The chromatic symmetric function of a graph centred at a vertex

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Abstract

We discover new linear relations between the chromatic symmetric functions of certain sequences of graphs and apply these relations to find new families of e-positive unit interval graphs. Motivated by the results of Gebhard and Sagan, we revisit their ideas and reinterpret their equivalence relation in terms of a new quotient algebra of NCSym. We investigate the projection of the chromatic symmetric function Y_G in noncommuting variables in this quotient algebra, which defines $y_{G:v}$, the chromatic symmetric function of a graph G centred at a vertex v. We then apply our methods to $y_{G:v}$ and find new families of unit interval graphs that are (e)-positive, a stronger condition than classical e-positivity, thus confirming new cases of the $(3+1)$-free conjecture of Stanley and Stembridge.

In our study of $y_{G:v}$, we also describe methods of constructing new e-positive graphs from given (e)-positive graphs and classify the (e)-positivity of trees and cut vertices. We moreover construct a related quotient algebra of NCQSym to prove theorems relating the coefficients of $y_{G:v}$ to acyclic orientations of graphs, including a noncommutative refinement of Stanley's sink theorem.

Mathematics Subject Classifications: 05A18, 05C15, 05C20, 05C25, 05E05, 16T30

1 Introduction

The chromatic symmetric function X_G of a graph G was introduced by Stanley [25] in 1995 as a generalization of Birkhoff's chromatic polynomial [3]. Since then, it has inspired fruitful research mainly in two avenues. The first avenue is to determine whether two nonisomorphic trees can have the same chromatic symmetric function [2, 14, 17, 19]. Heil and Ji showed in [14] that there was no counterexample on $\leqslant 29$ vertices. However, the second and more prominent avenue of research is to prove the Stanley-Stembridge conjecture [26, Conjecture 5.5], which was formulated in terms of chromatic symmetric

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functions by Stanley in [25, Conjecture 5.1]. In 2013, Guay-Paquet [11] showed that to prove the Stanley-Stembridge conjecture it was sufficient to prove that all unit interval graphs were e-positive, namely that their chromatic symmetric functions expanded with nonnegative coefficients in the basis of elementary symmetric functions.

Much interest has also arisen due to a q-analogue of the conjecture, introduced by Shareshian and Wachs [23] in terms of the chromatic quasisymmetric functions of labelled unit interval graphs. Brosnan and Chow [4] and, independently, Guay-Paquet [12] proved that the chromatic quasisymmetric functions of labelled unit interval graphs were related to the cohomology of regular semisimple Hessenberg varieties, first conjectured by Shareshian and Wachs in [23], and this connection has been used to prove a special case of the q-analogue of the conjecture in [13]. Considerable progress has been made on the Stanley-Stembridge conjecture and Shareshian and Wachs' quasisymmetric refinement also due to the study of the modular law, which appears in several forms across [1, 8, 11, 15] and is a consequence of Orellana and Scott's triple-deletion rule [20] when $q = 1$.

A second approach to the Stanley-Stembridge conjecture was pioneered by Gebhard and Sagan in [10], where they introduced Y_G, the chromatic symmetric function of a graph G in noncommuting variables. Intriguingly, the natural labelling of unit interval graphs was also important to their approach, as it was for chromatic quasisymmetric functions. Gebhard and Sagan proved for a large family of labelled unit interval graphs that the graphs were (e)-positive at their last vertices (see Definition 13), a condition stronger than being e-positive arising from an equivalence relation on the algebra of NCSym. Dahlberg in [5] proved that triangular ladders were (e)-positive at their last vertices using a sign-reversing involution, resolving a special case of the conjecture identified by Stanley in [25]. Our paper makes progress on the Stanley-Stembridge conjecture by combining the modular law with (e)-positivity to prove new cases of the conjecture. We also investigate in-depth the equivalence relation introduced by Gebhard and Sagan, including, in particular, the equivalence class of Y_G. More precisely, our paper is structured as follows.

We cover the necessary background in Section 2. In Section 3 we describe in Proposition 6 sequences of graphs with the property that their chromatic symmetric functions form an arithmetic progression, which may be applied to deduce positivity. In Corollary 8, we specialize and obtain a tool to prove new cases of the Stanley-Stembridge conjecture, which we then apply in Proposition 11 to prove the e-positivity of a new family of unit interval graphs. In Section 4 we review the methods and results of Gebhard and Sagan from [10], and reinterpret them in terms of a new quotient algebra exhibiting "unbalanced commutativity", UBCSym, of NCSym, for which we define analogues of the elementary, power sum and monomial bases. In UBCSym we also define $y_{G:v}$, the chromatic symmetric function of G centred at v. We then meld the ideas of Gebhard and Sagan and the ideas of Section 3 to prove that many more families of labelled unit interval graphs are (e)-positive at their last vertices in Section 5. In Section 6 we introduce a technique to work with linear maps on UBCSym, which allows us to prove the validity of methods of constructing new e-positive graphs from given (e)-positive graphs in Theorems 32 and 35. In Section 7 we resolve the related questions of when trees are (e)-positive and which graphs are (e)-positive at a cut vertex. We construct a quotient algebra UBCQSym of NCQSym in Section 8 and prove in Theorem 42 a noncommutative refinement of Stanley's sink theorem [25, Theorem 3.3]. We conclude the paper with Section 9, in which we discuss the connections to a construction of Pawlowski [21], as well as possible further avenues of research.

2 Background

In this section we review the necessary background and notation that will be used in the rest of the paper. This section may be skipped, or referred back to later, by those familiar with algebraic combinatorics.

An (integer) *composition* $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ is a finite ordered list of positive integers, where $\ell(\alpha)$ is the *length* of α. We call the integers the *parts* of the composition. When $\alpha_{j+1} = \cdots = \alpha_{j+m} = i$, we often abbreviate this sublist to i^m. If $\alpha_1 + \cdots + \alpha_{\ell(\alpha)} = d$, we say that α is a composition of d. We will also write $\emptyset$ to denote the empty composition.

Let $[d] = \{1, \ldots, d\}$. If $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ is a composition of d, then we define $\text{set}(\alpha)$ to be the set $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)-1}\} \subseteq [d-1]$. This induces a natural one-to-one correspondence between the compositions of d and the subsets of $[d-1]$.

An (integer) *partition* $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is a composition with parts satisfying $\lambda_1 \geqslant \cdots \geqslant \lambda_{\ell(\lambda)}$. If $\lambda_1 + \cdots + \lambda_{\ell(\lambda)} = d$, then we say that λ is a partition of d and write $\lambda \vdash d$. We also define $\lambda!$ to mean the quantity $\lambda_1! \cdots \lambda_{\ell(\lambda)}!$. Given two partitions λ and μ, we write $\lambda \cup \mu$ to denote the partition obtained by combining the parts of λ and μ together in weakly decreasing order.

We next define Sym, the *algebra of symmetric functions*, which may be realized as a subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$, where the variables x_j commute, as follows. The *ith elementary symmetric function* e_i is defined by

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}.$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$, we define the *elementary symmetric function* e_λ to be

$$e_\lambda = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}.$$

Sym can be defined as the graded algebra

$$\text{Sym} = \text{Sym}^0 \oplus \text{Sym}^1 \oplus \cdots$$

where for each $d \in \mathbb{Z}_{\geqslant 0}$, the dth graded piece Sym^d is spanned by the basis $\{e_\lambda\}_{\lambda \vdash d}$.

Another basis of Sym consists of the power sum symmetric functions. The *ith power sum symmetric function* p_i is defined by

$$p_i = \sum_j x_j^i,$$

and we define the *power sum symmetric function* p_λ for a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ to be

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}.$$

Then the set $\{p_\lambda\}_{\lambda \vdash d}$ forms a basis for Sym^d.

We now turn our attention to graphs. All graphs in this paper will be *finite* and *simple*. That is, our graphs G will consist of a nonempty finite vertex set $V(G)$ and a finite edge set $E(G)$ consisting of pairs of distinct vertices. For $v, w \in V(G)$, we write vw to mean an edge connecting v and w. Given a graph G with vertices v, w, we write $G + vw$ to denote the graph obtained by adding an edge connecting v and w to G. The *order* $|G|$ of a graph G is the number of vertices of the graph.

Using the standard notation, given a vertex $v \in V(G)$, its *open neighbourhood* $N(v)$ is the set of all vertices of G connected by an edge to v. The *closed neighbourhood* $N[v]$ of v is the set $N(v) \cup \{v\}$.

A *proper colouring* of a graph G is a map $\kappa : V(G) \to \mathbb{Z}_{>0}$ such that $\kappa(v) \neq \kappa(w)$ whenever $vw \in E(G)$. In 1995, Stanley defined the chromatic symmetric function of G in commuting variables as follows.

Definition 1. [25, Definition 2.1] Let G be a graph with vertex set $\{v_1, \ldots, v_d\}$. Then the *chromatic symmetric function* of G is defined to be

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}$$

where the sum is over all proper colourings κ of G.

Given a basis $\{b_i\}_{i \in I}$ of a vector space over $\mathbb{Q}$, we say that an element of the space is *b-positive* if it expands in the b-basis with all coefficients nonnegative. We will say that a graph G is *e-positive* if and only if X_G is e-positive.

A *set partition* π of $[d]$ is a collection of disjoint nonempty sets $B_1, \ldots, B_{\ell(\pi)}$ whose union is $[d]$, and we denote this by

$$\pi = B_1 / \cdots / B_{\ell(\pi)} \vdash [d].$$

We call the B_j for $1 \leqslant j \leqslant \ell(\pi)$ the *blocks* of π and $\ell(\pi)$ the *length* of π. For ease of notation we usually omit the set parentheses and commas of the blocks. We also define $\lambda(\pi)$ to be the integer partition of d whose parts are $|B_1|, \ldots, |B_{\ell(\pi)}|$ sorted in weakly decreasing order. We will let $\pi!$ denote $\lambda(\pi)!$. For a set partition $\pi \vdash [d]$ we will use the notation $B_{\pi,i}$ for $i \in [d]$ to mean the block of π containing i.

For a finite set of integers S, define $S + n = \{s + n : s \in S\}$. Then for two set partitions $\pi \vdash [n]$ and $\sigma = B_1 / \cdots / B_{\ell(\sigma)} \vdash [m]$, their *slash product* is defined to be

$$\pi \mid \sigma = \pi / (B_1 + n) / \cdots / (B_{\ell(\sigma)} + n) \vdash [n + m].$$

We next define NCSym, the *algebra of symmetric functions in noncommuting variables*, which may be realized as a subalgebra of $\mathbb{Q}\langle\langle x_1, x_2, \dots\rangle\rangle$, where the variables x_j do not commute. NCSym can be defined as the graded algebra

$$\text{NCSym} = \text{NCSym}^0 \oplus \text{NCSym}^1 \oplus \cdots ,$$

where the dth graded piece is spanned by the bases $\{e_\pi\}_{\pi\vdash[d]}$, $\{p_\pi\}_{\pi\vdash[d]}$ and $\{m_\pi\}_{\pi\vdash[d]}$, which we define next.

The *elementary symmetric function* e_π in NCSym is given by

$$e_\pi = \sum_{(i_1,\dots,i_d)} x_{i_1} \cdots x_{i_d},$$

summed over all tuples $(i_1, \dots, i_d)$ with $i_j \neq i_k$ if $B_{\pi,j} = B_{\pi,k}$. For set partitions π, σ, we have $e_\pi e_\sigma = e_{\pi|\sigma}$, e.g. by [5, Lemma 2.1].

The *power sum symmetric function* p_π in NCSym is given by

$$p_\pi = \sum_{(i_1,\dots,i_d)} x_{i_1} \cdots x_{i_d},$$

summed over all tuples $(i_1, \dots, i_d)$ with $i_j = i_k$ if $B_{\pi,j} = B_{\pi,k}$.

Finally, the *monomial symmetric function* m_π in NCSym is given by

$$m_\pi = \sum_{(i_1,\dots,i_d)} x_{i_1} \cdots x_{i_d},$$

summed over all tuples $(i_1, \dots, i_d)$ with $i_j = i_k$ if and only if $B_{\pi,j} = B_{\pi,k}$.

There is an algebra map $\rho : \text{NCSym} \to \text{Sym}$ obtained by allowing the variables to commute. By parts (ii) and (iii) of [22, Theorem 2.1], we have $\rho(e_\pi) = \pi! e_{\lambda(\pi)}$ and $\rho(p_\pi) = p_{\lambda(\pi)}$.

We will also define an action of the symmetric group $\mathcal{S}_d$ on the dth graded piece of NCSym by permuting the positions of the variables. For $\delta \in \mathcal{S}_d$, we define the right action on monomials by

$$\delta \circ (x_{i_1} \cdots x_{i_d}) = x_{i_{\delta^{-1}(1)}} \cdots x_{i_{\delta^{-1}(d)}}$$

and extend linearly. For $\pi \vdash [d]$, we then have a left action $\delta \circ m_\pi = m_{\delta(\pi)}$, $\delta \circ e_\pi = e_{\delta(\pi)}$ and $\delta \circ p_\pi = p_{\delta(\pi)}$ by [10, Section 2], where the action of δ on set partitions of $[d]$ is by permuting the elements of the blocks.

Gebhard and Sagan in [10, Definition 3.4] defined a linear operation called *induction*, $\uparrow$, on NCSym^d for $d \in \mathbb{Z}_{>0}$, by defining it on monomials via

$$(x_{i_1} \cdots x_{i_d})\uparrow = x_{i_1} \cdots x_{i_d} x_{i_d},$$

and extending linearly. Similarly, for $j \leqslant d$, they also defined $\uparrow_j^{d+1}$ on NCSym^d by defining

$$(x_{i_1} \cdots x_{i_d})\uparrow_j^{d+1} = x_{i_1} \cdots x_{i_d} x_{i_j},$$

and extending linearly.

A *labelled graph* on d vertices is a graph with vertex set $[d]$. We can also define the action of $\delta \in \mathcal{S}_d$ on labelled graphs on d vertices, by letting δ act by permuting the vertex labels. The labelled graph $\delta(G)$ is then just a relabelling of G. We also define *reverse graph* G^r of a labelled graph G on d vertices to be the labelled graph $\delta(G)$, where $\delta \in \mathcal{S}_d$ is the permutation exchanging each i with $d+1-i$.

A *labelled unit interval graph* is a labelled graph G with the property that whenever $i \leqslant v < w \leqslant j$ and $ij \in E(G)$, then $vw \in E(G)$ as well.

Definition 2. Given a labelled unit interval graph on d vertices, we associate two weakly increasing sequences $(m_i)_{i=1}^d$ and $(w_i)_{i=1}^d$, where $m_i \geqslant i$ is the largest label in $N[i]$ and $w_i \leqslant i$ is the smallest label in $N[i]$.

Note the closed neighbourhood $N[i]$ is given by the set $\{w_i, w_i+1, \ldots, m_i\}$. Either of the two sequences is sufficient to uniquely determine the labelled unit interval graph.

Some labelled unit interval graphs we require familiarity with are the *path* P_d on d vertices with an edge between i and $i+1$ for each $i \in [d-1]$ and the *complete graph* K_d on d vertices with an edge between every pair of distinct vertices. The cycle C_d for $d \geqslant 3$ is obtained by adding an edge between 1 and d to the path P_d. We also define K_π for $\pi \vdash [d]$ to be the labelled graph on d vertices with an edge between $i \neq j$ if and only if $B_{\pi,i} = B_{\pi,j}$.

Given two labelled graphs G and H on n and m vertices, respectively, define $G \mid H$ to be the disjoint union of G and H, where the vertices corresponding to G have labels in $[n]$ in the same relative order as in G, and the vertices corresponding to H have labels in $[m]+n$ in the same relative order as in H. We also define the *concatenation* $G+H$ with vertex set $[n+m-1]$ to be the labelled graph obtained from $G \mid H$ by formally identifying vertices n and $n+1$ of $G \mid H$ and otherwise shifting labels so that the vertices of $G+H$ have labels in the same relative order as in $G \mid H$. For a sequence $(G_j)_{j=1}^k$ of labelled graphs, define $\sum_{j=1}^k G_j$ to mean $G_1 + \cdots + G_k$. If $k=0$, we take the convention that $\sum_{j=1}^k G_j = K_1$. Note when G, H are labelled unit interval graphs that G^r, $G \mid H$ and $G+H$ are all also labelled unit interval graphs.

Guay-Paquet showed in [11, Theorem 5.1] that the Stanley-Stembridge conjecture is equivalent to the following.

Conjecture 3. All labelled unit interval graphs are e-positive.

Gebhard and Sagan defined a noncommutative analogue of the chromatic symmetric function in NCSym, which they used to resolve cases of Conjecture 3.

Definition 4. [10, Definition 3.1] Let G be a labelled graph on d vertices. Then the *chromatic symmetric function in noncommuting variables* of G is defined to be

$$Y_G = \sum_{\kappa} x_{\kappa(1)} \cdots x_{\kappa(d)},$$

where the sum is over all proper colourings κ of G.

Note we have $\rho(Y_G) = X_G$, by [9, Proposition 3.5] $Y_{G|H} = Y_G Y_H$, by [10, Proposition 3.3] $Y_{\delta(G)} = \delta \circ Y_G$, and by [9, Lemma 4.9] $Y_{K_\pi} = e_\pi$.

Gebhard and Sagan showed in [10, Proposition 3.5] that Y_G satisfied a *deletion-contraction relation*. Given a labelled graph G on d vertices and an edge $jd \in E(G)$, we define $G \setminus jd$ to be the the labelled graph obtained by removing the edge jd from G, and G/jd to be the labelled graph on $d-1$ vertices obtained from G by formally identifying vertices j and d as the single vertex j in G/jd. Dahlberg gave in [5, Proposition 2.2] a slight generalization of Gebhard and Sagan's [10, Proposition 3.5], obtained by relabelling vertices.

Proposition 5. *[5, Proposition 2.2] If G is a labelled graph on d vertices with $jd \in E(G)$, then*

$$Y_G = Y_{G \setminus jd} - Y_{G/jd}{\uparrow}_j^d.$$

We end this section by defining NCQSym, the *algebra of quasisymmetric functions in noncommuting variables*, with bases indexed by set compositions. A *set composition* Φ of $[d]$, written $\Phi \vDash [d]$, is an ordered list of blocks of some set partition $\widetilde{\Phi} \vdash [d]$, which we write as

$$\Phi = B_1 \mathbin{/\!/} \cdots \mathbin{/\!/} B_{\ell(\Phi)},$$

where $\ell(\Phi)$ is the *length* of Φ. We also define $\alpha(\Phi)$ to be the integer composition $(|B_1|, \dots, |B_{\ell(\Phi)}|)$.

The graded algebra NCQSym may be realized as a subalgebra of $\mathbb{Q}\langle\langle x_1, x_2, \dots \rangle\rangle$, where the variables x_j do not commute, via

$$\mathrm{NCQSym} = \mathrm{NCQSym}^0 \oplus \mathrm{NCQSym}^1 \oplus \cdots ,$$

where the dth graded piece NCQSym^d is spanned by the basis $\{M_\Phi\}_{\Phi \vDash [d]}$. The *monomial quasisymmetric function* M_Φ is defined by

$$M_\Phi = \sum_{(i_1, \dots, i_d)} x_{i_1} \cdots x_{i_d},$$

summed over all tuples $(i_1, \dots, i_d)$ with $i_j = i_k$ if and only if $B_{\widetilde{\Phi},j} = B_{\widetilde{\Phi},k}$ and $i_j < i_k$ whenever $B_{\widetilde{\Phi},j}$ appears before $B_{\widetilde{\Phi},k}$ in Φ. Note NCSym is a subalgebra of NCQSym, via

$$m_\pi = \sum_{\widetilde{\Phi} = \pi} M_\Phi.$$

3 Arithmetic progressions of graphs

Our first result describes relations between the chromatic symmetric functions of certain sequences of graphs, and has applications toward positivity.

Proposition 6. *Suppose $v_1, \ldots, v_k$ are distinct vertices of a graph G satisfying $N[v_1] = \cdots = N[v_k]$. If w is another vertex of G not adjacent to any v_j, then the $(X_{G_j})_{j=0}^{k}$ form an arithmetic progression, where $G_0 = G$ and $G_j = G_{j-1} + v_j w$ for $1 \leqslant j \leqslant k$. In particular, if G_0 and G_k are b-positive for some basis $\{b_\lambda\}_{\lambda \vdash n \geqslant 0}$ of* Sym*, then so is every G_j for $0 \leqslant j \leqslant k$.*

Proof. We will show that the statement of the first part of the proposition holds for any number of vertices k by applying induction on k.

The base case $k = 1$ is immediate, because $(X_{G_j})_{j=0}^{1}$ will always be an arithmetic progression, being a sequence of length 2.

When $k > 1$, the vertices v_1, v_2, w are mutually adjacent in $G + v_1 w + v_2 w$ (since $\{v_1, v_2\} \subseteq N[v_1] = N[v_2]$ implies that v_1 and v_2 are joined by an edge in G). Then by triple-deletion [20, Theorem 3.1],

$$X_{G+v_1w+v_2w} = X_{G+v_1w} + X_{G+v_2w} - X_G.$$

Since $N[v_1] = N[v_2]$, we obtain a graph automorphism of G by exchanging v_1 and v_2, and so $G + v_1 w$ and $G + v_2 w$ are isomorphic. Our equation thus becomes

$$X_{G+v_1w+v_2w} = 2X_{G+v_1w} - X_G,$$

or equivalently,

$$X_{G_2} - X_{G_1} = X_{G_1} - X_{G_0}.$$

We next note that the graph G_1 satisfies the hypotheses of the proposition with vertex w not adjacent to the $k-1$ vertices $v_2, \ldots, v_k$. By the inductive hypothesis, the $(X_{G_j})_{j=1}^{k}$ form an arithmetic progression. Then, since $X_{G_2} - X_{G_1} = X_{G_1} - X_{G_0}$ is its common difference, we can extend it to obtain the arithmetic progression $(X_{G_j})_{j=0}^{k}$.

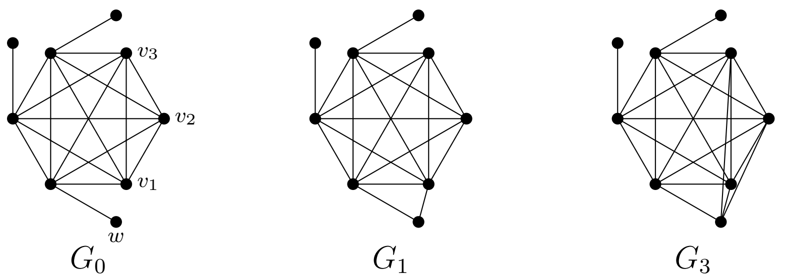
The second part of the proposition then follows because if X_{G_0} and X_{G_k} are b-positive, then

$$X_{G_j} = \frac{k-j}{k} X_{G_0} + \frac{j}{k} X_{G_k}$$

must also be b-positive for any $0 \leqslant j \leqslant k$. □

Example 7. Let $G = G_0$ be the left graph in Figure 1 below. Then G, together with vertices v_1, v_2, v_3 and w, as labelled, satisfies the hypotheses of Proposition 6. Let G_1 and G_3 denote $G + v_1 w$ and $G + v_1 w + v_2 w + v_3 w$, respectively.

Figure 1



Then we have, in the basis of Schur functions (see [18, Section 1.3] for an introduction),

$$\begin{aligned}
X_{G_0} &= 5760s_{(1^9)} + 7200s_{(2,1^7)} + 3168s_{(2^2,1^5)} + 468s_{(2^3,1^3)} + 2880s_{(3,1^6)} \\
&\quad + 864s_{(3,2,1^4)} + 360s_{(4,1^5)}, \\
X_{G_3} &= 14400s_{(1^9)} + 12960s_{(2,1^7)} + 3888s_{(2^2,1^5)} + 288s_{(2^3,1^3)} + 2880s_{(3,1^6)} + 432s_{(3,2,1^4)},
\end{aligned}$$

which are both Schur-positive.

By Proposition 6, the graph G_1 is also Schur-positive, since

$$\begin{aligned}
X_{G_1} &= \frac{2}{3}X_{G_0} + \frac{1}{3}X_{G_3} \\
&= 8640s_{(1^9)} + 9120s_{(2,1^7)} + 3408s_{(2^2,1^5)} + 408s_{(2^3,1^3)} + 2880s_{(3,1^6)} \\
&\quad + 720s_{(3,2,1^4)} + 240s_{(4,1^5)}.
\end{aligned}$$

We can specialise the previous proposition and obtain a tool to help prove the e-positivity of labelled unit interval graphs, making progress toward Conjecture 3.

Corollary 8. *Suppose G is a labelled unit interval graph. Then the following hold.*

(a) If $i < |G|$ is a vertex such that $m_i + 1 \leqslant m_{i+1}$ and $(w_{m_i+1}, m_{m_i+1}) = \cdots = (w_{m_i+k}, m_{m_i+k})$, then the $(X_{G_j})_{j=0}^k$ form an arithmetic progression, where $G_j = G + \{ib \mid m_i + 1 \leqslant b \leqslant m_i + j\}$ for $0 \leqslant j \leqslant k$. In particular, if G_0 and G_k are e-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

(b) Alternatively, if $i > 1$ is a vertex such that $w_i - 1 \geqslant w_{i-1}$ and $(w_{w_i-1}, m_{w_i-1}) = \cdots = (w_{w_i-k}, m_{w_i-k})$, then the $(X_{G_j})_{j=0}^k$ form an arithmetic progression, where $G_j = G + \{bi \mid w_i - 1 \geqslant b \geqslant w_i - j\}$ for $0 \leqslant j \leqslant k$. In particular, if G_0 and G_k are e-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

Proof. Part (a) follows immediately from applying Proposition 6 to G on vertices i and $m_i + 1, \ldots, m_i + k$ in the e-basis. Part (b) follows from applying part (a) to the reverse graph G^r of G. □

Observation 9. *When the hypotheses of either part (a) or (b) of Corollary 8 are satisfied, the labelled graphs G_j for $0 \leqslant j \leqslant k$ are labelled unit interval graphs.*

Remark 10. Special cases of the relations in Corollary 8 were studied by Dahlberg and van Willigenburg in [8] to give a proof of the e-positivity of the *lollipop graphs* $L_{m,n}$ for $m, n \geqslant 1$ in [8, Theorem 8], which are the labelled unit interval graphs associated with the sequence $(w_i)_{i=1}^{m+n}$ where

$$w_i = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant m, \\ i - 1 & \text{if } m + 1 \leqslant i \leqslant m + n. \end{cases}$$

In [15], Huh, Nam and Yoo further studied these relations for the chromatic quasisymmetric function of labelled unit interval graphs, and proved in [15, Theorem 4.9] the

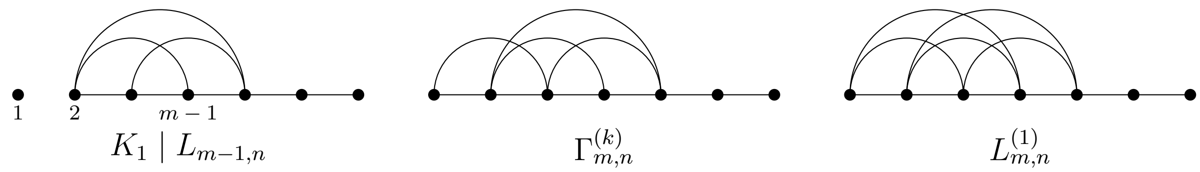
e-positivity of *melting lollipop graphs* $L_{m,n}^{(k)}$ for $m, n \geqslant 1$ and $0 \leqslant k \leqslant m-1$, obtained by deleting the edges between vertex m and vertices $1, \ldots, k$ from $L_{m,n}$. (The definition given differs slightly from that of Huh, Nam and Yoo's in that these are actually the reverse graphs of what they call $L_{m,n}^{(k)}$. See the rightmost graph of Figure 2 for the example of $L_{5,2}^{(1)}$.) In fact, Corollary 8 is equivalent to [15, Theorem 3.4(b')] for $q = 1$. The quasisymmetric case of these relations is also studied in [1] by Abreu and Nigro.

We will refer to the graphs $L_{m,n}^{(k)}$ as *type I melting lollipop graphs* to distinguish them from a related family of graphs introduced in the next proposition, which we will prove are e-positive by an application of Corollary 8. Melting lollipop graphs are interesting because there exists an induction scheme from which their e-positivity can be deduced only from the e-positivity argument in Corollary 8 and the e-positivity of disjoint unions of complete graphs.

Proposition 11. Type II melting lollipop graphs $\Gamma_{m,n}^{(k)}$ *for* $m \geqslant 3$, $n \geqslant 1$ *and* $1 \leqslant k \leqslant m-1$*, obtained by deleting the edges between vertex* 1 *and vertices* $m, \ldots, m-k+1$ *from* $L_{m,n}$*, are* e*-positive.*

Proof. Apply Corollary 8(a) to the labelled unit interval graph $K_1 \mid L_{m-1,n}$ on vertices 1 and $2, \ldots, m-1$. Figure 2 illustrates the case of $m = 5$, $n = 2$ and $k = 2$.

Figure 2



The graph $K_1 \mid L_{m-1,n}$ is e-positive because lollipop graphs are e-positive, e.g. by [8, Theorem 8]. The type I melting lollipop graph $L_{m,n}^{(1)}$ is e-positive by [15, Theorem 4.9]. Therefore since

$$X_{\Gamma_{m,n}^{(k)}} = \frac{k-1}{m-2} X_{K_1} X_{L_{m-1,n}} + \frac{m-k-1}{m-2} X_{L_{m,n}^{(1)}},$$

the graph $\Gamma_{m,n}^{(k)}$ is e-positive. □

4 UBCSym and graph concatenations

Applying Corollary 8 to deduce the e-positivity of certain labelled unit interval graphs requires the e-positivity of a pair of labelled unit interval graphs to be known ahead of time. It will be useful to review some of the known e-positive labelled unit interval graphs in the literature to find more graphs on which we can apply the technique from Corollary 8. Gebhard and Sagan in [10, Corollary 7.7] showed the e-positivity of all labelled unit interval graphs obtained from concatenating a sequence of complete graphs, which they called K_α-*chains*. We will also review some of the ideas they used to prove e-positivity, stemming from an equivalence relation in NCSym.

In [10, Section 6], Gebhard and Sagan noted that “even for some of the simplest graphs, Y_G is usually not e-positive.” As an example, they gave $Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$. Dahlberg and van Willigenburg later showed in [9, Theorem 4.14] that Y_G is e-positive if and only if $G = K_\pi$ for some set partition π.

However, after defining

$$e_{\pi_1} \equiv_i e_{\pi_2} \quad \text{if and only if} \quad \lambda(\pi_1) = \lambda(\pi_2) \text{ and } |B_{\pi_1,i}| = |B_{\pi_2,i}|,$$

and extending linearly, we then have

$$Y_{P_3} \equiv_3 \frac{1}{2}e_{12/3} + \frac{1}{2}e_{123},$$

where the right-hand side is e-positive. This is because $\lambda(13/2) = (2,1) = \lambda(1/23)$ and the size of a block containing 3 in each of $13/2$ and $1/23$ is 2, hence the $-\frac{1}{2}e_{13/2}$ and $+\frac{1}{2}e_{1/23}$ in Y_{P_3} cancel. Gebhard and Sagan called this relation *congruence modulo* i.

Gebhard and Sagan in [10] and, later, Dahlberg in [5] together found several families of labelled unit interval graphs G that were congruent modulo $|G|$ to an e-positive function in NCSym. We will say then that a labelled graph G is (e)-*positive* to mean that Y_G is congruent modulo $|G|$ to an e-positive function, e.g. the labelled graph P_3 is (e)-positive. We next give an equivalent formulation of these ideas in terms of a new quotient algebra of NCSym.

For $d \in \mathbb{Z}_{>0}$ and any set partition $\pi \vdash [d]$, write type(π) to mean the pair (λ, b), where $b = |B_{\pi,d}|$ and λ is the partition whose parts are the sizes of the other parts of π, e.g.

$$\text{type}(1/24/35) = ((2,1),2).$$

When π is the empty set partition, write type$(\pi) = (\emptyset)$.

Recall that one basis of NCSym consists of the e_π over all set partitions π. We can define UBCSym first as the free vector space spanned by elements $e_{\text{type}(\pi)}$ over all set partitions π. Then UBCSym is naturally a quotient vector space of NCSym via the linear projection map

$$\begin{aligned} \nu : \text{NCSym} &\to \text{UBCSym} \\ e_\pi &\mapsto e_{\text{type}(\pi)}. \end{aligned}$$

The kernel of ν is given by

$$\ker \nu = \text{span}\{e_{\pi_1} - e_{\pi_2} \mid \text{type}(\pi_1) = \text{type}(\pi_2)\}.$$

If set partitions π_1, π_2 satisfy type$(\pi_1) =$ type(π_2), then for any other set partition σ, we still have type$(\pi_1 \mid \sigma) =$ type$(\pi_2 \mid \sigma)$ and type$(\sigma \mid \pi_1) =$ type$(\sigma \mid \pi_2)$, so the kernel of ν is in fact a two-sided ideal of NCSym, via the equalities

$$\nu((e_{\pi_1} - e_{\pi_2})e_\sigma) = \nu(e_\sigma(e_{\pi_1} - e_{\pi_2})) = 0$$

and extending bilinearly. Moreover, it is a graded ideal of NCSym. This makes UBCSym a graded quotient algebra of NCSym, with

$$\mathrm{UBCSym} = \mathrm{NCSym} \,/\, \mathrm{span}\{e_{\pi_1} - e_{\pi_2} \mid \mathrm{type}(\pi_1) = \mathrm{type}(\pi_2)\}.$$

We will write $\mathrm{UBCSym}^d = \nu(\mathrm{NCSym}^d)$ to denote the homogeneous part of degree d in UBCSym. The kernel of ν is contained in the kernel of ρ, so the induced map $\bar{\rho}$: UBCSym $\to$ Sym is well-defined, and Sym is a quotient algebra of UBCSym.

Note for $d \in \mathbb{Z}_{>0}$, for all $\pi \vdash [d]$ and $\delta \in \mathcal{S}_d$ fixing d we have that $\nu(\delta \circ e_\pi) = \nu(e_{\delta(\pi)}) = \nu(e_\pi)$. Extending linearly, for any $f \in \mathrm{NCSym}^d$ and $\delta \in \mathcal{S}_d$ fixing d, we also have $\nu(\delta \circ f) = \nu(f)$. This is the content of [10, Lemma 6.6].

If $\mathrm{type}(\pi_1) = \mathrm{type}(\pi_2)$ for set partitions $\pi_1, \pi_2 \vdash [d]$ with $d \in \mathbb{Z}_{>0}$, then there exists $\delta \in \mathcal{S}_d$ fixing d such that $\pi_1 = \delta(\pi_2)$, and so $\nu(p_{\pi_1}) = \nu(p_{\pi_2})$. So we can define $p_{\mathrm{type}(\pi)} = \nu(p_\pi)$ for each set partition π. Since the $p_{\mathrm{type}(\pi)}$ over all $\pi \vdash [d]$ span UBCSym^d, which has dimension equal to the number of distinct types of set partitions of $[d]$, it follows that the $p_{\mathrm{type}(\pi)}$ form another basis for UBCSym. Similarly, we can define $m_{\mathrm{type}(\pi)} = \nu(m_\pi)$ for each set partition π, and they form a third basis for UBCSym.

In [10, Lemma 6.2], Gebhard and Sagan state that if $f, g \in \mathrm{NCSym}$ are homogeneous of degree $d \in \mathbb{Z}_{>0}$ satisfying $\nu(f) = \nu(g)$, then $\nu(f\uparrow) = \nu(g\uparrow)$. We will define the linear operation *induction*, $\uparrow$, on UBCSym^d to be the induced map sending $\nu(f) \mapsto \nu(f\uparrow)$ for every $f \in \mathrm{NCSym}^d$.

Our main object of study will be $\nu(Y_G)$ of a labelled graph G. Since for any labelled graph G on d vertices and $\delta \in \mathcal{S}_d$ fixing d we have

$$\nu(Y_{\delta(G)}) = \nu(\delta \circ Y_G) = \nu(Y_G),$$

the value of $\nu(Y_G)$ depends only on the (unlabelled) graph G and the choice of vertex labelled last.

Definition 12. Given a labelled graph G, we define

$$y_G = \nu(Y_G).$$

If G is a graph with a distinguished vertex v, the *chromatic symmetric function of G centred at v* is

$$y_{G:v} = y_G,$$

where G is given a labelling with v as the last vertex.

We will call an arbitrary function in UBCSym (e)*-positive* if all coefficients are non-negative in its expansion in the e-basis. Note that this is consistent with the notation of Gebhard and Sagan in that the following gives an equivalent definition of (e)-positivity of a labelled graph.

Definition 13. A labelled graph G is (e)*-positive* if and only if y_G is (e)-positive. We also say that a graph G is (e)*-positive at a vertex v* if and only if $y_{G:v}$ is (e)-positive.

As an example, we saw earlier that the labelled graph P_3 is (e)-positive, and

$$y_{P_3} = \frac{1}{2} e_{((2),1)} + \frac{1}{2} e_{(\emptyset,3)}.$$

Note since

$$\bar{\rho}(e_{(\lambda,b)}) = \lambda! b! e_{\lambda \cup (b)},$$

any graph that is (e)-positive at some vertex is then necessarily also e-positive.

Gebhard and Sagan in [10] and Dahlberg in [5] found results showing for certain families of labelled graphs H, the concatenation $G + H$ is (e)-positive whenever G is (e)-positive, motivating the following definition.

Definition 14. A labelled graph H is *appendable (e)-positive* if and only if $G + H$ is (e)-positive for all (e)-positive labelled graphs G.

Appendable (e)-positive labelled graphs H are necessarily (e)-positive, because K_1 is (e)-positive, and so $H = K_1 + H$ must be (e)-positive by definition of appendable (e)-positivity.

We next briefly list the known (e)-positive and appendable (e)-positive labelled graphs from [10] and [5]. Results that follow by some combination of the listed propositions are omitted.

Proposition 15. *[10, Proposition 6.8] Cycle graphs C_n for $n \geqslant 3$ are (e)-positive.*

Proposition 16. *[10, Theorem 7.6] Complete graphs K_n for $n \geqslant 1$ are appendable (e)-positive.*

Proposition 17. *[5, Theorem 5.3]* Triangular ladder graphs *TL_n for $n \geqslant 1$, given by the sequence $(m_i)_{i=1}^{n}$ where each $m_i = \min\{i+2, n\}$, are appendable (e)-positive.*

These alone already prove the e-positivity of a plethora of labelled unit interval graphs. In particular, any labelled unit interval graph obtained by concatenating a sequence of complete graphs and triangular ladder graphs is (e)-positive (and therefore e-positive), as noted by Dahlberg in [5, Corollary 5.4]. Dahlberg suggests in [5, Section 5] that the following conjecture, which is a strengthening of the Stanley-Stembridge conjecture, may hold, and has verified it for all labelled unit interval graphs on up to 7 vertices.

Conjecture 18. [5, Section 5] All labelled unit interval graphs are (e)-positive.

An even more optimistic conjecture is the following.

Conjecture 19. All labelled unit interval graphs are appendable (e)-positive.

One of our goals in the next section will be to lend credence to this conjecture, by finding more families of appendable (e)-positive labelled unit interval graphs.

5 New (e)-positive labelled unit interval graphs

In this section we will combine the ideas in Section 3 with the ideas of Gebhard and Sagan to find new (e)-positive and appendable (e)-positive labelled unit interval graphs. We begin this section by modifying Corollary 8 to obtain versions that apply to (e)-positivity and appendable (e)-positivity.

Proposition 20. *Suppose G is a labelled unit interval graph. Then the following hold.*

(a) If $i < |G|$ is a vertex such that $m_i + 1 \leqslant m_{i+1}$ and $(w_{m_i+1}, m_{m_i+1}) = \cdots = (w_{m_i+k}, m_{m_i+k})$ with $m_i + k < |G|$, then the $(y_{G_j})_{j=0}^{k}$ form an arithmetic progression, where $G_j = G + \{ib \mid m_i + 1 \leqslant b \leqslant m_i + j\}$. In particular, if G_0 and G_k are (e)-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

(b) Alternatively, if $i > 1$ is a vertex such that $w_i - 1 \geqslant w_{i-1}$ and $(w_{w_i-1}, m_{w_i-1}) = \cdots = (w_{w_i-k}, m_{w_i-k})$, then the $(y_{G_j})_{j=0}^{k}$ form an arithmetic progression, where $G_j = G + \{bi \mid w_i - 1 \geqslant b \geqslant w_i - j\}$ for $0 \leqslant j \leqslant k$. In particular, if G_0 and G_k are (e)-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

Proof. We will only give a proof of part (a). The proof of (b) is similar.

It suffices to show that the $(y_{G_j})_{j=0}^{k}$ form an arithmetic progression in UBCSym, since the second part of the statement would then follow from

$$y_{G_j} = \frac{k-j}{k} y_{G_0} + \frac{j}{k} y_{G_k}.$$

We will proceed by induction on k.

The base case $k = 1$ is immediate, because $(y_{G_j})_{j=0}^{1}$ will always be an arithmetic progression, being a sequence of length 2.

When $k > 1$, note since $(w_{m_i+1}, m_{m_i+1}) = (w_{m_i+2}, m_{m_i+2})$, there is a graph automorphism of G obtained by exchanging vertices $m_i + 1$ and $m_i + 2$. Let $\delta \in \mathcal{S}_{|G|}$ be the permutation swapping $m_i + 1$ and $m_i + 2$. Then the labelled graph obtained from adding an edge joining i and $m_i + 2$ to G is the labelled graph $\delta(G_1)$. Note δ fixes $|G|$, since $m_i + k < |G|$.

The vertices $m_i + 1$, $m_i + 2$, i are mutually adjacent in G_2. By noncommutative triple-deletion [9, Proposition 3.6],

$$Y_{G_2} - Y_{\delta(G_1)} - Y_{G_1} + Y_{G_0} = 0.$$

Applying the projection map ν, we obtain

$$y_{G_2} - 2y_{G_1} + y_{G_0} = 0.$$

We next note that the labelled unit interval graph G_1 satisfies the hypotheses of part (a) with vertex i and the $k - 1$ vertices $m_i + 2, \ldots, m_i + k$ (where m_i is defined by the sequence of G). By the inductive hypothesis, the $(y_{G_j})_{j=1}^{k}$ form an arithmetic progression with common difference $y_{G_2} - y_{G_1} = y_{G_1} - y_{G_0}$. Therefore it can be extended to obtain the arithmetic progression $(y_{G_j})_{j=0}^{k}$, as desired. ☐

Corollary 21. *Suppose G is a labelled unit interval graph. Then the following hold.*

(a) If $i < |G|$ is a vertex such that $m_i + 1 \leqslant m_{i+1}$ and $(w_{m_i+1}, m_{m_i+1}) = \cdots = (w_{m_i+k}, m_{m_i+k})$ with $m_i + k < |G|$, then define the labelled graphs $G_j = G + \{ib \mid m_i + 1 \leqslant b \leqslant m_i + j\}$ for $0 \leqslant j \leqslant k$. If G_0 and G_k are appendable (e)-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

(b) Alternatively, if $i > 1$ is a vertex such that $w_i - 1 \geqslant w_{i-1}$ and $(w_{w_i-1}, m_{w_i-1}) = \cdots = (w_{w_i-k}, m_{w_i-k})$ with $w_i - k > 1$, then define the labelled graphs $G_j = G + \{bi \mid w_i - 1 \geqslant b \geqslant w_i - j\}$ for $0 \leqslant j \leqslant k$. If G_0 and G_k are appendable (e)-positive, then so is every G_j for $0 \leqslant j \leqslant k$.

Proof. For part (a), note if G_0 and G_k are appendable (e)-positive, then for every $n \in \mathbb{Z}_{>0}$, the labelled graphs $K_n + G_0$ and $K_n + G_k$ are (e)-positive. The labelled unit interval graph $K_n + G_0$ satisfies the hypotheses of Proposition 20(a) on the images in $K_n + G_0$ of the vertices i and $m_i + 1, \ldots, m_i + k$ of G_0. Therefore,

$$y_{K_n+G_j} = \frac{k-j}{k} y_{K_n+G_0} + \frac{j}{k} y_{K_n+G_k},$$

and so $K_n + G_j$ is (e)-positive for every $n \in \mathbb{Z}_{>0}$ and every $0 \leqslant j \leqslant k$. By Proposition 30, the labelled graphs G_j for $0 \leqslant j \leqslant k$ are all appendable (e)-positive.

The proof of part (b) is the same, except part (b) of Proposition 20 is applied instead of part (a). □

Remark 22. Note the symmetry between the statements of parts (a) and (b) of Corollary 21. In particular, if the labelled unit interval graphs $(G_j)_{j=0}^k$ are considered in either part of the corollary, then the appendable (e)-positivity of the $(G_j^r)_{j=0}^k$ follow from the appendable (e)-positivity of G_0^r and G_k^r by applying the opposite part of the corollary.

We next prove the (e)-positivity and appendable (e)-positivity of several families of labelled unit interval graphs. The appendable (e)-positivity of our first family of graphs will follow from the computations of Gebhard and Sagan in [10, Section 7].

Proposition 23. Twin peaks graphs TP_{n+1} *for $n \geqslant 2$, obtained by removing the edge between 1 and $n+1$ in K_{n+1}, are appendable (e)-positive.*

Proof. Let G be a labelled graph with

$$y_G = \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{(\lambda,b)}.$$

Then by [10, Lemma 7.3] and [10, Lemma 7.5], which describe the expansions of $y_{G+K_{n+1}}$

and $y_{G+K_n:|G|}\uparrow$ in the e-basis,

$$\begin{aligned}
y_{G+K_{n+1}} = & \sum_{|\lambda|+b=|G|}\sum_{i=0}^{n-1} c_{(\lambda,b)}\frac{(n-1)!(b-1)!}{(n-i-1)!(b+i)!}\big((b-n+i)e_{(\lambda\cup(b+i),n-i)} \\
& +(i+1)e_{(\lambda\cup(n-i-1),b+i+1)}\big)\,, \\
y_{G+K_n:|G|}\uparrow = & \sum_{|\lambda|+b=|G|}\sum_{i=0}^{n-1} c_{(\lambda,b)}\frac{(n-1)!(b-1)!}{(n-i-1)!(b+i)!}\big(e_{(\lambda\cup(b+i),n-i)} - e_{(\lambda\cup(n-i-1),b+i+1)}\big)\,.
\end{aligned}$$

By the deletion-contraction relation in Proposition 5,

$$\begin{aligned}
y_{G+TP_{n+1}} &= y_{G+K_{n+1}} + y_{G+K_n:|G|}\uparrow \\
&= \sum_{|\lambda|+b=|G|}\sum_{i=0}^{n-1} c_{(\lambda,b)}\frac{(n-1)!(b-1)!}{(n-i-1)!(b+i)!}\big((b-n+i+1)e_{(\lambda\cup(b+i),n-i)} \\
&+ ie_{(\lambda\cup(n-i-1),b+i+1)}\big)\,.
\end{aligned}$$

Now suppose G is (e)-positive, i.e. each coefficient $c_{(\lambda,b)}$ is $\geqslant 0$. The contribution of $c_{(\lambda,b)}e_{(\lambda,b)}$ to $y_{G+TP_{n+1}}$ is

$$\sum_{i=0}^{n-1} c_{(\lambda,b)}\frac{(n-1)!(b-1)!}{(n-i-1)!(b+i)!}\big((b-n+i+1)e_{(\lambda\cup(b+i),n-i)} + ie_{(\lambda\cup(n-i-1),b+i+1)}\big)\,,$$

which has nonnegative coefficients in the e-basis, except possibly at the $e_{(\lambda\cup(b+i),n-1)}$ when both $0\leqslant i\leqslant n-1$ and $b-n+i+1<0$. In that case, $j=n-i-b-1$ satisfies $0\leqslant j\leqslant n-1$ and $(\lambda\cup(n-j-1),b+j+1)=(\lambda\cup(b+i),n-i)$.

So the coefficient of $e_{(\lambda\cup(b+i),n-i)}$ in the contribution of $c_{(\lambda,b)}e_{(\lambda,b)}$ to $y_{G+TP_{n+1}}$ when $0\leqslant i\leqslant n-1$ and $b-n+i+1<0$ is

$$c_{(\lambda,b)}\left(\frac{(n-1)!(b-1)!}{(n-i-1)!(b+i)!}(b-n+i+1) + \frac{(n-1)!(b-1)!}{(b+i)!(n-i-1)!}(n-i-b-1)\right) = 0.$$

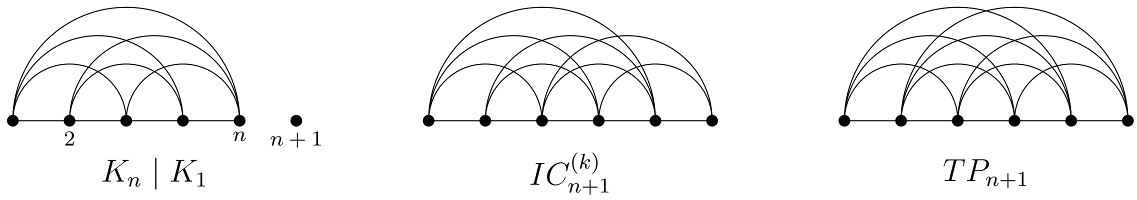
Therefore, if G is (e)-positive, then so is $G+TP_{n+1}$. That is, TP_{n+1} is appendable (e)-positive. □

We can now find new (e)-positive and appendable (e)-positive families of graphs by applying Proposition 20 and Corollary 21 to all the known (e)-positive and appendable (e)-positive labelled unit interval graphs we already have.

Proposition 24. Melting ice cream scoop graphs $IC_{n+1}^{(k)}$ *for* $n\geqslant 2$ *and* $1\leqslant k\leqslant n$, *obtained by deleting the edges between vertex* $n+1$ *and vertices* $1,\ldots,k$ *from* K_{n+1}, *and their reverse graphs are appendable* (e)*-positive.*

Proof. To prove the appendable (e)-positivity of $IC_{n+1}^{(k)}$, we will apply Corollary 21(b) to $K_n \mid K_1$ on vertex $n+1$ and vertices $n, \ldots, 2$. Figure 3 illustrates the case of $n = 5$ and $k = 2$.

Figure 3



Note $K_n \mid K_1$ is appendable (e)-positive because K_n is appendable (e)-positive by Proposition 16 and because for any labelled graph G we have $y_{G+K_n|K_1} = y_{G+K_n} e_{(\emptyset,1)}$. Additionally, TP_{n+1} is appendable (e)-positive by Proposition 23. Therefore, $IC_{n+1}^{(k)}$ is appendable (e)-positive by Corollary 21(b) setting $G_0 = K_n \mid K_1$, $G_{n-1} = TP_{n+1}$ and $i = n+1$.

Next note that $K_1 \mid K_n$, the reverse graph of $K_n \mid K_1$, is appendable (e)-positive, because for any labelled graph G we have $y_{G+K_1|K_n} = y_G e_{(\emptyset,n)}$. The reverse graph of TP_{n+1}, which is TP_{n+1} itself, is appendable (e)-positive by Proposition 23. By Remark 22, the reverse graph of $IC_{n+1}^{(k)}$ is also appendable (e)-positive. $\square$

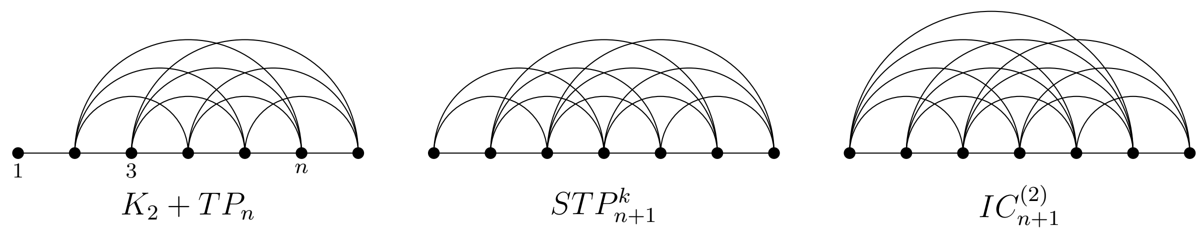
Proposition 25. Snowy twin peaks graphs STP_{n+1}^k *for* $n \geqslant 3$ *and* $1 \leqslant k \leqslant n-2$*, given by the sequence* $(m_i)_{i=1}^{n+1}$ *where*

$$m_i = \begin{cases} k+1 & \text{if } i = 1, \\ n & \text{if } i = 2, \\ n+1 & \text{if } 3 \leqslant i \leqslant n+1, \end{cases}$$

and their reverse graphs are appendable (e)*-positive.*

Proof. To prove the appendable (e)-positivity of STP_{n+1}^k, we will apply Corollary 21(a) to $K_2 + TP_n$ on vertex 1 with vertices $3, \ldots, n$. Figure 4 illustrates the case of $n = 6$ and $k = 3$.

Figure 4



Note $K_2 + TP_n$ is appendable (e)-positive because both K_2 and TP_n are appendable (e)-positive by Propositions 16 and 23, respectively, and so for any (e)-positive labelled graph G, the labelled graph $G + K_2$ is (e)-positive, and therefore the labelled graph $G + K_2 + TP_n$ is (e)-positive. Additionally, $IC^{(2)}_{n+1}$ is appendable (e)-positive by Proposition 24. Therefore, STP^k_{n+1} is appendable (e)-positive by applying Corollary 21(a) setting $G_0 = K_2 + TP_n$, $G_{n-2} = IC^{(2)}_{n+1}$ and $i = 1$.

Next note that $TP_n + K_2$, the reverse graph of $K_2 + TP_n$, is appendable (e)-positive because TP_n and K_2 are appendable (e)-positive by Propositions 23 and 16, and the reverse graph of $IC^{(2)}_{n+1}$ is appendable (e)-positive by Proposition 24. By Remark 22, the reverse graph of STP^k_{n+1} is also appendable (e)-positive. □

Proposition 26. Wide melting lollipop graphs $WL^{(k)}_{m,n}$ *for* $m \geqslant 4$, $n \geqslant 0$ *and* $1 \leqslant k \leqslant m-2$, *given by the sequence* $(w_i)_{i=1}^{m+n}$ *where*

$$w_i = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant m-1, \\ k+1 & \text{if } i = m, \\ i-2 & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

are (e)*-positive.*

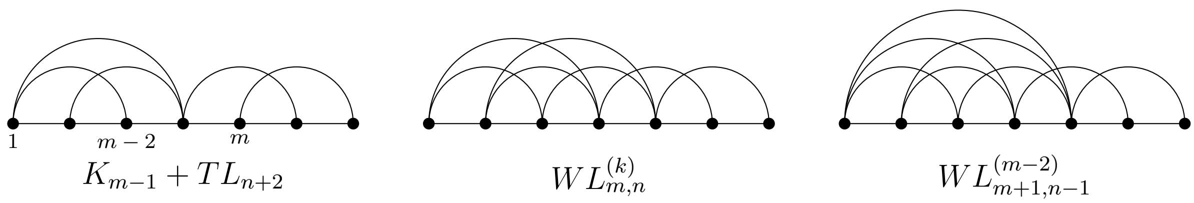
Proof. We will proceed by induction on n. For the base case $n = 0$, note that in Proposition 24 we can equivalently define $IC^{(k)}_m$ to be given by the sequence $(w_i)_{i=1}^{m}$ where

$$w_i = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant m-1, \\ k+1 & \text{if } i = m. \end{cases}$$

Hence $WL^{(k)}_{m,0} = IC^{(k)}_m$, which is (e)-positive by Proposition 24.

For $n > 0$, we will apply Corollary 21(b) to $K_{m-1} + TL_{n+2}$ on vertex m with vertices $m-2, \ldots, 1$. Figure 5 illustrates the case of $m = 5$, $n = 2$ and $k = 1$.

Figure 5



Note $K_{m-1} + TL_{n+2}$ is (e)-positive because triangular ladders are appendable (e)-positive by Proposition 17, and $WL^{(m-2)}_{m+1,n-1}$ is (e)-positive by the inductive hypothesis. Therefore since

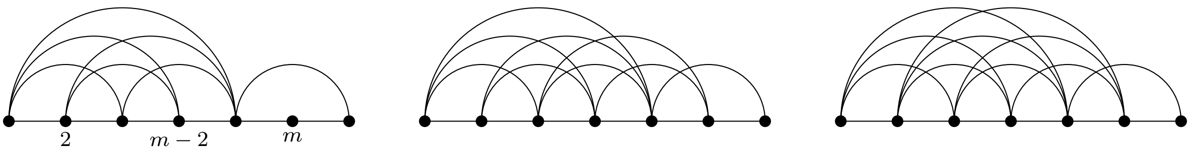
$$y_{WL^{(k)}_{m,n}} = \frac{k}{m-2} y_{K_{m-1}+TL_{n+2}} + \frac{m-k-2}{m-2} y_{WL^{(m-2)}_{m+1,n-1}},$$

the labelled graph $WL^{(k)}_{m,n}$ is (e)-positive. □

Proposition 27. *Wide melting lollipop graphs $WL^{(k)}_{m,1}$ with $n = 1$, for $m \geqslant 4$ and $1 \leqslant k \leqslant m-2$, and their reverse graphs are appendable (e)-positive.*

Proof. To prove the appendable (e)-positivity of $WL^{(k)}_{m,1}$, we will apply Corollary 21(b) to $K_{m-1} + K_3$ on vertex m and vertices $m-2, \ldots, 2$. Figure 6 illustrates the case of $m = 6$ and $k = 2$.

Figure 6



$K_{m-1} + K_3$ $\quad$ $WL^{(k)}_{m,1}$ $\quad$ $(STP^2_{m+1})^r$

Note $K_{m-1}+K_3$ is appendable (e)-positive because both K_{m-1} and K_3 are appendable (e)-positive by Proposition 16, and the reverse graph of STP^2_{m+1} is appendable (e)-positive by Proposition 25. Therefore $WL^{(k)}_{m,1}$ is appendable (e)-positive by Corollary 21 setting $G_0 = K_{m-1} + K_3$, $G_{m-3} = (STP^2_{m+1})^r$ and $i = m$.

Next note that $K_3 + K_{m-1}$, the reverse graph of $K_{m-1} + K_3$, is appendable (e)-positive because both K_3 and K_{m-1} are appendable (e)-positive by Proposition 16, and STP^2_{m+1}, the reverse graph of $(STP^2_{m+1})^r$, is appendable (e)-positive by Proposition 25. By Remark 22, the reverse graph of $WL^{(k)}_{m,1}$ is also appendable (e)-positive. ☐

6 The reduction to complete graphs

In this section we will highlight a simple but useful idea. For any linear map on UBCSym^d with a special interpretation when $y_{G:v}$ is substituted in, we can compute the image of any function in UBCSym^d by writing the function in the e-basis and then noting each $e_{(\lambda,b)} = y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b}$. In particular, to understand the map on UBCSym^d, it is enough to study its behaviour on disjoint unions of complete graphs.

Our first illustration of this concept is a shorter derivation of the first part of [10, Corollary 6.1], which computes the result of the induction operation on the e-basis of UBCSym. The second part of [10, Corollary 6.1] also follows by symmetry.

Lemma 28. *[10, Corollary 6.1] For any partition λ and positive integer b,*

$$e_{(\lambda,b)}\uparrow = \frac{1}{b} e_{(\lambda\cup(b),1)} - \frac{1}{b} e_{(\lambda,b+1)}.$$

Proof. By Proposition 5, deletion-contraction,

$$y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+K_2} = y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b|K_1} - y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b}\uparrow.$$

Therefore,

$$e_{(\lambda,b)}\uparrow = y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b}\uparrow = y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b|K_1} - y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+K_2}.$$

To compute $y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+K_2} = e_{((\lambda_1,\ldots,\lambda_{\ell(\lambda)-1}),\lambda_{\ell(\lambda)})} y_{K_b+K_2}$, we note by Proposition 20(b) (applied to $K_b \mid K_1$ and vertex $b+1$ together with vertices $b,\ldots,1$) that

$$y_{K_b+K_2} = \frac{b-1}{b} y_{K_b|K_1} + \frac{1}{b} y_{K_{b+1}} = \frac{b-1}{b} e_{((b),1)} + \frac{1}{b} e_{(\emptyset,b+1)}.$$

Hence we conclude that

$$\begin{aligned} e_{(\lambda,b)}\uparrow &= y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b|K_1} - y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+K_2} \\ &= e_{(\lambda\cup(b),1)} - \left(\frac{b-1}{b} e_{(\lambda\cup(b),1)} + \frac{1}{b} e_{(\lambda,b+1)}\right) \\ &= \frac{1}{b} e_{(\lambda\cup(b),1)} - \frac{1}{b} e_{(\lambda,b+1)}. \end{aligned}$$

□

To give more applications of the idea introduced in this section, we will require more linear maps on UBCSym^d with special interpretations when $y_{G:v}$ is substituted in. Our next result shows for any labelled graph H that the map sending each $y_G \mapsto y_{G+H}$ is linear, which also explains why graph concatenation is natural to study in UBCSym.

Theorem 29. *If H is a labelled graph, then for every $d \in \mathbb{Z}_{>0}$, there exists a linear map $\overline{T_H} : \mathrm{UBCSym}^d \to \mathrm{UBCSym}^{d+|H|-1}$ sending each $y_G \mapsto y_{G+H}$ for all labelled graphs G on d vertices.*

Proof. For each subset $S_2 \subseteq E(H)$, define $T_{S_2} : \mathrm{NCSym}^d \to \mathrm{UBCSym}^{d+|H|-1}$ to be the linear map taking each $p_\pi \mapsto p_{\mathrm{type}(\pi')}$, where given a set partition $\pi \vdash [d]$, we construct the set partition $\pi' \vdash [d+|H|-1]$ describing the connected components of the labelled graph $K_\pi + (V(H), S_2)$.

We next show that T_{S_2} induces a linear map on UBCSym^d. If $\pi_1, \pi_2 \vdash [d]$ are of the same type, then there exists $\delta \in \mathcal{S}_d$ fixing d such that $\pi_1 = \delta(\pi_2)$. If we interpret δ as an element of $\mathcal{S}_{d+|H|-1}$ acting only on the first d positions, we see that δ fixes the last $|H|$ positions and satisfies $\pi_1' = \delta(\pi_2')$. So $\mathrm{type}(\pi_1') = \mathrm{type}(\pi_2')$ and

$$T_{S_2}(p_{\pi_1}) = p_{\mathrm{type}(\pi_1')} = p_{\mathrm{type}(\pi_2')} = T_{S_2}(p_{\pi_2})$$

whenever π_1, π_2 are of the same type. This induces a linear map $\overline{T_{S_2}} : \mathrm{UBCSym}^d \to \mathrm{UBCSym}^{d+|H|-1}$.

Next we define

$$\overline{T_H} = \sum_{S_2 \subseteq E(H)} (-1)^{|S_2|} \overline{T_{S_2}}.$$

Let G be any labelled graph on d vertices. By [10, Theorem 3.6],

$$\begin{aligned} Y_{G+H} &= \sum_{S\subseteq E(G+H)} (-1)^{|S|} p_{\pi(S)} \\ &= \sum_{S_1\subseteq E(G)} (-1)^{|S_1|} \sum_{S_2\subseteq E(H)} (-1)^{|S_2|} p_{\pi(S_1\cup S_2)}, \end{aligned}$$

where $\pi(S)$ is the set partition of $[d+|H|-1]$ describing the connected components of the labelled graph $(V(G+H), S)$. Also,

$$Y_G = \sum_{S_1\subseteq E(G)} (-1)^{|S_1|} p_{\pi(S_1)},$$

where in this case $\pi(S_1) \vdash [d]$ describes the connected components of $(V(G), S_1)$.

Then

$$\begin{aligned} \overline{T_H}(y_G) &= \sum_{S_2\subseteq E(H)} (-1)^{|S_2|} \overline{T_{S_2}}(y_G) \\ &= \sum_{S_2\subseteq E(H)} (-1)^{|S_2|} T_{S_2} \left(\sum_{S_1\subseteq E(G)} (-1)^{|S_1|} p_{\pi(S_1)} \right) \\ &= \sum_{S_2\subseteq E(H)} (-1)^{|S_2|} \sum_{S_1\subseteq E(G)} (-1)^{|S_1|} p_{\text{type}(\pi(S_1\cup S_2))} = y_{G+H}, \end{aligned}$$

proving the statement of the proposition. □

We can apply our techniques to the map in Theorem 29 to obtain an equivalent condition to appendable (e)-positivity.

Proposition 30. *A labelled graph H is appendable (e)-positive if and only if $K_d + H$ is (e)-positive for all $d \in \mathbb{Z}_{>0}$.*

Proof. The forward direction follows immediately from the definition of appendable (e)-positivity, since each K_d is (e)-positive.

For the reverse direction, suppose each K_d+H is (e)-positive. Let G be any (e)-positive labelled graph. By Theorem 29,

$$\begin{aligned}
y_{G+H} &= \overline{T_H}(y_G) \\
&= \overline{T_H}\left(\sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{(\lambda,b)}\right) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \overline{T_H}(e_{(\lambda,b)}) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \overline{T_H}(y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b}) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+H} \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{((\lambda_1,\ldots,\lambda_{\ell(\lambda)-1}),\lambda_{\ell(\lambda)})} y_{K_b+H}
\end{aligned}$$

for some nonnegative coefficients $c_{(\lambda,b)}$, and so y_{G+H} is (e)-positive. Since this holds for all (e)-positive G, it follows that H is appendable (e)-positive. □

Corollary 31. *Conjectures 18 and 19 are equivalent.*

Proof. Conjecture 19 implies Conjecture 18, because every appendable (e)-positive labelled graph is (e)-positive.

For the other direction, suppose Conjecture 18 held. Let H be any labelled unit interval graph. For each $d \in \mathbb{Z}_{>0}$, the labelled graph $K_d + H$ is also a labelled unit interval graph, and so is (e)-positive, by assumption. Then by Proposition 30, every labelled unit interval graph H is appendable (e)-positive. So Conjecture 19 would follow from Conjecture 18. □

Another application of the idea in this section gives a method of constructing new e-positive graphs from a pair of (e)-positive graphs.

Theorem 32. *If G and H are (e)-positive labelled graphs, then $G + H^r$ is e-positive.*

Proof. By Theorem 29, there exists a linear map $\overline{T_{H^r}} : \mathrm{UBCSym}^{|G|} \to \mathrm{UBCSym}^{|G|+|H|-1}$ sending $y_{G'} \mapsto y_{G'+H^r}$ for all labelled graphs G' on $|G|$ vertices.

Since G is (e)-positive, there exist nonnegative coefficients $c_{(\lambda,b)}$ such that

$$\begin{aligned}
X_{G+H^r} &= \bar{\rho}\left(\overline{T_{H^r}}\left(y_G\right)\right) \\
&= \bar{\rho}\left(\overline{T_{H^r}}\left(\sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{(\lambda,b)}\right)\right) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \bar{\rho}\left(\overline{T_{H^r}}(e_{(\lambda,b)})\right) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \bar{\rho}\left(\overline{T_{H^r}}(y_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b})\right) \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} X_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_b+H^r} \\
&= \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \lambda! e_\lambda X_{H+K_b}.
\end{aligned}$$

Note since H is (e)-positive, by Proposition 16 each $H+K_b$ is (e)-positive (and hence e-positive). So $G+H^r$ is e-positive. □

We next give a summary of the positivity results we obtain from concatenating (e)-positive and appendable (e)-positive graphs.

Corollary 33. *If G, G' are (e)-positive and $(H_i)_{i=1}^k$ are appendable (e)-positive, then*

(a) $\sum_{i=1}^k H_i$ *is appendable (e)-positive,*

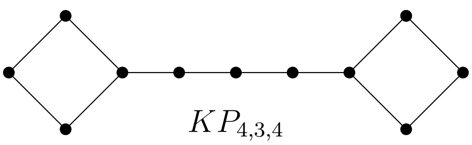
(b) $G+\sum_{i=1}^k H_i$ *is (e)-positive, and*

(c) $G+\sum_{i=1}^k H_i + G'^r$ *is e-positive.*

Proof. Part (a) follows from definition of appendable (e)-positivity and induction on k. Part (b) follows from part (a). Part (c) follows from part (b) together with Theorem 32. □

We demonstrate Corollary 33 in the following proposition, which proves the e-positivity of *kayak paddle graphs* $KP_{m,\ell-1,n} = C_m + P_{\ell+1} + C_n$ for $m, n \geqslant 3$ and $\ell \geqslant 0$. See Figure 7 below for the example of $KP_{4,3,4}$.

Figure 7



Proposition 34. *Kayak paddle graphs $KP_{m,\ell-1,n}$ for $m,n \geqslant 3$ and $\ell \geqslant 0$, are e-positive.*

Proof. Note $KP_{m,\ell-1,n} = C_m + \sum_{i=1}^{\ell} K_2 + C_n^r$, where the cycles C_m and C_n are (e)-positive by Proposition 15, and the complete graph K_2 is appendable (e)-positive by Proposition 16. By Corollary 33(c), the graph $KP_{m,\ell-1,n}$ is e-positive. ☐

Our last result of the section is a theorem relating the (e)-positivity of $y_{G:v}$ to the e-positivity of X_{G-v}, where $G-v$ is the graph G with vertex v and all incident edges deleted.

Theorem 35. *If a graph G is (e)-positive at a vertex v, then $G-v$ is e-positive.*

Proof. Define the linear map $\vartheta : \mathrm{UBCSym}^{|G|} \to \mathrm{Sym}^{|G|-1}$ satisfying

$$\vartheta(p_{(\lambda,b)}) = \begin{cases} p_\lambda & \text{if } b=1, \\ 0 & \text{otherwise.} \end{cases}$$

Give G a labelling in which v is labelled last. By [10, Theorem 3.6],

$$y_{G:v} = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\mathrm{type}(\pi(S))},$$

where $\pi(S)$ is the set partition describing the connected components of $(V(G), S)$. Note $\mathrm{type}(\pi(S)) = (\lambda, b)$ has $b = 1$ if and only if $S \subseteq E(G-v)$. So

$$y_{G:v} = \sum_{S \subseteq E(G-v)} (-1)^{|S|} p_{(\lambda(S),1)} + \sum_{\substack{S \subseteq E(G) \\ S \nsubseteq E(G-v)}} (-1)^{|S|} p_{\mathrm{type}(\pi(S))},$$

where for $S \subseteq E(G-v)$ the integer partition $\lambda(S)$ describes the connected components of $(V(G-v), S)$, and $\mathrm{type}(\pi(S)) = (\lambda, b)$ has $b > 1$ for all $S \subseteq E(G)$ satisfying $S \nsubseteq E(G-v)$.

Therefore,

$$\vartheta(y_{G:v}) = \sum_{S \subseteq E(G-v)} (-1)^{|S|} p_{\lambda(S)} = X_{G-v},$$

with the right equality following by [25, Theorem 2.5]. The equation above holds for all graphs on $|G|$ vertices with a distinguished vertex so in particular,

$$\vartheta(e_{(\lambda,b)}) = \vartheta(y_{K_{\lambda_1}|\cdots|K_{\ell(\lambda)}|K_b}) = X_{K_{\lambda_1}|\cdots|K_{\lambda_{\ell(\lambda)}}|K_{b-1}} = \lambda!(b-1)! e_{\lambda \cup (b-1)},$$

where for a labelled graph G we take $G \mid K_0$ to mean G, and for a partition λ we take $\lambda \cup (0)$ to mean λ.

So if

$$y_{G:v} = \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{(\lambda,b)},$$

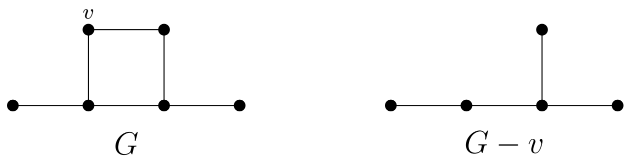
then we have

$$X_{G-v} = \vartheta(y_{G:v}) = \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} \lambda!(b-1)! e_{\lambda \cup (b-1)}.$$

In particular if G is (e)-positive at v, then $G-v$ is e-positive. ☐

Example 36. Let G denote the left graph in Figure 8 below, with vertex v as labelled.

Figure 8



Then

$$\begin{aligned} y_{G:v} = \frac{1}{12}e_{((4,1),1)} + \frac{1}{60}e_{((5),1)} + \frac{1}{4}e_{((2^2),2)} + \frac{1}{6}e_{((3,1),2)} + \frac{1}{24}e_{((4),2)} \\ + \frac{1}{6}e_{((1^2),4)} + \frac{1}{12}e_{((2),4)} + \frac{1}{6}e_{((1),5)} + \frac{1}{40}e_{(\emptyset,6)}, \end{aligned}$$

which is (e)-positive. By Theorem 35, the graph $G - v$ is therefore e-positive.

7 The (e)-positivity of trees and cut vertices

In [6], Dahlberg, She and van Willigenburg studied the positivity of chromatic symmetric functions of trees in the Schur and e-bases. It will be of interest to study which trees are (e)-positive and at which vertices, because we can construct more e-positive trees from given (e)-positive trees by applying Theorem 32 or the appendable (e)-positivity of paths.

In their study of positivity of trees, it was also important for Dahlberg, She and van Willigenburg to understand how cut vertices affect positivity. It is also natural to ask if and when a graph can be (e)-positive at a cut vertex, noting that none of the (e)-positivity results from the previous sections demonstrate (e)-positivity at a cut vertex. In this short section, we will resolve both questions.

Proposition 37. *If G is a graph with cut vertex v, then G is not (e)-positive at v.*

Proof. Since v is a cut vertex of G, the graph $G-v$ has at least 2 connected components, each of order $< |G| - 1$.

Give G a labelling where v is ordered last. Consider the graph $G + G^r$ on $2|G| - 1$ vertices. The image of v in $G+G^r$ is a cut vertex whose deletion gives the disjoint union of 2 copies of $G-v$, the connected components of which each have order $< |G|-1 = \lfloor \frac{2|G|-1}{2} \rfloor$. By [6, Theorem 35], the graph $G + G^r$ is not e-positive.

Then by Theorem 32, the graph G cannot be (e)-positive at cut vertex v. □

Remark 38. In Section 4, we defined appendable (e)-positivity only for labelled graphs, although there is a notion of appendable (e)-positivity that depends only on a graph and a pair of distinguished vertices. (Namely, we may want to ask for which graphs H with distinguished vertices v, w is it true that for all graphs G (e)-positive at a vertex u the concatenation obtained by identifying vertex u of G and vertex v of H is (e)-positive at

w.) In our definition of appendable (e)-positivity for labelled graphs, these vertices are the first and last vertices of the labelled graph. However, for graphs on $\geqslant 2$ vertices, there is no labelling that allows a single vertex to be both the first and last vertex of the labelling. This raises the possibility that there may exist nontrivial cases in which a graph and a pair of distinguished vertices satisfies the more general notion of appendable (e)-positivity, but in a way that cannot be encapsulated by the notion of appendable (e)-positivity for labelled graphs.

Proposition 37 combined with Proposition 30 resolves this concern, because for any connected labelled graph G on $\geqslant 2$ vertices, $K_d + G$ for $d \geqslant 2$ cannot be (e)-positive at the cut vertex d. So every connected graph with a pair of distinguished vertices that can be considered appendable (e)-positive admits a labelling in which it is appendable (e)-positive as a labelled graph.

Corollary 39. *A tree T is (e)-positive at vertex v if and only if T is a path with v as one of its endpoints.*

Proof. For the forward direction, suppose the contrary. Then let T be a minimal tree that is (e)-positive at a vertex v such that T is not a path with v as one of its endpoints. By Proposition 37, v cannot be a cut vertex of the tree T, and so must be a leaf. Let w denote its unique neighbour.

Note T is obtained from $T - v$ by concatenating a copy of K_2 at w so that the other endpoint of the K_2 becomes the vertex v in T. By [10, Lemma 7.5], if

$$y_{T-v:w} = \sum_{|\lambda|+b=|T|-1} c_{(\lambda,b)} e_{(\lambda,b)}$$

then

$$y_{T:v} = \sum_{|\lambda|+b=|T|-1} c_{(\lambda,b)} \left(\frac{b-1}{b} e_{(\lambda\cup(b),1)} + \frac{1}{b} e_{(\lambda,b+1)} \right).$$

Note for (λ, b) satisfying $|\lambda| + b = |T| - 1$ in the above equations that the coefficient of $e_{(\lambda,b)}$ in $y_{T-v:w}$ is b times the coefficient of $e_{(\lambda,b+1)}$ in $y_{T:v}$. Since T is (e)-positive at v, it follows then that $T - v$ is (e)-positive at w. But then $T - v$ is a smaller tree that is (e)-positive at a vertex w such that either $T - v$ is not a path or w is not one of its endpoints, contradicting the minimality of T.

The reverse direction is given by [10, Proposition 6.4], where Gebhard and Sagan show that paths are (e)-positive at their endpoints. $\square$

8 UBCQSym and acyclic orientations

In this section we will define the quotient algebra UBCQSym of NCQSym by giving a construction analogous to that of the quotient algebra UBCSym of NCSym. Working in UBCQSym will aid us in proving theorems relating the coefficients of $y_{G:v}$ in the e-basis and acyclic orientations.

A *marked composition* $\hat{\alpha}$ of $d \in \mathbb{Z}_{>0}$ is a composition of d with a distinguished part, which we identify by writing a caret above it. For example, $(2, \hat{2}, 3)$ is a marked composition of 7 with underlying composition $(2, 2, 3)$ and distinguished part chosen to be the second 2. Given a set composition $\Phi \vDash [d]$ with $d \in \mathbb{Z}_{>0}$, define $\operatorname{type}(\Phi)$ to be the marked composition with underlying composition $\alpha(\Phi)$ and distinguished part corresponding to the part of Φ containing d, e.g. $\operatorname{type}(13 \sslash 45 \sslash 2) = (2, \hat{2}, 1)$. When Φ is the empty set composition, write $\operatorname{type}(\Phi) = (\emptyset)$.

Recall that one basis of NCQSym consists of the M_Φ over all set compositions Φ. We can define UBCQSym first as the free vector space spanned by elements $M_{\operatorname{type}(\Phi)}$ over all set compositions Φ. It is naturally a quotient vector space of NCQSym via the linear projection map $\nu : \mathrm{NCQSym} \to \mathrm{UBCQSym}$ sending each $M_\Phi \mapsto M_{\operatorname{type}(\Phi)}$.

We define an action of $\mathcal{S}_d$ on NCQSym^d by permuting the positions of the variables, analogous to the action of $\mathcal{S}_d$ on NCSym^d. It can be verified from the definitions of the monomial basis of NCQSym and the action of $\delta \in \mathcal{S}_d$ on NCQSym^d that

$$\delta \circ M_\Phi = M_{\delta(\Phi)},$$

where the action of δ on set compositions of $[d]$ is by permuting the elements of the blocks.

Note for $\Phi \vDash [d]$ and $\delta \in \mathcal{S}_d$ fixing d that $\nu(\delta \circ M_\Phi) = \nu(M_{\delta(\Phi)}) = \nu(M_\Phi)$. Extending linearly, for any $f \in \mathrm{NCQSym}^d$ and $\delta \in \mathcal{S}_d$ fixing d, we also have $\nu(\delta \circ f) = \nu(f)$. The kernel of ν is given by

$$\ker \nu = \operatorname{span}\{\delta \circ f - f \mid f \in \mathrm{NCQSym}^d,\ \delta \in \mathcal{S}_d,\ \delta(d) = d \in \mathbb{Z}_{>0}\}.$$

For homogeneous $f \in \mathrm{NCQSym}^d$ and $g \in \mathrm{NCQSym}^{d'}$ with $\delta \in \mathcal{S}_d$ fixing $d \in \mathbb{Z}_{>0}$, we have

$$(\delta \circ f - f)g = \delta \circ fg - fg,$$

where in the right-hand side we extend $\delta \in \mathcal{S}_d$ to a permutation in $\mathcal{S}_{d+d'}$ fixing the last $1 + d'$ positions, and

$$g(\delta \circ f - f) = \delta' \circ gf - gf,$$

where $\delta' \in \mathcal{S}_{d+d'}$ fixes the first d' positions and permutes positions $1 + d', \dots, d + d'$ according to $\delta \in \mathcal{S}_d$ (and, in particular, fixes $d + d'$). Extending bilinearly, we see that $\ker \nu$ is a two-sided graded ideal of NCQSym, so UBCQSym is a graded quotient algebra of NCQSym with

$$\mathrm{UBCQSym} = \mathrm{NCQSym} \,/\, \operatorname{span}\{\delta \circ f - f \mid f \in \mathrm{NCQSym}^d,\ \delta \in \mathcal{S}_d,\ \delta(d) = d \in \mathbb{Z}_{>0}\}.$$

We will write $\mathrm{UBCQSym}^d = \nu(\mathrm{NCQSym}^d)$ to denote the homogeneous part of degree d in UBCQSym.

Note that the $\nu(m_\pi)$ over set partitions π of distinct types are linearly independent in UBCQSym, because $\operatorname{type}(\pi) = (\lambda, b)$ if and only if the coefficient of $M_{(\lambda_1, \dots, \lambda_{\ell(\lambda)}, \hat{b})}$ is nonzero in the expansion of $\nu(m_\pi)$ in the M-basis. Therefore UBCSym is a subalgebra of UBCQSym.

A *labelled composition* of $d \in \mathbb{Z}_{>0}$ is a pair (δ, α) consisting of a permutation in $\mathcal{S}_d$ and a composition of d. We may also write a labelled composition more compactly by writing the permutation in one-line notation and including commas to indicate the parts of the composition. For example, we write $(14, 52, 3)$ to mean the labelled composition with underlying permutation 14523 and underlying composition $(2, 2, 1)$.

Given a labelled composition (δ, α) of d, define the quasisymmetric function in noncommuting variables

$$F_{(\delta,\alpha)} = \sum_{\substack{i_{\delta(1)} \leqslant \cdots \leqslant i_{\delta(d)} \\ i_{\delta(j)} < i_{\delta(j+1)} \text{ if } j \in \operatorname{set}(\alpha)}} x_{i_1} \cdots x_{i_d}.$$

For example,

$$F_{(14,52,3)} = \sum_{i_1 \leqslant i_4 < i_5 \leqslant i_2 < i_3} x_{i_1} \cdots x_{i_5}.$$

Note for $\delta, \delta' \in \mathcal{S}_d$ that

$$F_{(\delta\delta',\alpha)} = \delta \circ F_{(\delta',\alpha)}.$$

For a set composition $\Phi = \Phi_1 /\!\!/ \cdots /\!\!/ \Phi_{\ell(\Phi)}$, define

$$Q_\Phi = F_{(\Phi_1^r, \ldots, \Phi_{\ell(\Phi)}^r)},$$

where Φ_j^r denotes the elements in jth part of Φ written in descending order. As an example, $Q_{13/\!\!/45/\!\!/2} = F_{(31,54,2)}$. The Q_Φ for $\Phi \vDash [d]$ form a basis for NCQSym^d by an upper-triangularity argument against the M_Φ, after ordering set compositions by refinement.

Note if set compositions $\Phi, \Psi \vDash [d]$ are of the same type $\hat{\alpha}$ with underlying composition α and kth part marked, then $\nu(Q_\Phi) = \nu(Q_\Psi)$. This follows because $(\Phi_1^r, \ldots, \Phi_{\ell(\Phi)}^r)$ and $(\Psi_1^r, \ldots, \Psi_{\ell(\Psi)}^r)$ both have underlying composition α, and their underlying permutations δ, δ' satisfy $\delta(\sum_{j=1}^{k-1} \alpha_j + 1) = \delta'(\sum_{j=1}^{k-1} \alpha_j + 1) = d$, and so $\delta\delta'^{-1} \in \mathcal{S}_d$ fixes d, and therefore

$$\nu(Q_\Phi) = \nu(F_{(\delta,\alpha)}) = \nu(\delta\delta'^{-1} \circ F_{(\delta',\alpha)}) = \nu(F_{(\delta',\alpha)}) = \nu(Q_\Psi).$$

Define then $Q_{\operatorname{type}(\Phi)} = \nu(Q_\Phi)$. Because the Q_Φ over all set compositions of $[d]$ span NCQSym^d, it follows that the $Q_{\operatorname{type}(\Phi)}$ over all set compositions of $[d]$ span $\mathrm{UBCQSym}^d$. Moreover, since the dimension of $\mathrm{UBCQSym}^d$ is exactly the number of marked compositions of d, it follows that the $Q_{\operatorname{type}(\Phi)}$ over all set compositions of d form a basis for $\mathrm{UBCQSym}^d$.

A *labelled poset* on $d \in \mathbb{Z}_{>0}$ elements is a poset given by the set $[d]$ with a partial ordering $<_P$. For a labelled poset P on d elements, define the quasisymmetric function in noncommuting variables

$$Y_P = \sum_\kappa x_{\kappa(1)} \ldots x_{\kappa(d)},$$

where the sum is over all maps $\kappa : [d] \to \mathbb{Z}_{>0}$ satisfying $\kappa(i) < \kappa(j)$ whenever $i <_P j$. Also let y_P denote $\nu(Y_P)$. We can define the action of $\delta \in \mathcal{S}_d$ on labelled posets on d elements by permuting labels, e.g. the labelled poset $\delta(P)$ is a relabelling of P. It follows by the definition of Y_P and the action of δ on NCQSym^d and on labelled posets on d elements that $Y_{\delta(P)} = \delta \circ Y_P$.

We will require the following lemma, which follows by [24, Lemma 4.5.3(b)] from the theory of P-partitions.

Lemma 40. *Let P be a labelled poset on d elements, and let s be a fixed linear extension of P. If w is another linear extension of P, given by $i_1 <_w \cdots <_w i_d$, then define $\delta_w \in \mathcal{S}_d$ to be the permutation $i_1 \cdots i_d$, and α_w^s to be the composition of d satisfying $\mathrm{set}(\alpha_w^s) = \{j \mid i_j <_s i_{j+1}\}$.*

Then

$$Y_P = \sum_w F_{(\delta_w, \alpha_w^s)},$$

where the sum is over all linear extensions w of P.

Corollary 41. *Let P be a labelled poset on d elements. Let s be a linear extension of P satisfying $i >_s d$ if and only if $i >_P d$, and let $\varepsilon \in \mathcal{S}_d$ be the permutation satisfying*

$$\varepsilon\delta_s(i) = \begin{cases} i & \text{if } i < \delta_s^{-1}(d), \\ d & \text{if } i = \delta_s^{-1}(d), \\ i-1 & \text{if } i > \delta_s^{-1}(d). \end{cases}$$

Then the coefficient of $Q_{\hat{\alpha}}$, where $\hat{\alpha}$ has underlying composition α and kth part marked, in the Q-expansion of y_P counts the number of linear extensions w of $\varepsilon(P)$ satisfying $\alpha_w^{\varepsilon(s)} = \alpha$ and $\delta_w(\sum_{j=1}^{k-1} \alpha_j + 1) = d$. In particular, y_P is Q-positive.

Proof. Observe that ε fixes d, since $\varepsilon(d) = \varepsilon\delta_s\delta_s^{-1}(d) = d$. Then

$$y_P = \nu(Y_P) = \nu(\varepsilon \circ Y_P) = \nu(Y_{\varepsilon(P)}) = y_{\varepsilon(P)}.$$

Next consider the linear extension $\varepsilon(s)$ of $\varepsilon(P)$, given by

$$1 <_{\varepsilon(s)} \cdots <_{\varepsilon(s)} \delta_s^{-1}(d) - 1 <_{\varepsilon(s)} d <_{\varepsilon(s)} \delta_s^{-1}(d) <_{\varepsilon(s)} \cdots <_{\varepsilon(s)} d-1.$$

Note that we have $i >_{\varepsilon(s)} d$ if and only if $i >_{\varepsilon(P)} d$.

For any linear extension w of $\varepsilon(P)$, we note $i \notin \mathrm{set}(\alpha_w^{\varepsilon(s)})$ if and only if $\delta_w(i) >_{\varepsilon(s)} \delta_w(i+1)$, which occurs only if $\delta_w(i) > \delta_w(i+1)$ in $\mathbb{Z}_{>0}$ or if $\delta_w(i+1) = d$. The latter case cannot occur, because $\delta_w(i) >_{\varepsilon(s)} d$ if and only if $\delta_w(i) >_{\varepsilon(P)} d$, and so $\delta_w(i) >_w d$ since w is a linear extension of $\varepsilon(P)$. So $i \notin \mathrm{set}(\alpha_w^{\varepsilon(s)})$ only if $\delta_w(i) > \delta_w(i+1)$ in $\mathbb{Z}_{>0}$, implying that the marked composition $(\delta_w, \alpha_w^{\varepsilon(s)})$ is of the form $(\Phi_1^r, \ldots, \Phi_{\ell(\Phi)}^r)$ for some set composition $\Phi \vDash [d]$.

The result then follows by applying Lemma 40 to $\varepsilon(P)$ with the linear extension $\varepsilon(s)$ and studying the projection in NCQSym^d, since then each $F_{(\delta_w, \alpha_w^{\varepsilon(s)})}$ is equal to some Q_Φ. ∎

We can now state a noncommutative analogue of Stanley's [25, Theorem 3.3]. It is proved by adapting Stanley's original proof using P-partitions and using the linear map $\varphi : \mathrm{UBCQSym}^{|G|} \to \mathbb{Q}[t]$ satisfying

$$\varphi(Q_{\hat{\alpha}}) = \begin{cases} t(t-1)^{k-1} & \text{if } \hat{\alpha} = (1^{|G|-k}, \hat{k}), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 42. *Suppose*

$$y_{G:v} = \sum_{|\lambda|+b=|G|} c_{(\lambda,b)} e_{(\lambda,b)}.$$

Let $\mathrm{sink}_v(G,j)$ *count the number of acyclic orientations of* G *with* j *sinks, including a sink at* v*. Then*

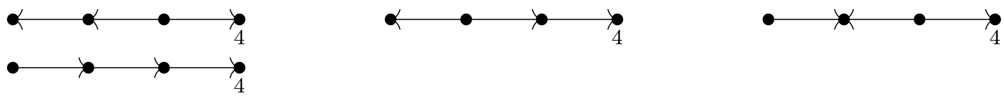
$$\mathrm{sink}_v(G,j) = \sum_{\substack{|\lambda|+b=|G| \\ \ell(\lambda)+1=j}} c_{(\lambda,b)} \lambda!(b-1)!.$$

Example 43. We will apply Theorem 42 to the path P_4 and its last vertex. We can compute

$$y_{P_4} = \frac{1}{3} e_{((3),1)} + \frac{1}{2} e_{((2),2)} + \frac{1}{6} e_{(\emptyset,4)}.$$

By the theorem, there are exactly $\frac{1}{3}3!0! + \frac{1}{2}2!1! = 3$ acyclic orientations of P_4 with two sinks, one of which is at the last vertex, and $\frac{1}{6}3! = 1$ acyclic orientation of P_4 with a unique sink at the last vertex. These are shown in Figure 9 below.

Figure 9



Remark 44. Gebhard and Sagan's [10, Theorem 4.4] is Theorem 42 in the special case of $j = 1$.

From Theorem 42 we can recover Stanley's original [25, Theorem 3.3].

Corollary 45. *[25, Theorem 3.3] Suppose*

$$X_G = \sum_{\lambda \vdash |G|} c_\lambda e_\lambda.$$

Let $\mathrm{sink}(G,j)$ *count the number of acyclic orientations of* G *with* j *sinks. Then*

$$\mathrm{sink}(G,j) = \sum_{\substack{\lambda \vdash |G| \\ \ell(\lambda)=j}} c_\lambda.$$

Remark 46. In [16], the acyclic orientation polynomial of a graph is introduced and defined to be the generating function for the sinks of a graph's acyclic orientations. It is shown in [16, Theorem 3.2] that the acyclic orientation polynomial satisfies a subgraph expansion analogous to [25, Theorem 2.5] for X_G, and so the authors of [16] were able to prove the existence of a linear map sending $X_G \mapsto \sum_{j=1}^{|G|} \operatorname{sink}(G,j)t^j$, giving a new proof in [16, Theorem 4.5] of Stanley's sink theorem (Corollary 45) without using P-partitions.

In fact, there is a linear map sending Y_G to the acyclic orientation polynomial of a labelled graph G, e.g. by comparing [10, Theorem 3.6] with [16, Theorem 3.2]. After composing that with the linear map sending the acyclic orientation polynomial of G to $\sum_{j=1}^{|G|} \operatorname{sink}_v(G,j)t^j$, one can construct the linear map induced on the quotient, sending $y_{G:v} \mapsto \sum_{j=1}^{|G|} \operatorname{sink}_v(G,j)t^j$, which gives another proof of Theorem 42.

9 The pointed chromatic symmetric function and further avenues

In [21], Pawlowski defines *the pointed chromatic symmetric function* $X_{G,v} \in \mathrm{Sym}[t]$ for a graph G with distinguished vertex v. From [10, Theorem 3.6] and [21, Definition 3.1], one can show that the injective linear map

$$\begin{aligned} \eta : \mathrm{Sym}[t] &\to \mathrm{UBCSym} \\ p_\lambda t^j &\mapsto p_{(\lambda,j+1)} \end{aligned}$$

sends each $X_{G,v} \mapsto y_{G:v}$. That is, results for $X_{G,v}$ lift to results for $y_{G:v}$, and vice-versa. For example, the proof of Theorem 35 says that there is a linear map $\vartheta\eta : \mathrm{Sym}[t] \to \mathrm{Sym}$ sending each $X_{G,v} \mapsto X_{G-v}$ by evaluating at $t = 0$.

Pawlowski studied the expansion of $X_{G,v}$ in the basis of $\mathrm{Sym}[t]$ consisting of *pointed Schur functions*, first considered by Strahov in [27]. By applying η, one can construct a basis of Schur functions for UBCSym and define (s)-positivity and appendable (s)-positivity in a way analogous to our definitions of (e)-positivity and appendable (e)-positivity. Then [21, Theorem 3.15] equivalently states that the paths P_n for $n \geqslant 1$ are appendable (s)-positive. It is known that the (s)-positivity of $X_{G,v}$ implies the Schur-positivity of X_G. It can be shown that X_{G-v} is also Schur-positive, by applying the map $\vartheta\eta$ together with [27, Theorem 6.4.1].

Another basis of $\mathrm{Sym}[t]$ considered by Pawlowski is the basis of *pointed elementary symmetric functions*. One can deduce by [21, Definition 4.1] and [21, Theorem 4.6] that the image of a pointed elementary symmetric function under η is an element in UBCSym of the form $\frac{1}{\lambda!(b-1)!}e_{(\lambda,b)}$. (Note that the same scaling coefficients arise in the proof of Theorem 35 and in Theorem 42.) Therefore, $X_{G,v}$ is pointed e-positive if and only if $y_{G:v}$ is (e)-positive, and Pawlowski's [21, Conjecture 2] is equivalent to Conjecture 18. Additionally, by [21, Corollary 4.3], any (e)-positive function is necessarily (s)-positive, so for any conjecture of (e)-positivity, one can also study the weaker conjecture of (s)-positivity for the same functions.

From the framework introduced in this paper, there are various further avenues of study, especially as possible approaches to the Stanley-Stembridge conjecture. It would be interesting if there existed a q-analogue of Conjecture 18. Naively defining a chromatic quasisymmetric function centred at a vertex by a construction similar to [23, Definition 1.2] gives a function in UBCQSym[q] but not UBCSym[q] even just for the labelled unit interval graph K_2. It may, however, be possible to define such a function implicitly for labelled unit interval graphs at their rightmost vertex by requiring the function satisfy a q-analogue of the relations in Proposition 20 and taking the function evaluated at $K_{\lambda_1} \mid \cdots \mid K_{\ell(\lambda)} \mid K_b$ to be $\frac{[\lambda_1]_q! \cdots [\lambda_{\ell(\lambda)}]_q! [b]_q!}{\lambda! b!} e_{(\lambda,b)}$. Abreu and Nigro showed something similar holds for the chromatic quasisymmetric functions of labelled unit interval graphs in [1, Theorem 1.1].

In the first part of [7, Conjecture 6.1], Dahlberg, She and van Willigenburg conjectured that for any connected labelled unit interval graph G, if one constructs a second labelled unit interval graph G' by following a certain procedure, then $G \geqslant_e G'$ in the chromatic e-positivity poset. In [7, Remark 6.3], they note that this conjecture would imply the Stanley-Stembridge conjecture. It may be fruitful to also study either a q-analogue of this newer conjecture or a version of it in UBCSym (or a version combining the two, if an appropriate definition for the chromatic quasisymmetric function centred at a vertex exists for labelled unit interval graphs at their rightmost vertex). The relations in Corollary 8 and Proposition 20, as well as their q-analogues, can help reduce the number of cases needed to prove various versions of the conjecture of Dahlberg, She and van Willigenburg.

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