alpha$ become $\operatorname{rev}\alpha$ and α, respectively, while the length $\ell(\alpha)$ stays unchanged. Hence, the claim we must prove becomes

$$S\left(\eta^{(q)}_{\operatorname{rev}\alpha}\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta^{(q)}_{\beta}. \tag{18}$$

It is this equality that we will be proving.

First, we observe that every $\beta \in \operatorname{Comp}_n$ satisfies

$$S(M_\beta) = (-1)^{\ell(\beta)} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\operatorname{rev}\beta)}} M_\gamma. \tag{19}$$

(Indeed, this is just the formula (10), applied to β instead of α and restated using Definition 15.) Substituting $\operatorname{rev}\beta$ for β in (19), we obtain the following: Every $\beta \in \operatorname{Comp}_n$ satisfies

$$\begin{aligned} S\left(M_{\operatorname{rev}\beta}\right) &= (-1)^{\ell(\operatorname{rev}\beta)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\operatorname{rev}(\operatorname{rev}\beta))}} M_\gamma \qquad (\text{since } \operatorname{rev}\beta\in\operatorname{Comp}_n) \\ &= (-1)^{\ell(\beta)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} M_\gamma \end{aligned} \tag{20}$$

(since $\operatorname{rev}(\operatorname{rev}\beta) = \beta$ and $\ell(\operatorname{rev}\beta) = \ell(\beta)$).

Next, we recall a simple fact ([GriVas23b, Proposition 3.11]), which says that if $\beta \in \operatorname{Comp}_n$ is arbitrary, then we have the logical equivalence

$$(D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \iff (D(\beta) \subseteq D(\alpha)). \tag{21}$$

The definition of $\eta^{(q)}_{\operatorname{rev}\alpha}$ yields

$$\begin{aligned} \eta^{(q)}_{\operatorname{rev}\alpha} &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\operatorname{rev}\alpha)}} r^{\ell(\beta)} M_\beta = \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\operatorname{rev}\beta)\subseteq D(\operatorname{rev}\alpha)}}}_{\substack{=\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}}\\ (\text{by the equivalence (21)})}} \underbrace{r^{\ell(\operatorname{rev}\beta)}}_{\substack{=r^{\ell(\beta)}\\ (\text{since } \ell(\operatorname{rev}\beta)=\ell(\beta))}} M_{\operatorname{rev}\beta} \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } \operatorname{rev}\beta \text{ for } \beta \text{ in the sum,}\\ \text{since the map } \operatorname{Comp}_n\to\operatorname{Comp}_n,\ \delta\mapsto\operatorname{rev}\delta\\ \text{is a bijection}\end{array}\right) \\ &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} M_{\operatorname{rev}\beta}. \end{aligned}$$

Applying the $\mathbf{k}$-linear map S to both sides of this equality, we obtain

$$\begin{aligned} S\left(\eta^{(q)}_{\operatorname{rev}\alpha}\right) &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} S\left(M_{\operatorname{rev}\beta}\right) \\ &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} (-1)^{\ell(\beta)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} M_\gamma \qquad (\text{by (20)}) \\ &= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)\subseteq D(\alpha)}} \underbrace{r^{\ell(\beta)} (-1)^{\ell(\beta)}}_{=(-r)^{\ell(\beta)}} M_\gamma \\ &= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)\subseteq D(\alpha)}} (-r)^{\ell(\beta)}}_{\substack{=(-r+1)^{\ell(\alpha)-\ell(\gamma)}(-r)^{\ell(\gamma)}\\ (\text{by Lemma 18 } \mathbf{(b)})}} M_\gamma \end{aligned}$$

$$= \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} (-r+1)^{\ell(\alpha)-\ell(\gamma)} (-r)^{\ell(\gamma)} M_\gamma$$

$$= \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} \underbrace{(-q)^{\ell(\alpha)-\ell(\gamma)} (-r)^{\ell(\gamma)}}_{=(-1)^{\ell(\alpha)} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)}} M_\gamma \qquad \left(\text{since } -\underbrace{r}_{=q+1} + 1 = -q\right)$$

$$= (-1)^{\ell(\alpha)} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma.$$

In view of

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \underbrace{\eta_\beta^{(q)}}_{\substack{= \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta)}} r^{\ell(\gamma)} M_\gamma \\ \text{(by (4))}}}$$

$$= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta)}} r^{\ell(\gamma)} M_\gamma$$

$$= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta)}} (q-1)^{\ell(\alpha)-\ell(\beta)} r^{\ell(\gamma)} M_\gamma$$

$$= \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} \underbrace{\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\beta) \subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)}}_{\substack{=(1+(q-1))^{\ell(\alpha)-\ell(\gamma)} \\ \text{(by Lemma 18 } \mathbf{(c)}\text{)}}} r^{\ell(\gamma)} M_\gamma$$

$$= \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} \left(\underbrace{1+(q-1)}_{=q}\right)^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma = \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma,$$

we can rewrite this as

$$S\left(\eta_{\operatorname{rev}\alpha}^{(q)}\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta_\beta^{(q)}.$$

Thus, (18) is proved. As we explained, this proves Theorem 17. ☐

3.5 The coproduct of $\eta_\alpha^{(q)}$

We begin with a definition that we will use several times:

Definition 19. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ are two compositions, then the composition $\alpha\beta$ is defined by

$$\alpha\beta = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_k).$$

This composition $\alpha\beta$ is called the *concatenation* of α and β. The operation of concatenation (sending any two compositions α and β to $\alpha\beta$) is associative, and the empty composition $\varnothing$ is a neutral element for it; thus, the set of all compositions is a monoid under this operation.

The coproduct of the Hopf algebra QSym is a $\mathbf{k}$-linear map

$$\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$$

that can be described by the formula

$$\Delta(M_\alpha) = \sum_{\substack{\beta,\gamma \in \mathrm{Comp};\\ \alpha = \beta\gamma}} M_\beta \otimes M_\gamma, \tag{22}$$

which holds for all $\alpha \in \mathrm{Comp}$. (See [GriRei20, §5.1] for the definition of Δ, and see [GriRei20, Proposition 5.1.7] for a proof of (22).)

We claim the following simple formula for $\Delta\left(\eta_\alpha^{(q)}\right)$ (analogous to (22)):

Theorem 20. *Let* $\alpha \in \mathrm{Comp}$. *Then,*

$$\Delta\left(\eta_\alpha^{(q)}\right) = \sum_{\substack{\beta,\gamma \in \mathrm{Comp};\\ \alpha = \beta\gamma}} \eta_\beta^{(q)} \otimes \eta_\gamma^{(q)}.$$

This theorem generalizes [Hsiao07, Corollary 2.7]. We shall prove it using the following notion:

Definition 21. Let α be a composition. Then, $C(\alpha)$ shall denote the set of all compositions $\beta \in \mathrm{Comp}_{|\alpha|}$ satisfying $D(\beta) \subseteq D(\alpha)$. (The compositions belonging to $C(\alpha)$ are often called the *coarsenings* of α.)

For instance, $C(2,1,3) = \{(2,1,3),\ (3,3),\ (2,4),\ (6)\}$.

Using the notion of $C(\alpha)$, we can restate (4) as follows: For any $\alpha \in \mathrm{Comp}$, we have

$$\eta_\alpha^{(q)} = \sum_{\beta \in C(\alpha)} r^{\ell(\beta)} M_\beta. \tag{23}$$

We shall also use a simple summation formula ([GriVas23b, Proposition 5.17]) that relies on the combinatorics of compositions and their coarsenings:

Proposition 22. *Let $(A, +, 0)$ be an abelian group. Let $u_{\mu,\nu}$ be an element of A for each pair $(\mu, \nu) \in \text{Comp} \times \text{Comp}$ of two compositions. Let $\alpha \in \text{Comp}$. Then,*

$$\sum_{\substack{\mu,\nu\in\text{Comp};\\ \mu\nu\in C(\alpha)}} u_{\mu,\nu} = \sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} u_{\mu,\nu}.$$

We are now ready to prove Theorem 20:

Proof of Theorem 20. From (23), we obtain

$$\eta_\alpha^{(q)} = \sum_{\beta\in C(\alpha)} r^{\ell(\beta)} M_\beta = \sum_{\lambda\in C(\alpha)} r^{\ell(\lambda)} M_\lambda.$$

Applying the **k**-linear map Δ to both sides of this equality, we find

$$\begin{aligned}
\Delta\left(\eta_\alpha^{(q)}\right) &= \sum_{\lambda\in C(\alpha)} r^{\ell(\lambda)} \underbrace{\Delta\left(M_\lambda\right)}_{\substack{=\sum_{\substack{\mu,\nu\in\text{Comp};\\ \lambda=\mu\nu}} M_\mu\otimes M_\nu\\ \text{(by (22))}}} = \underbrace{\sum_{\lambda\in C(\alpha)} \sum_{\substack{\mu,\nu\in\text{Comp};\\ \lambda=\mu\nu}}}_{=\sum_{\substack{\mu,\nu\in\text{Comp};\\ \mu\nu\in C(\alpha)}}} \underbrace{r^{\ell(\lambda)}}_{\substack{=r^{\ell(\mu\nu)}\\ \text{(since } \lambda=\mu\nu)}} M_\mu\otimes M_\nu \\
&= \sum_{\substack{\mu,\nu\in\text{Comp};\\ \mu\nu\in C(\alpha)}} r^{\ell(\mu\nu)} M_\mu\otimes M_\nu \\
&= \sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} r^{\ell(\mu\nu)} M_\mu\otimes M_\nu \qquad (24)
\end{aligned}$$

(by Proposition 22, applied to $A = \text{QSym}\otimes\text{QSym}$ and $u_{\mu,\nu} = r^{\ell(\mu\nu)} M_\mu\otimes M_\nu$).

Now,

$$\begin{aligned}
&\underbrace{\sum_{\substack{\beta,\gamma\in\text{Comp};\\ \alpha=\beta\gamma}}}_{=\sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}}} \underbrace{\eta_\beta^{(q)}}_{\substack{=\sum_{\mu\in C(\beta)} r^{\ell(\mu)} M_\mu\\ \text{(by (23))}}} \otimes \underbrace{\eta_\gamma^{(q)}}_{\substack{=\sum_{\nu\in C(\gamma)} r^{\ell(\nu)} M_\nu\\ \text{(by (23))}}} \\
&= \sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}} \left(\sum_{\mu\in C(\beta)} r^{\ell(\mu)} M_\mu\right) \otimes \left(\sum_{\nu\in C(\gamma)} r^{\ell(\nu)} M_\nu\right) \\
&= \sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} \underbrace{r^{\ell(\mu)} r^{\ell(\nu)}}_{\substack{=r^{\ell(\mu)+\ell(\nu)}=r^{\ell(\mu\nu)}\\ \text{(since the definition}\\ \text{of concatenation}\\ \text{yields } \ell(\mu)+\ell(\nu)=\ell(\mu\nu))}} M_\mu\otimes M_\nu \\
&= \sum_{\substack{\beta,\gamma\in\text{Comp};\\ \beta\gamma=\alpha}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} r^{\ell(\mu\nu)} M_\mu\otimes M_\nu = \Delta\left(\eta_\alpha^{(q)}\right)
\end{aligned}$$

(by (24)). This proves Theorem 20. □

Another proof of Theorem 20 can be found in [GriVas23a, Section 3.7].

3.6 The coalgebra morphism T_r

We define a **k**-linear map $T_r : \mathrm{QSym} \to \mathrm{QSym}$ by setting

$$T_r\left(M_\alpha\right) = r^{\ell(\alpha)} M_\alpha \qquad \text{for each } \alpha \in \mathrm{Comp}\,.$$

This definition is legitimate, since $\left(M_\alpha\right)_{\alpha\in\mathrm{Comp}}$ is a basis of the **k**-module QSym. The map T_r is usually not a **k**-algebra homomorphism, but always a **k**-coalgebra homomorphism:

Proposition 23. *The map* $T_r : \mathrm{QSym} \to \mathrm{QSym}$ *is a* **k***-coalgebra homomorphism.*

Proof of Proposition 23. Easy consequence of (22); see [GriVas23a] for details. ☐

To us, the map T_r becomes useful thanks to the following slick expression for $\eta_\alpha^{(q)}$ that it allows:

Theorem 24. *Let* $S : \mathrm{QSym} \to \mathrm{QSym}$ *be the antipode of the Hopf algebra* QSym. *Let* $\alpha \in \mathrm{Comp}$. *Then,*

$$\eta_\alpha^{(q)} = (-1)^{\ell(\alpha)}\, T_r\left(S\left(M_{\operatorname{rev}\alpha}\right)\right).$$

Proof of Theorem 24. Follows readily from (10); see [GriVas23a] for details. ☐

4 The dual eta basis of NSym

4.1 NSym and the duality pairing

Let NSym denote the free **k**-algebra with generators $H_1, H_2, H_3, \ldots$ (that is, the tensor algebra of the free **k**-module with basis $(H_1, H_2, H_3, \ldots)$). This **k**-algebra NSym is known as the *ring of noncommutative symmetric functions* over **k**. We refer to [GriRei20, §5.4], [GKLLRT94] and [Meliot17, §6.1] for more about this **k**-algebra[4]; we will only need a few basic properties.

We set $H_0 := 1 \in \mathrm{NSym}$. Thus, an element H_n of NSym is defined for each $n \in \mathbb{N}$. For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}$, we set

$$H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_k} \in \mathrm{NSym}\,.$$

The family $\left(H_\alpha\right)_{\alpha\in\mathrm{Comp}}$ is then a basis of the **k**-module NSym. (Note that $H_{(n)} = H_n$ for each $n > 0$.)

The **k**-algebra NSym is graded, with each generator H_n being homogeneous of degree n (and thus each basis element H_α being homogeneous of degree $|\alpha|$). It becomes a connected graded **k**-bialgebra if we define its coproduct $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ and its counit $\varepsilon : \mathrm{NSym} \to \mathbf{k}$ as follows:

[4] We note some notational differences: What we call H_α is called S_α in [GKLLRT94] and in [Meliot17]. Furthermore, the algebra NSym is denoted by NCSym in [Meliot17] (unfortunately, since NCSym also has a different meaning).

- The coproduct $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ is the $\mathbf{k}$-algebra homomorphism that sends each generator H_n to $\sum_{i=0}^{n} H_i \otimes H_{n-i}$.
- The counit $\varepsilon : \mathrm{NSym} \to \mathbf{k}$ is the $\mathbf{k}$-algebra homomorphism that sends each generator H_n (with $n > 0$) to 0.

Therefore, NSym becomes a Hopf algebra (since any connected graded $\mathbf{k}$-bialgebra is a Hopf algebra). Its antipode S is described in [GriRei20, (5.4.12)].

Most importantly to us, the Hopf algebra NSym is isomorphic to the graded dual of QSym. Specifically, we can define a $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle : \mathrm{NSym} \times \mathrm{QSym} \to \mathbf{k}$ by requiring that

$$\langle H_\alpha, M_\beta \rangle = [\alpha = \beta] \tag{25}$$

for all $\alpha, \beta \in \mathrm{Comp}$ (where we are using Convention 2)[5]. It can be seen that this $\mathbf{k}$-bilinear form produces a canonical isomorphism

$$\begin{aligned} \mathrm{NSym} &\to \mathrm{QSym}^o, \\ f &\mapsto \langle f, \cdot \rangle \end{aligned}$$

of graded Hopf algebras, where QSym^o is the graded dual of the Hopf algebra QSym. Thus, we can identify NSym with the graded dual of the Hopf algebra QSym. (In [GriRei20, §5.4], this is used as a definition of NSym, while the properties that we used to define NSym above are stated as [GriRei20, Theorem 5.4.2].)

4.2 The dual eta basis

We shall now construct a basis of NSym that is dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \mathrm{Comp}}$ of QSym. This requires the assumption that r is invertible (since this assumption ensures that $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \mathrm{Comp}}$ is a basis of QSym in the first place[6]). Thus, we make the following convention:

Convention 3. For the rest of Section 4, we assume that r is invertible in $\mathbf{k}$.

Definition 25. For each $n \in \mathbb{N}$ and each composition α of n, we define an element

$$\eta_\alpha^{*(q)} := \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta \in \mathrm{NSym}\,.$$

Example 26. We have

$$\eta_{()}^{*(q)} = H_{()} = 1_{\mathrm{NSym}};$$

[5] This bilinear form $\langle \cdot, \cdot \rangle$ is denoted by $(\cdot, \cdot)$ in [GriRei20, §5.4].

[6] by Theorem 11

$$\eta^{*(q)}_{(1)} = \frac{1}{r} H_{(1)};$$
$$\eta^{*(q)}_{(2)} = \frac{1}{r} H_{(2)} - \frac{1}{r^2} H_{(1,1)};$$
$$\eta^{*(q)}_{(1,1)} = \frac{1}{r^2} H_{(1,1)}.$$

We now claim the following:

Proposition 27.

(a) *The family* $\left(\eta^{*(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}}$ *is the basis of* NSym *dual to the basis* $\left(\eta^{(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}}$ *of* QSym *with respect to the bilinear form* $\langle\cdot,\cdot\rangle$.

Here, the notion of a "dual basis" should be understood in the graded sense, as explained in [GriRei20, §1.6]. Concretely, our claim is saying that $\left(\eta^{*(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}}$ *is a graded basis of* NSym *and satisfies*

$$\left\langle \eta^{*(q)}_\alpha, \eta^{(q)}_\beta \right\rangle = [\alpha = \beta] \tag{26}$$

for all $\alpha, \beta \in \mathrm{Comp}$.

(b) *Let* $n \in \mathbb{N}$. *Consider the* n*-th graded components* QSym_n *and* NSym_n *of the graded* $\mathbf{k}$*-modules* QSym *and* NSym. *Then, the family* $\left(\eta^{*(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}_n}$ *is the basis of* NSym_n *dual to the basis* $\left(\eta^{(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}_n}$ *of* QSym_n *with respect to the bilinear form* $\langle\cdot,\cdot\rangle$.

Proof of Proposition 27. (See [GriVas23a] for details.)

(b) Proposition 8 (divided by $r^{\ell(\beta)}$) shows that

$$M_\beta = \sum_{\substack{\alpha\in\mathrm{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta^{(q)}_\alpha \qquad \text{for each } \beta \in \mathrm{Comp}_n.$$

Definition 25 yields that

$$\eta^{*(q)}_\alpha = \sum_{\substack{\beta\in\mathrm{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} H_\beta \qquad \text{for each } \alpha \in \mathrm{Comp}_n.$$

Comparing these two equalities, we see that the coefficients in the expansion of $\eta^{*(q)}_\alpha$ in terms of the H_β are precisely the coefficients in the expansion of M_β in terms of the $\eta^{(q)}_\alpha$. In other words, the change-of-basis matrix from the basis $(M_\alpha)_{\alpha\in\mathrm{Comp}_n}$ of QSym_n to the basis $\left(\eta^{(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}_n}$ is the transpose of the change-of-basis matrix from the alleged basis $\left(\eta^{*(q)}_\alpha\right)_{\alpha\in\mathrm{Comp}_n}$ of NSym_n to the basis $(H_\alpha)_{\alpha\in\mathrm{Comp}_n}$. Thus, since the basis

$(H_\alpha)_{\alpha \in \mathrm{Comp}_n}$ of NSym_n is dual to the basis $(M_\alpha)_{\alpha \in \mathrm{Comp}_n}$ of QSym_n, we conclude that the family $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \mathrm{Comp}_n}$ is dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \mathrm{Comp}_n}$ of QSym_n (and is itself a basis)[7]. This proves Proposition 27 **(b)**.

(a) This follows from part **(b)** by taking the direct sum over all n. □

4.3 The dual eta basis: product

We shall now study the multiplicative structure of the dual eta basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \mathrm{Comp}}$. First, we introduce a notation for the simplest entries of this basis:

Definition 28. For each positive integer n, we let

$$\eta_n^{*(q)} := \eta_{(n)}^{*(q)} = \sum_{\beta \in \mathrm{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta \qquad (27)$$
$$\in \mathrm{NSym}.$$

(The second equality sign here follows from Definition 25, since $\ell((n)) = 1$ and since $D((n)) = \varnothing$ is a subset of every $D(\beta)$.)

It turns out that we can easily express $\eta_\alpha^{*(q)}$ for any composition α using these $\eta_n^{*(q)}$:

Proposition 29. *We have*

$$\eta_\alpha^{*(q)} = \eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)} \qquad \textit{for each composition } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k).$$

The main idea of the proof of Proposition 29 is to recognize that if $n = |\alpha|$, then the compositions $\beta \in \mathrm{Comp}_n$ satisfying $D(\alpha) \subseteq D(\beta)$ are precisely the compositions obtained from α by breaking up each entry of α into pieces. A slicker way to formalize this proof proceeds using the notion of concatenation (Definition 19).

First, let us show a proposition that says (in the jargon of combinatorial Hopf algebras) that the basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \mathrm{Comp}}$ of NSym is multiplicative:

Proposition 30. *Let α and β be two compositions. Then,*

$$\eta_\alpha^{*(q)} \eta_\beta^{*(q)} = \eta_{\alpha\beta}^{*(q)}.$$

Proof sketch. It is a folklore result in the theory of combinatorial Hopf algebras that the dual basis to a multiplicative basis (i.e., a basis $(f_\alpha)_{\alpha \in \Lambda}$ indexed by the elements of some monoid Λ and satisfying $f_\alpha f_\beta = f_{\alpha\beta}$ for all $\alpha, \beta \in \Lambda$) is a comultiplicative basis (i.e.,

[7] The underlying fact we are using here is a consequence of the classical linear-algebraic truism that the dual of a linear map is represented by the transpose of the corresponding matrix. This truism entails that when B and C are two bases of the same finite-dimensional vector space, then the change-of-basis matrix between B and C is the transpose of the change-of-basis matrix between their dual bases C^* and B^* (not B^* and C^*).

a basis $(g_\alpha)_{\alpha\in\Lambda}$ satisfying $\Delta(g_\alpha) = \sum_{\substack{\beta,\gamma\in\Lambda;\\ \alpha=\beta\gamma}} g_\beta \otimes g_\lambda$ for all $\alpha \in \Lambda$). Conversely, the dual basis to a comultiplicative basis is a multiplicative basis. Since Theorem 20 shows that the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}_n}$ of QSym_n is comultiplicative, we can thus conclude that its dual basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\mathrm{Comp}_n}$ is multiplicative. That is, Proposition 30 holds. A more detailed version of this proof, without reference to folklore, can be found in the Appendix. $\square$

Corollary 31. *Let $\beta_1, \beta_2, \ldots, \beta_k$ be finitely many compositions. Then,*

$$\eta_{\beta_1}^{*(q)}\eta_{\beta_2}^{*(q)}\cdots\eta_{\beta_k}^{*(q)} = \eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)}.$$

Proof. This follows by induction on k using Proposition 30. (The base case, $k=0$, follows from $\eta_{()}^{*(q)} = 1$.) $\square$

Proof of Proposition 29. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition. Then, applying Corollary 31 to the 1-element compositions $\beta_i = (\alpha_i)$, we obtain

$$\eta_{(\alpha_1)}^{*(q)}\eta_{(\alpha_2)}^{*(q)}\cdots\eta_{(\alpha_k)}^{*(q)} = \eta_{(\alpha_1)(\alpha_2)\cdots(\alpha_k)}^{*(q)} = \eta_\alpha^{*(q)}.$$

Thus,

$$\eta_\alpha^{*(q)} = \eta_{(\alpha_1)}^{*(q)}\eta_{(\alpha_2)}^{*(q)}\cdots\eta_{(\alpha_k)}^{*(q)} = \eta_{\alpha_1}^{*(q)}\eta_{\alpha_2}^{*(q)}\cdots\eta_{\alpha_k}^{*(q)}$$

(since $\eta_{(n)}^{*(q)} = \eta_n^{*(q)}$ for each $n > 0$). This proves Proposition 29. $\square$

4.4 The dual eta basis: generating function

Following a classical method in the theory of symmetric functions, we shall now study the generating functions

$$H(t) = \sum_{n\geqslant 0} H_n t^n \qquad \text{and} \qquad G(t) = \sum_{n\geqslant 1} \eta_n^{*(q)} t^n.$$

However, in order to avoid the technicalities surrounding tensor products of power series rings, we shall not adjoin t as an indeterminate to the ring NSym, but rather assume that t is a nilpotent element of our base ring $\mathbf{k}$. This will ensure that all our infinite sums have only finitely many nonzero addends, and thus can be treated like finite sums. The downside of this trick is that we never obtain fully-fledged power series in this way; however, we can still obtain anything we want from them (mainly: comparing coefficients) by picking an appropriate t in an appropriate ring extension of $\mathbf{k}$. (See [GriVas23a] for a version of our argument that avoids this trick and uses "honest" power series instead.)

Definition 32. For any nilpotent element $t \in \mathbf{k}$, we define the elements

$$H(t) := \sum_{n\geqslant 0} H_n t^n \in \mathrm{NSym} \qquad \text{and} \qquad G(t) := \sum_{n\geqslant 1} \eta_n^{*(q)} t^n \in \mathrm{NSym}\,.$$

Now, it is easy to see the following:

Proposition 33. *Let $t \in \mathbf{k}$ be a nilpotent element. Then,*

$$G(t) = 1 - \frac{1}{1 + \dfrac{H(t) - 1}{r}} = \frac{H(t) - 1}{H(t) + q}.$$

In particular, $H(t) + q$ is invertible.

Proof. We have

$$H(t) = \sum_{n \geqslant 0} H_n t^n = \underbrace{H_0}_{=1} \underbrace{t^0}_{=1} + \sum_{n \geqslant 1} H_n t^n = 1 + \sum_{n \geqslant 1} H_n t^n.$$

Hence,

$$H(t) - 1 = \sum_{n \geqslant 1} H_n \underbrace{t^n}_{=tt^{n-1}} = t \sum_{n \geqslant 1} H_n t^{n-1}$$

is nilpotent (since t is nilpotent and commutes with everything in NSym). Thus, $\dfrac{H(t) - 1}{r}$ is nilpotent as well, so that $1 + \dfrac{H(t) - 1}{r}$ is invertible (since adding 1 to a nilpotent element always yields an invertible element).

If $v \in \text{NSym}$ is a nilpotent element, then the geometric series formula yields $\dfrac{1}{1 - v} = \sum\limits_{k \geqslant 0} v^k$, from which we easily obtain

$$\sum_{k \geqslant 1} v^k = \frac{1}{1 - v} - 1. \tag{28}$$

We shall use this result in a somewhat modified form: If $u \in \text{NSym}$ is a nilpotent element, then $-u/r$ is also nilpotent, and we have

$$\begin{aligned}\sum_{k \geqslant 1} \underbrace{\frac{1}{r^k} (-1)^k u^k}_{=(-u/r)^k} &= \sum_{k \geqslant 1} (-u/r)^k = \frac{1}{1 - (-u/r)} - 1 \qquad \text{(by (28) for } v = -u/r\text{)} \\ &= \frac{1}{1 + u/r} - 1.\end{aligned}$$

Multiplying both sides of this equality by -1, we obtain

$$\sum_{k \geqslant 1} \frac{1}{r^k} (-1)^{k-1} u^k = -\left(\frac{1}{1 + u/r} - 1\right) = 1 - \frac{1}{1 + u/r}. \tag{29}$$

The definition of $G(t)$ yields

$$G(t) = \sum_{n \geqslant 1} \eta_n^{*(q)} t^n = \sum_{n \geqslant 1} \left(\sum_{\beta \in \text{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta) - 1} H_\beta \right) t^n \qquad \text{(by (27))}$$

$$= \sum_{n \geqslant 1} \sum_{\beta \in \operatorname{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta t^n = \sum_{\substack{\beta \in \operatorname{Comp};\\ \beta \neq \varnothing}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta t^{|\beta|}$$

(here, we have folded the two summation signs into one)

$$= \sum_{k \geqslant 1} \underbrace{\sum_{(n_1,n_2,\ldots,n_k) \in \operatorname{Comp}}}_{= \sum_{n_1,n_2,\ldots,n_k \geqslant 1}} \frac{1}{r^k} (-1)^{k-1} \underbrace{H_{(n_1,n_2,\ldots,n_k)} t^{n_1+n_2+\cdots+n_k}}_{\substack{= \left(H_{n_1} H_{n_2} \cdots H_{n_k}\right)\left(t^{n_1} t^{n_2} \cdots t^{n_k}\right) \\ = \left(H_{n_1} t^{n_1}\right)\left(H_{n_2} t^{n_2}\right) \cdots \left(H_{n_k} t^{n_k}\right)}}$$

(here, we have renamed the composition β as $(n_1, n_2, \ldots, n_k)$)

$$= \sum_{k \geqslant 1} \frac{1}{r^k} (-1)^{k-1} \underbrace{\sum_{n_1,n_2,\ldots,n_k \geqslant 1} (H_{n_1} t^{n_1})(H_{n_2} t^{n_2}) \cdots (H_{n_k} t^{n_k})}_{\substack{= \left(\sum_{n \geqslant 1} H_n t^n\right)^k \\ \text{(by the product rule)}}}$$

$$= \sum_{k \geqslant 1} \frac{1}{r^k} (-1)^{k-1} \left(\underbrace{\sum_{n \geqslant 1} H_n t^n}_{= H(t) - 1} \right)^k = \sum_{k \geqslant 1} \frac{1}{r^k} (-1)^{k-1} (H(t) - 1)^k$$

$$= 1 - \frac{1}{1 + (H(t) - 1)/r} \qquad \text{(by (29), applied to } u = H(t) - 1\text{)}$$

$$= 1 - \frac{r}{H(t) + r - 1} = \frac{H(t) - 1}{H(t) + r - 1} = \frac{H(t) - 1}{H(t) + q}$$

(since $r - 1 = q$ (because $r = q + 1$)). The invertibility of $H(t) + q$ has been shown implicitly by the above computation. Thus, Proposition 33 is proved. ☐

Proposition 34. *Let $t \in \mathbf{k}$ be a nilpotent element. Let $k \in \mathbb{N}$. Then,*

$$G(t)^k = \sum_{\substack{\beta \in \operatorname{Comp};\\ \ell(\beta) = k}} \eta_\beta^{*(q)} t^{|\beta|}. \tag{30}$$

Proof. From $G(t) = \sum_{n \geqslant 1} \eta_n^{*(q)} t^n$, we obtain

$$G(t)^k = \left(\sum_{n \geqslant 1} \eta_n^{*(q)} t^n \right)^k = \sum_{n_1,n_2,\ldots,n_k \geqslant 1} \left(\eta_{n_1}^{*(q)} t^{n_1}\right)\left(\eta_{n_2}^{*(q)} t^{n_2}\right) \cdots \left(\eta_{n_k}^{*(q)} t^{n_k}\right)$$

(by the product rule)

$$= \sum_{n_1,n_2,\ldots,n_k \geqslant 1} \eta_{n_1}^{*(q)} \eta_{n_2}^{*(q)} \cdots \eta_{n_k}^{*(q)} t^{n_1+n_2+\cdots+n_k}$$

$$= \sum_{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathrm{Comp}} \underbrace{\eta^{*(q)}_{\beta_1}\eta^{*(q)}_{\beta_2}\cdots\eta^{*(q)}_{\beta_k}}_{\substack{=\eta^{*(q)}_\beta \\ \text{(by Proposition 29)}}} \underbrace{t^{\beta_1+\beta_2+\cdots+\beta_k}}_{=t^{|\beta|}}$$

(here, we have renamed $n_1, n_2, \ldots, n_k$ as $\beta_1, \beta_2, \ldots, \beta_k$)

$$= \sum_{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathrm{Comp}} \eta^{*(q)}_\beta t^{|\beta|} = \sum_{\substack{\beta\in\mathrm{Comp};\\ \ell(\beta)=k}} \eta^{*(q)}_\beta t^{|\beta|}.$$

This proves Proposition 34. □

4.5 The dual eta basis: coproduct

Consider the comultiplication $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ of the Hopf algebra NSym. We again recall the Iverson bracket notation (Convention 2).

Theorem 35. *For any positive integer n, we have*

$$\Delta\left(\eta^{*(q)}_n\right) = \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\beta|+|\gamma|=n;\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \eta^{*(q)}_\beta \otimes \eta^{*(q)}_\gamma.$$

Example 36. For $n = 2$, there are exactly three pairs (β, γ) of compositions $\beta, \gamma \in \mathrm{Comp}$ satisfying $|\beta| + |\gamma| = n$ and $|\ell(\beta) - \ell(\gamma)| \leqslant 1$: namely, the pairs $(\varnothing, (2))$, $((1), (1))$ and $((2), \varnothing)$. Hence, Theorem 35 (applied to $n = 2$) yields

$$\begin{aligned}\Delta\left(\eta^{*(q)}_2\right) &= (-q)^{1-1}(q-1)^0 \eta^{*(q)}_\varnothing \otimes \eta^{*(q)}_{(2)} + (-q)^{1-1}(q-1)^1 \eta^{*(q)}_{(1)} \otimes \eta^{*(q)}_{(1)} \\ &\qquad + (-q)^{1-1}(q-1)^0 \eta^{*(q)}_{(2)} \otimes \eta^{*(q)}_\varnothing \\ &= \eta^{*(q)}_\varnothing \otimes \eta^{*(q)}_{(2)} + (q-1)\eta^{*(q)}_{(1)} \otimes \eta^{*(q)}_{(1)} + \eta^{*(q)}_{(2)} \otimes \eta^{*(q)}_\varnothing \\ &= 1 \otimes \eta^{*(q)}_2 + (q-1)\eta^{*(q)}_1 \otimes \eta^{*(q)}_1 + \eta^{*(q)}_2 \otimes 1\end{aligned}$$

(since $\eta^{*(q)}_{(2)} = \eta_2$ and $\eta^{*(q)}_{(1)} = \eta_1$ and $\eta^{*(q)}_\varnothing = 1$).

Similar computations show that

$$\Delta\left(\eta^{*(q)}_1\right) = 1 \otimes \eta^{*(q)}_1 + \eta^{*(q)}_1 \otimes 1$$

and

$$\begin{aligned}\Delta\left(\eta^{*(q)}_3\right) &= 1 \otimes \eta^{*(q)}_3 + (q-1)\eta^{*(q)}_1 \otimes \eta^{*(q)}_2 - q\eta^{*(q)}_1 \otimes \left(\eta^{*(q)}_1\right)^2 \\ &\qquad - q\left(\eta^{*(q)}_1\right)^2 \otimes \eta^{*(q)}_1 + (q-1)\eta^{*(q)}_2 \otimes \eta^{*(q)}_1 + \eta^{*(q)}_3 \otimes 1\end{aligned}$$

(since Proposition 29 yields $\eta^{*(q)}_{(1,1)} = \left(\eta^{*(q)}_1\right)^2$).

Proof of Theorem 35. Let $t \in \mathbf{k}$ be a nilpotent element. (We will later construct a specific nilpotent element to which we will apply the following.)

Define the elements

$$\mathbf{G} := G(t) \in \operatorname{NSym} \qquad \text{and} \qquad \mathbf{H} := H(t) \in \operatorname{NSym}.$$

Proposition 33 then says that

$$\mathbf{G} = \frac{\mathbf{H} - 1}{\mathbf{H} + q}, \tag{31}$$

and that $\mathbf{H} + q$ is invertible.

Next, we observe the following:

Claim 1: We have

$$\Delta(\mathbf{H}) = \mathbf{H} \otimes \mathbf{H}. \tag{32}$$

[*Proof of Claim 1:* From $\mathbf{H} = H(t) = \sum_{n \in \mathbb{N}} H_n t^n$, we obtain both

$$\begin{aligned} \mathbf{H} \otimes \mathbf{H} &= \left(\sum_{i \in \mathbb{N}} H_i t^i \right) \otimes \left(\sum_{j \in \mathbb{N}} H_j t^j \right) = \sum_{i,j \in \mathbb{N}} (H_i \otimes H_j) t^{i+j} \\ &= \sum_{n \in \mathbb{N}} \left(\sum_{\substack{i,j \in \mathbb{N}; \\ i+j=n}} H_i \otimes H_j \right) t^n \end{aligned} \tag{33}$$

and

$$\Delta(\mathbf{H}) = \Delta\left(\sum_{n \in \mathbb{N}} H_n t^n \right) = \sum_{n \in \mathbb{N}} \Delta(H_n) t^n. \tag{34}$$

However, for each $n \in \mathbb{N}$, we have $\Delta(H_n) = \sum_{\substack{i,j \in \mathbb{N}; \\ i+j=n}} H_i \otimes H_j$ (by [GriRei20, (5.4.2)]). Thus, the right hand sides of the equalities (34) and (33) are equal. Therefore, so are their left hand sides. This proves Claim 1.]

Since Δ is a $\mathbf{k}$-algebra homomorphism, the element $\Delta(\mathbf{H} + q)$ is invertible (because $\mathbf{H} + q$ is invertible, but $\mathbf{k}$-algebra homomorphisms preserve invertibility). However,

$$\begin{aligned} \Delta(\mathbf{H} + q) &= \Delta(\mathbf{H}) + q && \text{(again since } \Delta \text{ is a } \mathbf{k}\text{-algebra homomorphism)} \\ &= \mathbf{H} \otimes \mathbf{H} + q && \text{(by Claim 1)}. \end{aligned}$$

Thus, we have shown that $\mathbf{H} \otimes \mathbf{H} + q$ is invertible.

Define four elements h_1, h_2, g_1 and g_2 of $\operatorname{NSym} \otimes \operatorname{NSym}$ by

$$\begin{aligned} h_1 &= \mathbf{H} \otimes 1 \qquad \text{and} \qquad h_2 = 1 \otimes \mathbf{H} \qquad \text{and} \\ g_1 &= \mathbf{G} \otimes 1 \qquad \text{and} \qquad g_2 = 1 \otimes \mathbf{G}. \end{aligned}$$

The map

$$\iota_1 : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym},$$
$$z \mapsto z \otimes 1$$

is a $\mathbf{k}$-algebra homomorphism. Applying it to both sides of (31), we find

$$\iota_1(\mathbf{G}) = \iota_1\left(\frac{\mathbf{H}-1}{\mathbf{H}+q}\right) = \frac{\iota_1(\mathbf{H})-1}{\iota_1(\mathbf{H})+q}$$

(since ι_1 is a $\mathbf{k}$-algebra homomorphism). Since $\iota_1(\mathbf{G}) = \mathbf{G} \otimes 1 = g_1$ and $\iota_1(\mathbf{H}) = \mathbf{H} \otimes 1 = h_1$, we can rewrite this as

$$g_1 = \frac{h_1 - 1}{h_1 + q}. \tag{35}$$

Likewise we can obtain

$$g_2 = \frac{h_2 - 1}{h_2 + q}. \tag{36}$$

Implicitly, this also shows that the denominators $h_1 + q$ and $h_2 + q$ here are invertible. Moreover,

$$\underbrace{h_1}_{=\mathbf{H}\otimes 1} \underbrace{h_2}_{=1\otimes\mathbf{H}} = (\mathbf{H} \otimes 1)(1 \otimes \mathbf{H}) = \mathbf{H} \otimes \mathbf{H}.$$

Recall that $\mathbf{H} \otimes \mathbf{H} + q$ is invertible. In other words, $h_1 h_2 + q$ is invertible (since $h_1 h_2 = \mathbf{H} \otimes \mathbf{H}$).

The elements $h_1 = \mathbf{H} \otimes 1$ and $h_2 = 1 \otimes \mathbf{H}$ clearly commute. The elements $\frac{1}{h_1 + q}$, $\frac{1}{h_2 + q}$ and $\frac{1}{h_1 h_2 + q}$ (which are well-defined, since we have shown that the denominators are invertible) are rational functions in these commuting elements h_1 and h_2, and therefore also commute with them (and with each other). Thus, the five elements h_1, h_2, $\frac{1}{h_1 + q}$, $\frac{1}{h_2 + q}$ and $\frac{1}{h_1 h_2 + q}$ generate a commutative $\mathbf{k}$-subalgebra of $\mathrm{NSym} \otimes \mathrm{NSym}$. Let us denote this commutative $\mathbf{k}$-subalgebra by $\mathcal{H}$. Clearly, the elements $h_1 + q$, $h_2 + q$ and $h_1 h_2 + q$ are invertible in $\mathcal{H}$. Also, the element $q + 1 = r$ is invertible in $\mathcal{H}$ (since it is invertible in $\mathbf{k}$ already).

The equalities (35) and (36) show that the elements g_1 and g_2 also belong to the commutative $\mathbf{k}$-algebra $\mathcal{H}$. These two equalities also entail

$$1 + q g_1 g_2 = \frac{(q+1)(h_1 h_2 + q)}{(h_1 + q)(h_2 + q)}$$

(by some straightforward computations using the commutativity of $\mathcal{H}$). Thus, $1 + q g_1 g_2$ is invertible in $\mathcal{H}$ (since $q + 1$, $h_1 h_2 + q$, $h_1 + q$ and $h_2 + q$ are invertible in $\mathcal{H}$).

Recall that $h_1 h_2 = \mathbf{H} \otimes \mathbf{H}$. Comparing this with (32), we obtain

$$\Delta(\mathbf{H}) = h_1 h_2. \tag{37}$$

Hence,

$$\frac{\Delta(\mathbf{H})-1}{\Delta(\mathbf{H})+q}=\frac{h_1h_2-1}{h_1h_2+q}$$
$$=\frac{g_1+g_2+(q-1)\,g_1g_2}{1+qg_1g_2}. \tag{38}$$

(Indeed, the last equality sign can easily be verified by straightforward computations in the commutative $\mathbf{k}$-algebra $\mathcal{H}$, using the equalities (35) and (36).)

From $g_1=\mathbf{G}\otimes 1$ and $g_2=1\otimes\mathbf{G}$, we obtain

$$g_1g_2=(\mathbf{G}\otimes 1)(1\otimes\mathbf{G})=\mathbf{G}\otimes\mathbf{G}. \tag{39}$$

Note that this tensor is a multiple of t (since $\mathbf{G}=G(t)=\sum_{n\geqslant 1}\eta_n^{*(q)}\underbrace{t^n}_{=tt^{n-1}}=t\sum_{n\geqslant 1}\eta_n^{*(q)}t^{n-1}$ is a multiple of t), and thus is nilpotent. Hence, qg_1g_2 is nilpotent as well.

Applying the map Δ to both sides of (31), we find

$$\begin{aligned}
\Delta(\mathbf{G})&=\Delta\left(\frac{\mathbf{H}-1}{\mathbf{H}+q}\right)=\frac{\Delta(\mathbf{H})-1}{\Delta(\mathbf{H})+q} && \text{(since Δ is a $\mathbf{k}$-algebra homomorphism)}\\
&=\frac{g_1+g_2+(q-1)\,g_1g_2}{1+qg_1g_2} && \text{(by (38))}\\
&=\underbrace{\frac{1}{1+qg_1g_2}}_{\substack{=\sum_{i\in\mathbb{N}}(-qg_1g_2)^i\\ \text{(by the geometric series formula)}}}\cdot\left(g_1+g_2+(q-1)\,g_1g_2\right)\\
&=\sum_{i\in\mathbb{N}}\underbrace{(-qg_1g_2)^i}_{=(-q)^i(g_1g_2)^i}\left(g_1+g_2+(q-1)\,g_1g_2\right)\\
&=\sum_{i\in\mathbb{N}}(-q)^i\Bigg(\underbrace{g_1g_2}_{\substack{=\mathbf{G}\otimes\mathbf{G}\\ \text{(by (39))}}}\Bigg)^i\Bigg(\underbrace{g_1}_{=\mathbf{G}\otimes 1}+\underbrace{g_2}_{=1\otimes\mathbf{G}}+(q-1)\underbrace{g_1g_2}_{\substack{=\mathbf{G}\otimes\mathbf{G}\\ \text{(by (39))}}}\Bigg)\\
&=\sum_{i\in\mathbb{N}}(-q)^i(\mathbf{G}\otimes\mathbf{G})^i\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+(q-1)\,\mathbf{G}\otimes\mathbf{G}\right).
\end{aligned}$$

In order to simplify the right hand side, we need two further claims:

Claim 2: Let $u,v\in\mathbb{N}$. Then,

$$\mathbf{G}^u\otimes\mathbf{G}^v=\sum_{\substack{\beta,\gamma\in\text{Comp};\\ \ell(\beta)=u\text{ and }\ell(\gamma)=v}}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right)t^{|\beta|+|\gamma|}.$$

[*Proof of Claim 2:* From $\mathbf{G} = G(t)$, we obtain

$$\mathbf{G}^u = G(t)^u = \sum_{\substack{\beta \in \operatorname{Comp}; \\ \ell(\beta)=u}} \eta_\beta^{*(q)} t^{|\beta|} \qquad \text{(by (30), applied to } k = u\text{)}$$

and similarly

$$\mathbf{G}^v = \sum_{\substack{\gamma \in \operatorname{Comp}; \\ \ell(\gamma)=v}} \eta_\gamma^{*(q)} t^{|\gamma|}.$$

Tensoring these two equalities together, we obtain

$$\begin{aligned} \mathbf{G}^u \otimes \mathbf{G}^v &= \left(\sum_{\substack{\beta \in \operatorname{Comp}; \\ \ell(\beta)=u}} \eta_\beta^{*(q)} t^{|\beta|} \right) \otimes \left(\sum_{\substack{\gamma \in \operatorname{Comp}; \\ \ell(\gamma)=v}} \eta_\gamma^{*(q)} t^{|\gamma|} \right) \\ &= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=u \text{ and } \ell(\gamma)=v}} \underbrace{\eta_\beta^{*(q)} t^{|\beta|} \otimes \eta_\gamma^{*(q)} t^{|\gamma|}}_{= \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|}} = \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=u \text{ and } \ell(\gamma)=v}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|}. \end{aligned}$$

This proves Claim 2.]

Claim 3: Let $i \in \mathbb{N}$. Then,

$$\begin{aligned} &(\mathbf{G} \otimes \mathbf{G})^i (\mathbf{G} \otimes 1 + 1 \otimes \mathbf{G} + (q-1)\, \mathbf{G} \otimes \mathbf{G}) \\ &= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ |\ell(\beta)-\ell(\gamma)| \leqslant 1; \\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|}. \end{aligned}$$

[*Proof of Claim 3:* From $(\mathbf{G} \otimes \mathbf{G})^i = \mathbf{G}^i \otimes \mathbf{G}^i$, we obtain

$$\begin{aligned} &(\mathbf{G} \otimes \mathbf{G})^i (\mathbf{G} \otimes 1 + 1 \otimes \mathbf{G} + (q-1)\, \mathbf{G} \otimes \mathbf{G}) \\ &= \left(\mathbf{G}^i \otimes \mathbf{G}^i \right) (\mathbf{G} \otimes 1 + 1 \otimes \mathbf{G} + (q-1)\, \mathbf{G} \otimes \mathbf{G}) \\ &= \underbrace{\mathbf{G}^{i+1} \otimes \mathbf{G}^i}_{\substack{= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|} \\ \text{(by Claim 2)}}} + \underbrace{\mathbf{G}^i \otimes \mathbf{G}^{i+1}}_{\substack{= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|} \\ \text{(by Claim 2)}}} \\ &\qquad + (q-1) \underbrace{\mathbf{G}^{i+1} \otimes \mathbf{G}^{i+1}}_{\substack{= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|} \\ \text{(by Claim 2)}}} \\ &= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|} + \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)} \right) t^{|\beta|+|\gamma|} \end{aligned}$$

$$+ (q-1) \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}. \tag{40}$$

On the other hand, let us observe that two integers u and v satisfy the two conditions

$$|u-v| \leqslant 1 \qquad \text{and} \qquad \max\{u,v\} = i+1$$

if and only if they satisfy one of the three mutually exclusive conditions

$$(u = i+1 \text{ and } v = i), \quad (u = i \text{ and } v = i+1) \quad \text{and} \quad (u = i+1 \text{ and } v = i+1).$$

Hence, two compositions $\beta, \gamma \in \operatorname{Comp}$ satisfy the two conditions

$$|\ell(\beta) - \ell(\gamma)| \leqslant 1 \qquad \text{and} \qquad \max\{\ell(\beta), \ell(\gamma)\} = i+1$$

if and only if they satisfy one of the three mutually exclusive conditions

$$(\ell(\beta) = i+1 \text{ and } \ell(\gamma) = i), \quad (\ell(\beta) = i \text{ and } \ell(\gamma) = i+1) \quad \text{and} \quad (\ell(\beta) = i+1 \text{ and } \ell(\gamma) = i+1).$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=1\\ (\text{since } \ell(\beta)=i+1>i=\ell(\gamma))}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\quad + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=1\\ (\text{since } \ell(\beta)=i<i+1=\ell(\gamma))}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\quad + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=q-1\\ (\text{since } \ell(\beta)=i+1=\ell(\gamma))}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\quad + (q-1) \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.
\end{aligned}$$

Comparing this with (40), we obtain

$$\begin{aligned}
&(\mathbf{G}\otimes\mathbf{G})^{i}\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+(q-1)\,\mathbf{G}\otimes\mathbf{G}\right)\\
&=\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}.
\end{aligned}$$

This proves Claim 3.]

Now, we can finish our computation of $\Delta(\mathbf{G})$: As we know,

$$\begin{aligned}
\Delta(\mathbf{G})&=\sum_{i\in\mathbb{N}}(-q)^{i}\underbrace{(\mathbf{G}\otimes\mathbf{G})^{i}\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+(q-1)\,\mathbf{G}\otimes\mathbf{G}\right)}_{\substack{=\sum\limits_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\ \text{(by Claim 3)}}}\\
&=\sum_{i\in\mathbb{N}}\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}}(-q)^{i}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\
&=\sum_{j>0}\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=j}}(-q)^{j-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\
&\qquad\text{(here, we have substituted } j-1 \text{ for } i \text{ in the outer sum)}\\
&=\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0}}(-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}
\end{aligned}$$

(here, we have folded the two summation signs into one). Comparing this with

$$\begin{aligned}
\Delta(\mathbf{G})&=\Delta\left(\sum_{n\geqslant 1}\eta_{n}^{*(q)}t^{n}\right)&&\left(\text{since } \mathbf{G}=G(t)=\sum_{n\geqslant 1}\eta_{n}^{*(q)}t^{n}\right)\\
&=\sum_{n\geqslant 1}\Delta\left(\eta_{n}^{*(q)}\right)t^{n}&&(\text{since } \Delta \text{ is } \mathbf{k}\text{-linear}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sum_{n\geqslant 1}\Delta\left(\eta_{n}^{*(q)}\right)t^{n}\\
&=\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0}}(-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}
\end{aligned}$$

$$= \sum_{n \geqslant 0} \sum_{\substack{\beta,\gamma \in \text{Comp}; \\ |\ell(\beta)-\ell(\gamma)| \leqslant 1; \\ \max\{\ell(\beta),\ell(\gamma)\}>0; \\ |\beta|+|\gamma|=n}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^n \tag{41}$$

(here, we have collected all addends with equal values of $|\beta| + |\gamma|$ in our sum).

Now, forget that we fixed t. We thus have proved the equality (41) for each nilpotent element $t \in \mathbf{k}$.

Now, let m be a positive integer. Let us adjoin a nilpotent element t satisfying $t^{m+1} = 0$ (but $t^m \neq 0$) to the ring $\mathbf{k}$. More precisely, let $\mathbf{k}'$ be the quotient ring $\mathbf{k}[T] / (T^{m+1})$ of the polynomial ring $\mathbf{k}[T]$, and let $t \in \mathbf{k}'$ be the residue class of the polynomial T. Then, the ring $\mathbf{k}'$ is commutative, and its element t is nilpotent (with $t^{m+1} = 0$). Moreover, the $m+1$ elements $t^0, t^1, \ldots, t^m$ form a basis of the $\mathbf{k}$-module $\mathbf{k}'$. Hence, if M is any $\mathbf{k}$-module, and if $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$ are arbitrary elements of M satisfying $\sum_{n=0}^{m} a_n \otimes t^n = \sum_{n=0}^{m} b_n \otimes t^n$, then

$$a_n = b_n \qquad \text{for each } n \in \{0, 1, \ldots, m\},$$

and thus, in particular,

$$a_m = b_m. \tag{42}$$

We have proved the equality (41) for our base ring $\mathbf{k}$, but we can equally well prove it for $\mathbf{k}'$ instead (since $\mathbf{k}'$ is again a commutative ring). Thus, the equality (41) holds for our nilpotent element $t \in \mathbf{k}'$. The two sides of this equality then belong to $\text{NSym}_{\mathbf{k}'} \otimes_{\mathbf{k}'} \text{NSym}_{\mathbf{k}'}$ (where $\text{NSym}_{\mathbf{k}'}$ denotes the ring of noncommutative symmetric functions over $\mathbf{k}'$). However, we have canonical isomorphisms $\text{NSym}_{\mathbf{k}'} \cong \text{NSym} \otimes \mathbf{k}'$ (where NSym still denotes $\text{NSym}_{\mathbf{k}}$) and thus

$$\begin{aligned} \text{NSym}_{\mathbf{k}'} \otimes_{\mathbf{k}'} \text{NSym}_{\mathbf{k}'} &\cong (\text{NSym} \otimes \mathbf{k}') \otimes_{\mathbf{k}'} (\text{NSym} \otimes \mathbf{k}') \\ &\cong (\text{NSym} \otimes \text{NSym}) \otimes \mathbf{k}'. \end{aligned}$$

Applying the latter isomorphism to both sides of (41), we obtain

$$\begin{aligned} &\sum_{n \geqslant 1} \Delta\left(\eta_n^{*(q)}\right) \otimes t^n \\ &= \sum_{n \geqslant 0} \sum_{\substack{\beta,\gamma \in \text{Comp}; \\ |\ell(\beta)-\ell(\gamma)| \leqslant 1; \\ \max\{\ell(\beta),\ell(\gamma)\}>0; \\ |\beta|+|\gamma|=n}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) \otimes t^n \end{aligned}$$

in the tensor product $(\text{NSym} \otimes \text{NSym}) \otimes \mathbf{k}'$. Since $t^{m+1} = 0$, we see that all addends for $n \geqslant m+1$ on both sides of this equality must vanish, and we can thus restrict both sums to range from $n = 0$ to m (or from $n = 1$ to m) only. Hence we obtain

$$\sum_{n=1}^{m} \Delta\left(\eta_n^{*(q)}\right) \otimes t^n$$

$$= \sum_{n=0}^{m} \sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0;\\ |\beta|+|\gamma|=n}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_{\beta}^{*(q)} \otimes \eta_{\gamma}^{*(q)}\right) \otimes t^n.$$

Thus, comparing the coefficients of t^m on both sides of this equality (or, to be more formal: applying (42)), we conclude that

$$\Delta\left(\eta_m^{*(q)}\right) = \sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0;\\ |\beta|+|\gamma|=m}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \eta_{\beta}^{*(q)} \otimes \eta_{\gamma}^{*(q)}. \qquad (43)$$

However, any two compositions β and γ that satisfy $|\beta| + |\gamma| = m$ will automatically satisfy $\max\{\ell(\beta), \ell(\gamma)\} > 0$ (since m is positive). Hence, in the summation sign on the right hand side of (43), the condition "$\max\{\ell(\beta), \ell(\gamma)\} > 0$" is redundant. We can thus rewrite this summation sign as follows:

$$\sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0;\\ |\beta|+|\gamma|=m}} = \sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1;\\ |\beta|+|\gamma|=m}} = \sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\beta|+|\gamma|=m;\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1}} .$$

Hence, (43) rewrites as

$$\Delta\left(\eta_m^{*(q)}\right) = \sum_{\substack{\beta,\gamma\in\text{Comp};\\ |\beta|+|\gamma|=m;\\ |\ell(\beta)-\ell(\gamma)|\leqslant 1}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \eta_{\beta}^{*(q)} \otimes \eta_{\gamma}^{*(q)}.$$

Renaming m as n in this result, we obtain precisely the claim of Theorem 35. ☐

Using Theorem 35, we can easily compute the coproduct of any $\eta_\alpha^{*(q)}$:[8]

Corollary 37. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any composition. Then,*

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\text{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\text{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leqslant 1 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k}$$
$$\cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \left(\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)} \otimes \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\right).$$

[8] The symbol "#" means "number". Thus, e.g., we have (# of odd numbers $i \in [2n]$) $= n$ for each $n \in \mathbb{N}$.

Proof. The comultiplication Δ of the $\mathbf{k}$-bialgebra NSym is a $\mathbf{k}$-algebra homomorphism (indeed, this is true for any $\mathbf{k}$-bialgebra). However, Proposition 29 yields

$$\eta_{\alpha}^{*(q)} = \eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)}.$$

Hence,

$$\Delta\left(\eta_{\alpha}^{*(q)}\right) = \Delta\left(\eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)}\right) = \Delta\left(\eta_{\alpha_1}^{*(q)}\right) \Delta\left(\eta_{\alpha_2}^{*(q)}\right) \cdots \Delta\left(\eta_{\alpha_k}^{*(q)}\right)$$

(since Δ is a $\mathbf{k}$-algebra homomorphism). Now, we can use Theorem 35 to compute each factor $\Delta\left(\eta_{\alpha_s}^{*(q)}\right)$ on the right hand side, and expand the resulting product of sums into one single sum (using the product rule). The result is

$$\Delta\left(\eta_{\alpha}^{*(q)}\right) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \mathrm{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leqslant 1 \text{ for all } s}} (-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k} (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}$$
$$\cdot \left(\eta_{\beta_1}^{*(q)} \eta_{\beta_2}^{*(q)} \cdots \eta_{\beta_k}^{*(q)} \otimes \eta_{\gamma_1}^{*(q)} \eta_{\gamma_2}^{*(q)} \cdots \eta_{\gamma_k}^{*(q)}\right).$$

Since Corollary 31 yields $\eta_{\beta_1}^{*(q)} \eta_{\beta_2}^{*(q)} \cdots \eta_{\beta_k}^{*(q)} = \eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)}$ and $\eta_{\gamma_1}^{*(q)} \eta_{\gamma_2}^{*(q)} \cdots \eta_{\gamma_k}^{*(q)} = \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}$, this formula becomes precisely the claim of Corollary 37. $\square$

5 The product rule for $\eta_{\alpha}^{(q)}$

We now approach the most intricate of the rules for the $\eta_{\alpha}^{(q)}$ functions: the product rule, i.e., the expression of a product $\eta_{\delta}^{(q)} \eta_{\varepsilon}^{(q)}$ as a $\mathbb{Z}[q]$-linear combination of other $\eta_{\alpha}^{(q)}$'s. We shall give three different versions of this rule, all equivalent but using somewhat different indexing sets. Only the first version will be proved in any detail, as it suffices for the applications we have in mind.

5.1 The product rule in terms of compositions

Our first version of the product rule is as follows:[9]

Theorem 38. *Let δ and ε be two compositions. Then,*

$$\eta_{\delta}^{(q)} \eta_{\varepsilon}^{(q)} = \sum_{\substack{k\in\mathbb{N};\\ \beta_1,\beta_2,\ldots,\beta_k \in \mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leqslant 1 \textit{ for all } s;\\ \ell(\beta_s)+\ell(\gamma_s)>0 \textit{ for all } s}} (-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k}$$

[9] The symbol "#" means "number" (so that, e.g., we have $(\#\text{ of odd numbers } i \in [2n]) = n$ for each $n \in \mathbb{N}$).

$$\cdot (q-1)^{(\#\ of\ all\ s\in[k]\ such\ that\ \ell(\beta_s)=\ell(\gamma_s))}$$
$$\cdot \eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}.$$

Remark 39. The compositions $\beta_1, \beta_2, \ldots, \beta_k$ and $\gamma_1, \gamma_2, \ldots, \gamma_k$ in the sum on the right hand side of Theorem 38 are allowed to be empty. Nevertheless, the sum is finite. Indeed, if $k \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots, \beta_k \in \text{Comp}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \text{Comp}$ satisfy

$$\beta_1\beta_2\cdots\beta_k = \delta \qquad \text{and} \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon \qquad \text{and}$$
$$|\ell(\beta_s) - \ell(\gamma_s)| \leqslant 1 \text{ for all } s \qquad \text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s,$$

then $k \leqslant \ell(\delta) + \ell(\varepsilon)$, because

$$\begin{aligned}
&\underbrace{\ell(\delta)}_{\substack{=\ell(\beta_1)+\ell(\beta_2)+\cdots+\ell(\beta_k) \\ (\text{since } \delta=\beta_1\beta_2\cdots\beta_k)}} + \underbrace{\ell(\varepsilon)}_{\substack{=\ell(\gamma_1)+\ell(\gamma_2)+\cdots+\ell(\gamma_k) \\ (\text{since } \varepsilon=\gamma_1\gamma_2\cdots\gamma_k)}} \\
&= (\ell(\beta_1) + \ell(\beta_2) + \cdots + \ell(\beta_k)) + (\ell(\gamma_1) + \ell(\gamma_2) + \cdots + \ell(\gamma_k)) \\
&= \sum_{s=1}^{k} \ell(\beta_s) + \sum_{s=1}^{k} \ell(\gamma_s) = \sum_{s=1}^{k} \underbrace{(\ell(\beta_s) + \ell(\gamma_s))}_{\substack{\geqslant 1 \\ (\text{since our above assumptions} \\ \text{yield } \ell(\beta_s)+\ell(\gamma_s)>0, \\ \text{but } \ell(\beta_s)+\ell(\gamma_s) \text{ is an integer})}} \geqslant \sum_{s=1}^{k} 1 = k.
\end{aligned}$$

This narrows down the options for k to the finite set $\{0, 1, \ldots, \ell(\delta) + \ell(\varepsilon)\}$, and thus leaves only finitely many options for $\beta_1, \beta_2, \ldots, \beta_k$ (since there are only finitely many ways to decompose the composition δ as a concatenation $\delta = \beta_1\beta_2\cdots\beta_k$ when k is fixed) and for $\gamma_1, \gamma_2, \ldots, \gamma_k$ (similarly). Thus, the sum is finite.

Example 40. Let δ and ε be two compositions of the form $\delta = (a, b)$ and $\varepsilon = (c)$ for some positive integers a, b, c. Then, Theorem 38 expresses the product $\eta^{(q)}_\delta \eta^{(q)}_\varepsilon = \eta^{(q)}_{(a,b)} \eta^{(q)}_{(c)}$ as a sum over all choices of $k \in \mathbb{N}$ and of k compositions $\beta_1, \beta_2, \ldots, \beta_k \in \text{Comp}$ and of k further compositions $\gamma_1, \gamma_2, \ldots, \gamma_k \in \text{Comp}$ satisfying

$$\beta_1\beta_2\cdots\beta_k = \delta \qquad \text{and} \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon \qquad \text{and}$$
$$|\ell(\beta_s) - \ell(\gamma_s)| \leqslant 1 \text{ for all } s \qquad \text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s.$$

These choices are

1. having $k = 1$ and $\beta_1 = \delta = (a, b)$ and $\gamma_1 = \varepsilon = (c)$;

2. having $k = 2$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\gamma_1 = \varnothing$ and $\gamma_2 = (c)$;

3. having $k = 2$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\gamma_1 = (c)$ and $\gamma_2 = \varnothing$;

4. having $k = 3$ and $\beta_1 = \varnothing$ and $\beta_2 = (a)$ and $\beta_3 = (b)$ and $\gamma_1 = (c)$ and $\gamma_2 = \varnothing$ and $\gamma_3 = \varnothing$;

5. having $k = 3$ and $\beta_1 = (a)$ and $\beta_2 = \varnothing$ and $\beta_3 = (b)$ and $\gamma_1 = \varnothing$ and $\gamma_2 = (c)$ and $\gamma_3 = \varnothing$;
6. having $k = 3$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\beta_3 = \varnothing$ and $\gamma_1 = \varnothing$ and $\gamma_2 = \varnothing$ and $\gamma_3 = (c)$.

Thus, Theorem 38 yields

$$\begin{aligned}
&\eta^{(q)}_{(a,b)}\eta^{(q)}_{(c)} \\
&= (-q)^{2-1}(q-1)^0\,\eta^{(q)}_{(a+b+c)} + (-q)^{1+1-2}(q-1)^1\,\eta^{(q)}_{(a,\ b+c)} \\
&\qquad + (-q)^{1+1-2}(q-1)^1\,\eta^{(q)}_{(a+c,\ b)} + (-q)^{1+1+1-3}(q-1)^0\,\eta^{(q)}_{(c,a,b)} \\
&\qquad + (-q)^{1+1+1-3}(q-1)^0\,\eta^{(q)}_{(a,c,b)} + (-q)^{1+1+1-3}(q-1)^0\,\eta^{(q)}_{(a,b,c)} \\
&= -q\eta^{(q)}_{(a+b+c)} + (q-1)\,\eta^{(q)}_{(a,\ b+c)} + (q-1)\,\eta^{(q)}_{(a+c,\ b)} + \eta^{(q)}_{(c,a,b)} + \eta^{(q)}_{(a,c,b)} + \eta^{(q)}_{(a,b,c)}.
\end{aligned}$$

Note that the last three addends $\eta^{(q)}_{(c,a,b)}$, $\eta^{(q)}_{(a,c,b)}$, $\eta^{(q)}_{(a,b,c)}$ here come from those choices in which $\min\{\ell(\beta_s), \ell(\gamma_s)\} = 0$ for each $s \in [k]$ (that is, for each $s \in [k]$, one of the two compositions β_s and γ_s is empty). In these choices, the two powers

$$(-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \qquad \text{and} \qquad (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}$$

are equal to 1 (because the exponents are easily seen to be 0), whereas the composition $(|\beta_1| + |\gamma_1|,\ |\beta_2| + |\gamma_2|,\ \ldots,\ |\beta_k| + |\gamma_k|)$ is a shuffle of δ with ε. Thus, these choices contribute terms of the form $\eta^{(q)}_{\varphi}$, where φ is a shuffle of δ with ε, to the right hand side of Theorem 38, and these terms all have coefficient 1. These are the only choices of $k,\ \beta_1, \beta_2, \ldots, \beta_k,\ \gamma_1, \gamma_2, \ldots, \gamma_k$ that have $k = \ell(\delta) + \ell(\varepsilon)$. All other choices have $k < \ell(\delta) + \ell(\varepsilon)$, and these choices lead to addends that involve either a nontrivial power of $-q$ or a nontrivial power of $q-1$ (or both). In this sense, we can view Theorem 38 as a deformation of the overlapping shuffle product formula for $M_\delta M_\varepsilon$ (see, e.g., [GriRei20, Proposition 5.1.3]), although the concept of a "deformation" must be understood in a wide sense (we cannot obtain the latter just by specializing the former).

We will derive Theorem 38 from Corollary 37. For this, we will again use the duality between NSym and QSym:

Lemma 41. *Let $f, g \in \mathrm{QSym}$ and $h \in \mathrm{NSym}$ be arbitrary. Let the tensor $\Delta(h) \in \mathrm{NSym} \otimes \mathrm{NSym}$ be written in the form $\Delta(h) = \sum_{i\in I} s_i \otimes t_i$, where I is a finite set and where $s_i, t_i \in \mathrm{NSym}$ for each $i \in I$. Then,*

$$\langle h, fg\rangle = \sum_{i\in I} \langle s_i, f\rangle \langle t_i, g\rangle .$$

Proof. This is analogous to Lemma 75, except that the roles of QSym and NSym have now been switched. ☐

For the sake of convenience, let us extend Lemma 41 to infinite sums with only finitely many infinite addends:

Lemma 42. *Let $f, g \in \operatorname{QSym}$ and $h \in \operatorname{NSym}$ be arbitrary. Let the tensor $\Delta(h) \in \operatorname{NSym} \otimes \operatorname{NSym}$ be written in the form $\Delta(h) = \sum_{i \in I} s_i \otimes t_i$, where I is a set and where $s_i, t_i \in \operatorname{NSym}$ for each $i \in I$ are chosen such that only finitely many $i \in I$ satisfy $s_i \neq 0$. Then,*

$$\langle h, fg \rangle = \sum_{i \in I} \langle s_i, f \rangle \langle t_i, g \rangle .$$

Proof. This is easily reduced to Lemma 41 (just replace the set I by its subset $I' := \{i \in I \mid s_i \neq 0\}$). ☐

Proof of Theorem 38. Forget that we fixed δ and ε. For any three compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, δ and ε, we define an element

$$d_{\delta,\varepsilon}^{\alpha}(q) = \sum_{\substack{\beta_1, \beta_2, \ldots, \beta_k \in \operatorname{Comp}; \\ \gamma_1, \gamma_2, \ldots, \gamma_k \in \operatorname{Comp}; \\ \beta_1 \beta_2 \cdots \beta_k = \delta; \\ \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon; \\ |\ell(\beta_s) - \ell(\gamma_s)| \leqslant 1 \text{ for all } s; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for all } s}} (-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s), \ell(\gamma_s)\} - k} \cdot (q-1)^{(\#\text{ of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s))} \in \mathbf{k}. \tag{44}$$

Note that the sum on the right hand side here is finite (because for a given $k \in \mathbb{N}$ and given compositions δ and ε, there are only finitely many ways to decompose δ as $\delta = \beta_1 \beta_2 \cdots \beta_k$, and only finitely many ways to decompose ε as $\varepsilon = \gamma_1 \gamma_2 \cdots \gamma_k$), and thus is a polynomial in q with integer coefficients. Moreover, this polynomial is the same for all rings $\mathbf{k}$ and all values of q.

The claim that we must prove (i.e., the claim of Theorem 38) can now be rewritten as

$$\eta_{\delta}^{(q)} \eta_{\varepsilon}^{(q)} = \sum_{\alpha \in \operatorname{Comp}} d_{\delta,\varepsilon}^{\alpha}(q)\, \eta_{\alpha}^{(q)} \tag{45}$$

for all $\delta, \varepsilon \in \operatorname{Comp}$ (because this is what we obtain if we collect like terms on the right hand side of Theorem 38). Here, the right hand side is a polynomial in q with coefficients in QSym (because $d_{\delta,\varepsilon}^{\alpha}(q) = 0$ for all compositions α satisfying $|\alpha| \neq |\delta| + |\varepsilon|$, and thus only finitely many addends of the sum have any chance of being nonzero).

If we expand both sides of (45) in the monomial basis $(M_\beta)_{\beta \in \operatorname{Comp}}$ of QSym, then each specific coefficient will be a polynomial in q with coefficients in $\mathbb{Z}$, and again, the polynomial will be the same for every $\mathbf{k}$ and every q. Thus, the claim we must prove can be rewritten as a set of identities between polynomials with coefficients in $\mathbb{Z}$.

When proving such a claim, we can always WLOG assume that $\mathbf{k}$ is the polynomial ring $\mathbb{Z}[X]$ and that q is the indeterminate X. Even better, we can WLOG assume that $\mathbf{k}$ is the field of rational functions $\mathbb{Q}(X)$ and that q is the indeterminate X (since the polynomial ring $\mathbb{Z}[X]$ is canonically a subring of $\mathbb{Q}(X)$, and we lose nothing by passing to a larger ring). Let us make this latter assumption. Then, $r = q + 1 = X + 1$ is an invertible element of $\mathbf{k}$ (since $\mathbf{k} = \mathbb{Q}(X)$ is a field).

Hence, the dual basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \text{Comp}}$ of NSym is well-defined. Moreover, any composition α satisfies

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{\delta,\varepsilon \in \text{Comp}} d_{\delta,\varepsilon}^{\alpha}(q) \left(\eta_\delta^{*(q)} \otimes \eta_\varepsilon^{*(q)}\right). \tag{46}$$

(Indeed, this is just Corollary 37, rewritten by collecting like addends on the right hand side and simplifying the resulting expression using (44).)

Now, our claim (45) follows easily from (46) by dualization (i.e., using Lemma 42). The details are similar to our proof of Proposition 30, except that we are now drawing conclusions about QSym from NSym instead of the other way around. See [GriVas23a] for details. ☐

5.2 The product rule in terms of stufflers

We will next rewrite Theorem 38 in a somewhat different language, using certain surjective maps instead of factorizations of compositions. First, we introduce several pieces of notation:

Definition 43. Let i and j be two integers. Then, we write $i \approx j$ (and say that i is *nearly equal* to j) if and only if $|i - j| \leqslant 1$.

(Of course, $\approx$ is not an equivalence relation.)

Definition 44. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ be two compositions.

Fix two chains (i.e., totally ordered sets) $P = \{p_1 < p_2 < \cdots < p_\ell\}$ and $Q = \{q_1 < q_2 < \cdots < q_m\}$, and let

$$U = P \sqcup Q$$

be their disjoint union. This U is a poset with $\ell + m$ elements $p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_m$, whose relations are given by $p_1 < p_2 < \cdots < p_\ell$ and $q_1 < q_2 < \cdots < q_m$ (while each p_i is incomparable to each q_j).

If $f : U \to X$ is a map from U to any set X, and if $s \in X$ is any element, then we define the two sets

$$f_P^{-1}(s) := \{u \in [\ell] \mid f(p_u) = s\} \qquad \text{and}$$
$$f_Q^{-1}(s) := \{v \in [m] \mid f(q_v) = s\}.$$

(Essentially, $f_P^{-1}(s)$ and $f_Q^{-1}(s)$ are the sets of the preimages of s in P and Q, respectively, except that they consist of numbers instead of actual elements of P and Q.)

A *stufufuffler* for δ and ε shall mean a surjective and weakly order-preserving map

$$f : U \to \{1 < 2 < \cdots < k\} \qquad \text{for some } k \in \mathbb{N}$$

with the property that each $s \in \{1 < 2 < \cdots < k\}$ satisfies

$$\left|f_P^{-1}(s)\right| \approx \left|f_Q^{-1}(s)\right|. \tag{47}$$

("Weakly order-preserving" means that if u and v are two elements of the poset U satisfying $u < v$, then $f(u) \leqslant f(v)$.)

If $f : U \to \{1 < 2 < \cdots < k\}$ is a stufufuffler for δ and ε, then we define three further concepts:

- We define the *weight* $\operatorname{wt}(f)$ of f to be the composition

$$(\operatorname{wt}_1(f), \operatorname{wt}_2(f), \ldots, \operatorname{wt}_k(f)),$$

where

$$\begin{aligned}\operatorname{wt}_s(f) &= \sum_{u \in f_P^{-1}(s)} \delta_u + \sum_{v \in f_Q^{-1}(s)} \varepsilon_v \\ &= \sum_{\substack{u \in [\ell];\\ f(p_u)=s}} \delta_u + \sum_{\substack{v \in [m];\\ f(q_v)=s}} \varepsilon_v \qquad \text{for each } s \in [k].\end{aligned}$$

(Note that (47) ensures that the two sums on the right hand side here have nearly equal numbers of addends. Moreover, the surjectivity of f ensures that at least one of these two sums has at least one addend, and thus $\operatorname{wt}_s(f)$ is a positive integer; therefore, $\operatorname{wt}(f)$ is a composition.)

- We define the *loss* of f to be the nonnegative integer

$$\operatorname{loss}(f) := \sum_{s=1}^{k} \max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\} - k.$$

(This really is a nonnegative integer, since the surjectivity of f yields that $\max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\} \geqslant 1$ for each $s \in [k]$, and thus we obtain $\operatorname{loss}(f) = \sum_{s=1}^{k} \underbrace{\max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\}}_{\geqslant 1} - k \geqslant \underbrace{\sum_{s=1}^{k} 1}_{=k} - k = 0$.)

- We define the *poise* of f to be the nonnegative integer

$$\operatorname{poise}(f) := \left(\# \text{ of all } s \in [k] \text{ such that } \left|f_P^{-1}(s)\right| = \left|f_Q^{-1}(s)\right|\right).$$

Example 45. Let $\delta = (a, b)$ and $\varepsilon = (c, d, e)$ be two compositions. Then, the poset U in Definition 44 is $U = \{p_1 < p_2\} \sqcup \{q_1 < q_2 < q_3\}$. The following maps (written in two-line notation) are stufufufflers for δ and ε:

$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

(The list is not exhaustive – there are many more stufufufflers for δ and ε.)

On the other hand, here are some maps (in two-line notation) that are not stufufufflers for δ and ε:

- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix}$ is not a stufufuffler, since it violates (47) for $s = 1$.
- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix}$ is not a stufufuffler, since it is not weakly increasing ($f(q_1) > f(q_2)$).
- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}$ is not a stufufuffler, since it fails to be surjective onto $\{1 < 2 < \cdots < k\}$ whatever k is.

Here are the weights of the eight stufufufflers listed above:

$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (a, b, c, d, e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = (c, a, d, e, b),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix} = (a + b + c,\ d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} = (a,\ b + c + d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{pmatrix} = (c,\ a + b + d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = (a + b + c + d + e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} = (a + b + c + d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix} = (a + c + d,\ b + e).$$

The losses of these stufufufflers are 0, 0, 1, 1, 1, 2, 1 and 1, respectively. Their poises are 0, 0, 0, 0, 0, 0, 1 and 1, respectively.

Intuitively, the composition $\operatorname{wt}(f)$ in Definition 44 can be thought of as a variant of a stuffle[10] of δ with ε, but instead of adding an entry of δ with an entry of ε, it allows adding i consecutive entries of δ and j consecutive entries of ε whenever i and j are integers satisfying $i \approx j$. (Such a sum can be obtained by starting with 0 and taking turns at adding the next available entry from δ or from ε; thus the name "stufufuffle".) The poise statistic $\operatorname{poise}(f)$ tells us how often this $i \approx j$ relation becomes an equality. The loss statistic $\operatorname{loss}(f)$ tells how much is being added, i.e., how far this "stufufuffle" deviates from a stuffle.

Now we can restate the multiplication rule for $\eta_\delta^{(q)}\eta_\varepsilon^{(q)}$ in terms of stufufufflers:

Theorem 46. *Let δ and ε be two compositions. Then,*

$$\eta_\delta^{(q)}\eta_\varepsilon^{(q)} = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta_{\operatorname{wt}(f)}^{(q)}.$$

Example 47. Let $\delta = (a, b)$ and $\varepsilon = (c, d)$ be two compositions of length 2. Let us compute $\eta_{(a,b)}^{(q)}\eta_{(c,d)}^{(q)}$ using Theorem 46. The stufufufflers for δ and ε are the maps (written here in two-line notation)

$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 4 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 2 \end{pmatrix}.$$

Their respective weights are

$$\begin{array}{lll} (a, b, c, d), & (a, c, b, d), & (a, c, d, b), \\ (c, a, b, d), & (c, a, d, b), & (c, d, a, b), \\ (a,\ b+c+d), & (c,\ a+b+d), & (a+b+c,\ d), \\ (a+c+d,\ b), & (a+c,\ b+d), & (a+b+c+d), \\ (a+c,\ b,\ d), & (a+c,\ d,\ b), & (a,\ c,\ b+d), \\ (c,\ a,\ b+d), & (a,\ b+c,\ d), & (c,\ a+d,\ b); \end{array}$$

[10] "Stuffles" are also known as "overlapping shuffles"; see [GriRei20, Proposition 5.1.3 and Example 5.1.4] for the meaning of this concept (and [DEMT17] for more).

their respective losses are

$$\begin{gathered}0,0,0,\\0,0,0,\\1,1,1,\\1,0,1,\\0,0,0,\\0,0,0,\end{gathered}$$

whereas their respective poises are

$$\begin{gathered}0,0,0,\\0,0,0,\\0,0,0,\\0,2,1,\\1,1,1,\\1,1,1.\end{gathered}$$

Thus, Theorem 46 yields

$$\begin{aligned}\eta^{(q)}_{(a,b)}\eta^{(q)}_{(c,d)} = \eta^{(q)}_{(a,b,c,d)} &+ \eta^{(q)}_{(a,c,b,d)} + \eta^{(q)}_{(a,c,d,b)}\\ &+ \eta^{(q)}_{(c,a,b,d)} + \eta^{(q)}_{(c,a,d,b)} + \eta^{(q)}_{(c,d,a,b)}\\ &- q\eta^{(q)}_{(a,\ b+c+d)} - q\eta^{(q)}_{(c,\ a+b+d)} - q\eta^{(q)}_{(a+b+c,\ d)}\\ &- q\eta^{(q)}_{(a+c+d,\ b)} + (q-1)^2\,\eta^{(q)}_{(a+c,\ b+d)} - q\,(q-1)\,\eta^{(q)}_{(a+b+c+d)}\\ &+ (q-1)\,\eta^{(q)}_{(a+c,\ b,\ d)} + (q-1)\,\eta^{(q)}_{(a+c,\ d,\ b)} + (q-1)\,\eta^{(q)}_{(a,\ c,\ b+d)}\\ &+ (q-1)\,\eta^{(q)}_{(c,\ a,\ b+d)} + (q-1)\,\eta^{(q)}_{(a,\ b+c,\ d)} + (q-1)\,\eta^{(q)}_{(c,\ a+d,\ b)}.\end{aligned}$$

Let us now outline how Theorem 46 can be derived from Theorem 38.

Proof. Fix a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Let $\mathbf{P}$ be the set of all pairs

$$((\beta_1, \beta_2, \ldots, \beta_k),\ (\gamma_1, \gamma_2, \ldots, \gamma_k))$$

satisfying the six conditions

$$\begin{gathered}\beta_1, \beta_2, \ldots, \beta_k \in \mathrm{Comp}; \qquad \gamma_1, \gamma_2, \ldots, \gamma_k \in \mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k = \delta; \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon;\\ |\ell(\beta_s) - \ell(\gamma_s)| \leqslant 1 \text{ for each } s;\\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s.\end{gathered}$$

On the other hand, let $\mathbf{S}$ be the set of all stufufufflers f for δ and ε satisfying $\mathrm{wt}(f) = \alpha$.

Then, there is a bijection $\Phi : \mathbf{S} \to \mathbf{P}$, which sends any stufufuffler $f \in \mathbf{S}$ to the pair

$$((\beta_1, \beta_2, \ldots, \beta_k), \ (\gamma_1, \gamma_2, \ldots, \gamma_k)),$$

where

$$\begin{aligned} \beta_s = \big(&\text{the composition consisting of the } \delta_u \text{ for all } u \in f_P^{-1}(s) \\ &(\text{in the order of increasing } u)\big) \qquad \text{and} \\ \gamma_s = \big(&\text{the composition consisting of the } \varepsilon_v \text{ for all } v \in f_Q^{-1}(s) \\ &(\text{in the order of increasing } v)\big) \qquad \text{for all } s \in [k]. \end{aligned}$$

We leave it to the reader to verify that this map Φ really is well-defined and bijective (see [GriVas23a] for a few hints on this). It also has a further useful property: If Φ sends a stufufuffler f to a pair $((\beta_1, \beta_2, \ldots, \beta_k), \ (\gamma_1, \gamma_2, \ldots, \gamma_k))$, then

$$\sum_{s=1}^{k} \max\{\ell(\beta_s), \ell(\gamma_s)\} - k = \operatorname{loss}(f) \qquad \text{and}$$
$$(\# \text{ of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s)) = \operatorname{poise}(f).$$

Thus, we can use the bijection Φ to convince ourselves that the sum on the right hand side of Theorem 46 equals the sum on the right hand side of Theorem 38. Hence, the former theorem follows from the latter. ☐

5.3 The product rule in terms of subsets

Finally, let us state the product rule for the $\eta_\alpha^{(q)}$ (Theorem 38) in yet another form, using classical shuffles ([GriVas22, Corollary 1]):

Definition 48. If T is any set of integers, then $T-1$ shall denote the set $\{t-1 \mid t \in T\}$.

Definition 49. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition with n entries. For any $i \in [n-1]$, we let $\alpha^{\downarrow i}$ denote the following composition with $n-1$ entries:

$$\alpha^{\downarrow i} := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n).$$

Furthermore, for any subset $I \subseteq [n-1]$, we set

$$\alpha^{\downarrow I} := \left(\left(\cdots \left(\alpha^{\downarrow i_k} \right) \cdots \right)^{\downarrow i_2} \right)^{\downarrow i_1},$$

where $i_1, i_2, \ldots, i_k$ are the elements of I in increasing order.

Finally, if I and J are two subsets of $[n-1]$, then we set

$$\alpha^{\downarrow I \downarrow\downarrow J} := \alpha^{\downarrow K}, \qquad \text{where } K = I \cup J \cup (J-1).$$

Example 50. Let $\alpha = (a, b, c, d, e, f, g)$ be a composition with 7 entries. Then,

$$\alpha^{\downarrow 2} = (a,\ b+c,\ d,\ e,\ f,\ g)\,;$$
$$\alpha^{\downarrow \{2,4,5\}} = (a,\ b+c,\ d+e+f,\ g)\,;$$
$$\alpha^{\downarrow \{2\} \downarrow\downarrow \{6\}} = \alpha^{\downarrow \{2,5,6\}} = (a,\ b+c,\ d,\ e+f+g)\,.$$

Theorem 51. *Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ be two compositions.*

*If T is any m-element subset of $[n+m]$, then we define the T-*shuffle *of δ with ε to be the composition*

$$\delta \lfloor T \rfloor \varepsilon := (\gamma_1, \gamma_2, \ldots, \gamma_{n+m})\,,$$

where

$$\gamma_k := \begin{cases} \delta_i, & \text{if } k \text{ is the } i\text{-th smallest element of } [n+m] \setminus T; \\ \varepsilon_j, & \text{if } k \text{ is the } j\text{-th smallest element of } T. \end{cases}$$

Furthermore, if T is any subset of $[n+m]$, then we define a further subset

$$T' := (T \setminus (T-1)) \setminus \{n+m\}\,.$$

Then,

$$\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \sum_{\substack{\text{triples } (T,I,J); \\ T \subseteq [n+m]; \\ |T| = m; \\ I \subseteq T'; \\ J \subseteq T' \setminus \{1\}; \\ I \cap J = \varnothing}} (-q)^{|J|} (q-1)^{|I|} \eta^{(q)}_{(\delta \lfloor T \rfloor \varepsilon)^{\downarrow I \downarrow\downarrow J}}.$$

Example 52. Let $\delta = (a)$ and $\varepsilon = (b, c)$ be two compositions. Then, applying Theorem 51 (with $n = 1$ and $m = 2$), we see that $\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \eta_{(a)}^{(q)} \eta_{(b,c)}^{(q)}$ is a sum over all triples (T, I, J) satisfying

$$T \subseteq [3]\,, \qquad |T| = 2, \qquad I \subseteq T', \qquad J \subseteq T' \setminus \{1\}\,, \qquad I \cap J = \varnothing.$$

There are exactly six such triples (T, I, J), namely

$$(\{1,2\}\,,\ \varnothing,\ \varnothing)\,,\ (\{1,2\}\,,\ \varnothing,\ \{2\})\,,\ (\{1,2\}\,,\ \{2\}\,,\ \varnothing)\,,$$
$$(\{1,3\}\,,\ \varnothing,\ \varnothing)\,,\ (\{1,3\}\,,\ \{1\}\,,\ \varnothing)\,,\ (\{2,3\}\,,\ \varnothing,\ \varnothing)\,.$$

Thus, the claim of Theorem 51 becomes

$$\eta_{(a)}^{(q)} \eta_{(b,c)}^{(q)} = \eta_{(b,c,a)}^{(q)} - q\eta_{(a+b+c)}^{(q)} + (q-1)\,\eta_{(b,\ a+c)}^{(q)} + \eta_{(b,a,c)}^{(q)} + (q-1)\,\eta_{(a+b,\ c)}^{(q)} + \eta_{(a,b,c)}^{(q)}$$

(here, we have listed the addends in the same order in which the corresponding triples were listed above).

Theorem 51 can be derived from Theorem 46 by constructing a bijection between the stufufufflers of δ and ε and the triples (T, I, J) from Theorem 51. The details of this bijection are somewhat bothersome, so we shall omit them, not least because Theorem 51 can also be proved in a different way (using enriched P-partitions). The latter proof has been outlined in [GriVas22, Corollary 1] and will be elaborated upon in forthcoming work.

6 Applications

We shall now discuss some applications of the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ and its features.

6.1 Hopf subalgebras of QSym

The $q = 1$ case in particular is useful for constructing Hopf subalgebras of QSym, such as the peak subalgebra Π introduced by Stembridge [Stembr97, §3] (without its Hopf structure) and later studied by various authors ([AgBeSo06, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5], [Hsiao07] etc.). We shall now briefly survey some Hopf subalgebras that can be obtained in this way.

Convention 4. For the rest of Subsection 6.1, we fix a set T of compositions (i.e., a subset T of Comp).

We let $\mathrm{QSym}_T^{(q)}$ be the $\mathbf{k}$-submodule of QSym spanned by the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in T}$.

When is this $\mathbf{k}$-submodule $\mathrm{QSym}_T^{(q)}$ a subcoalgebra of QSym? The answer is simple:[11]

Proposition 53. *For any subset Y of $\{1, 2, 3, \ldots\}$, we let*

$$\begin{aligned} Y^* &:= \{\textit{all compositions whose entries all belong to } Y\} \\ &= \{(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp} \mid \alpha_i \in Y \textit{ for each } i\}. \end{aligned}$$

(a) *If $T = Y^*$ for some subset Y of $\{1, 2, 3, \ldots\}$, then $\mathrm{QSym}_T^{(q)}$ is a subcoalgebra of* QSym*.*

(b) *If $\mathbf{k}$ is a field and $r \neq 0$, then the converse holds as well: If $\mathrm{QSym}_T^{(q)}$ is a subcoalgebra of* QSym*, then $T = Y^*$ for some subset Y of $\{1, 2, 3, \ldots\}$.*

Proof sketch. **(a)** This follows from Theorem 20.

(b) Use the graded dual NSym of QSym and Proposition 30. (The orthogonal complement of a subcoalgebra is an ideal.) ☐

Proposition 53 allows us to restrict ourselves to sets T of the form Y^* for $Y \subseteq \{1, 2, 3, \ldots\}$ if we want $\mathrm{QSym}_T^{(q)}$ to be a Hopf subalgebra of QSym. However, not every set T of this form Y^* results in a Hopf subalgebra. For generic q, this happens fairly rarely:

[11] We are being sloppy: For us here, a "subcoalgebra" of a coalgebra C means a $\mathbf{k}$-submodule D of C that satisfies

$$\Delta(D) \subseteq (\text{image of the canonical map } D \otimes D \to C \otimes C).$$

This is **not** the algebraically literate definition of a "subcoalgebra", as it does not imply that D itself becomes a $\mathbf{k}$-coalgebra (after all, the canonical map $D \otimes D \to C \otimes C$ might fail to be injective, and then it is not clear how to "restrict" Δ to a map $D \to D \otimes D$). Fortunately, the two definitions are equivalent when $\mathbf{k}$ is a field (or when D is a direct addend of C as a $\mathbf{k}$-module).

Proposition 54. *Let Y be a subset of $\{1,2,3,\ldots\}$ that is closed under addition (i.e., satisfies $y+z \in Y$ for every $y, z \in Y$). Let $T := Y^*$. Then, $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of* QSym.

Proof sketch. Clearly, $1 = \eta_{\varnothing}^{(q)} \in \operatorname{QSym}_T^{(q)}$, and Proposition 53 **(a)** shows that $\operatorname{QSym}_T^{(q)}$ is a subcoalgebra of QSym. Next, we will show that $\operatorname{QSym}_T^{(q)}$ is closed under multiplication. In view of Theorem 38, this will follow once we can show the following claim:

> *Claim 1:* Let $k \in \mathbb{N}$. Let $\delta \in Y^*$ and $\varepsilon \in Y^*$ be two compositions all of whose entries are $\in Y$. Let $\beta_1, \beta_2, \ldots, \beta_k \in \operatorname{Comp}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \operatorname{Comp}$ be $2k$ compositions satisfying
>
> $$\beta_1 \beta_2 \cdots \beta_k = \delta \qquad \text{and} \qquad \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon$$
> $$\text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s.$$
>
> Then,
> $$(|\beta_1| + |\gamma_1|,\ |\beta_2| + |\gamma_2|,\ \ldots,\ |\beta_k| + |\gamma_k|) \in Y^*.$$

Proof of Claim 1. We need to show that $|\beta_s| + |\gamma_s| \in Y$ for each $s \in [k]$. To do so, we fix $s \in [k]$. Then, $\ell(\beta_s) + \ell(\gamma_s) > 0$ (by assumption). In other words, at least one of the compositions β_s and γ_s is nonempty.

However, all entries of the composition β_s are entries of the composition $\beta_1 \beta_2 \cdots \beta_k = \delta$, and thus belong to Y (since $\delta \in Y^*$). Thus, the sum of all entries of β_s either equals 0 or belongs to Y (since Y is closed under addition). In other words, the size $|\beta_s|$ either equals 0 or belongs to Y. Similarly, $|\gamma_s|$ either equals 0 or belongs to Y. Hence, the sum $|\beta_s| + |\gamma_s|$ either equals 0 or belongs to Y as well (since Y is closed under addition). Since $|\beta_s| + |\gamma_s|$ cannot equal 0 (because at least one of the compositions β_s and γ_s is nonempty), we thus conclude that $|\beta_s| + |\gamma_s|$ belongs to Y. In other words, $|\beta_s| + |\gamma_s| \in Y$. As we said, this completes the proof of Claim 1. ☐

Now, Claim 1 (together with $1 \in \operatorname{QSym}_T^{(q)}$) shows that $\operatorname{QSym}_T^{(q)}$ is a $\mathbf{k}$-subalgebra of QSym. As we saw above, $\operatorname{QSym}_T^{(q)}$ is a $\mathbf{k}$-subcoalgebra of QSym as well, and thus is a $\mathbf{k}$-subbialgebra of QSym. This bialgebra $\operatorname{QSym}_T^{(q)}$ is connected graded, and therefore a Hopf algebra (by Takeuchi's famous result [GriRei20, Proposition 1.4.16]). The inclusion map $\operatorname{QSym}_T^{(q)} \to \operatorname{QSym}$ is a bialgebra morphism between two Hopf algebras, and thus automatically a Hopf algebra morphism (by another well-known result: [GriRei20, Corollary 1.4.27]). Hence, $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym. This proves Proposition 54. ☐

Example 55. The subset $\{2,4,6,8,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under addition. Thus, Proposition 54 shows that $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym for $Y := \{2,4,6,8,\ldots\}$ and $T := Y^*$. This Hopf subalgebra can be viewed as a copy of QSym in the indeterminates $x_1^2, x_2^2, x_3^2, \ldots$, and thus is isomorphic to QSym.

Example 56. The subset $\{2,3,4,5,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under addition. Thus, Proposition 54 shows that $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym for $Y := \{2,3,4,5,\ldots\}$ and $T := Y^*$.

Proposition 54 is not very surprising. In fact, (4) shows that (under the assumptions of Proposition 54) the space $\operatorname{QSym}_T^{(q)}$ is just the **k**-linear span of the functions $r^{\ell(\alpha)}M_\alpha$ with $\alpha \in Y^*$; but the latter span is easily seen to be a Hopf subalgebra (using [GriRei20, Proposition 5.1.3] and (22)).

If $q \neq 1$ and if r is invertible, then Proposition 54 has a converse (i.e., $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym only when Y is closed under addition), since it is easy to see that

$$\eta_{(a)}^{(q)}\eta_{(b)}^{(q)} = (q-1)\,\eta_{(a+b)}^{(q)} + \eta_{(a,b)}^{(q)} + \eta_{(b,a)}^{(q)} \qquad \text{for any } a,b \geqslant 1.$$

However, more interesting behavior emerges when $q = 1$:

Proposition 57. *Let Y be a subset of $\{1,2,3,\ldots\}$ that is closed under ternary addition (i.e., satisfies $y+z+w \in Y$ for every $y,z,w \in Y$). Let $T := Y^*$. Then, $\operatorname{QSym}_T^{(1)}$ is a Hopf subalgebra of* QSym*.*

Proof sketch. This is similar to Proposition 54, but now we set $q = 1$ and observe that all addends on the right hand side of Theorem 38 that satisfy

$$\ell(\beta_s) = \ell(\gamma_s) \qquad \text{for at least one } s \in [k]$$

are 0 (because they include the factor $(1-1)^{\text{a positive integer}}$, which vanishes), and all the remaining addends have the property that $|\ell(\beta_s) - \ell(\gamma_s)| = 1$ for all s (since we have $|\ell(\beta_s) - \ell(\gamma_s)| \leqslant 1$ and $\ell(\beta_s) \neq \ell(\gamma_s)$). Hence, the following claim now replaces Claim 1:

> *Claim 1':* Let $k \in \mathbb{N}$. Let $\delta \in Y^*$ and $\varepsilon \in Y^*$ be two compositions all of whose entries are $\in Y$. Let $\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp}$ and $\gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp}$ be $2k$ compositions satisfying
>
> $$\beta_1\beta_2\cdots\beta_k = \delta \qquad \text{and} \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon$$
> $$\text{and} \qquad |\ell(\beta_s) - \ell(\gamma_s)| = 1 \text{ for all } s.$$
>
> Then,
>
> $$(|\beta_1| + |\gamma_1|,\ |\beta_2| + |\gamma_2|,\ \ldots,\ |\beta_k| + |\gamma_k|) \in Y^*.$$

Proof of Claim 1'. We need to show that $|\beta_s| + |\gamma_s| \in Y$ for each $s \in [k]$. To do so, we fix $s \in [k]$. Then, $|\ell(\beta_s) - \ell(\gamma_s)| = 1$ (by assumption), and thus $\ell(\beta_s) + \ell(\gamma_s)$ is odd. Hence, $|\beta_s| + |\gamma_s|$ is a sum of an odd number of entries of δ and ε, and therefore a sum of an odd number of elements of Y (since δ and ε belong to Y^*). But Y is closed under ternary addition, and therefore any sum of an odd number of elements of Y must belong to Y (easy induction exercise). Hence, $|\beta_s| + |\gamma_s| \in Y$, and thus Claim 1' is proved. □

The rest of the proof proceeds as for Proposition 54. □

Example 58. The subset $\{1, 3, 5, 7, \ldots\}$ of $\{1, 2, 3, \ldots\}$ is closed under ternary addition. Thus, Proposition 57 shows that $\mathrm{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym for $Y := \{1, 3, 5, 7, \ldots\}$ and $T := Y^*$. This Hopf subalgebra is precisely the peak algebra Π of [Stembr97, §3], [AgBeSo06, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5] and [Hsiao07] (since [Hsiao07, (2.1) and (2.2)] shows that the θ_α for α odd have the same span as the η_α for α odd, but [Hsiao07, Proposition 2.1] shows that the latter η_α are precisely our $\eta_\alpha^{(1)}$ up to sign).

Example 59. The subset $\{\text{positive integers } \neq 2\} = \{1, 3, 4, 5, \ldots\}$ of $\{1, 2, 3, \ldots\}$ is closed under ternary addition. Thus, Proposition 57 shows that $\mathrm{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym for $Y := \{\text{positive integers } \neq 2\}$ and $T := Y^*$. This Hopf subalgebra is the Hopf subalgebra Ξ constructed in [BMSW00, Theorem 5.7]. (Indeed, both Hopf subalgebras have the same orthogonal complement: the principal ideal of NSym generated by $\eta_2^* = \dfrac{1}{4} X_2 = \dfrac{1}{4}(2H_2 - H_1 H_1)$.)

Example 60. More generally, if we pick a positive integer k and set

$$Y := \{\text{odd positive integers}\} \cup \{k, k+1, k+2, \ldots\}$$

and $T := Y^*$, then Proposition 57 shows that $\mathrm{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym (since Y is closed under ternary addition).

The reader can find more examples without trouble. When $\mathbf{k}$ is nontrivial and 2 is invertible in $\mathbf{k}$, Proposition 57 is easily seen to have a converse (using Example 40).

6.2 A new shuffle algebra

Next, we shall use the enriched q-monomial quasisymmetric functions to realize a certain deformed version of the shuffle product, which has appeared in recent work of [BoNoTh22] by Bouillot, Novelli and Thibon (generalizing the "block shuffle product" of Hirose and Sato [HirSat22, $\Diamond$]).

Shuffle products are a broad and deep subject with a long history and many applications (e.g., to multiple zeta values, algebraic topology and stochastic differential equations). An overview of known variants (such as the stuffles, the "muffles", the infiltrations and many more) can be found in [DEMT17, Table 1]. In the following, we shall discuss a variant that does not directly fit into the framework of [DEMT17], but is sufficiently similar to enjoy some of the same behavior. To our knowledge, this variant has first appeared in [BoNoTh22]. We will use the letters a and b for what was called α and β in [BoNoTh22], as we prefer to use Greek letters for compositions.

Let $\mathcal{F}$ be the free $\mathbf{k}$-algebra with generators $x_1, x_2, x_3, \ldots$. It has a basis consisting of all words over the alphabet $\{x_1, x_2, x_3, \ldots\}$; these words are in bijection with the compositions. In fact, let us set

$$x_\gamma := x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_k} \tag{48}$$

for every composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Then, the bijection sends each composition γ to the word x_γ.

For any $k \in \mathbb{N}$, we let $\zeta_k : \mathcal{F} \to \mathcal{F}$ be the $\mathbf{k}$-linear operator defined by

$$\zeta_k(1) = 0;$$
$$\zeta_k(x_i w) = x_{i+k} w \qquad \text{for each } i \geqslant 1 \text{ and any word } w.$$

(Thus, explicitly, the map ζ_k sends 1 to 0, and transforms any nonempty word by adding k to the subscript of its first letter. For example, $\zeta_k(x_u x_v x_w) = x_{u+k} x_v x_w$ for any $u, v, w \geqslant 1$.)

Fix two elements a and b of the base ring $\mathbf{k}$.

Let $\# : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be the $\mathbf{k}$-bilinear map on $\mathcal{F}$ defined recursively by the requirements

$$1 \# w = w \qquad \text{for any word } w;$$
$$w \# 1 = w \qquad \text{for any word } w;$$
$$(x_i u) \# (x_j v) = x_i (u \# (x_j v)) + x_j ((x_i u) \# v) + a x_{i+j} (u \# v) + b \zeta_{i+j} (u \# v)$$
$$\text{for any } i, j \geqslant 1 \text{ and any words } u \text{ and } v.$$

We call this bilinear map $\#$ the *stufufuffle*[12]. Explicitly, we can compute this operation as follows:

Proposition 61. *Let δ and ε be two compositions. Then, using the notation of (48), we have*

$$x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}.$$

Proof sketch. Write δ and ε as $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$. Use strong induction on $\ell + m$.

Induction step: If $\delta = \varnothing$ or $\varepsilon = \varnothing$, then the claim is easy to check. Thus, assume WLOG that neither δ nor ε is $\varnothing$. Let $i = \delta_1$ and $j = \varepsilon_1$ and $\overline{\delta} = (\delta_2, \delta_3, \ldots, \delta_\ell)$ and $\overline{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m)$. Hence, $x_\delta = x_i x_{\overline{\delta}}$ and $x_\varepsilon = x_j x_{\overline{\varepsilon}}$, so that

$$\begin{aligned} x_\delta \# x_\varepsilon &= (x_i x_{\overline{\delta}}) \# (x_j x_{\overline{\varepsilon}}) \\ &= x_i \left(x_{\overline{\delta}} \# \underbrace{(x_j x_{\overline{\varepsilon}})}_{= x_\varepsilon} \right) + x_j \left(\underbrace{(x_i x_{\overline{\delta}})}_{= x_\delta} \# x_{\overline{\varepsilon}} \right) + a x_{i+j} (x_{\overline{\delta}} \# x_{\overline{\varepsilon}}) + b \zeta_{i+j} (x_{\overline{\delta}} \# x_{\overline{\varepsilon}}) \\ &\qquad \text{(by the recursive definition of } \#) \\ &= x_i (x_{\overline{\delta}} \# x_\varepsilon) + x_j (x_\delta \# x_{\overline{\varepsilon}}) + a x_{i+j} (x_{\overline{\delta}} \# x_{\overline{\varepsilon}}) + b \zeta_{i+j} (x_{\overline{\delta}} \# x_{\overline{\varepsilon}}). \end{aligned} \tag{49}$$

On the other hand, the stufufufflers f for δ and ε can be classified into four types:

[12] This is a riff on the notion of "stuffle" (which is recovered when $a = 1$ and $b = 0$) and the fact that multiple letters of both words u and v can get combined into one in $u \# v$.

1. *Type 1* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 1$ and $\left|f_Q^{-1}(1)\right| = 0$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 = i$).
2. *Type 2* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 0$ and $\left|f_Q^{-1}(1)\right| = 1$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\varepsilon_1 = j$).
3. *Type 3* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 1$ and $\left|f_Q^{-1}(1)\right| = 1$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 + \varepsilon_1 = i + j$).
4. *Type 4* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| + \left|f_Q^{-1}(1)\right| > 2$ (so that both numbers $\left|f_P^{-1}(1)\right|$ and $\left|f_Q^{-1}(1)\right|$ are positive[13], and one of them is at least 2, and therefore the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 + \varepsilon_1 +$ (some further numbers)).

A type-1 stufufuffler f for δ and ε becomes a stufufuffler for $\overline{\delta}$ and ε if we decrease all its values by 1 and remove p_1 from P. This is furthermore a bijection from $\{\text{type-1 stufufufflers for } \delta \text{ and } \varepsilon\}$ to $\left\{\text{stufufufflers for } \overline{\delta} \text{ and } \varepsilon\right\}$, and this bijection preserves both loss and poise while removing the first entry from the weight. Hence, we obtain

$$\begin{aligned}
\sum_{\substack{f \text{ is a type-1 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \overline{\delta} \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_i x_{\operatorname{wt}(f)} \\
&= x_i \cdot \underbrace{\sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \overline{\delta} \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}}_{\substack{= x_{\overline{\delta}} \# x_\varepsilon \\ \text{(by the induction hypothesis,} \\ \text{since } \overline{\delta} \text{ has length } \ell - 1 < \ell)}} \\
&= x_i \left(x_{\overline{\delta}} \# x_\varepsilon\right).
\end{aligned}$$

Similar reasoning leads to

$$\begin{aligned}
\sum_{\substack{f \text{ is a type-2 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= x_j \left(x_\delta \# x_{\overline{\varepsilon}}\right); \\
\sum_{\substack{f \text{ is a type-3 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right); \\
\sum_{\substack{f \text{ is a type-4 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right).
\end{aligned}$$

Adding these four equalities together (and recalling that each stufufuffler for δ and ε belongs to exactly one of the four types 1, 2, 3 and 4), we obtain

$$\sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}$$

[13] by (47), applied to $s = 1$

$$= x_i \left(x_{\overline{\delta}} \# x_\varepsilon\right) + x_j \left(x_\delta \# x_{\overline{\varepsilon}}\right) + a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right) + b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right).$$

Comparing this with (49), we obtain

$$x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}.$$

This completes the induction step, and thus Proposition 61 is proved. ☐

Theorem 62. *The bilinear map # is commutative and associative, and the element* $1 \in \mathcal{F}$ *is a neutral element for it. Thus, the* $\mathbf{k}$*-module* $\mathcal{F}$*, equipped with the operation # (as multiplication), becomes a commutative* $\mathbf{k}$*-algebra with unity* 1.

It appears possible to prove Theorem 62 by induction, but the most convenient method at this point is to deduce this from the properties of the enriched q-monomial basis of QSym. To wit, the following proposition connects the map # to the latter basis:

Proposition 63. *Let* q *and* u *be two elements of* $\mathbf{k}$ *such that* $a = (q-1)\,u$ *and* $b = -qu^2$. *(Such* q *and* u *do not always exist, of course.)*

Let $\operatorname{eta} : \mathcal{F} \to \operatorname{QSym}$ *be the* $\mathbf{k}$*-linear map that sends the word* $x_\alpha = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \in \mathcal{F}$ *to* $u^{\ell(\alpha)} \eta^{(q)}_\alpha \in \operatorname{QSym}$ *for each composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. *Then,* $\operatorname{eta}(g \# h) = (\operatorname{eta} g) \cdot (\operatorname{eta} h)$ *for any* $g, h \in \mathcal{F}$.

Proof sketch. Let $g, h \in \mathcal{F}$. We WLOG assume that $g = x_\delta$ and $h = x_\varepsilon$ for two compositions δ and ε. Consider these δ and ε. Thus,

$$\operatorname{eta} g = \operatorname{eta} x_\delta = u^{\ell(\delta)} \eta^{(q)}_\delta \qquad \text{(by the definition of eta)}$$

and similarly $\operatorname{eta} h = u^{\ell(\varepsilon)} \eta^{(q)}_\varepsilon$. Multiplying these two equalities, we find

$$\begin{aligned} &(\operatorname{eta} g) \cdot (\operatorname{eta} h) \\ &= u^{\ell(\delta)} \eta^{(q)}_\delta \cdot u^{\ell(\varepsilon)} \eta^{(q)}_\varepsilon = u^{\ell(\delta)+\ell(\varepsilon)} \eta^{(q)}_\delta \eta^{(q)}_\varepsilon \\ &= u^{\ell(\delta)+\ell(\varepsilon)} \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta^{(q)}_{\operatorname{wt}(f)} \end{aligned} \tag{50}$$

(by Theorem 46).

On the other hand, from $g = x_\delta$ and $h = x_\varepsilon$, we obtain

$$g \# h = x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}$$

(by Proposition 61). Hence, by the definition of eta, we obtain

$$\operatorname{eta}(g \# h) = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} u^{\ell(\operatorname{wt}(f))} \eta^{(q)}_{\operatorname{wt}(f)}. \tag{51}$$

We must prove that the left hand sides of (51) and (50) are equal. Of course, it suffices to show that the right hand sides are equal. For that purpose, it suffices to show that

$$b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} u^{\ell(\operatorname{wt}(f))} = u^{\ell(\delta)+\ell(\varepsilon)} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)}$$

whenever f is a stufufuffler for δ and ε. Recalling that $a = (q-1)\,u$ and $b = -qu^2$, we can easily boil this down to the fact that every stufufuffler f for δ and ε satisfies

$$2 \operatorname{loss}(f) + \operatorname{poise}(f) + \ell(\operatorname{wt}(f)) = \ell(\delta) + \ell(\varepsilon);$$

but this fact is easily verified combinatorially. □

Proof of Theorem 62 (sketched). All claims of this theorem boil down to polynomial identities in a and b. For example, associativity of # is saying that the elements $(u\#v)\#w$ and $u\#(v\#w)$ of $\mathcal{F}$ have the same t-coefficient whenever u, v, w, t are four words; but this is easily revealed (upon expanding all products) to be an equality between two polynomials in a and b (when u, v, w, t are fixed). Note that all relevant polynomials have integer coefficients.

Thus, in order to prove Theorem 62, we can WLOG assume that a and b are two distinct indeterminates in a polynomial ring over $\mathbb{Z}$ (for example, $a = X$ and $b = Y$ in the polynomial ring $\mathbb{Z}[X, Y]$). Even better, we can WLOG assume that a and b are two algebraically independent elements of a $\mathbb{Z}$-algebra.

However, in the ring $\mathbb{Z}[X, Y]$, the two elements $X + Y$ and XY are algebraically independent (since they are the elementary symmetric polynomials in the indeterminates X and Y). Thus, we can WLOG assume that $\mathbf{k} = \mathbb{Z}[X, Y]$ and that $a = X + Y$ and $b = XY$. Moreover, we can extend the base ring $\mathbf{k}$ to its quotient field $\mathbb{Q}(X, Y)$. So we assume that $\mathbf{k} = \mathbb{Q}(X, Y)$ and $a = X + Y$ and $b = XY$.

Set $q := -XY^{-1}$ and $u := -Y$ in $\mathbf{k}$. Then, simple computations confirm that $a = (q-1)\,u$ and $b = -qu^2$. Hence, the map $\operatorname{eta} : \mathcal{F} \to \operatorname{QSym}$ constructed in Proposition 63 satisfies

$$\operatorname{eta}(g\#h) = (\operatorname{eta} g) \cdot (\operatorname{eta} h) \qquad \text{for any } g, h \in \mathcal{F} \tag{52}$$

(by Proposition 63). Moreover, the element $u = -Y \in \mathbf{k}$ is invertible (since $\mathbf{k}$ is a field), and so is the element $r := q + 1 = -XY^{-1} + 1 \in \mathbf{k}$ (for the same reason, since $r \neq 0$). Thus, the family $\left(u^{\ell(\alpha)} \eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym (by Theorem 11 **(a)**). Hence, the map eta is a $\mathbf{k}$-module isomorphism (since it sends the basis $(x_\alpha)_{\alpha \in \operatorname{Comp}}$ of $\mathcal{F}$ to the basis $\left(u^{\ell(\alpha)} \eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ of QSym). The equality (52) shows that this isomorphism eta transfers the multiplication of QSym to the binary operation # on $\mathcal{F}$. Since the former multiplication is associative, we thus conclude that the latter operation # is associative as well. Similarly, we can see that # is commutative. Finally, it is clear that 1 is a neutral element for #. Thus, Theorem 62 is proved. □

In view of Theorem 62, we can restate Proposition 63 as follows:

Theorem 64. *Let q and u be two elements of $\mathbf{k}$ such that $a = (q-1)\,u$ and $b = -qu^2$.*

Let eta : $\mathcal{F} \to \operatorname{QSym}$ *be the* $\mathbf{k}$*-linear map that sends the word* $x_\alpha = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \in \mathcal{F}$ *to* $u^{\ell(\alpha)} \eta_\alpha^{(q)} \in \operatorname{QSym}$ *for each composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$*. Then,* eta *is a* $\mathbf{k}$*-algebra homomorphism from the* $\mathbf{k}$*-algebra* $(\mathcal{F}, \#)$ *to the* $\mathbf{k}$*-algebra* QSym.

We can also turn $\mathcal{F}$ into a coalgebra. In fact, let $\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ be the $\mathbf{k}$-linear map that sends each word $w_1 w_2 \cdots w_n$ to $\sum_{i=0}^{n} w_1 w_2 \cdots w_i \otimes w_{i+1} w_{i+2} \cdots w_n$. This map Δ is called the *deconcatenation coproduct* (or the *cut coproduct*). This coproduct turns $\mathcal{F}$ into a coalgebra (with counit $\varepsilon : \mathcal{F} \to \mathbf{k}$ sending each word $w_1 w_2 \cdots w_n$ to $\begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$). The map eta : $\mathcal{F} \to \operatorname{QSym}$ from Theorem 64 is then easily seen to be a $\mathbf{k}$-coalgebra homomorphism (by Theorem 20).

The stufufuffle product $\#$ on $\mathcal{F}$ respects the deconcatenation coproduct Δ of $\mathcal{F}$, in the following sense:

Theorem 65. *The* $\mathbf{k}$*-algebra* $(\mathcal{F}, \#)$*, equipped with the coproduct* Δ *and the counit* ε *constructed above, is a commutative connected graded Hopf algebra.*

Theorem 66. *Let q and u be two elements of $\mathbf{k}$ such that $a = (q-1)\,u$ and $b = -qu^2$.*

Let eta : $\mathcal{F} \to \operatorname{QSym}$ *be the* $\mathbf{k}$*-linear map from Theorem 64. Then,* eta *is a Hopf algebra homomorphism from the Hopf algebra* $(\mathcal{F}, \#, \Delta, \varepsilon)$ *to the Hopf algebra* QSym.

We leave the proofs of these two theorems to the reader. (They follow the same mold as our above proof of Theorem 62.)

Likewise, using Theorem 17 and the proof method of Theorem 62 above, we can prove the following:

Theorem 67. *Let S be the antipode of the Hopf algebra $(\mathcal{F}, \#)$ constructed in Theorem 65. Let $n \in \mathbb{N}$ and $\alpha \in \operatorname{Comp}_n$. Then, in $\mathcal{F}$, we have*

$$S(x_\alpha) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\operatorname{rev} \alpha)}} a^{\ell(\alpha) - \ell(\beta)} x_\beta.$$

The recent work [BoNoTh22, Theorem 5.2] constructs another basis (X_I) of QSym (indexed by subsets I of $[n-1]$ instead of compositions α, but this difference is insubstantial) that multiplies according to the stufufuffle product (thus obtaining another $\mathbf{k}$-algebra homomorphism from $\mathcal{F}$ to QSym, and with it another proof of Theorem 62). While similar to ours, it uses the alphabet-transformed functions $H_k((s-t)\,A)$ instead of the plain H_k, which lead to a basis of QSym that does not appear to have a simple combinatorial formula like our $\eta_\alpha^{(q)}$.

Remark 68. Assume that $\mathbf{k}$ is a field of characteristic 0. Then, Leray's theorem ([GriRei20, Theorem 1.7.29(f)]) shows that any commutative connected graded $\mathbf{k}$-bialgebra A is isomorphic as a graded $\mathbf{k}$-algebra to the symmetric algebra of a certain graded $\mathbf{k}$-module (namely, of $(\ker\varepsilon) / (\ker\varepsilon)^2$, where ε is the counit of A). In other words, any such A is isomorphic as a graded **k**-algebra to a polynomial ring whose generators are homogeneous of various positive degrees, with exactly $\dim\left((\ker\varepsilon) / (\ker\varepsilon)^2\right)_i$ many generators of degree i. This applies, in particular, to our connected graded **k**-bialgebra $(\mathcal{F}, \#)$. Moreover, using standard Hilbert-series arguments, it is easy to see that the number of generators of each given degree does not depend on the parameters a and b. Hence, as a graded **k**-algebra, our $(\mathcal{F}, \#)$ is isomorphic to the usual shuffle algebra (which is obtained for $a = 0$ and $b = 0$).

However, this is not a **k**-coalgebra isomorphism; nor is it canonical (although we suspect that a canonical **k**-algebra isomorphism may exist); nor does it extend to fields of positive characteristic.

A Appendix: The map R_q

We finish by stating yet another formula for $\eta_\alpha^{(q)}$, which may eventually prove useful. This formula relies on some more notations. We first define a simple combinatorial operation on compositions:

Definition 69. Let $\alpha \in \operatorname{Comp}$, and let $n = |\alpha|$. Then, $\overline{\alpha}$ shall denote the unique composition γ of n such that $D(\gamma) = [n-1] \setminus D(\alpha)$. (This γ is indeed unique, since the map D is a bijection.) This composition $\overline{\alpha}$ is called the *complement* of α.

For example, $\overline{(2,5,1,1)} = (1,2,1,1,1,3)$. We observe some simple properties of complements of compositions:

Proposition 70.

- ***(a)*** *Every composition α satisfies $\overline{\overline{\alpha}} = \alpha$.*
- ***(b)*** *For each $n \in \mathbb{N}$, the map $\operatorname{Comp}_n \to \operatorname{Comp}_n,\ \beta \mapsto \overline{\beta}$ is a bijection.*
- ***(c)*** *If α and β are two compositions of n for some $n \in \mathbb{N}$, then the statements "$D(\beta) \subseteq D(\alpha)$" and "$D\left(\overline{\beta}\right) \supseteq D(\overline{\alpha})$" are equivalent.*
- ***(d)*** *If α is a composition of a positive integer n, then $\ell(\overline{\alpha}) + \ell(\alpha) = n + 1$.*

Proof. See [GriVas23a] for the (very simple) proof. □

We now define a linear endomorphism of QSym:

Definition 71. We let R_q be the **k**-linear map from QSym to QSym that sends each M_α (with $\alpha \in \operatorname{Comp}$) to $r^{\ell(\overline{\alpha})} M_{\overline{\alpha}}$. (This is well-defined, since $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym.)

This map R_q is neither an algebra endomorphism nor a coalgebra endomorphism of QSym (not even when $r = 1$), but it is exactly what we need for our formula. First, let us observe that the map R_q is "close to an involution" in the following sense:

Proposition 72. *Let n be a positive integer. Let $f \in \operatorname{QSym}$ be homogeneous of degree n. Then, $(R_q \circ R_q)(f) = r^{n+1} f$.*

Proof. By linearity, it suffices to check this for $f = M_\alpha$ for all compositions $\alpha \in \operatorname{Comp}_n$. But this case follows easily from Proposition 70 **(a)** and **(d)**. ☐

Now we can state our final formula for $\eta_\alpha^{(q)}$:

Theorem 73. *Let $\alpha \in \operatorname{Comp}$. Then,*

$$\eta_\alpha^{(q)} = R_q\left(L_{\overline{\alpha}}\right).$$

Proof. Let $n = |\alpha|$, so that $\alpha \in \operatorname{Comp}_n$. Therefore, $\overline{\alpha} \in \operatorname{Comp}_n$ as well, so that (3) yields

$$L_{\overline{\alpha}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\overline{\alpha})}} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D\left(\overline{\beta}\right) \supseteq D(\overline{\alpha})}} M_{\overline{\beta}} \tag{53}$$

(here, we have substituted $\overline{\beta}$ for β in the sum, since Proposition 70 **(b)** shows that the map $\operatorname{Comp}_n \to \operatorname{Comp}_n$, $\beta \mapsto \overline{\beta}$ is a bijection). By Proposition 70 **(c)**, we can replace the condition "$D\left(\overline{\beta}\right) \supseteq D(\overline{\alpha})$" under the summation sign by the equivalent condition "$D(\beta) \subseteq D(\alpha)$". Hence, we can rewrite (53) as

$$L_{\overline{\alpha}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} M_{\overline{\beta}}.$$

Applying the linear map R_q to both sides of this equality, we find

$$\begin{aligned} R_q\left(L_{\overline{\alpha}}\right) &= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \underbrace{R_q\left(M_{\overline{\beta}}\right)}_{\substack{= r^{\ell\left(\overline{\beta}\right)} M_{\overline{\overline{\beta}}} \\ \text{(by the definition of } R_q)}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \underbrace{r^{\ell\left(\overline{\overline{\beta}}\right)} M_{\overline{\overline{\beta}}}}_{\substack{= r^{\ell(\beta)} M_\beta \\ \text{(by Proposition 70 (a))}}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \\ &= \eta_\alpha^{(q)} \qquad \text{(by (4))}. \end{aligned}$$

This proves Theorem 73. ☐

Remark 74. Let n be a positive integer, and let $\alpha \in \operatorname{Comp}_n$. Combining Theorem 73 with Proposition 72, we can easily see that $R_q\left(\eta_\alpha^{(q)}\right) = r^{n+1} L_{\overline{\alpha}}$. Contrasting this equality with Theorem 73 reveals a symmetry of sorts between the $\eta_\alpha^{(q)}$ and $L_{\overline{\alpha}}$. This symmetry explains the similarity between Proposition 12 and Proposition 13 (and allows one to derive one of these propositions from the other with a bit of work).

B Appendix: Some proofs omitted from the above

As promised, we shall now include some straightforward or otherwise simple proofs omitted from the text above.

Proof of Proposition 6. Essentially, this is obtained by substituting the definition of M_β into (4) and expanding. Let us elaborate:

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a composition of n. Then, M_β is defined to be the sum of all monomials of the form $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ with $i_1 < i_2 < \cdots < i_\ell$. But these monomials are precisely the monomials $x_{g_1} x_{g_2} \cdots x_{g_n}$ that satisfy

$$\begin{aligned} & g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n \\ & \text{and } \ (g_i < g_{i+1} \text{ for each } i \in D(\beta)) \\ & \text{and } \ (g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\beta)). \end{aligned}$$

Hence, we obtain

$$M_\beta = \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in D(\beta); \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\beta)}} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Multiplying this by $r^{\ell(\beta)}$, we find

$$\begin{aligned} r^{\ell(\beta)} M_\beta &= \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in D(\beta); \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\beta)}} \underbrace{r^{\ell(\beta)}}_{\substack{= r^{|\{g_1, g_2, \ldots, g_n\}|} \\ \text{(since the conditions on } g_1, g_2, \ldots, g_n \\ \text{ensure that the set } \{g_1, g_2, \ldots, g_n\} \\ \text{has exactly } \ell = \ell(\beta) \text{ many distinct elements)}}} x_{g_1} x_{g_2} \cdots x_{g_n} \\ &= \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in D(\beta); \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\beta)}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}. \end{aligned}$$

Now, forget that we fixed β. Summing the equality that we just obtained over all compositions $\beta \in \operatorname{Comp}_n$ that satisfy $D(\beta) \subseteq D(\alpha)$, we obtain

$$\begin{aligned} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta &= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in D(\beta); \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\beta)}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \\ &= \sum_{\substack{I \subseteq [n-1]; \\ I \subseteq D(\alpha)}} \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in I; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus I}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \end{aligned}$$

(here, we have substituted I for $D(\beta)$ in the first sum, since the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection). In view of (4), we can rewrite this as

$$\eta_\alpha^{(q)} = \sum_{\substack{I \subseteq [n-1]; \\ I \subseteq D(\alpha)}} \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in I; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus I}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}$$

$$= \sum_{I \subseteq D(\alpha)} \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in I; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus I}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \tag{54}$$

(since every $I \subseteq D(\alpha)$ also satisfies $I \subseteq [n-1]$).

However, we have the following equality of summation signs:

$$\sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha)}} = \sum_{\substack{I \subseteq [n-1]; \\ I \subseteq D(\alpha)}} \sum_{\substack{g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n; \\ g_i < g_{i+1} \text{ for each } i \in I; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus I}}$$

(because for any given n-tuple $(g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n)$, the condition "$g_i = g_{i+1}$ for each $i \in [n-1] \setminus D(\alpha)$" is equivalent to the existence of a subset $I \subseteq D(\alpha)$ satisfying $(g_i < g_{i+1}$ for each $i \in I)$ and $(g_i = g_{i+1}$ for each $i \in [n-1] \setminus I)$, and moreover, such a subset I is necessarily unique if it exists). Hence, the right hand side of (5) equals the right hand side of (54). Thus, (5) follows from (54), and this proves Proposition 6. ☐

Proof of Proposition 7. Let $n = |\alpha|$, so that $\alpha \in \mathrm{Comp}_n$. Clearly, it suffices to show that the right hand sides of (5) and (6) are identical.

But this is true, since they are just the same sum with a different way of indexing its addends. Indeed, the monomials $x_{g_1} x_{g_2} \cdots x_{g_n}$ for all n-tuples $(g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n)$ satisfying $(g_i = g_{i+1}$ for each $i \in [n-1] \setminus D(\alpha))$ are precisely the monomials of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for all ℓ-tuples $(i_1 \leqslant i_2 \leqslant \cdots \leqslant i_\ell)$. Moreover, the coefficients $r^{|\{g_1, g_2, \ldots, g_n\}|}$ in front of the former monomials equals the coefficients $r^{|\{i_1, i_2, \ldots, i_\ell\}|}$ in front of the latter ones (because either coefficient can be rewritten as r^k, where k is the number of distinct indeterminates appearing in the given monomial). This shows that the right hand sides of the equalities (5) and (6) are identical. Hence, (6) follows from (5). ☐

Next, let us prove Proposition 30 in detail. To do so, we will use the comultiplication $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of the Hopf algebra QSym as well as the duality between NSym and QSym:

Lemma 75. *Let $f, g \in \mathrm{NSym}$ and $h \in \mathrm{QSym}$ be arbitrary. Let the tensor $\Delta(h) \in \mathrm{QSym} \otimes \mathrm{QSym}$ be written in the form $\Delta(h) = \sum_{i \in I} s_i \otimes t_i$, where I is a finite set and where $s_i, t_i \in \mathrm{QSym}$ for each $i \in I$. Then,*

$$\langle fg, h \rangle = \sum_{i \in I} \langle f, s_i \rangle \langle g, t_i \rangle .$$

Proof. Recall that the $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle$ identifies NSym with the graded dual QSym^o as Hopf algebras. Thus, in particular, the multiplication of NSym and the comultiplication of QSym are mutually adjoint with respect to this form. In other words, if $f, g \in \mathrm{NSym}$ and $h \in \mathrm{QSym}$, then

$$\langle fg, h \rangle = \sum_{(h)} \left\langle f, h_{(1)} \right\rangle \left\langle g, h_{(2)} \right\rangle ,$$

where we are using the Sweedler notation $\sum_{(h)} h_{(1)} \otimes h_{(2)}$ for $\Delta(h)$ (see, e.g., [GriRei20, (1.2.3)]). Lemma 75 is just restating this fact without using the Sweedler notation. $\square$

Proof of Proposition 30. This follows by dualization from Theorem 20. Here are the details:

Forget that we fixed α and β. Proposition 27 **(a)** shows that the families $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\mathrm{Comp}}$ and $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ are mutually dual bases of NSym and QSym with respect to the bilinear form $\langle\cdot,\cdot\rangle$. This shows that

$$\left\langle \eta_\lambda^{*(q)}, \eta_\mu^{(q)} \right\rangle = [\lambda = \mu] \tag{55}$$

for all $\lambda, \mu \in \mathrm{Comp}$. But another consequence of this duality is that the bilinear form $\langle\cdot,\cdot\rangle$ is nondegenerate, and that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ is a basis of QSym. Hence, in order to prove that two elements $f, g \in \mathrm{NSym}$ are equal, it suffices to show that $\left\langle f, \eta_\gamma^{(q)} \right\rangle = \left\langle g, \eta_\gamma^{(q)} \right\rangle$ holds for each $\gamma \in \mathrm{Comp}$.

We shall use this strategy to prove $\eta_\alpha^{*(q)}\eta_\beta^{*(q)} = \eta_{\alpha\beta}^{*(q)}$ for all $\alpha, \beta \in \mathrm{Comp}$. Thus, we need to show that $\left\langle \eta_\alpha^{*(q)}\eta_\beta^{*(q)}, \eta_\gamma^{(q)} \right\rangle = \left\langle \eta_{\alpha\beta}^{*(q)}, \eta_\gamma^{(q)} \right\rangle$ holds for all $\alpha, \beta, \gamma \in \mathrm{Comp}$.

To show this, we fix $\alpha, \beta, \gamma \in \mathrm{Comp}$. Theorem 20 then says that

$$\Delta\left(\eta_\gamma^{(q)}\right) = \sum_{\substack{\varphi,\psi\in\mathrm{Comp};\\ \gamma=\varphi\psi}} \eta_\varphi^{(q)} \otimes \eta_\psi^{(q)}.$$

Hence, Lemma 75 yields

$$\begin{aligned}\left\langle \eta_\alpha^{*(q)}\eta_\beta^{*(q)}, \eta_\gamma^{(q)} \right\rangle &= \sum_{\substack{\varphi,\psi\in\mathrm{Comp};\\ \gamma=\varphi\psi}} \underbrace{\left\langle \eta_\alpha^{*(q)}, \eta_\varphi^{(q)} \right\rangle}_{\substack{=[\alpha=\varphi]\\ \text{(by (55))}}} \underbrace{\left\langle \eta_\beta^{*(q)}, \eta_\psi^{(q)} \right\rangle}_{\substack{=[\beta=\psi]\\ \text{(by (55))}}} = \sum_{\substack{\varphi,\psi\in\mathrm{Comp};\\ \gamma=\varphi\psi}} [\alpha=\varphi]\cdot[\beta=\psi] \\ &= [\gamma = \alpha\beta]\end{aligned}$$

(since the only possible nonzero addend that the sum $\sum_{\substack{\varphi,\psi\in\mathrm{Comp};\\ \gamma=\varphi\psi}} [\alpha=\varphi]\cdot[\beta=\psi]$ could have is the addend for $\varphi = \alpha$ and $\psi = \beta$, but this addend only exists when $\gamma = \alpha\beta$). Comparing this with

$$\begin{aligned}\left\langle \eta_{\alpha\beta}^{*(q)}, \eta_\gamma^{(q)} \right\rangle &= [\alpha\beta = \gamma] \qquad \text{(by (55))} \\ &= [\gamma = \alpha\beta],\end{aligned}$$

we obtain $\left\langle \eta_\alpha^{*(q)}\eta_\beta^{*(q)}, \eta_\gamma^{(q)} \right\rangle = \left\langle \eta_{\alpha\beta}^{*(q)}, \eta_\gamma^{(q)} \right\rangle$.

Forget that we fixed γ. We thus have shown that $\left\langle \eta_\alpha^{*(q)}\eta_\beta^{*(q)}, \eta_\gamma^{(q)} \right\rangle = \left\langle \eta_{\alpha\beta}^{*(q)}, \eta_\gamma^{(q)} \right\rangle$ for each $\gamma \in \mathrm{Comp}$. Since $\left(\eta_\gamma^{(q)}\right)_{\gamma\in\mathrm{Comp}}$ is a basis of QSym, and since the bilinear form $\langle\cdot,\cdot\rangle$ is nondegenerate, we thus conclude that $\eta_\alpha^{*(q)}\eta_\beta^{*(q)} = \eta_{\alpha\beta}^{*(q)}$. This proves Proposition 30. $\square$

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