Self-Avoiding Walks on Cayley Graphs Through the Lens of Symbolic Dynamics

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Abstract

We study dynamical and computational properties of the set of bi-infinite self-avoiding walks on Cayley graphs, as well as ways to compute, approximate and bound their connective constant. To do this, we introduce the skeleton $\mathbb{X}_{G,S}$ of a finitely generated group G relative to a generating set S, which is a one-dimensional subshift made of configurations on S that avoid all words that reduce to the identity. We provide a characterization of groups which have SFT skeletons and sofic skeletons: first, there exists a finite generating set S such that $\mathbb{X}_{G,S}$ is a subshift of finite type if and only if G is a plain group; second, there exists S such that $\mathbb{X}_{G,S}$ is sofic if and only if G is a plain group, $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty\times\mathbb{Z}/2\mathbb{Z}$. We also characterize finitely generated torsion groups as groups whose skeletons are aperiodic.

For connective constants, using graph height functions and bridges, we show that Cayley graphs of finitely generated torsion groups do not admit graph height functions, and that for groups that admit transitive graph height functions, the connective constant is equal to the growth rate of periodic points of the skeleton. Finally, we take a brief look at the set of bi-infinite geodesics and introduce an analog of the connective constant for the geodesic growth.

Mathematics Subject Classifications: 37B10, 20F10, 05C25, 68Q45

1 Introduction

In this article we study bi-infinite self-avoiding walks on Cayley graphs of finitely generated groups from the point of view of symbolic dynamics and group theory. A self-avoiding walk is a path on a graph that visits a vertex at most once. Figure 1 shows an example of a self-avoiding walk on the hexagonal grid. These walks were originally introduced by Flory for the study of long-chain polymers [19]. Although the original setting was the infinite square grid, self-avoiding walks are more generally studied in the context of infinite quasi-transitive graphs, intersecting with areas such as combinatorics, probability and statistical

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physics. The fundamental problem in this area is the study of the asymptotic growth rate of the number of self-avoiding walks of a given length, called the connective constant. See [30] for a recent survey on this problem. Recently, there has been increasing interest in the study of the set of all self-avoiding walks on edge-labeled graphs from the point of view of formal language theory [51, 46]. We take this study further by focusing on both bi-infinite self-avoiding walks and bi-infinite geodesics on Cayley graphs of finitely generated groups.

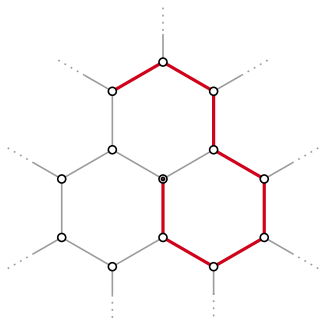


Figure 1: A self-avoiding walk, marked in red, on the hexagonal grid.

Given a finitely generated group G, and a symmetric finite generating set S, we introduce the skeleton of G with respect to S, denoted $\mathbb{X}_{G,S}$, as the set of labels of bi-infinite self-avoiding walks on the Cayley graph $\Gamma(G,S)$. The name, *skeleton*, comes from the study of the decidability of tiling problems on groups, particularly the infinite snake problem [4]. The skeleton $\mathbb{X}_{G,S}$ is equivalently defined as the set of bi-infinite words $x \in S^{\mathbb{Z}}$ that contain no factors representing the identity in G (see Sections 2.3 and 2.4 for formal definitions). This set admits a $\mathbb{Z}$-action through the shift operation, making it a symbolic dynamical system, commonly refered to as a subshift. Our present goal is to establish connections between the dynamical properties $\mathbb{X}_{G,S}$, and geometric and algebraic properties of the underlying group and Cayley graph. We will explore how the skeleton provides a way to translate results from groups to shift spaces.

The article is divided into three parts. First, we look at the relation between the skeletons' dynamical properties and the underlying group's properties. We study which groups admit skeletons that are subshifts of finite type, sofic, effective, or minimal, and study their periodic points. We completely characterize the first two properties and provide partial results for the rest. Second, we look at the skeleton's entropy. Using a result independently established by Rosenfeld [57] for subshifts and Grimmet et al. [24] for self-avoiding walks, we show the topological entropy of the skeleton is given by

$$h(\mathbb{X}_{G,S}) = \log(\mu(G,S)),$$

where $\mu(G,S)$ is the connective constant of the Cayley graph $\Gamma(G,S)$. This answers one of the questions from Problem 108 posed by Rufus Bowen in his notebook of problems [9]. With this connection, we give some results on the approximation of connective constants

and the existence of graph height functions. Lastly, we focus our attention on a subset of the skeleton composed of all bi-infinite geodesics, which we call the geodesic skeleton. We study its dynamic properties and introduce a geodesic analog of the connective constant.

Characterizing Classes of Subshifts

An important class of subshifts in the theory of symbolic dynamics is the class of subshifts of finite type. These subshifts are defined as sets of bi-infinite sequences that avoid a finite set of forbidden patterns. In this article, we completely classify groups that admit a skeleton that is a subshift of finite type.

Theorem A. *Let G be a finitely generated group. Then, there exists a finite generating set S such that $\mathbb{X}_{G,S}$ is an SFT if and only if G is a plain group.*

Plain groups are a subset of virtually free groups defined as free products of a finite number of finite groups and a free group (Definition 25). The connective constants of some Cayley graphs of plain groups were previously studied by Gilch and Müller [22]. We refine their result by showing that, when the skeleton is a subshift of finite type, the connective constant is a non-negative rational power of a Perron number.

A bigger class of subshifts is the class of sofic shifts. A subshift is said to be sofic if the language of its finite factors is regular. For the skeleton, this is equivalent to the language of labels of bi-infinitely extendable self-avoiding walks being regular. Lindorfer and Woess showed that the set of labels of finite length self-avoiding walks on a connected quasi-transitive locally finite deterministically-labeled infinite graph is regular if and only if the graph has only thin ends, all of them of size 1 [51]. For Cayley graphs, through a theorem of Haring-Smith (Theorem 27), this can be shown to be exactly the class of plain groups. Nevertheless, when working with bi-infinitely extendable self-avoiding walks, there exist Cayley graphs (in particular the bi-infinite ladder graph) with ends of size 2 where the language of bi-infinite extendable walks is regular, i.e. its skeleton $\mathbb{X}_{G,S}$ is sofic. We classify all groups that admit a Cayley graph for which the skeleton is sofic.

Theorem B. *Let G be a finitely generated group. There exists S such that $\mathbb{X}_{G,S}$ is sofic if and only if G is a plain group, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$.*

This result should be contrasted with Proposition 47 that shows that every group admits a generating set such that the skeleton is not sofic. In other words, the property of having a sofic skeleton is dependent on the Cayley graph. As is the case for subshifts of finite type, when the skeleton is sofic its connective constant is a non-negative rational power of a Perron number.

Lastly, from our study of effective skeletons, we provide an alternative proof of the existence of effective one-dimensional subshift that has no computable configurations (Proposition 36).

Connective Constants and Graph Height Functions

The next part of the article is concerned with finding ways to approximate connective constants through periodic configurations and finding lower bounds when they are not available.

We show that if a Cayley graph $\Gamma(G, S)$ admits a graph height function (see Definition 58), then the skeleton contains periodic points. As shown in [4], torsion groups can be characterized as groups whose skeletons never contain periodic points, independently of the generating set. This allows us to state the following result.

Theorem C. *The Cayley graphs of infinite torsion finitely generated groups do not admit graph height functions.*

This theorem generalizes a result by Grimmet and Li who showed the Grigorchuk group does not admit graph height functions, and more generally, Cayley graphs of torsion groups with certain conditions on the stabilizer of the identity [28].

In contrast, using the General Bridge Theorem by Lindorfer [50], we can extend a result of Clisby [14] on the approximation of the connective constant through periodic self-avoiding walks for the square lattice, to groups admitting a particular type of graph height function.

Theorem D. *Let G be a finitely generated group and S a finite generating set. If $\Gamma(G, S)$ admits a graph height function (h, H) such that H acts transitively on $\Gamma(G, S)$, then*

$$\mu(G, S) = \lim_{n\to\infty} \sqrt[n]{e_n},$$

where e_n denotes the number of periodic points of period $n \in \mathbb{N}$ in $\mathbb{X}_{G,S}$.

The next step is to look at torsion groups to find ways of approximating their connective constants without the use of periodic points. To do this, we use a counting method popularized by Rosenfeld [56, 57], that provides us with a method to find lower bounds on any graph by finding solutions to an inequality dependent on the number of simple cycles on the graph (see Proposition 65).

Bi-infinite Geodesics

A sub-class of self-avoiding walks that are of special interest are geodesics. A geodesic is a shortest path between two points on the Cayley graph. We define the geodesic skeleton, $\mathbb{X}^g_{G,S}$ of a group G with respect to the generating set S as the set of all bi-infinite geodesics on the corresponding Cayley graph. We obtain partial results on the classification of groups that admit geodesic skeletons that are subshifts of finite type, sofic and effective. In addition, we show that the characterization of torsion groups through periodic points also holds in this case.

Theorem E. *Let G be a finitely generated group. The following are equivalent,*

- *G is a torsion group,*
- *$\mathbb{X}_{G,S}$ is aperiodic for all (some) generating sets S,*
- *$\mathbb{X}^g_{G,S}$ is aperiodic for all (some) generating sets S.*

Finally, we introduce a geodesic analog of the connective constant. If we take $\Gamma_{G,S}$ the geodesic growth function of G with respect to S, we define the geodesic connective constant of a Cayley graph as the limit,

$$\mu^g(G,S) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{G,S}(n)}.$$

The geodesic growth of groups has been extensively studied, especially in the case of virtually nilpotent groups [10, 7, 6, 8]. As was the case with the skeleton, the entropy of the geodesic skeleton is the logarithm of the geodesic connective constant. We are able to explicitly compute the geodesic connective constants for lattices with known (or well approximated) connective constants such as the square grid, ladder graph ($\mathbb{L}$) and hexagonal grid ($\mathbb{H}$):

- $\mu^g(\mathbb{Z}^2) = 2$,
- $\mu^g(\mathbb{L}) = 1$,
- $\mu^g(\mathbb{H}) = \sqrt{2}$.

Structure of the Article The paper is organized as follows. Section 2 is devoted to definitions and background on symbolic dynamics, combinatorial group theory ans self-avoiding walks. Section 3 surveys general properties of the skeleton subshift, and shows how its entropy corresponds to the logarithm of the connective constant of the corresponding Cayley graph. In Section 4 we investigate how dynamical and computational properties of the skeleton subshift – existence of periodic configurations in $\mathbb{X}_{G,S}$, minimality of $\mathbb{X}_{G,S}$, being SFT or effective – relate to properties on the group G itself. Next, in Section 5 we provide a characterization of groups that admit sofic skeletons. To do this we also introduce notions from the study of thin and thick ends of graphs, and automorphisms of graphs. In Section 6 we use the skeleton to get new results on entropy and connective constant. We begin by looking at graph height functions and bridges, and their relation to periodicity in the skeleton. Then, we use Rosenfeld's counting method to provide lower bounds on the connective constant based on the number of simple cycles on the Cayley graph of a given length. Finally, Section 7 is devoted to the study of the geodesic skeleton and the geodesic connective constant.

2 Background and Definitions

Given an alphabet A, we denote by A^n the set of words on A of length n, $A^{\leqslant n}$ the set of words of length at most n, and A^* the set of all finite length words including the empty word ϵ. Furthermore, we denote by $A^+ = A^* \setminus \{\epsilon\}$ the set of non-empty finite words over A. A factor v of a word w is a contiguous subword of w; we denote this by $v \sqsubseteq w$. For a bi-infinite word $x \in A^{\mathbb{Z}}$, given $i, j \in \mathbb{Z}$, $x_{[i,j]}$ denotes the word $x_i x_{i+1} \cdots x_j$, $x_{[j,+\infty)}$ the infinite word stating at j, and $x_{(-\infty,i]}$ the infinite word finishing at i. For a word $w \in A^*$, the expression w^∞ denotes the infinite word obtained by repeating w. We denote the free group defined by the free generating set of size n by $\mathbb{F}_n$, and $\mathbb{F}_S$ the free group generated by S. The commutator of two group elements g, h is denoted by $[g, h] = ghg^{-1}h^{-1}$.

2.1 Symbolic Dynamics

Given a finite alphabet A, we define the **full-shift** over A as the set of maps $A^{\mathbb{Z}} = \{x : \mathbb{Z} \to A\}$. We call maps $x : \mathbb{Z} \to A$ **configurations**. There is a natural $\mathbb{Z}$-action on the full-shift called the **shift**, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, given by $\sigma(x)_i = x_{i+1}$. The full-shift is also endowed with the prodiscrete topology, making it a compact space.

Let $F \subseteq \mathbb{Z}$ be a finite connected subset. A **pattern** of support F is an element $p \in A^F$. We say a pattern $p \in A^F$ appears in a configuration $x \in A^{\mathbb{Z}}$ if there exists $k \in \mathbb{Z}$ such that $x_{k+i} = p_i$ for all $i \in F$. Given a set of patterns $\mathcal{F}$, we define the set of configurations where no pattern from $\mathcal{F}$ appears as,

$$\mathcal{X}_{\mathcal{F}} = \{x \in A^{\mathbb{Z}} \mid \forall p \in \mathcal{F},\ p \text{ does not appear in } x\}.$$

A **subshift** is a subset of the full-shift $X \subseteq A^{\mathbb{Z}}$ such that there exists a set of patterns $\mathcal{F}$ that verifies $X = \mathcal{X}_{\mathcal{F}}$. Subshifts are equivalently defined as closed σ-invariant subsets of the full-shift. We say a subshift X is

- a **subshift of finite type** (SFT) if there exists a finite $\mathcal{F}$ such that $X = \mathcal{X}_{\mathcal{F}}$,
- **sofic** if there exists a regular $\mathcal{F}$ such that $X = \mathcal{X}_{\mathcal{F}}$,
- **effective** if there exists a decidable $\mathcal{F}$ such that $X = \mathcal{X}_{\mathcal{F}}$.

Each class is strictly contained in the next.

The **language** of a subshift X, denoted $\mathcal{L}(X)$, is defined as the set of all contiguous patterns that appear within some configuration from X. Formally,

$$\mathcal{L}(X) = \{w \in A^* \mid \exists x \in X, w \sqsubseteq x\}.$$

Any subshift can be defined by taking the complement of its language as forbidden patterns, that is, $X = \mathcal{X}_{\mathcal{L}(X)^c}$. Furthermore, a subshift is sofic if and only if its language is regular. Similarly, for a set of forbidden patterns we define the language of **locally admissible patterns**, $\mathcal{L}_{loc}(\mathcal{F})$, as the set of words $w \in A^*$ which contain no patterns

from $\mathcal{F}$. Notice that $\mathcal{L}(\mathcal{X}_{\mathcal{F}}) \subseteq \mathcal{L}_{loc}(\mathcal{F})$, but is often not equal, and $\mathcal{X}_{\mathcal{F}} = \mathcal{X}_{\mathcal{L}_{loc}(\mathcal{F})^c}$.

An important measure of the combinatorial properties of the subshift is its **complexity**, p_X, defined as $p_X(n) = |\mathcal{L}(X) \cap A^n|$, which counts the amount of words of a given length present in the subshift's language. Because the complexity is a submultiplicative function, that is, $p_X(m+n) \leqslant p_X(m)p_X(n)$, Fekete's Lemma allows us to define the **asymptotic word growth rate** as

$$\alpha^{\infty}(X) = \lim_{n\to\infty} \sqrt[n]{p_X(n)}.$$

Analogously, we define the function $q(n) = |\mathcal{L}_{loc}(\mathcal{F}) \cap A^n|$, which is also submultiplicative, and thus $\alpha(\mathcal{F}) = \lim_{n\to\infty} \sqrt[n]{q(n)}$ exists. Rosenfeld showed this quantity is equal to the asymptotic word growth rate.

Lemma 1 ([57])**.** *For $\mathcal{F} \subseteq A^+$, $\alpha(\mathcal{F}) = \alpha^{\infty}(\mathcal{X}_{\mathcal{F}})$.*

An important quantity in symbolic dynamics is the (topological) **entropy** of the subshift, defined as

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log(p_X(n)) = \log(\alpha^{\infty}(X)).$$

For example, in the case of the full-shift on A, $h(A^{\mathbb{Z}}) = \log(|A|)$. Entropy informs many dynamical properties of the subshift, and is invariant under shift-commuting continuous bijections. Furthermore, for two subshifts X and Y such that $Y \subseteq X$, $h(Y) \leqslant h(X)$. A classical result by Lind [48] shows that the entropies of SFTs and sofic subshifts are exactly the set of non-negative rational multiple of Perron numbers. In contrast, the set of entropies of effective subshifts is the set of right computable real numbers [38].

We say a configuration $x \in X$ is **periodic** if there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $\sigma^k(x) = x$. We say the subshift X is **aperiodic** if it contains no periodic configurations. If a non-empty subshift is sofic, it always contains periodic configurations.

We say a subshift is **minimal** if it does not contain non-empty proper subshifts. Equivalently, a subshift is minimal if every orbit under the shift action is dense in the subshift. Finally, a minimal subshift with a periodic configuration is always finite.

For a comprehensive introduction to one-dimensional symbolic dynamics we refer the reader to [49], were proof of our assertions can be found.

2.2 Combinatorial Group Theory

Let G be a finitely generated (f.g.) group and S a finite generating set. In this article we will only consider finite **symmetric** generating sets, that is, generating sets that verify $S = S^{-1}$, that never contain the identity. Elements in the group are represented as words on the alphabet S through the evaluation function $w \mapsto \overline{w}$. Two words w and v represent the same element in G when $\overline{w} = \overline{v}$, and we denote this by $w =_G v$. We say a word is

reduced if it contains no factor of the form ss^{-1} or $s^{-1}s$ with $s \in S$. We denote the identity of a group G by 1_G.

Definition 2. Let G be a group. We say (S, R) is a **presentation** of G, denoted $G = \langle S \mid R \rangle$, if the group is isomorphic to $\langle S \mid R \rangle = \mathbb{F}_S / \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the normal closure of R, i.e. the smallest normal subgroup containing R. We say G is **recursively presented** if there exists a presentation (S, R) such that S is finite and R is recursively enumerable.

For a group G and a generating set S, we define:

$$\mathrm{WP}(G, S) = \{w \in S^* \mid w =_G \epsilon\}.$$

Definition 3. The **word problem** of a group G with respect to a set of generators S asks to determine, given a word $w \in S^*$, if $w \in \mathrm{WP}(G, S)$.

We say a word $w \in S^+$ is G**-reduced** if w contains no factor in $\mathrm{WP}(G, S)$. We say a word $w \in S^*$ is a **geodesic** if for all words $v \in S^*$ such that $\overline{w} = \overline{v}$ we have $|w| \leqslant |v|$. For a given group G and generating set S, we denote its **language of geodesics** by $\mathrm{Geo}(G, S)$. The length of an element $g \in G$ with respect to S is defined as $\|g\|_S = |w|$ where w is any geodesic representing g. This length also defines a G-invariant metric $d_S : G \times G \to \mathbb{N}$ given by $d_S(g, h) = \|g^{-1}h\|_S$.

Definition 4. Let G be a finitely generated group, with generating set S. The **growth function** $\gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ of G with respect to S is defined for a given $n \in \mathbb{N}$ as the amount of elements of length at most n. In other words, $\gamma_{G,S}(n) = |\{g \in G \mid \|g\|_S \leqslant n\}|$.

We say an element $g \in G$ has **torsion** if there exists $n \geqslant 1$ such that $g^n = 1_G$. If there is no such n, we say g is **torsion-free**. Analogously, we say G is a **torsion group** if all of its elements have torsion. Otherwise, if the only torsion element is the identity, we say the group is **torsion-free**.

The **free product** of two groups G and H given by presentations $\langle S_G \mid R_G \rangle$ and $\langle S_H \mid R_H \rangle$ respectively, is the group given by the presentation,

$$G * H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle.$$

Finally, let $\mathcal{P}$ be a class of groups (for example abelian groups, free groups, etc). We say a group G is **virtually** $\mathcal{P}$, if there exists a finite index subgroup $H \leqslant G$ that is in $\mathcal{P}$.

2.3 The Skeleton, Infinite Domino Snakes and Analogies between Groups and Subshifts

Let G be a finitely generated group with S a set of generators. We define the **skeleton** of G with respect to S as the subshift,

$$\mathbb{X}_{G,S} = \{x \in S^{\mathbb{Z}} \mid \forall w \sqsubseteq x,\ w \notin \mathrm{WP}(G, S)\} = \mathcal{X}_{\mathrm{WP}(G,S)}.$$

In other words, $\mathbb{X}_{G,S}$ is the set of all bi-infinite words that do not contain factor that evaluate to the group's identity. This subshift was originally defined in the context of the **infinite snake problem** [4], where its properties where shown to inform the decidability of the problem. In particular, if $\mathbb{X}_{G,S}$ is sofic, then the infinite snake problem for (G, S) is decidable.

Example 5. Take $\mathbb{Z}^2$ with its standard generating set $S = \{\mathtt{a}^{\pm}, \mathtt{b}^{\pm}\}$. Its skeleton is given by

$$\mathbb{X}_{\mathbb{Z}^2,S} = \{x \in S^{\mathbb{Z}} \mid \forall w \sqsubseteq x,\ |w|_{\mathtt{a}} \neq 0 \ \vee \ |w|_{\mathtt{b}} \neq 0\},$$

where $|w|_s$ is the sum of exponents of the generator s.

Example 6. Let $\mathcal{D}_\infty$ be the infinite dihedral group. The skeletons of this group can be radically different depending on the generating set. For instance, if we take the presentation

$$\mathcal{D}_\infty = \langle \mathtt{a}, \mathtt{b} \mid \mathtt{a}^2, \mathtt{b}^2 \rangle,$$

the corresponding skeleton is the finite subshift $\{(\mathtt{ab})^\infty, (\mathtt{ba})^\infty\}$. On the other hand, if we take the presentation,

$$\mathcal{D}_\infty = \langle \mathtt{r}, \mathtt{s} \mid \mathtt{s}^2, \mathtt{srsr} \rangle,$$

the skeleton is infinite: for every $n \in \mathbb{Z}$ it contains a configuration x defined by $x(n) = \mathtt{s}$ and $x(k) = \mathtt{r}$ for all $k \neq n$.

The skeleton is also present in Rufus Bowen's notebook of problems [9, Problem 108], where he asks what can be said about $\mathbb{X}_{G,S}$, what is its entropy and if it is intrisically ergodic. In Section 3.2 we tackle the second question. Further still, this subshift is inserted in the larger project of understanding the analogies between multidimensional subshifts and finitely generated groups. Jeandel and Vanier observed [44] that the forbidden patterns of a subshift play a similar role to relations of a group. A summary of this comparison is shown in Table 1. These comparison has been further strengthened

Group	Subshift
Group with n generators	Subshift on n symbols
Word problem $\mathrm{WP}(G)$	co-language $\mathcal{L}(X)^c$
Finitely presented group	SFT
Recursively presented group	Effectively closed subshift
Simple group	Minimal subshift
H is a quotient of G	$Y \subseteq X$

Table 1: A part of the Jeandel-Vanier dictionary between groups and subshifts as introduced in [44]

through results such as Higman and Boone-Higman Theorems for subshifts [44].

The skeleton is an attempt to establish these analogies explicitly, by using the generators as an alphabet, and $\mathrm{WP}(G, S)$ as the set of forbidden patterns. We will see that in this case some of these analogies hold and some do not.

2.4 Cayley Graphs and Self-Avoiding Walks

Let G be a finitely generated group along with a finite symmetric generating set S. The **Cayley graph** of G with respect to S, denoted $\Gamma(G,S)$, is defined by the set of vertices $V_\Gamma = G$ and the set of labeled edges $E_\Gamma = \{(g,s,gs) \mid g \in G, s \in S\} \subseteq G \times S \times G$. Each edge $e = (g,s,h) \in E_\Gamma$ has an initial vertex $\mathfrak{i}(e) = g$, a terminal vertex $\mathfrak{t}(e) = h$ and a label $\lambda(e) = s$. The graph is also endowed with an involution $e \mapsto e^{-1} = (h, s^{-1}, g) \in E_\Gamma$. If a generator has order 2, that is, if $s \in S$ satisfies $s^2 = 1_G$, we take a unique edge between g and gs for every $g \in G$. Notice that every Cayley graph is $|S|$-regular, locally finite, transitive and deterministically labeled, that is, for every vertex there is a unique out-going edge for each label S. The group G acts by translation on $\Gamma(G,S)$ by left multiplication, in other words, the action of $g \in G$ over a vertex $h \in V_\Gamma$ is given by $g \cdot h = gh$. Through this action, we can identify G with a subgroup of the automorphism group of the Cayley graph.

We also consider the **undirected Cayley graph** $\hat{\Gamma}(G,S)$, where we collapse each edge e and e^{-1} to a single undirected edge between $\mathfrak{i}(e)$ to $\mathfrak{t}(e)$. In other words, $\hat{\Gamma}(G,S)$ is the graph with vertex set G such that $g,h \in G$ are adjacent if $gh^{-1} \in S$.

Example 7. The hexagonal grid $\mathbb{H}$ is the Cayley graph of the affine Coxeter group $\tilde{A}_2$ given by the presentation,

$$\tilde{A}_2 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^3, (\mathtt{ac})^3, (\mathtt{bc})^3 \rangle.$$

This can be seen in Figure 2.

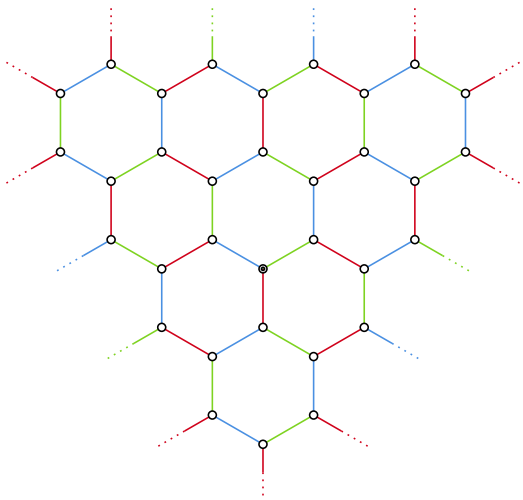


Figure 2: A Cayley graph of the affine Coxeter group $\tilde{A}_2$. The red edges represent $\mathtt{a}$, blue edges represent $\mathtt{b}$, and green edges $\mathtt{c}$.

Example 8. The ladder graph $\mathbb{L}$ is the Cayley graph of $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the presentation,

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle \mathtt{t}, \mathtt{s} \mid \mathtt{s}^2, \mathtt{tst}^{-1}\mathtt{s} \rangle.$$

Figure 3: A Cayley graph of the group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The generator $\mathtt{t}$ defines the horizontal right-pointing edges, and the generator $\mathtt{s}$ defines the vertical undirected edges.

This can be seen in Figure 3. This is not the only group that admits the ladder graph as a Cayley graph, this is also the case for the groups $\mathcal{D}_\infty$ and $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$.

A **path** on $\Gamma(G, S)$ is a sequence of edges $\pi = (e_1, \ldots, e_n)$ such that for all $i \in \{1, \ldots, n-1\}$ we have $\mathfrak{i}(e_{i+1}) = \mathfrak{t}(e_i)$. We denote the initial vertex of the path by $\mathfrak{i}(\pi) = \mathfrak{i}(e_1)$ and its terminal vertex as $\mathfrak{t}(\pi) = \mathfrak{t}(e_n)$. The length of the path is given by $\ell(\pi) = n$, and its label is $\lambda(\pi) = \lambda(e_1) \cdots \lambda(e_n) \in S^*$. We also define the sequence of vertices visited by π as the sequence $V(\pi) = (g_0, \ldots, g_n)$ with $g_i = \mathfrak{i}(e_{i+1})$ for all $i \in \{0, \ldots, n-1\}$ and $g_n = \mathfrak{t}(e_n)$. This formalism gives us a one-to-one correspondence between paths starting at 1_G and words in S^*. In particular, a path π satisfies $\mathfrak{i}(\pi) = \mathfrak{t}(\pi)$ if and only if $\lambda(\pi) \in \mathrm{WP}(G, S)$.

A path π is a **self-avoiding walk** (SAW) if it never visits the same vertex twice. We define the language of self-avoiding walks over $\Gamma(G, S)$ as the set

$$L_{SAW}(G, S) = \{\lambda(\pi) \mid \pi \text{ is a SAW with } \mathfrak{i}(\pi) = 1_G\}.$$

Remark that the language remains the same if we change the initial vertex from the identity to any other group element because the graph is transitive. Furthermore, because Cayley graphs are deterministically labeled, no two SAWs share the same label. A **bi-infinite SAW** centered at $g \in G$ is a sequence of edges $\pi = (e_i)_{i \in \mathbb{Z}} \in E^{\mathbb{Z}}$ such that $\mathfrak{i}(e_{i+1}) = \mathfrak{t}(e_i)$, and $g = \mathfrak{i}(e_0)$ such that π never visits the same vertex twice. We can thus state the following.

Lemma 9. *Let G be a group and S a generating set. Then,*

$$\begin{aligned} \mathbb{X}_{G,S} &= \{\lambda(\pi) \in S^{\mathbb{Z}} \mid \pi \textit{ is a bi-infinite SAW centered at } 1_G\}, \\ &= \mathcal{X}_{L_{SAW}(G,S)^c}. \end{aligned}$$

Moreover, $\mathcal{L}_{loc}(\mathbb{X}_{G,S}) = L_{SAW}(G, S)$.

Proof. This is a direct consequence of the definitions. □

Once again, as the graph is transitive, we can change the center for any other element of the group. Notice that any finite subwalk of a bi-infinite SAW is a SAW. The converse is not always true, that is, there are SAWs that do not appear in any bi-infinite SAW (see Figure 4).

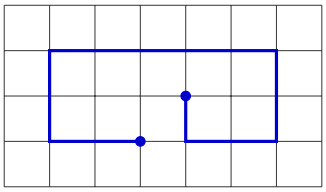


Figure 4: On $\mathbb{Z}^2$ with standard generating set: a finite SAW that does not appear in a bi-infinite SAW.

Remark 10. For the undirected Cayley graph $\hat{\Gamma}(G,S)$ we can define a labelling function λ' from the set of self-avoiding walks over $\hat{\Gamma}(G,S)$ to S^*. Take a self-avoiding walk π that passes through the sequence of vertices $(g_0, g_1, \ldots, g_n)$, its label $\lambda'(\pi) = s_0 \cdots s_{n-1}$ is such that $s_i = g_i^{-1} g_{i+1}$. This way, both $L_{\mathrm{SAW}}(G,S)$ and $\mathbb{X}_{G,S}$ are equal to the corresponding labels of self-avoiding walks on the *undirected* Cayley graph. Because of this, in what follows we do not distinguish between the directed and undirected Cayley graphs.

2.5 Connective Constants

Let c_n be the number of self-avoiding walks of length n in the Cayley graph $\Gamma(G,S)$ starting at the identity. This sequence is submultiplicative, i.e. $c_{n+m} \leqslant c_n c_m$ for all $m, n \in \mathbb{N}$, so that by Fekete's Lemma, the limit of $\sqrt[n]{c_n}$ exists:

$$\mu(G,S) = \lim_{n\to\infty} \sqrt[n]{c_n} = \inf_{n\in\mathbb{N}} \sqrt[n]{c_n} \in [1,\infty).$$

This limit is known as the **connective constant** of the Cayley graph. For general quasi-transitive graphs, this limit was proved to be independent of the starting vertex by Hammersly and Morton [35].

In general, connective constants are hard to compute. Nevertheless, the exact value of some connective constants is known. For instance, for the hexagonal grid (which as we saw, is a Cayley graph for $\tilde{A}_2$) its value is $\sqrt{2+\sqrt{2}}$ [15], for the bi-infinite ladder (as in Figure 3) it is the golden mean $\frac{1}{2}(1+\sqrt{5})$ [1], and for some Cayley graphs of free products of finite groups it is the zero of a polynomial [22]. On the other hand, giving a closed form for the connective constant of $\mathbb{Z}^2$ with standard generators is still an open problem. The best estimate as of writing is

$$\mu(\mathbb{Z}^2) \approx 2.63815853032790(3),$$

obtained by Jacobsen, Scullard, and Guttman [43].

There are, however, bounds on the connective constant. We translate the following results from Grimmet and Li – which were stated for larger classes of graphs – to our Cayley graph context.

Theorem 11 ([25, 26])**.** *Let G be an infinite finitely generated group and S a generating set.*

- $\mu(G,S) \geqslant \sqrt{|S|-1}$,
- *For* $w \neq_G 1_G$ *and* N *its normal closure in* G, $\mu(G/N,S) < \mu(G,S)$,
- *For* $g \notin S$ *a non-identity element of* G *and* $S' = S \cup \{g^{\pm}\}$, $\mu(G,S) < \mu(G,S')$.

For more bounds and details, see [30].

3 General Properties

Let us begin by establishing properties of the skeleton that are common for all groups and generating sets.

3.1 Bi-infinite SAWs through Group Elements and Computability

A first observation is that $\mathbb{X}_{G,S} = \varnothing$ for a generating set if and only if G is a finite group; this is a consequence of Konig's Lemma (see [61]). As we only consider infinite finitely generated groups, unless explicitly stated, the skeletons are never empty. Next, if $\pi = (e_i)_{i\in\mathbb{Z}}$ is a bi-infinite SAW on the Cayley graph $\Gamma(G,S)$, its inverse $\pi^{-1} = (e_i^{-1})_{i\in\mathbb{Z}}$ is also a bi-infinite SAW. Therefore, for each configuration $x \in \mathbb{X}_{G,S}$, its inverse configuration $x^{-1}(k) = x(k)^{-1}$ belongs to $\mathbb{X}_{G,S}$.

Next, as shown in Figure 4, the word $w = a^{-2}b^2a^5b^{-2}a^{-2}b$ in $\mathbb{Z}^2$ avoids factors from the word problem but does not define a configuration. Nevertheless, the word ab defines the same group element, and can be extended to a complete configuration. Is this always the case? Is it true that for every non-trivial group element there is a word evaluating to it in $\mathcal{L}(\mathbb{X}_{G,S})$? For every finitely generated group G and every group element $g \in G$, one can find a generating set S such that this is true (Proposition 12).

Proposition 12. *Let* G *be a finitely generated group. Then there exists* S *a generating set for* G *such that for every non trivial group element* $g \in G$*, there exists a word* $w \in S^*$ *such that* $\overline{w} = g$ *and* $w \in \mathcal{L}(\mathbb{X}_{G,S})$.

Proof. A theorem by Seward [58, Theorem 1.8] states that for every finitely generated group G, there exists a finite generating set S such that the Cayley graph $\Gamma(G,S)$ has a regular spanning tree. In particular, this tree has no leaves, thus each path leading to a vertex can be continued (see Figure 5). Consider such an S and one associated regular spanning tree of $\Gamma(G,S)$. Take a non trivial element $g \in G$ and consider the path connecting the identity 1_G to g in the regular spanning tree. Then this finite simple path can be extended to an infinite simple path inside the spanning tree, leading to an infinite simple path going through 1_G and g. Translating this path into a bi-infinite sequence of elements in S give a configuration of the skeleton $\mathbb{X}_{G,S}$. □

This can be made more precise in the case of one and two ended groups. Cayley graph $\Gamma = \Gamma(G,S)$ has k ends if k is the supremum of the number of infinite connected components of the induced subgraph $\Gamma[V_\Gamma \setminus A]$ over every finite subset $A \subseteq V$ (see Section 5.1).

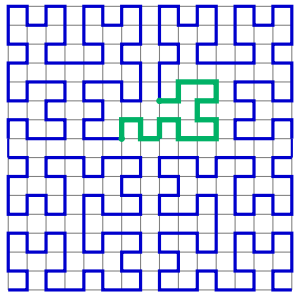


Figure 5: In blue, an example of regular spanning tree of degree 2 – thus a bi-infinite Hamiltonian path – for $\mathbb{Z}^2$ with its standard presentation. In green a path from $(0,0)$ to $(2,2)$ extracted from this regular spanning tree. Note that this path is highly non-geodesic.

Proposition 13. *Let G be a finitely generated group with one or two ends, with S a generating set. Define $S' = \{g \in G \mid \|g\|_S \leqslant 3\}$. Then, for every non trivial group element $g \in G$, there exists a word $w \in (S')^*$ such that $\overline{w} = g$ and $w \in \mathcal{L}(\mathbb{X}_{G,S'})$.*

Proof. In [11, Theorem 1.3] Carrasco-Vargas showed that for one and two ended groups, Sewards' Theorem holds for S', that is, there is a Hamiltonian path on the Cayley graph $\Gamma(G, S')$. This directly implies our statement. □

By translating and joining Lemmas 4.4 and 3.7 from [11], we can state results on the decidability of the language of the skeleton for particular sets of generators.

Proposition 14. *Let G be a finitely generated group with one or two ends, with S a generating set. Suppose G has decidable word problem, and define $S' = \{g \in G \mid \|g\|_S \leqslant 3\}$. Then, $\mathcal{L}(\mathbb{X}_{G,S'})$ is computable.*

For a skeleton, having a computable language means that there is an algorithm that determines if a finite SAW is bi-infinitely extendable. A particular class of subshifts that have computable language are sofic subshifts. In Sections 4 and 5 we will explore when skeletons belong to this class.

3.2 Entropy

As seen in Lemma 9, the skeleton shift is the set of labels of bi-infinite SAWs over a Cayley graph. Consequently, its complexity function counts the number of infinitely bi-extendable SAWs of length n, with $\alpha^\infty(\mathbb{X}_{G,S})$ being their asymptotic growth rate. Furthermore, the number of locally admissible words of length n is exactly c_n, the number of finite SAWs of length n. Therefore, by Lemma 1 we obtain the following.

Lemma 15. *For a finitely generated group G and a generating set S*

$$h(\mathbb{X}_{G,S}) = \log(\mu(G, S)).$$

This equality can also be deduced from [24], where Grimmet et al. show that the connective constant is equal to the growth rate of infinitely bi-extendable SAWs.

Let (G, S) be a f.g. group and $\gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ its growth function. We define its asymptotic growth rate as the value,

$$\mathfrak{H}_{G,S} = \lim_{n\to\infty} \frac{1}{n} \log(\gamma_{G,S}(n)).$$

As the growth function is sub-multiplicative (see [12]), $\mathfrak{H}_{G,S}$ exists by Fekete's Lemma.

Remark 16. An alternative way to look at the growth of a group is the **strict growth function** $\sigma_{G,S}$, where $\sigma_{G,S}(n)$ is the number of elements of length *exactly* n. As is the case for the growth function, $\sigma_{G,S}$ is sub-multiplicative. Its asymptotic growth rate is the same as that of $\gamma_{G,S}$, namely $\mathfrak{H}_{G,S}$. This can be seen through their generating functions. Take $F, f : \mathbb{C} \to \mathbb{C}$ defined as

$$F(z) = \sum_{n\in\mathbb{N}} \gamma_{G,S}(n) z^n, \ f(z) = \sum_{n\in\mathbb{N}} \sigma_{G,S}(n) z^n.$$

By the Cauchy-Hadamard theorem we know that the asymptotic growth rate of $\gamma_{G,S}$ (resp. $\sigma_{G,S}$) is the reciprocal of the radius of convergence of F (resp. f). Because strict growth can be expressed as $\sigma_{G,S}(n) = \gamma_{G,S}(n) - \gamma_{G,S}(n-1)$, with the convention that $\gamma_{G,S}(-1) = 0$, we get that $f(x) = (1-x)F(x)$. As the term $(1-x)$ does not change the radius of convergence of the series, we have that

$$\lim_{n\to\infty} \frac{\log(\sigma_{G,S}(n))}{n} = \mathfrak{H}_{G,S}.$$

Proposition 17. *Let G be a f.g. group with S a finite generating set. Then,*

$$\mathfrak{H}_{G,S} \leqslant h(\mathbb{X}_{G,S}) \leqslant \log(|S| - 1).$$

Proof. Let $p(n)$ be the complexity function for $\mathbb{X}_{G,S}$ and $k = |S|$. For the upper bound, notice that the total number of reduced words of length n over S is exactly the number of elements of length n in free group $\mathbb{F}_m$, with $m = \lceil \frac{k}{2} \rceil$. Therefore,

$$p(n) \leqslant \gamma_{\mathbb{F}_m}(n) - \gamma_{\mathbb{F}_m}(n-1) = k(k-1)^{n-1}.$$

On the other hand, every element of length n has a geodesic representative of length n, which by definition is G-reduced. In particular, this representative is a SAW of length n. Thus, $\sigma_{G,S}(n) \leqslant c_n$ and

$$\mathfrak{H}_{G,S} = \lim_{n\to\infty} \frac{\log(\sigma_{G,S}(n))}{n} \leqslant \log(\mu(G, S)).$$ □

Remark 18. The bounds from Proposition 17 are tight in general, as free groups with free generating sets satisfy $\mathfrak{H}_{\mathbb{F}_m,S} = h(\mathbb{X}_{\mathbb{F}_m,S}) = \log(2m-1)$. Nevertheless, by Theorem 11 we know that for non-free groups $h(\mathbb{X}_{G,S}) < \log(2m-1)$ for all generating sets. This same theorem also tells us that for groups with polynomial growth the lower bound is strict, as

$$0 = \mathfrak{H}_{G,S} < \frac{1}{2} \log(|S| - 1) \leqslant h(\mathbb{X}_{G,S}).$$

Another straightforward bound we find from algebraic considerations is the following.

Proposition 19. *Take G a finitely generated group and S generating set. If $\{s_1, \ldots, s_n\} \subseteq S$ is a subset of generators such that there induced semigroup $\langle s_1, \ldots, s_n\rangle_+$ does not contain the identity, then $h(\mathbb{X}_{G,S}) \geqslant \log(n)$.*

Proof. If $\langle s_1, \ldots, s_n\rangle_+$ does not contain the identity, any combination of these generators will give a word that does not contain factors that evaluate to the identity. In other words, the skeleton contains the full-shift $\{s_1, \ldots, s_n\}^{\mathbb{Z}}$. Consequently, $h(\mathbb{X}_{G,S_n}) \geqslant \log(n)$. □

Example 20. Take $\mathbb{Z}^2$ with its standard generating set $\{\mathtt{a}^{\pm}, \mathtt{b}^{\pm}\}$. Then, the semigroup generated by $\mathtt{a}$ and $\mathtt{b}$ does not contain the identity. Then, $h(\mathbb{X}_{\mathbb{Z}^2,\{\mathtt{a}^{\pm},\mathtt{b}^{\pm}\}}) \geqslant \log(2)$. Similarly, if we take the discrete Heisenberg group H_3 with generating set $\{\mathtt{a}^{\pm}, \mathtt{b}^{\pm}, \mathtt{c}^{\pm}\}$ through the presentation,

$$H_3 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid [\mathtt{a}, \mathtt{c}], [\mathtt{b}, \mathtt{c}], [\mathtt{a}, \mathtt{b}]\mathtt{c}^{-1}\rangle,$$

the semigroup given by the three generators $\mathtt{a}, \mathtt{b}$ and $\mathtt{c}$ does not contain the identity. Then, by the previous proposition $h(\mathbb{X}_{H_3,\{\mathtt{a}^{\pm},\mathtt{b}^{\pm},\mathtt{c}^{\pm}\}}) \geqslant \log(3)$.

Remark 21. Given a group G, the entropy of its skeleton can be made arbitrarily large. This can be done be taking bigger and bigger generating sets and using the lower bound $\sqrt{|S| - 1}$ given by Theorem 11. This can also be done in torsion-free groups by taking a torsion-free element $g \in G$, a generating set containing $\{g, g^2, \ldots, g^n\}$ and using the previous proposition.

In Section 6 we will see methods to approximate entropy and connective constants for different classes of groups.

4 Dynamic and Computational Aspects

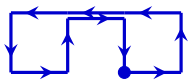
The goal of this section is to explore the multiple dynamical and computational properties of skeletons, and how they interact with the algebraic properties of the underlying group. We look at groups that admits SFT, sofic, effective or minimal skeletons, as well as their periodic points.

A subshift can be defined by various different sets of forbidden patterns. We saw, from its definition and Lemma 9, that $\mathbb{X}_{G,S}$ is defined by at least two different sets, namely $L_{SAW}(G, S)$ and $\mathrm{WP}(G, S)$. We begin by describing an additional set that will help us better understand the structure of forbidden patterns.

We begin by looking at the set of patterns that define simple cycles (also called embedded cycles) in the Cayley graph. We define the set of labels of simple cycles of a group G with respect to a finite generating set S as,

$$\begin{aligned}\mathcal{O}_{G,S} &= \{w \in \mathrm{WP}(G, S) \mid w \text{ defines a simple cycle in } \Gamma(G, S)\}\\ &= \{w \in \mathrm{WP}(G, S) \mid \forall w' \sqsubset w,\ w' \notin \mathrm{WP}(G, S)\}.\end{aligned}$$

Example 22. Consider $\mathbb{Z}^2$ with its standard presentation $\langle \mathtt{a}, \mathtt{b} \mid [\mathtt{a}, \mathtt{b}] \rangle$. Then the word $\mathtt{aba}^{-3}\mathtt{b}^{-1}\mathtt{abab}^{-1}$ is in $\mathrm{WP}(\mathbb{Z}^2, \{\mathtt{a}, \mathtt{b}\})$ but not in $\mathcal{O}_{\mathbb{Z}^2,\{\mathtt{a},\mathtt{b}\}}$ since there are repeated vertices in the path it represents in $\Gamma(\mathbb{Z}^2, \{\mathtt{a}, \mathtt{b}\})$.



We call elements of $\mathcal{O}_{G,S}$ **self-avoiding polygons** (SAPs) of the Cayley graph of $\Gamma(G, S)$.

Lemma 23. *Let $\mathcal{F} = \mathcal{O}_{G,S} \cup \{ss^{-1} \mid s \in S\}$. Then, $\mathbb{X}_{G,S} = \mathcal{X}_{\mathcal{F}}$.*

Proof. Since $\mathcal{O}_{G,S} \subseteq \mathrm{WP}(G, S)$ and $\{ss^{-1} \mid s \in S\} \subseteq \mathrm{WP}(G, S)$, we have that $\mathcal{F} \subseteq \mathrm{WP}(G, S)$. So, the subshifts defined by two sets respect the reciprocal inclusion, and we have $\mathbb{X}_{G,S} \subseteq \mathcal{X}_{\mathcal{F}}$.

Reciprocally, take some configuration $x \in \mathcal{X}_{\mathcal{F}}$ and assume it contains some pattern $w \in \mathrm{WP}(G, S)$. Without loss of generality we assume that $w = s_1 \dots s_n$ for some $n \in \mathbb{N}$, so that $\overline{s_1 \dots s_n} = 1_G$. Consider the group elements g_i defined by $g_i = \overline{s_1 \dots s_i}$ for $i \in \{1 \dots n\}$ and $g_0 = 1_G$. Since $x \in \mathcal{X}_{\mathcal{F}}$ the pattern w does not belong to $\mathcal{O}_{G,S} \cup \{ss^{-1} \mid s \in S\}$. So necessarily $n > 1$ and there are some repetitions among the g_i's in addition to $g_0 = g_n$. Take two indices i, j such that $i < j$ and $\{i, j\} \neq \{0\}, \{n\}, \{0, n\}$ (at least one of the two indices is neither 0 nor n) and i, j are minimal. Then the word $s_i \dots s_j$ defines a cycle in $\Gamma(G, S)$, which contradicts our original assumption. Finally, $x \in \mathbb{X}_{G,S}$, which concludes the proof. $\square$

This alternative set of forbidden patterns for $\mathbb{X}_{G,S}$ will be particularly helpful in the proof of Theorem 26, where we characterize groups G which admit a generating set S such that $\mathbb{X}_{G,S}$ is an SFT and also in Section 6.2.

4.1 SFT Skeletons

To find SFTs, we start with a warm-up lemma that contains the central idea used in our classification of groups that admit skeleton SFTs.

Lemma 24. *$\mathbb{X}_{\mathbb{Z}^d,S}$ is not an SFT for $d \geqslant 2$ and any generating set S.*

Proof. Let S be a generating set for $\mathbb{Z}^d$ and suppose $\mathcal{F}'$ is a finite set of forbidden patterns such that $\mathbb{X}_{\mathbb{Z}^d,S} = \mathcal{X}_{\mathcal{F}'}$. Then, as S generates the group, there must exist $s_1, s_2 \in S$ such that $\langle s_1 \rangle \cap \langle s_2 \rangle = \{1_{\mathbb{Z}^d}\}$, and $\langle s_1, s_2 \rangle \simeq \mathbb{Z}^2$. Let us denote $N = \max_{w \in \mathcal{F}'} |w|$. Take the SAP defined by the square of length $2N$ on the first two generators $w = s_1^{2N} s_2^{2N} s_1^{-2N} s_2^{-2N}$. Notice that no factors of w^2 of length N belong to $\mathcal{F}'$ as they are all globally admissible in $\mathbb{X}_{\mathbb{Z}^d,S}$. Let $x = w^\infty$. Clearly $x \notin \mathbb{X}_{\mathbb{Z}^d,S}$, as it contains w which satisfies $\overline{w} = 1_G$. Nevertheless, no factor of x of length N is contained in $\mathcal{F}'$. Therefore $x \in \mathcal{X}_{\mathcal{F}'}$, which is a contradiction. $\square$

The main idea of this lemma is using arbitrarily large cycles that are locally self-avoiding. This way, it is not possible to detect that the path eventually crosses itself using a finite window. Which groups admit generating sets that define SFT skeletons then? Let us show that this is the case of a specific class of virtually free groups.

Definition 25. A group G is **plain** if there exist finite groups $\{G_i\}_{i=1}^k$ and $m \geqslant 1$ such that G is isomorphic to the free product

$$\left(\mathop{\ast}_{i=1}^{k} G_i\right) * \mathbb{F}_m.$$

We say a finite generating set S for such G is **standard** if it can be written as the disjoint union $S = S_1 \cup \cdots \cup S_k \cup S_{k+1}$ where S_i is a generating set for G_i and S_{k+1} is a free generating set for $\mathbb{F}_m$.

Theorem 26 (Theorem A)**.** *Let G be a finitely generated group. Then, there exists a finite generating set S such that $\mathbb{X}_{G,S}$ is an SFT if and only if G is a plain group.*

In order to prove this theorem we use a characterization of plain groups with respect to their simple cycles. The **diameter** of a simple cycle is the greatest distance between to vertices in the cycle. A vertex v in a graph Γ is said to be a **cut vertex** if $\Gamma \setminus \{v\}$ is disconnected. A graph is said to be **2-connected** if it contains no cut vertices. A maximal 2-connected subgraph is called a **block**.

Theorem 27 ([37])**.** *Let G be a group and $m \in \mathbb{N}$. Then, the following are equivalent*

- *G admits a finite generating set S such that all simple cycles in the undirected Cayley graph $\Gamma(G, S)$ have diameter at most m,*
- *G admits a finite generating set S such that all blocks in the undirected Cayley graph $\Gamma(G, S)$ have diamater at most m,*
- *G is a plain group.*

Proofs of this Theorem can be found in [37, 16].

Proof of Theorem 26. Let G be a plain group decomposed as $\left(\mathop{\ast}_{i=1}^{k} G_i\right) * \mathbb{F}_m$ with $S = S_1 \cup \cdots \cup S_{k+1}$ a standard generating set. Due to its free product structure, any word $w \in S^*$ can be uniquely decomposed as $w = w_1 w_2 \cdots w_r$ where,

- $w_j \in S_l^*$ for some l, for all $j \in \{1, \ldots, r\}$;
- w_j and w_{j+1} are words over different alphabets for all $j \in \{1, \ldots, r-1\}$.

If $\overline{w} = 1_G$, by our decomposition, $\overline{w}_j = 1_G$ for every j. This means every SAP from G must be entirely contained in one of the finite groups G_i, as $\mathbb{F}_m$ has no SAPs with its free generating set. Therefore, $\mathcal{O}_{G,S}$ is finite because the number of SAPs in each finite group

is finite. By Lemma 23, $\mathbb{X}_{G,S}$ is an SFT.

Now, let G be a finitely generated group with S such that $\mathbb{X}_{G,S}$ is an SFT, defined by the finite set of forbidden patterns $\mathcal{F}$. If G is not a plain group, by Theorem 27, the Cayley graph $\Gamma(G,S)$ contains arbitrarily large simple cycles, and therefore arbitrarily large SAPs. Next, we can assume without loss of generality that every word in $\mathcal{F}$ has the same length, say $N \geqslant 1$. If $\mathcal{F} \subseteq \mathrm{WP}(G,S)$, take a SAP W of length greater than $N+1$. Because SAPs contain no strict factors that belong to $\mathrm{WP}(G,S)$ and every cyclic permutation of the word defining a SAP is itself a SAP, the configuration $x = W^\infty$ does not contain any word from $\mathcal{F}$. Therefore $x \in \mathcal{X}_\mathcal{F} \setminus \mathbb{X}_{G,S}$, which is a contradiction. Suppose there are elements in $\mathcal{F}$ that are not in $\mathrm{WP}(G,S)$. As $\mathcal{F}$ contains forbidden patterns and $\mathcal{X}_\mathcal{F} = \mathbb{X}_{G,S}$, for every word $w \in \mathcal{F}$ there exists $N_w \in \mathbb{N}$ such that for every $u, v \in S^{N_w}$ the word uwv contains a factor from $\mathrm{WP}(G,S)$. Let $M = \max_{w\in\mathcal{F}} N_w$ and take W a SAP of length $2M+N+4$. Once again, because every SAP contains no strict factors that belong to $\mathrm{WP}(G,S)$ and every cyclic permutation of the word defining the SAP is itself a SAP, the configuration $x = W^\infty$ contains no factors from $\mathcal{F}$. Indeed, if there is a $w \in \mathcal{F}$ such that $w \sqsubseteq W$ we can take the cyclic permutation of W such that w is at the middle. Thus, w can be extended by words u, v of length $M+1$ such that uwv contains no factor in $\mathrm{WP}(G,S)$. As a consequence $x \in \mathcal{X}_\mathcal{F} \setminus \mathbb{X}_{G,S}$, which is a contradiction. □

As plain groups admit SFT skeletons, we have an effective procedure to calculate the connective constant of their Cayley graphs. As mentioned in Section 2.1, entropies of SFTs are non-negative rational multiples of logarithms of Perron numbers (see Theorem 4.4.4 [49]). Thus, we can slightly improve Corollary 3.4 from [22] in the case of (plain) groups.

Corollary 28. *Let G be a plain group with S a standard set of generators. Then, $\mu(G,S)$ is a non-negative rational power of a Perron number.*

Let us sketch how to compute the connective constant using SFTs. Let $\mathbb{X}_{G,S}$ be the skeleton of the plain group $G = (\ast_{i=1}^k G_i) * \mathbb{F}_m$ and $\mathcal{F}$ be the finite set of patterns defining it. Recall from Theorem 26 that this set corresponds to the SAPs on each individual group G_i as well as the words ss^{-1} for all $s \in S$. Let N be the length of the biggest word in $\mathcal{F}$. We can extend $\mathcal{F}$ to $\mathcal{F}'$ so that all words have length N. The Rauzy graph $R_N(G,S)$ of order N of $\mathbb{X}_{G,S}$ is the finite directed graph whose vertices are labeled by the language of size N of the skeleton $\mathcal{L}_N(\mathbb{X}_{G,S})$, and edges are labeled with $\mathcal{L}_{N+1}(\mathbb{X}_{G,S})$. There is an edge labeled by w from u to v if u is the prefix of length N of w and v is the suffix of length N of w. We denote the adjacency matrix of the graph R_N by M_N, that is, if $\mathcal{L}_N(\mathbb{X}_{G,S}) = \{u_1, \ldots, u_\ell\}$, the entry $M_N(i,j)$ represents the number of edges in R_N from u_i to u_j. Then the connective constant of $\Gamma(G,S)$ is the logarithm of the dominant eigenvalue of M_N, which exists by Perron-Frobenius' Theorem.

Example 29. Take S_3 the symmetric group on 3 elements with generating set $\mathtt{s}_1 = (1\ 2)$ and $\mathtt{s}_2 = (1\ 3)$, and the cyclic group $\mathbb{Z}/3\mathbb{Z} = \langle \mathtt{t} \rangle$. Then, the skeleton of the plain group

$G = S_3 * \mathbb{Z}/3\mathbb{Z}$ with respect to $S = \{\mathtt{s}_1, \mathtt{s}_2, \mathtt{t}^{\pm 1}\}$ is defined by the forbidden patterns,

$$\mathcal{F} = \{\mathtt{s}_1^2, \mathtt{s}_2^2, (\mathtt{s}_1\mathtt{s}_2)^3, \mathtt{t}^3, \mathtt{t}\mathtt{t}^{-1}, \mathtt{t}^{-1}\mathtt{t}\}.$$

We obtain that the connective constant $\mu = \mu(G, \{\mathtt{s}_1, \mathtt{s}_2, \mathtt{t}^{\pm 1}\})$ is the solution of the polynomial equation $x^7 - 4x^5 - 8x^4 - 8x^3 - 8x^2 - 8x - 4 = 0$ obtained from the characteristic polynomial of the matrix described above, which is approximately $\mu \approx 2.8698315$.

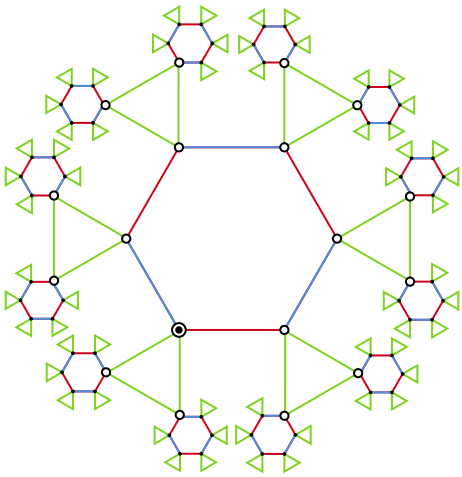


Figure 6: A portion of the Cayley graph of the plain group $S_3 * \mathbb{Z}/3\mathbb{Z}$. The two generators for S_3 are pictured in red and blue and alternate; the generator for $\mathbb{Z}/3\mathbb{Z}$ is pictured in green.

The skeletons of plain groups with respect to their standard generating sets also have nice dynamical properties. We say a subshift $X \subseteq A^{\mathbb{Z}}$ is **irreducible** if for every $w_1, w_2 \in \mathcal{L}(X)$, there exists some $w \in \mathcal{L}(X)$ such that $w_1 w w_2 \in \mathcal{L}(X)$.

Proposition 30. *Let G be a plain group with standard generating set S. Then, $\mathbb{X}_{G,S}$ is irreducible.*

Proof. Decompose G as $\left(\ast_{i=1}^{k} G_i\right) * \mathbb{F}_m$ with $S = S_1 \cup \cdots \cup S_{k+1}$ a standard generating set. Take $w_1, w_2 \in \mathcal{L}(\mathbb{X}_{G,S})$ appearing at position 0 of the configurations $x^{(1)}, x^{(2)} \in \mathbb{X}_{G,S}$ respectively. There is a unique decomposition $w_i = w_1^i w_2^i \cdots w_{r_i}^i$ where,

- $w_j^i \in S_l^+$ for some l, for all $j \in \{1, \ldots, r_i\}$;
- w_j^i and w_{j+1}^i are words over different alphabets for all $j \in \{1, \ldots, r-1\}$.

If $w_{r_1}^1, w_1^2 \in S_i^*$, take any generator $s \in S_j$ for $j \neq i$ and define $x = x^{(1)}_{(-\infty,0]} w_1 s w_2 x^{(2)}_{[|w_2|,+\infty)}$. Because we chose a generator that does not belong to G_i, and $x^{(1)}$ and $x^{(2)}$ belong to the skeleton, x must also belong to the skeleton. This implies, $w_1 s w_2 \in \mathcal{L}(\mathbb{X}_{G,S})$. If instead $w_r^1 \in S_i^*$ and $w_1^2 \in S_j^*$ for $i \neq j$, define $y = x^{(1)}_{(-\infty,0]} w_1 s s' w_2 x^{(2)}_{[|w_2|,+\infty)}$. As before, y must belong to $\mathbb{X}_{G,S}$. This means, $w_1 w_2 \in \mathcal{L}(\mathbb{X}_{G,S})$. $\square$

Corollary 31. *For G a plain group with standard generating set S, the set of periodic configurations of $\mathbb{X}_{G,S}$ is dense in $\mathbb{X}_{G,S}$. In other words, any bi-infinitely extendable SAW on $\Gamma(G,S)$ appears in a periodic bi-infinite SAW. Furthermore,*

$$\mu(G,S) = \lim_{n\to\infty} \sqrt[n]{e_n},$$

where e_n denotes the number of periodic points in $\mathbb{X}_{G,S}$ of period $n \in \mathbb{N}$.

This corollary states a general property of irreducible subshifts of finite type, namely, its set of periodic configurations is dense and its entropy is approximated through its periodic points [49]. We obtain a similar expression for the connective constants of Cayley graphs whose skeletons is not an SFT in Section 6.

4.2 Effective Skeletons

Let us briefly look at the case of effective skeletons. We know that recursively presented groups have recursively enumerable word problem. $\mathrm{WP}(G,S)$ is thus recursively enumerable for all finite generating sets. This enumeration gives us an enumeration of the forbidden patterns of our skeleton.

Lemma 32 ([4])**.** *Let G be a recursively presented group. Then, for every generating set S, the subshift $\mathbb{X}_{G,S}$ is an effective subshift. In particular, $\mu(G,S)$ is a right computable real number.*

In order to approach the converse, we give a computational upper bound of the word problem of the group in terms of the computability of finite SAWs on the Cayley graph.

Lemma 33. *The word problem for G with respect to a generating set S is decidable given an algorithm for $L_{SAW}(G,S)^c$.*

Proof. We describe a procedure to compute the word problem of G given an algorithm that determines if a word belongs to $L = L_{SAW}(G,S)^c$. We begin with an algorithm that computes all words $w \in S^{\leqslant n}$ such that $w \in \mathrm{WP}(G,S)$ given n. This algorithm, which we call $\mathcal{M}$, is shown in Algorithm 1.

Let us show the output of $\mathcal{M}$ on n is $\mathrm{WP}(G,S) \cap S^{\leqslant n}$. Let T_i be the set T in the algorithm after the first i iterations of the **for** loop, for $i \in \{2,\dots,n\}$. We claim $T_i = \mathrm{WP}(G,S) \cap S^{\leqslant i}$. First off, every non-self avoiding path of length two must represent the identity. Thus, $T_2 = \mathrm{WP}(G,S) \cap S^2$. Now, suppose we have the equality for T_i. Take $w \in \mathrm{WP}(G,S) \cap S^{i+1}$. This implies $w \in L$, and as seen in Lemma 23, it must represent a simple cycle, contain a shorter simple cycle, or a word of the form ss^{-1}. In the first case, w contains no factors from T_i and is therefore added to T_{i+1}. In the other two cases, it contains a factor from T_i that after being deleted creates a word that belongs to $\mathrm{WP}(G,S) \cap S^{\leqslant i} = T_i$. Therefore, $w \in T_{i+1}$. Conversely, if $w' \in T_{i+1} \setminus T_i$ we know $w \in L$. If w was added to T_{i+1} because it contains no factors from T_i, it must represent a simple loop and is therefore in $\mathrm{WP}(G,S)$. On the other hand, if w was added after deleting a factor from T_i, w is made up of a word representing the identity with a

Algorithm 1: $\mathcal{M}$

```
Input: n ⩾ 2
T ← ∅;
for w ∈ S^2 do
    if w ∈ L then
        T ← T ∪ {w};
    end
end
for i ∈ {3, ..., n} do
    for w ∈ S^i do
        if w ∈ L then
            if w contains no factors from T then
                T ← T ∪ {w};
            end
            for v ∈ T do
                Delete v from w if present, to obtain w';
                if w' ∈ T then
                    T ← T ∪ {w};
                end
            end
        end
    end
end
return T;
```

factor representing the identity inserted into it. This means, $w \in \mathrm{WP}(G,S)$ and therefore $T_{i+1} = \mathrm{WP}(G,S) \cap S^{\leqslant i+1}$.

Finally, to determine if a given word w belongs to $\mathrm{WP}(G,S)$, we run $\mathcal{M}$ on the input $|w|$, and see if it is present in T. □

As a consequence, if $L_{SAW}(G,S)$ is co-recursively enumerable, the word problem of G must be in Δ_2^0 on the arithmetical hierarchy. This is the case when $\mathbb{X}_{G*H,S}$ is effective for H any f.g. group, and $S = S_G \cup S_H$ with S_G, S_H generating sets for G and H respectively.

Conjecture 34. A group is recursively presented if and only if there exists a finite generating set S such that $\mathbb{X}_{G,S}$ is effective.

Even though recursively presented groups define subshifts that are effective, if the structure of the underlying group is computationally complex, the configurations of the skeleton may be uncomputable. We say a configuration $x \in S^{\mathbb{Z}}$ is **computable** if there is an algorithm that on input $n \in \mathbb{Z}$ computes $x_n \in S$.

Definition 35. A finitely generated group G and generating set S are said to be **algorithmically finite** if for every infinite recursively enumerable set $L \subseteq \mathbb{F}_S$, there exist

infinitely many pairs of distinct words $u, v \in L$ such that $\pi(u) = \pi(v)$, where $\pi : \mathbb{F}_S \to G$ is the canonical projection. We say G is a **Dehn Monster** if it is infinite, recursively presented and algorithmically finite.

This class of groups was introduced by Myasnikov and Osin in [53], where they showed that Dehn Monsters exist. Furthermore, they showed that being algorithmically finite does not depend on the generating set.

Proposition 36. *Let G be a Dehn Monster. Then $\mathbb{X}_{G,S}$ is effective for any finite generating set S, but no configuration in $\mathbb{X}_{G,S}$ is computable.*

Proof. As the properties of being infinite, recursively presented and algorithmically finite are independent of the generating set, we take any generating set S for G. If there existed a computable configuration $x \in \mathbb{X}_{G,S}$, we could recursively enumerate the set of words $L = \{x_{[0,n-1]} \in S^* \mid n \geqslant 1\}$. Then for any $u, v \in L$, $\pi(u) \neq \pi(v)$. If not, we would arrive at $x_{[0,n-1]} =_G x_{[0,m-1]}$ for some $n > m \geqslant 1$, which implies $x_{[n,m-1]} =_G \epsilon$. Therefore, any pair of elements in L maps to a different element through π, which contradicts the algorithmic finiteness of G. □

4.3 Periodic Configurations

Configurations of particular importance in the study of subshifts are periodic configurations. Recall that a configuration $x \in \mathbb{X}_{G,S}$ is periodic if there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $x_{i+k} = x_i$ for all $i \in \mathbb{Z}$. Such a configuration has a rigid structure, if we take $w = x_{[0,k-1]}$ the configuration x is equal to the bi-infinite repetition of w, i.e., $x = w^\infty$. We will see that the existence of periodic points in the skeleton imposes strong restrictions on the structure of the underlying group.

In [4] it was shown that for any finitely generated group with a torsion-free element, the skeleton contains a periodic point. The periodic configuration was obtained by iterating any geodesic of the torsion-free element with the smallest length in the group. By reinterpreting the proof of [33, Theorem 7] we obtain the following generalization.

Proposition 37. *Let G be a finitely generated group. Take a generating set S and a torsion-free element $g \in G$, and $k = \operatorname{argmin}\{\|g^n\|_S \mid n \geqslant 1\}$. Then, for any geodesic $w \in S^*$ representing g^k, $w^\infty \in \mathbb{X}_{G,S}$ and is a bi-infinite geodesic.*

Proof. Fix a generating set S and $g \in G$ torsion-free. Let $k \geqslant 1$ be as in the statement of the result, and denote $h = g^k$. Take a geodesic $w \in S^*$ for h and let $\pi = (e_0, \ldots, e_{n-1})$ be the (self-avoiding) walk starting at the identity in the Cayley graph, of label w and $|w| = n$. Let Π be the bi-infinite walk made by concatenating the paths $h^m \cdot \pi$, for all $m \in \mathbb{Z}$. Thus, $\lambda(\Pi) = w^\infty$. We claim Π is self-avoiding. Suppose it is not, and take the smallest $m \in \mathbb{N}$ such that w^m does not represent a SAW. Let π_i denote the walk $h^i \cdot \pi$ where $\mathfrak{i}(\pi_i) = h^i$ and $\mathfrak{t}(\pi_i) = h^{i+1}$. As m is minimal, we know the concatenated walks $\pi_0 \cdots \pi_{m-1}$ and $\pi_1 \cdots \pi_m$ are self-avoiding, and therefore the first intersection must occur between π_0 and π_m. Then, there exists $v, u \sqsubseteq w$ prefixes, and $f \in \pi_0 \cap \pi_m$ such

that $f = \overline{v} = h^{m-1}\overline{u}$. Once again, because m is minimal, $f \neq h, h^{m-1}$. If we compute the distance,

$$d_S(f, h^{m-1}f) = d_S(\overline{u}, \overline{v}) \leqslant |w|,$$

as k was chosen to minimize $\|g^k\|_S$, the distance between f and $h^{m-1}f$ must be $|w|$. As both vertices are in π_m, this is only possible if $f = h^m$ and $h^{m-1}f = h^{m-1}$. Thus $h^m = 1_G$, which is a contradiction as h is torsion-free. Therefore, $w^\infty \in \mathbb{X}_{G,S}$. Finally, as we chose k to minimize the distance to the identity of powers of g, w^n must be a geodesic for all $n \in \mathbb{N}$. □

Theorem 38 ([4])**.** *Let G be a finitely generated group. Then, G is a torsion group if and only if $\mathbb{X}_{G,S}$ is aperiodic for every (any) generating set.*

Proof. Suppose G is a torsion group and let $x \in \mathbb{X}_{G,S}$ be a periodic configuration that infinitely repeats the word w. Let $g = \overline{w}$. By definition of the skeleton, $g^n = \overline{w^n} \neq 1_G$ for all $n \in \mathbb{N}$. This contradicts the fact that G is a torsion group. Conversely, if G has a torsion-element, by Proposition 37, $\mathbb{X}_{G,S}$ contains a periodic point. □

Corollary 39. *If G is a finitely generated torsion group, then for all generating sets $\mathbb{X}_{G,S}$ is not sofic.*

Proof. If G is a finitely generated torsion group, Theorem 38 tells us that none of its skeletons contain periodic configurations. Because non-empty sofic shifts always contain periodic configurations, no skeleton of G can be sofic. □

4.4 Minimality

Our next objective is to find sufficient and necessary properties for the skeleton to be minimal. We begin by identifying possible subshifts of $\mathbb{X}_{G,S}$.

Lemma 40. *Let G be a finitely generated group. Then,*

- *For a symmetric subset $S' \subseteq S$ and $H = \langle S' \rangle$, $\mathbb{X}_{H,S'}$ is a subshift of $\mathbb{X}_{G,S}$.*
- *For $N \trianglelefteq G$ a normal subgroup, $\mathbb{X}_{G/N,S'}$ is a subshift of $\mathbb{X}_{G,S}$, where $S' = S \setminus N$.*

Proof. The first statement follows from the fact that any configuration from $\mathbb{X}_{H,S'}$ avoids all words from $\mathrm{WP}(G,S)$, as H is a subgroup of G. For the second statement, let $x \in \mathbb{X}_{G/N,S}$ and $\{w_i\}_i \subseteq S^*$ a set of generators for N. Then by definition no factor $w \sqsubseteq x$ belongs to $\mathrm{WP}(G/N,S) = \langle\langle \mathrm{WP}(G,S) \cup \{w_i\} \rangle\rangle$. In particular, it does not belong to $\mathrm{WP}(G,S)$. Therefore $x \in \mathbb{X}_{G,S}$. □

Because every non-finite quotient gives us a non-empty subshift of $\mathbb{X}_{G,S}$, if we want to find a minimal skeleton, it is reasonable to look at the class of just infinite groups. A group G is said to be **just infinite** if it is infinite and every proper quotient is finite.

Proposition 41. *Let G be a finitely generated group with a generating set S. If $\mathbb{X}_{G,S}$ is minimal, then G is a just infinite group.*

Proof. If $\mathbb{X}_{G,S}$ is minimal, every subshift of the form $\mathbb{X}_{G/N,S}$ must be either empty or equal to $\mathbb{X}_{G,S}$. Let N be a proper normal subgroup, that is, non trivial and not equal to G. By Theorem 11, the connective constants satisfy $\mu(G/N, S) < \mu(G, S)$. Thus, the entropy of $\mathbb{X}_{G/N,S}$ is strictly less than that of $\mathbb{X}_{G,S}$, so they cannot be equal. Then $\mathbb{X}_{G/N,S} = \varnothing$, meaning G/N is finite. Therefore, G is just infinite. □

Proposition 42. *Let G be a finitely generated group with a generating set S. If $\mathbb{X}_{G,S}$ is minimal, for every symmetric subset $S' \subsetneq S$, the subgroup $\langle S' \rangle$ is finite. In particular, torsion-free groups do not admit minimal skeletons.*

Proof. If $\mathbb{X}_{G,S}$ is minimal, every subshift of the form $\mathbb{X}_{H,S'}$, for $H = \langle S' \rangle$, must be either empty or equal to $\mathbb{X}_{G,S}$. If $H = G$, then by Theorem 11, $\mu(G, S') < \mu(G, S)$ meaning $\mathbb{X}_{G,S'}$ is empty, which is a contradiction. Therefore, $H \lneq G$. Now, take $s \in S \setminus H$ and $x \in \mathbb{X}_{H,S'}$. Define the configuration $x' = x_{(-\infty,-1]} s x_{[0,+\infty)} \in S^{\mathbb{Z}}$. Because x is in H's skeleton, we know neither $x_{(-\infty,-1]}$ nor $x_{[0,+\infty)}$ contain subwords from $\mathrm{WP}(G, S)$. Next, if there exist $i, j \in \mathbb{N}$ such that $x_{[i,-1]} s x_{[0,j]} \in \mathrm{WP}(G, S)$, then $s =_G (x_{[i,-1]})^{-1} (x_{[0,j]})^{-1}$ which implies $s \in H$. This is a contradiction. Therefore, $x' \in \mathbb{X}_{G,S} \setminus \mathbb{X}_{H,S'}$. As $\mathbb{X}_{G,S}$ is minimal, $\mathbb{X}_{H,S'} = \varnothing$ and thus H is finite. Finally, if a group is torsion-free, each generator generates $\mathbb{Z}$ which is not possible if the skeleton is minimal. □

Remark 43. Both conditions are not sufficient to characterize minimal skeletons. Take the group $\tilde{A}_2$ with generating set $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ as defined in Section 2.4. This group is just infinite [52], every pair of different generators generates a subgroup isomorphic to the finite group S_3, and every generator generates a copy of $\mathbb{Z}/2\mathbb{Z}$. Nonetheless, its skeleton is not minimal. Take the periodic configuration $x = (\mathtt{abcb})^{\infty}$ which belongs to the skeleton. Then, the closure of the orbit of x is finite and contains exactly periodic configurations defined by cyclic permutations of $\mathtt{abcb}$. But, the skeleton also contains the periodic configuration $y = (\mathtt{bcac})^{\infty}$, which is not one of the cyclic permutations.

As the remark shows, if a minimal skeleton contains periodic configurations, it must be finite. This is the case of $\mathcal{D}_{\infty}$ with generating set $\{\mathtt{a}, \mathtt{b}\}$, as seen on Example 6, which defines a minimal skeleton.

5 Sofic Skeletons

Let us tackle the question of which groups admit skeletons that are sofic. Since SFTs are sofic subshifts, from Theorem 26 we already know that plain groups admit sofic skeletons. But are there groups that admit sofic skeletons which are not SFTs? The first naive strategy would be to ask when the word problem of the group is regular, as this is the set of forbidden patterns used in the definition of the skeleton. Unfortunately, Anisimov showed in [2] that $\mathrm{WP}(G, S)$ is regular if and only if G is a finite group. We must therefore find other sets of forbidden patterns to study. Lemma 9 tells us that we can look at the classes of groups where the language of SAWs is regular. The class of groups with such property have already been classified.

Theorem 44 ([51])**.** *Let G be a finitely generated group with S a finite generating set. Then, $L_{\mathrm{SAW}}(G,S)$ is regular if and only if $\Gamma(G,S)$ has more than one end and all ends are thin of size 1.*

As Lindorfer and Woess show, if $\Gamma(G,S)$ has only thin ends of size 1 its blocks are finite [51, Lemma 5.3]. Combining this fact with Haring-Smith's characterization of plain groups (Theorem 27), we see that groups where $L_{\mathrm{SAW}}(G,S)$ is regular are exactly plain groups. Nevertheless, when considering bi-infinitely extendable SAWs, the situation is different.

Lemma 45. *The group $G = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by the presentation $\langle s,t \mid s^2, [t,s]\rangle$ has a sofic skeleton.*

Proof. We will exhibit a regular set of forbidden patterns for $\mathbb{X}_{G,S}$, with $S = \{s^{\pm 1}, t^{\pm 1}\}$. Take the set of forbidden patterns

$$\mathcal{F} = \{st^{\pm n}st^{\mp 1} \mid n \in \mathbb{N}\} \cup \{t^{\pm 1}st^{\mp n}s \mid n \in \mathbb{N}\} \cup \{s^2, t^{\pm 1}t^{\mp 1}\}.$$

It is a simple exercise to show that $\mathcal{F}$ is a regular language. Let us show $\mathbb{X}_{G,S} = \mathcal{X}_{\mathcal{F}}$. Suppose there is a configuration $x \in \mathbb{X}_{G,S} \setminus \mathcal{X}_{\mathcal{F}}$. Because x is in the skeleton, we know it does not contain factors of the form s^2 or $t^{\pm 1}t^{\mp 1}$. Therefore it must contain a factor of the form $st^{\pm n}st^{\mp 1}$ or $t^{\pm 1}st^{\pm n}s$. Suppose, x contains the word $w = st^n st^{-1}$, for some $n \in \mathbb{N}$. There is no way to extend this word to the right, as ws contains the factor $tst^{-1}s$ which evaluates to the identity, extending by t^{-k} with $k \geqslant n$ creates the factor $st^n st^{-n}$ which evaluates to the identity, and extending by $t^{-k}s$ with $k \leqslant n-1$ creates the factor $t^k st^{-k}s$ which also evaluates to the identity. This leads to a contradiction. The other cases being analogous, we have $\mathbb{X}_{G,S} \subseteq \mathcal{X}_{\mathcal{F}}$.

Now, suppose there is a configuration $x \in \mathcal{X}_{\mathcal{F}} \setminus \mathbb{X}_{G,S}$. By Lemma 23 and the definition of $\mathcal{F}$, x must contain a SAP. Nevertheless, all SAPs in G are cyclic permutations of words of the form $st^n st^{-n}$ for some $n \in \mathbb{N}$. Thus, each SAP contains a factor from $\mathcal{F}$, leading to a contradiction and proving $\mathcal{X}_{\mathcal{F}} \subseteq \mathbb{X}_{G,S}$. □

The Cayley graph of $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with respect to the before mentioned generating set is the bi-infinite ladder, which is a graph with two thin ends of size 2. An analogous proof can be done for the Cayley graph of $\mathbb{Z}$, which is a plain group, with respect to the generating set $\{\pm 1, \pm 2\}$, which also has thin ends of size 2.

To characterize groups which admit sofic skeletons we will make use of the fact that the language of a sofic subshift is regular. Our main tool in this regard will be the following version of the Pumping Lemma.

Lemma 46 (Pumping Lemma)**.** *Let L be a regular language. Then, there exists $p > 0$ such that every word $w \in L$ with $|w| \geqslant p$ can be decomposed as $w = w'uv$ with $|u| > 1$ and $|uv| \leqslant p$, such that for all $n \in \mathbb{N}$, $w'u^n v \in L$.*

This allows us to show that being sofic is a property of skeletons that depends on the generating set.

Proposition 47. *Every group G admits a generating set S such that $\mathbb{X}_{G,S}$ is not sofic.*

Proof. By Corollary 39, if G is a torsion group, no skeleton is sofic. We can therefore suppose G has a torsion-free element. Let S' be any generating set for G, and g a torsion-free element. We denote $s = g^2$, $t = g^3$, and define $S = S' \cup \{s, t\}$. Suppose $\mathbb{X}_{G,S}$ is sofic. Then, its language $\mathcal{L}(\mathbb{X}_{G,S})$ is regular. Take $p > 0$ given by the pumping lemma. The word $w = ts^{p+1}t^{-1}s^{-p}$ is contained in $\mathcal{L}(\mathbb{X}_{G,S})$ as it is globally admissible through the configuration $s^\infty ts^{p+1}t^{-1}s^{-p}t^{-1}(s^{-1})^\infty$ (see Figure 7).

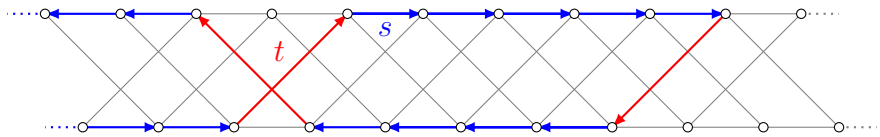


Figure 7: The configuration used for the Pumping Lemma (with $p = 4$) depicted in the Cayley graph of the subgroup $\langle s, t\rangle$. The blue edges represent s and the red edges t.

Now, by the Pumping Lemma we can decompose w as $w = w'uv$ with $|uv| \leqslant p$. Thus, $u = s^{-k}$ with $k \geqslant 1$. Therefore, the word $w'u^2v = ts^{p+1}t^{-1}s^{-(p+k)}$ belongs to $\mathcal{L}(\mathbb{X}_{G,S})$, which is a contradiction as $ts^{p+1}t^{-1}s^{-(p+1)} =_G \epsilon$. We conclude that $\mathbb{X}_{G,S}$ is not sofic. □

5.1 Ends and Automorphisms

To go towards a characterization we must make a brief detour through the theory of ends and automorphisms of infinite quasi-transitive graphs. Let us begin by taking a look at the theory of ends of connected graphs as introduced by Halin [32].

For a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$, and a subset of vertices $A \subseteq V_\Gamma$ we denote by $\Gamma \setminus A$ the graph obtained by removing the vertices from A and all their incident edges. We define a **ray** ρ to be an infinite sequence of distinct vertices $\pi = (v_0, v_1, \dots) \in V_\Gamma^{\mathbb{N}}$ such that there is an edge between v_i and v_{i+1}. Analogously, a **double ray** to be a bi-infinite sequence of distinct vertices $\pi = (\dots, v_{-1}, v_0, v_1, \dots) \in V_\Gamma^{\mathbb{Z}}$ such that each successive vertex is connected by an edge. Two rays are said to be **equivalent** if for any finite set $A \subseteq V_\Gamma$ all but finitely many of their vertices are contained in the same connected component of $\Gamma \setminus A$. The equivalence classes of this relation are called the **ends** of the graph. Given an end ω and a finite set $A \subseteq V_\Gamma$, we define $C(\omega, A)$ to be the connected component of $\Gamma \setminus A$ where all the rays defining ω eventually end up in.

A **defining sequence** for an end ω is a sequence of finite subsets $(A_i)_{i\in\mathbb{N}}$ such that for all $i \geqslant 1$, $A_i \cup C(\omega, A_i) \subseteq C(\omega, A_{i-1})$. We say that an end ω is **thin** if there exist $m \geqslant 1$ and a defining sequence $(A_i)_{i\in\mathbb{N}}$ such that $|A_i| = m$ for all $i \in \mathbb{N}$. The smallest m verifying this condition is called the **size** of ω. An end is called **thick** if its size is infinite. Thomassen and Woess [59] showed using Menger's Theorem that an end of size $m \in \mathbb{N} \cup \{\infty\}$, seen as an equivalence class of rays, contains a maximum of m vertex disjoint rays.

Let $\mathrm{Aut}(\Gamma)$ denote the set of automorphisms of Γ, that is, bijections $f : V_\Gamma \to V_\Gamma$ that preserve edge adjacency. We say a subgroup $G \leqslant \mathrm{Aut}(\Gamma)$ acts **quasi-transitively** on Γ if the set of orbits of the action $G \curvearrowright \Gamma$ is finite. We say G acts **transitively** if there is a unique orbit. Freudenthal and Hopf independently showed [20, 41] that a quasi-transitive graph has either 0, 1, 2 or an infinite amount of ends.

In our setting, all Cayley graphs $\Gamma(G, S)$ are transitive under the action of the group G by left translations. Furthermore, this action preserves the labeling given by the generating set.

Take Γ to be locally finite and connected. Following [33], automorphisms of Γ can be classified into three classes. An automorphism $g \in \mathrm{Aut}(\Gamma)$ is,

- **elliptic** if it fixes a finite subset of V_Γ,
- **parabolic** if it fixes a unique end, and
- **hyperbolic** if it fixes a unique pair of ends.

Halin showed [33] that for a non-elliptic automorphism $g \in \mathrm{Aut}(\Gamma)$ and vertex $v \in V$ the sequence $(v, g \cdot v, g^2 \cdot v, \dots)$ uniquely defines and fixes an end which we call the **direction** of g, and denote $D(g)$.

Theorem 48 (Halin, [33] Theorem 9)**.** *Let g be a non-elliptic automorphism acting on a connected locally finite graph Γ. Then,*

- *$D(g)$ and $D(g^{-1})$ have the same size m.*
- *$D(g) \neq D(g^{-1})$ if and only if $m < \infty$. In this case g is hyperbolic.*
- *There are m disjoint double rays $\{\pi_i\}_{i=1}^m$ that are invariant by some positive power of g.*
- *If g is hyperbolic, there exists a set $A \subseteq V_\Gamma$ of size m and $k \in \mathbb{N}$ such that $(g^{kn} \cdot A)_{n\in\mathbb{N}}$ and $(g^{-kn} \cdot A)_{n\in\mathbb{N}}$ are defining sequences for $D(g)$ and $D(g^{-1})$ respectively, that intersect each π_i in exactly one vertex.*

To precisely understand thin ends, we study the following graphs.

Definition 49. A connected locally finite graph is called a **strip** if it is two ended and quasi-transitive.

We present general facts about strips that can be found in [51] and can be partly deduced from Theorem 48. For every strip Q, there exits a hyperbolic automorphism $g \in \mathrm{Aut}(Q)$ that fixes both ends ω^+ and ω^-. Both ends have the same size, for instance m, which entails the existence of a finite set A of size m such that $(g^n \cdot A)$ and $(g^{-n} \cdot A)$ are defining sequence for ω^+ and ω^- respectively. In addition, there are m disjoint double rays intersecting every $g^n \cdot A$ at exactly one vertex. We call such a strip a g-strip of size m. When working with a g-strip, up to taking a power of g, we can assume that the subgraph induced by $C(\omega^+, A) \setminus C(\omega^+, g \cdot A)$, which we call $P(\omega^+)$, is connected and finite.

The following results show that quasi-transitive graphs contain strips, under conditions on their ends and automorphisms.

Lemma 50 (Lindorfer, Woess, [51] Lemma 3.3)**.** *Let Γ be a connected and locally finite graph where $G \leqslant \mathrm{Aut}(\Gamma)$ acts quasi-transitively. If Γ has a thin end of size m, then it contains a g-strip of size m for some $g \in G$.*

Lemma 51 (Lindorfer, Woess, [51] Lemma 3.4)**.** *Let Γ be a connected and locally finite graph where $G \leqslant \mathrm{Aut}(\Gamma)$ acts quasi-transitively. If G contains a parabolic element, then for every $m \geqslant 1$, Γ contains a g-strip of size at least m for some $g \in G$.*

5.2 Characterizing Sofic Skeletons

We provide the following characterization.

Theorem 52 (Theorem B)**.** *Let G be a finitely generated group. There exists S such that $\mathbb{X}_{G,S}$ is sofic if and only if G is a plain group, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$.*

The idea of the proof is as follows. First, we use the same constructions of Lindorfer and Woess [51] to find ladder-like structures on strips that will allow us to use the Pumping Lemma, and then conclude that all ends of the graph must be thin and of size at least 2. Next, by using similar ideas, we show that if the graph has an end of size two and the skeleton is sofic, then the group must be virtually $\mathbb{Z}$. Finally, we characterize virtually $\mathbb{Z}$ groups with sofic skeletons, completing the proof.

Lemma 53. *Let G be a finitely generated group with a generating set S, such that $\Gamma(G,S)$ contains an g-strip Q for some $g \in G$. If $\mathbb{X}_{G,S}$ is sofic, then Q has size at most 2.*

Proof. Suppose Q is of size greater or equal than 3. Then, Q contains three disjoint double rays which we call $\pi_1 = (v_i)_{i\in\mathbb{Z}}$, $\pi_2 = (u_i)_{i\in\mathbb{Z}}$ and $\pi_3 = (v'_i)_{i\in\mathbb{Z}}$, that are g-invariant. Recall we took our subgraph $P(\omega^+)$ to be connected and finite. Therefore, there is a path p_1 that connects two of the rays. Suppose without loss of generality that p_1 connects π_1 and π_2 from v_0 to u_0 with no other vertices from π_i for $i \in \{1,2,3\}$. Analogously, $g \cdot P(\omega^+)$ will connect π_3 with another of the rays through a path p_2. Up to rearranging indices, suppose p_2 connects π_2 to π_3 starting at u_k and ending at v'_k, for some $k \in \mathbb{N}$ such that there are no other vertices from π_i for $i \in \{1,2,3\}$. Because the vertex set of every element of the sequence $(g^n \cdot P(\omega^+))_{n\in\mathbb{N}}$ is pairwise disjoint, no walks in $\{g^{2n} \cdot p_1 \mid n \in \mathbb{Z}\} \cup \{g^{2n} \cdot p_2 \mid n \in \mathbb{Z}\}$ intersect. This way, the subgraph induced by the three paths $\{\pi_i\}_{i=1}^3$ and all $g^{2n} \cdot p_1$ and $g^{2n} \cdot p_2$, $Q' \subseteq Q$ is a periodic subdivision of the bi-infinite 3-ladder (see Figure 8).

Now, let us give names to the labels of the different portions of the subdivision. Denote λ_1 the label from u_k to $g \cdot u_k$, λ_2 the label from u_k to $g \cdot u_0$, λ_3 the label from u_0 to v_0, λ_4 the label from v_0 to $g \cdot v_0$, λ_5 the label from u_k to v'_k, and finally λ_6 the label from $g \cdot v'_k$ to v'_k. Then, for every $n \geqslant 1$ and $k < n$ the configuration

$$x = \lambda_1^\infty \lambda_2 . \lambda_3 \lambda_4^n \lambda_3^{-1} \lambda_1^{-k} \lambda_2^{-1} \lambda_5 \lambda_6^\infty,$$

belongs to the skeleton (See Figure 8). Thus, $\lambda_3 \lambda_4^n \lambda_3^{-1} \lambda_1^{-k} \in \mathcal{L}(\mathbb{X}_{G,S})$ for every $n \geqslant 1$ and $k < n$. Notice that the language $L = \{\lambda_3 \lambda_4^n \lambda_3^{-1} \lambda_1^{-k} \in S^* \mid k, n \in \mathbb{N}\}$ is regular. If $\mathbb{X}_{G,S}$ is sofic, its language $\mathcal{L}(\mathbb{X}_{G,S})$ is regular. Then by the closure properties of regular languages,

$$L' = L \cap \mathcal{L}(\mathbb{X}_{G,S}) = \{\lambda_3 \lambda_4^n \lambda_3^{-1} \lambda_1^{-k} \in S^* \mid k < n\},$$

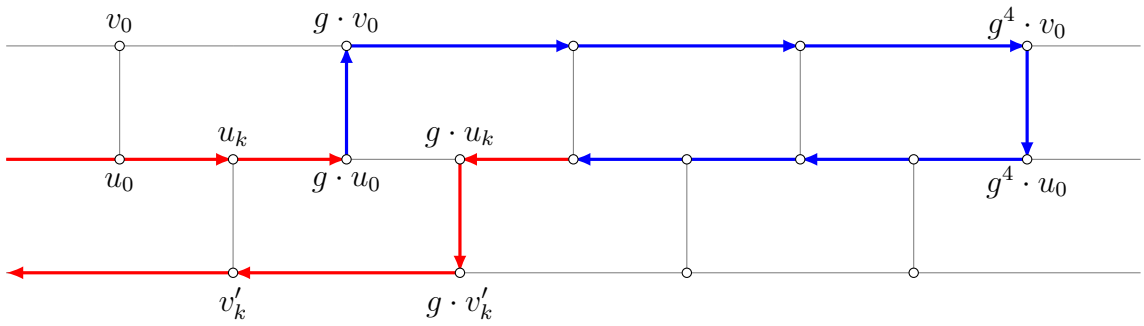


Figure 8: The periodic subdivision of the 3-ladder with the configuration x highlighted in red and blue. The word $\lambda_3\lambda_4^3\lambda_3^{-1}\lambda_1^{-2}$ is marked in blue, whereas the infinite prefix and suffix of x are marked in red.

is regular. By the Pumping Lemma, there exists a pumping length $p > 0$. Take $\lambda_3\lambda_4^{p+1}\lambda_3^{-1}\lambda_1^{-p} \in L'$. This word decomposes as $\tilde{w}ww'$ such that $|ww'| \leqslant p$. By the structure of our word, ww' is a suffix of λ_1^{-p}. Next, $\tilde{w}w^2w'$ belongs to L' and therefore has the form

$$\tilde{w}w^2w' = \lambda_3\lambda_4^n\lambda_3^{-1}\lambda_1^{-k} = \lambda_3\lambda_4^{p+1}\lambda_3^{-1}w_1w^2w',$$

for some $k, l \in \mathbb{N}$ and $w_1 \in S^*$. Because we are working over a Cayley graph, the labels of different edges starting from u_0 must be different and thus the first generators for λ_4 and λ_3^{-1} are different. Therefore, $n = p + 1$. This means, $\lambda_1^{-k} = w_1w^2w'$. Finally, as λ_1^{-k} is strictly longer than λ_1^{-p}, $k \geqslant p + 1$. But, this would imply $\lambda_3\lambda_4^{p+1}\lambda_3^{-1}\lambda_1^{-k}$ belongs to $\mathcal{L}(\mathbb{X}_{G,S})$ and is not self-avoiding, which is a contradiction.

□

Proposition 54. *Let G be a finitely generated group. If there exists S such that $\mathbb{X}_{G,S}$ is sofic, then G has more than one end, and $\Gamma(G, S)$ only has thin ends of size at most 2.*

Proof. Let G be a finitely generated group with generating set S such that $\mathbb{X}_{G,S}$ is sofic. By Theorem 38, G is not a torsion group and therefore contains non-elliptic elements when seen as a subgroup of $\mathrm{Aut}(\Gamma(G, S))$. If G is one-ended, then $\Gamma(G, S)$ has one end, which by Lemmas 51 and 53 is a contradiction. Thus, $\Gamma(G, S)$ has at least one thin end. By Lemma 50, every thin end of size m implies the existence of a strip of size m in $\Gamma(G, S)$. By Lemma 53, these strips – and consequently their corresponding ends – must have size at most 2. Finally, if $\Gamma(G, S)$ had a thick end, from the proof of Theorem 4.1 in [51] we know it contains a one-ended subgraph. As before, this contradicts Lemmas 51 and 53. □

The converse of this proposition is not true: the group $\mathbb{F}_2 \times \mathbb{Z}/2\mathbb{Z}$ along with the generating set $S = \{a^{\pm 1}, b^{\pm 1}, s\}$, given by the presentation $\langle a, b, s \mid s^2, [a, s], [b, s]\rangle$, has thin ends of size two, but its skeleton is not sofic. Similar to what we did in Proposition 47, we can use the Pumping Lemma on the words $sa^{n+1}sa^{-n}$, with $n \in \mathbb{N}$, which are in $\mathcal{L}(\mathbb{X}_{G,S})$ through the configuration $b^\infty sa^{n+1}sa^{-n}b^\infty$. The next Lemma captures this idea in the general setting.

Lemma 55. *Let G be a finitely generated group. If there exists S such that $\mathbb{X}_{G,S}$ is sofic and $\Gamma(G,S)$ has an end of size 2, then G is virtually $\mathbb{Z}$.*

Proof. Suppose $\Gamma(G,S)$ has more than two ends, and take ω^+ the end of size 2. By Lemma 50, there exists $g \in G$ and Q a g-strip of size 2. Then, there exist two g-invariant disjoint double rays $\pi_1 = (v_i)_{i\in\mathbb{Z}}$ and $\pi_2 = (u_i)_{i\in\mathbb{Z}}$. In the induced subgraph $P(\omega^+)$ we can find a path p linking, without loss of generality, v_0 and u_0 with no other vertices from π_1 and π_2. Furthermore, the walks belonging to $\{g^n \cdot p \mid n \in \mathbb{Z}\}$ do not intersect each other. This way, the graph spanned by π_1, π_2 and p is a periodic subdivision of the infinite 2-ladder, $Q' \subseteq Q$. Now, take an end $\omega_1 \neq \omega^\pm$ and $\pi_3 = (v'_i)_{i\in\mathbb{N}}$ a ray defining ω_1. As π_3 defines an end different from ω^+ there exists a smallest $N \in \mathbb{N}$ such that $v'_i \notin Q'$ for all $i > N$. Because $\Gamma(G,S)$ is transitive, we can take without loss of generality v'_N to be equal to some u_k with $k \in \mathbb{N}$, placed between $g \cdot u_0$ and $g^2 \cdot u_0$. This is all represented in Figure 9.

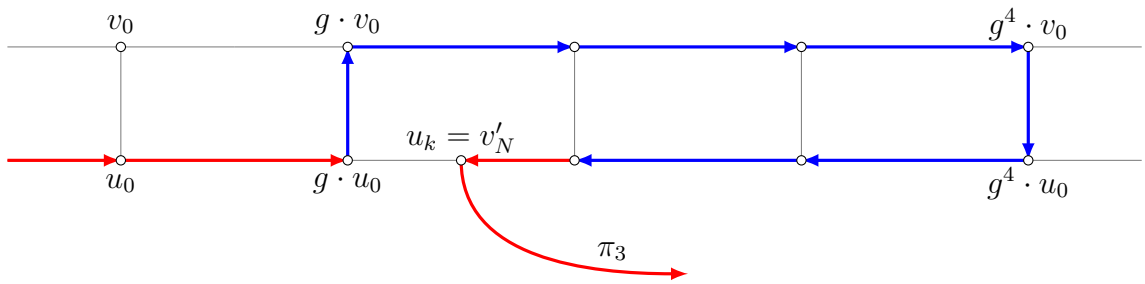


Figure 9: The periodic subdivision of the 2-ladder with the configuration x highlighted in red and blue. The word $\lambda_2\lambda_3^3\lambda_2^{-1}\lambda_1^{-2}$ is marked in blue, whereas the infinite prefix and suffix of x are marked in red.

Let us label the different sections of the bi-infinite ladder. We denote by λ_1 the label of the path from u_0 to $g \cdot u_0$, λ_2 the label from u_0 to v_0, λ_3 the label from v_0 to $g \cdot v_0$, λ_4 the label from $g^2 \cdot u_0$ to u_k, and $\lambda \in S^{\mathbb{N}}$ the label of the ray $(v'_{N+i})_{i\in\mathbb{N}}$. Then, for every $n \in \mathbb{N}$ and $k < n$ the configuration

$$x = \lambda_1^\infty.\lambda_2\lambda_3^n\lambda_2^{-1}\lambda_1^{-k}\lambda_4\lambda \in S^{\mathbb{Z}},$$

belongs to the skeleton. Then, $\lambda_2\lambda_3^n\lambda_2^{-1}\lambda_1^{-k} \in \mathcal{L}(\mathbb{X}_{G,S})$ for all $k < n$. Notice that the language given by $L = \{\lambda_2\lambda_3^n\lambda_2^{-1}\lambda_1^{-k} \mid n, k \in \mathbb{N}\}$ is regular. Therefore, $L' = L \cap \mathcal{L}(\mathbb{X}_{G,S})$ is regular as we assume $\mathbb{X}_{G,S}$ is sofic. Take $p > 0$ the pumping length of L' given by the Pumping Lemma. If we pump the word $\lambda_2\lambda_3^{p+1}\lambda_2^{-1}\lambda_1^{-p}$ is in L' as we did in the proof of Lemma 53, we conclude that there must exist $n, k \in \mathbb{N}$ with $k \geqslant n$ such that $\lambda_2\lambda_3^n\lambda_2^{-1}\lambda_1^{-k} \in L'$, which is a contradiction as it is not self-avoiding. □

Virtually $\mathbb{Z}$ groups have a very rigid structure. Epstein and Wall [17, 60] (see [47] for our current formulation) showed that a group is virtually $\mathbb{Z}$ if and only if it is of one of the following forms:

1. $\mathbb{Z} \ltimes_\phi F$, for some finite group F and $\phi \in \mathrm{Aut}(F)$,

2. $G_1 *_F G_2$, for G_1, G_2 and F finite groups such that $[G_1 : F] = [G_2 : F] = 2$.

Groups of the second type, $G_1 *_F G_2$, can be shown to be isomorphic to $\mathcal{D}_\infty \ltimes_\psi F$ for some homomorphism $\psi : \mathcal{D}_\infty \to \mathrm{Aut}(F)$ (see [21, Section 1.3]). Furthermore, every element $g \in \mathbb{Z} \ltimes_\phi F$ can be uniquely expressed as ft^n with $f \in F$, $n \in \mathbb{Z}$ and t the free generator of $\mathbb{Z}$. Similarly, every element $g \in \mathcal{D}_\infty \ltimes_\phi F$ can be uniquely expressed as $f\mathtt{r}^n\mathtt{s}^b$ with $f \in F$, $n \in \mathbb{Z}$, $b \in \{0, 1\}$, and $\mathtt{r}$ and $\mathtt{s}$ generators for $\mathcal{D}_\infty = \langle \mathtt{r}, \mathtt{s} \mid \mathtt{s}^2, \mathtt{rsrs} \rangle$.

Lemma 56. *Let $G = H \ltimes_\phi F$ be a group such that F is a finite group, and H is either $\mathbb{Z}$ or $\mathcal{D}_\infty$. Then, for any generating set S the ends of the Cayley graph $\Gamma(G, S)$ have size at least $|F|$.*

Proof. Take G as in the hypothesis. We will tackle the case when $H = \mathbb{Z}$ and $H = \mathcal{D}_\infty$ separately.

Case 1: $H = \mathbb{Z}$:
Let S be a generating set for G. Then, there must exist at least one generator that does not belong to F, which we call s. This generator, must have the form $s = gt^n$ for some $g \in F$ and $n \in \mathbb{Z}$, and is thus a torsion-free element of the group. For each element $f \in F$ we define the ray $\pi_f = (f, fs, \ \ldots, \ fs^i, \ \ldots)$. These rays are all pair-wise disjoint because s is torsion-free. Therefore, the end $D(s)$ has size at least $|F|$.

Case 2: $H = \mathcal{D}_\infty$:
Let S be a generating set for G. As before, there must exist at least one generator that does not belong to F, which we call s. If s is of the form $g\mathtt{r}^n$, it is a torsion free element, and by the argument for the previous case, $D(s)$ has size at least $|F|$. Suppose then that all elements $S \setminus F$ are of the form $g\mathtt{r}^n\mathtt{s}$. Because S is a generating set, $S \setminus F$ must contain at least two elements which we will name $s = g\mathtt{r}^n\mathtt{s}$ and $s' = g'\mathtt{r}^m\mathtt{s}$. Without loss of generality take $n > m$. Then, ss' is the torsion-free element $g_1\mathtt{r}^{n-m}$ for some $g_1 \in F$. As before, for each $f \in F$ define the ray $\pi_f = (f, fs, fss', \ \ldots, \ f(ss')^i, \ \ldots)$. Let us prove these rays are disjoint. If $f(ss')^k = f'(ss')^l$ for $f, f' \in F$ and $k, l \in \mathbb{N}$, then $(ss')^{k-l} \in F$ has torsion, which is a contradiction. On the other hand, if $f(ss')^k s = f'(ss')^l$ for $f, f' \in F$ and $k, l \in \mathbb{N}$, then $\mathtt{r}^{(n-m)(k+l)+n}\mathtt{s} \in F$, which is also a contradiction. Thus, the rays π_f are disjoint and therefore $D(ss')$ has size at least $|F|$. □

Proposition 57. *Let G be a virtually $\mathbb{Z}$ group. Then, there exists S such that $\mathbb{X}_{G,S}$ is sofic if and only if G is either $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty$.*

Proof. Let G be a virtually $\mathbb{Z}$ group. Then, G is of the form $H \ltimes_\phi F$ for $H \in \{\mathbb{Z}, \mathcal{D}_\infty\}$ and F a finite group. Joining Lemma 56 and Lemma 53, if $\mathbb{X}_{G,S}$ is sofic for some generating set S, $|F| \leqslant 2$. If $|F| = 1$, then G is either $\mathbb{Z}$ or $\mathcal{D}_\infty$. If $|F| = 2$, then $F \simeq \mathbb{Z}/2\mathbb{Z}$ and ϕ is the trivial automorphism. In this case G is either $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$.

Conversely, we already know $\mathbb{Z}$ and $\mathcal{D}_\infty$ admit sofic skeletons as they are plain groups. Similarly, by Lemma 45, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ admits a sofic skeleton. Finally, if we take the presentation for $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$ given by $\langle a, b, s \mid a^2, b^2, s^2, (sa)^2, (sb)^2 \rangle$ the corresponding Cayley graph is the bi-infinite ladder, and therefore Lemma 45 can be adapted to show its skeleton is sofic. □

We now have all the ingredients to characterize groups that admit a sofic skeleton.

Proof of Theorem 52. Let G be a finitely generated group that admits a sofic skeleton through the generating set S. From Lemma 9, $\Gamma(G, S)$ has only thin ends, all of size at most 2. If all ends are of size 1, G is a plain group. Next, if G has at least one end of size 2, it is virtually $\mathbb{Z}$ by Lemma 55. Then, by Proposition 57 G is either $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$. For the other direction, if G is a plain group by Theorem 26 it admits a sofic skeleton (as SFTs are sofic). Finally, if G is either $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$, Proposition 57 tells us G admits a sofic skeleton. □

6 Approximating Entropy and Connective Constants

6.1 Bridges and Periodic Points

We saw in Corollary 31 that the connective constant of SFT skeletons of plain groups can be approximated by their periodic points. This is also the case of irreducible sofic shifts [49, Theorem 4.3.6.]. The natural question that follows is if this is possible for skeletons that are not sofic. Clisby showed [14] that this is the case for $\mathbb{Z}^d$ with standard generating set, which by Theorem 52 do not define sofic skeletons. Instead of periodic configurations, Clisby used the term *endless SAWs*. By using the notion of a bridge, introduced by Hammersley and Welsh [36] and latter expanded upon by Grimmett and Li [29], we can generalize this result to any Cayley graph admitting a particular kind of graph height function.

Definition 58. Let Γ be an infinite, connected, locally finite, quasi-transitive graph. A **graph height function** (h, H) is composed of a function $h : V_\Gamma \to \mathbb{Z}$ and a subgroup $H \leqslant \mathrm{Aut}(\Gamma)$ acting quasi-transitively on Γ such that

- (H-difference-preserving) for all $u, v \in V_\Gamma$ and $g \in H$

$$h(g \cdot v) - h(g \cdot u) = h(v) - h(u),$$

- for all $u \in V_\Gamma$, there exists $v, v' \in V_\Gamma$ adjacent to u such that $h(v) < h(u) < h(v')$.

A **bridge** with respect to the height function (h, H) is a self-avoiding walk $\pi = (e_0, \ \ldots, \ e_{n-1})$ that verifies

$$h(\mathfrak{i}(e_0)) < h(\mathfrak{t}(e_i)) \leqslant h(\mathfrak{t}(e_{n-1})),$$

for all $i \in \{0, \ldots, n-1\}$.

Example 59. Take $G = \mathbb{Z}^2$ with the standard generating set $\{\mathtt{a}^{\pm}, \mathtt{b}^{\pm}\}$. If we look at the generators as $\mathtt{a} = (1,0)$ and $\mathtt{b} = (0,1)$, we define the map $h(g) = m$ for $g = (m,n) \in \mathbb{Z}^2$. This function defines a graph height function with respect to $H = \mathbb{Z}^2$ acting by left-translations. Further still, any elementary amenable group admits a graph height function [29].

Lemma 60. *Let G be a finitely generated group with generating set S. If $\Gamma(G,S)$ admits a graph height function (h,H), then $\mathbb{X}_{G,S}$ contains periodic configurations. Moreover, if π is a bridge such that $\mathfrak{i}(\pi)$ and $\mathfrak{t}(\pi)$ lie in the same H-orbit, then $\lambda(\pi)^{\infty} \in \mathbb{X}_{G,S}$.*

Proof. Let π and π' be two bridges such that $\mathfrak{t}(\pi) = \mathfrak{i}(\pi')$. Then, the concatenation of both paths, $\pi\pi'$, is a bridge. Furthermore, for every $g \in H$, $g \cdot \pi$ is also a bridge, as h is H-difference-preserving.

Now, let R be a finite right transversal for the action of H on $\Gamma(G,S)$. Take a bridge π such that $\mathfrak{i}(\pi), \mathfrak{t}(\pi) \in H \cdot r$ with $r \in R$. If $\mathfrak{i}(\pi) = h_1 \cdot r$ and $\mathfrak{t}(\pi) = h_2 \cdot r$, because h is H-difference-preserving, $h_2 h_1^{-1} \cdot \pi$ is a bridge starting at $h_2 \cdot r$. We can then concatenate π with $h_2 h_1^{-1} \cdot \pi$ to create a bridge, which we denote by π^2, whose label is given by $\lambda(\pi)^2$. This process can be iterated indefinitely to obtain a bi-infinite SAW whose label is given by $\lambda(\pi)^{\infty}$.

Next, take a bridge π such that $\mathfrak{i}(\pi) \in H \cdot r_1$ and $\mathfrak{t}(\pi) \in H \cdot r_2$, with $r_1, r_2 \in R$ distinct representatives. Up to translation by an element from H, we can take any bridge starting at a vertex in $H \cdot r_2$, say π_1 and concatenate to π to obtain a new bridge $\pi\pi_1$. Such a bridge exists by the definition of a graph height function as there must exist at least one vertex v next to r_2 such that $h(r_2) < h(v)$. Similarly, we can take any bridge in the H-orbit of $\mathfrak{t}(\pi_1)$, which we denote π_2, and concatenate it –up to translation by H– to $\pi\pi_1$. Iterating this process, for all $n \in \mathbb{N}$ we obtain a bridge $\pi\pi_1 \cdots \pi_n$. Because there is a finite number of H-orbits, we will have $i \leqslant j$ such that $\mathfrak{i}(\pi_i), \mathfrak{t}(\pi_j)$ belong to the same H-orbit. Then, as previously stated $\pi' = \pi_i \pi_{i+1} \cdots \pi_j$ is a bridge that can be iterated to obtain the periodic point $\lambda(\pi')^{\infty}$. □

We saw in Theorem 38 that torsion groups have aperiodic skeletons. By the previous lemma, graph height functions imply the existence of periodic points. Combining these two facts we can state the following.

Theorem 61 (Theorem C)**.** *The Cayley graphs of infinite torsion f.g. groups do not admit graph height functions.*

This generalizes a result from Grimmett and Li who showed that the Grigorchuk group (which is an infinite torsion group) does not admit a graph height function, and more generally, Cayley graphs of torsion groups with certain conditions on the stabilizer of the identity [28]. However, the converse of the previous theorem does not hold, as they also showed that the Higman group, which is torsion-free [39], does not admit graph height functions.

Bridges are particularly useful to compute the connective constant of graphs, and have been used to obtain exact expressions for the constant (for instance, [15]). Let us denote by

$b_{n,g}$ the number of bridges of length n starting at $g \in G$, and $b_n = \min_{g \in G} b_{n,g}$. As stated in the proof of Lemma 60, we can concatenate bridges with corresponding endpoints. Then, $b_n b_m \leqslant b_{n+m}$ and by Fekete's sub-additive Lemma, there exists a constant $\beta(\Gamma, h)$, where $\Gamma = \Gamma(G, S)$, such that

$$\beta(\Gamma, h) = \lim_{n \to \infty} \sqrt[n]{b_n}.$$

This process can be done for a larger class of graphs [29], and helps us compute connective constants.

Theorem 62 (General Bridge Theorem [50])**.** *Let Γ be an infinite, connected, locally finite, quasi-transitive graph. Then, if Γ admits a graph height function (h, H),*

$$\mu(\Gamma) = \max\{\beta(\Gamma, h), \beta(\Gamma, -h)\}.$$

Using this result, we can find conditions under which periodic points approximate the connective constant. In other words, periodic points from $\mathbb{X}_{G,S}$ approximate its entropy.

Theorem 63 (Theorem D)**.** *Let G be a finitely generated group and S a finite generating set. If $\Gamma(G, S)$ admits a graph height function (h, H) such that H acts transitively on $\Gamma(G, S)$, then*

$$\mu(G, S) = \lim_{n \to \infty} \sqrt[n]{e_n},$$

where e_n denotes the number of periodic points in $\mathbb{X}_{G,S}$ of period $n \in \mathbb{N}$.

Proof. Let us denote by $\bar{b}_n$ the minimum over all $g \in G$ of the number of bridges of length n starting at g for the graph height function $(H, -h)$. Because H acts transitively on $\Gamma(G, S)$, there is a single H-orbit. Thus, by Lemma 60, every bridge for h and $-h$ can be iterated to obtain a periodic point. This means,

$$\max\{b_n, \bar{b}_n\} \leqslant e_n \leqslant c_n.$$

By taking the nth root and limit, Theorem 62 implies,

$$\mu(G, S) = \max\{\beta(\Gamma, h), \beta(\Gamma, -h)\} \leqslant \lim_{n \to \infty} \sqrt[n]{e_n} \leqslant \mu(G, S). \qquad \square$$

Examples of Cayley graphs with a graph height function (h, H) such that H acts transitively are given by Cayley graphs that admit *strong* graph height function where $H = G$. Strong graph height functions are graph functions where we also ask for H to be a finite index subgroup of G, and to act by left translations [28]. A class of groups that admit such functions are groups with strictly positive first Betty number [27]. Other sufficient conditions can be found in [31].

6.2 Lower Bounds with Self-Avoiding Polygons

What other methods can we use when graph height functions are not available? We will make use of a counting argument popularized by Rosenfeld [56] to find lower bounds on the connective constant by studying the sets of forbidden patterns defining the skeleton. Rosenfeld found the following criterion for subshifts.

Theorem 64 ([57], Corollary 12)**.** *Let A be a finite alphabet and $\mathcal{F} \subseteq A^+$ a set of connected forbidden patterns. If there exists a positive real number $\beta > 1$ such that*

$$|A| \geqslant \beta + \sum_{n \geqslant 0} f_n \beta^{1-n},$$

then $\alpha(\mathcal{X}_{\mathcal{F}}) \geqslant \beta$, where f_n is the number of forbidden patterns of length n, that is, $f_n = |\mathcal{F} \cap A^n|$.

Therefore, we can use the different forbidden patterns we have found so far for the skeleton to find lower bounds for the connective constant. From Lemma 23, we know the set of SAPs along with words of the form ss^{-1} define a set of forbidden patterns for the skeleton.

Proposition 65. *Let (G, S) be an infinite finitely generated group. If there exists a positive real number β such that*

$$|S| - 1 \geqslant \beta + \sum_{n \geqslant 0} \rho_n \beta^{1-n},$$

then $\mu(G, S) \geqslant \beta$, where ρ_n the number of SAPs of length n, that is, $\rho_n = |\mathcal{O}_{G,S} \cap S^n|$.

The proof of the proposition is essentially the same as the one from [57], but we add it for completion.

Proof. Let L_k be the set of SAWs of length $k \in \mathbb{N}$. We prove by induction that $|L_k| \geqslant \beta |L_{k-1}|$, for $\beta > 1$ as in the statement. Notice $|L_0| = 1$ as it only contains the empty word, and $|L_1| = |S|$. By hypothesis, $\beta \leqslant |S|$, and therefore $|L_1| \geqslant \beta |L_0|$.

Suppose our statement is true up to some $k > 0$. In particular, for $j \leqslant k$

$$|L_{k-j}| \leqslant \frac{|L_k|}{\beta^j}.$$

Now, because every SAW from L_k can be extended in $|S| - 1$ ways, we have that

$$|L_{k+1}| = (|S| - 1)|L_k| - |B|,$$

where B is the set of SAWs that when extended generate a path of length $k + 1$ that self-intersects. Notice that if $u \in B$, it can be written in the form $u = u'v$, where v is a SAP, as u is the extension by one non-backtracking letter of a SAW. We define the sets $B_v = \{u \in B \mid u = u'v\}$ to obtain the upper bound $|B| \leqslant \sum_{v \in \mathcal{O}_{G,S}} |B_v|$. Then, every word in B_v is determined by a word from $L_{k+1-|v|}$, namely u'. Therefore,

$$|B_v| \leqslant |L_{k+1-|v|}| \leqslant \frac{|L_k|}{\beta^{|v|-1}},$$

and consequently,

$$|B| \leqslant |L_k| \sum_{v \in \mathcal{O}_{G,S}} \beta^{1-|v|} = |L_k| \sum_{n \geqslant 1} \rho_n \beta^{1-n}$$

Finally, joining all the formulas we obtain:

$$|L_{k+1}| \geqslant \left((|S| - 1) - \sum_{n \geqslant 1} \rho_n \beta^{1-n} \right) |L_k| \geqslant \beta |L_k|.$$

Our induction proven, we can iterate the identity to obtain $|L_k| \geqslant \beta^k$, and thus $\mu(G, S) \geqslant \beta$. □

This approach is different from the usual use of self-avoiding polygons to approximate $\mu(G, S)$ in the literature. We define the asymptotic growth rate for SAPs through,

$$\mu_{SAP} = \limsup_{n \to \infty} \sqrt{\rho_n}.$$

It has been shown that $\mu_{SAP} = \mu(G, S)$ for Euclidean lattices [34, 45], but $\mu_{SAP} < \mu(G, S)$ for many non-euclidean lattices, including some Cayley graphs of surface groups [55].

7 Geodesic skeletons

A geodesic is always a self-avoiding walk. It is then natural to see what changes when we restrict a group's skeleton to bi-infinite geodesics. We define the geodesic skeleton of G with respect to S by,

$$\mathbb{X}^g_{G,S} = \{x \in \mathbb{X}_{G,S} \mid \forall w \sqsubseteq x, w' =_G w : \ |w| \leqslant |w'|\}.$$

This subshift is contained in the skeleton $\mathbb{X}_{G,S}$, and the locally admissible language given by its defining forbidden patterns is $\mathrm{Geo}(G, S)$. In particular, $\mathbb{X}^g_{G,S}$ is generated by taking $\mathrm{Geo}(G, S)^c$ as the set of forbidden patterns. As was the case with the skeleton, $\mathbb{X}^g_{G,S}$ is empty if and only if the group is finite; this is due to Watkins who showed that every transitive infinite graph contains a bi-infinite geodesic [61].

SFT Geodesics We have a sufficient condition for the geodesic skeleton to be an SFT coming from a result by Gilman, Hermiller, Holt and Rees [23] that characterizes virtually free groups. They showed that for a finitely generated group G, there exists a finite generating set S such that $\mathrm{Geo}(G, S)$ is k-locally excluding, that is, there exists a set F of words of length k such that a word $w \in S^*$ is geodesic if no factor of length k belongs to F, if and only if G is virtually free. An immediate consequence is the following.

Proposition 66. *Let G be a virtually free group. Then, there exists S such that $\mathbb{X}^g_{G,S}$ is a SFT.*

Effective Geodesics

Lemma 67. *Let G be a recursively presented group. Then, $\mathbb{X}^g_{G,S}$ is effective for every finite generating set S.*

Proof. We describe a co-semi-algorithm for $\mathrm{Geo}(G,S)$. By using an enumeration for the word problem, we can test every word w' of length $|w'| < |w|$ to see if they define the same group element, i.e. $w'w^{-1} =_G 1_G$. If one such $w'w^{-1}$ appears in the enumeration, we know w is not geodesic and accept. If w is not geodesic, $w'w^{-1}$ will eventually be enumerated, for some w' of shorter length. When $w \in \mathrm{Geo}(G,S)$ the algorithm never stops. □

In other words, the effectiveness of $\mathbb{X}^g_{G,S}$ is a consequence of the fact that a recursively enumerable word problem implies that the language of geodesics is co-recursively enumerable.

Sofic Geodesics By their definition, we can easily obtain many examples of sofic geodesic skeletons.

Proposition 68. *Let G be a finitely generated group and S a generating set. If* $\mathrm{Geo}(G,S)$ *is regular, then $\mathbb{X}^g_{G,S}$ is sofic.*

Because the complement of a regular language is regular, when $\mathrm{Geo}(G,S)$ is regular then $\mathbb{X}^g_{G,S}$ is defined by a regular set of forbidden words, and is therefore sofic. We know that $\mathrm{Geo}(G,S)$ is regular for all generating sets in abelian groups [54] and hyperbolic groups [18]. Also, there exists at least one generating set such that $\mathrm{Geo}(G,S)$ is regular for virtually abelian groups [54], Coxeter groups [42] and other classes [13, 40, 3].

In order to find a characterization of groups that admit a geodesic skeleton that is sofic, we must look at geodesics that are not extendable. These elements are precisely the ones known as dead-ends. An element $g \in G$ is a **dead-end** with respect to the generating set S if for all $s \in S$ we have $d(1_G, gs) \leqslant d(1_G, g)$.

Proposition 69. *Let G be a finitely generated group along with a generating set S. Then, $\mathbb{X}^g_{G,S}$ is sofic and the language of geodesics defining dead-ends is regular if and only if* $\mathrm{Geo}(G,S)$ *is regular.*

Proof. Denote the language of geodesics defining dead-ends by D. If $\mathrm{Geo}(G,S)$ is regular, then D is regular: it suffices to take the minimal deterministic finite state automaton with a single sink state for $\mathrm{Geo}(G,S)$ and only keep accepting states where every outgoing transitions goes to the sink state. Furthermore, $\mathbb{X}^g_{G,S}$ is sofic by Proposition 68.

Conversely, suppose $\mathbb{X}^g_{G,S}$ is sofic and D is regular. To prove $\mathrm{Geo}(G,S)$ is regular in this case, we make use of the following notation: for a language $L \subseteq S^*$, consider the languages $L^{-1} = \{w \mid w^{-1} \in L\}$ and

$$mL = \{w \in S^* \mid \exists v, u \in S^* : \ vwu \in L\}.$$

Notice that if L is regular, then both L^{-1} and mL are regular. We claim that

$$\mathrm{Geo}(G,S) = \mathcal{L}(\mathbb{X}^g_{G,S}) \cup mD \cup m(D^{-1}).$$

Indeed, in this case $\mathrm{Geo}(G,S)$ is regular as it is the union of regular languages. Let us prove the equality. First, it is clear that $\mathcal{L}(\mathbb{X}^g_{G,S}) \subseteq \mathrm{Geo}(G,S)$. Next, because inverses of geodesics are geodesic, and factors of geodesics are geodesics, we have $mD \cup m(D^{-1}) \subseteq \mathrm{Geo}(G,S)$. This proves the left inclusion. For the converse, take $w \in \mathrm{Geo}(G,S)$. If w is bi-infinitely extendable as a geodesic, then $w \in \mathcal{L}(\mathbb{X}^g_{G,S})$. If w is not bi-extendable, there exist $v_1, v_2 \in S^*$ such that $v_1wv_2 \in \mathrm{Geo}(G,S)$, and $v_1wv_2s \notin \mathrm{Geo}(G,S)$ or for all $s \in S$ $sv_1wv_2 \notin \mathrm{Geo}(G,S)$. This means w is either a factor of dead-end (first case) or a factor of the inverse of a dead-end (second case). Therefore, $w \in mD \cup m(D^{-1})$. This proves our claim, and concludes the proof. □

Corollary 70. *Let G be any finitely generated group with generating set S, and $\mathbb{Z} = \langle t \rangle$. Then, $\mathbb{X}^g_{G*\mathbb{Z},S\cup\{t^{\pm1}\}}$ is sofic if and only if* $\mathrm{Geo}(G,S)$ *is regular.*

Proof. The language of dead-ends of $G * \mathbb{Z}$ is empty as any geodesic can be extended by $t^{\pm1}$. Furthermore, any geodesic in $G*\mathbb{Z}$ can be decomposed as geodesics on G separated by factors of the form $t^{\pm n}$. Therefore, $\mathrm{Geo}(G*\mathbb{Z}, S\cup\{t^{\pm1}\})$ is regular if and only if $\mathrm{Geo}(G,S)$ is regular. By Proposition 69 this happens if and only if $\mathbb{X}^g_{G*\mathbb{Z},S\cup\{t^{\pm1}\}}$ is sofic. □

We pose the following question for sofic geodesic skeleton.

Question 71. Is $\mathbb{X}^g_{G,S}$ sofic if and only if $\mathrm{Geo}(G,S)$ is regular?

Periodic Geodesics As was the case for the skeleton (Theorem 38), the aperiodicity of the geodesic skeleton also characterizes torsion groups.

Theorem 72. *Let G be a finitely generated group. Then, G is a torsion group if and only if $\mathbb{X}^g_{G,S}$ is aperiodic for every (any) generating set S.*

Proof. Suppose G contains a torsion-free element g. Then, by Proposition 37 for any generating set S, there exists $k \geqslant 1$ and $w \in S^*$ a geodesic for g^k such that $w^\infty \in \mathbb{X}^g_{G,S}$, which is a periodic configuration. Conversely, if there exists a periodic configuration $x = w^\infty \in \mathbb{X}^g_{G,S}$ for some generating set S, then $g = \overline{w}$ is a torsion-free element. □

7.1 Entropy and Connective Constant for Geodesics

The objective of this section is to define an analog of the connective constant for geodesics. This relies on finding the asymptotic growth rate of geodesics of a given length. The **geodesic growth** of G with respect to S is the map $\Gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ given by

$$\Gamma_{G,S}(n) = |\{w \in \mathrm{Geo}(G,S) \mid |w| \leqslant n\}|.$$

Because this function is sub-multiplicative we can define the **geodesic connective constant** of the Cayley graph $\Gamma(G,S)$ as

$$\mu^g(G,S) = \lim_{n\to\infty} \sqrt[n]{\Gamma_{G,S}(n)}.$$

As we saw in Remark 16, $\mu^g(G,S)$ can be shown to be equal to the growth rate of the number of geodesics of length exactly n. Thus, the geodesic growth is an upper bound on the complexity of $\mathbb{X}^g_{G,S}$. Because $\mathrm{Geo}(G,S)$ is the set of locally admissible words for the geodesic skeleton, we use Lemma 1 to obtain an expression for the entropy.

Lemma 73. *Let G be a finitely generated group along with a generating set S. Then,*

$$h(\mathbb{X}^g_{G,S}) = \log(\mu^g(G,S)).$$

In other words, the geodesic connective constant is equal to the connective constant of bi-extendable geodesics.

Same as with the connective constant, the geodesic version is a non-negative rational power of a Perron number when $\mathbb{X}^g_{G,S}$ is sofic, and a right-computable number when $\mathbb{X}^g_{G,S}$ is effective. It is also a lower bound of the connective constant, that is, $\mu^g(G,S) \leqslant \mu(G,S)$. This inequality may be strict: graphs may have geodesic connective constant equal to 1 without being finite. As shown in [10], the virtually $\mathbb{Z}^2$ group $H = \langle \mathtt{a}, \mathtt{t} \mid [\mathtt{a}, \mathtt{tat}^{-1}], \mathtt{t}^2 \rangle$, has geodesic growth of order $O(n^3)$ and therefore,

$$\mu^g(H, \{\mathtt{a}, \mathtt{t}\}) = 1 < \sqrt{3} \leqslant \mu(H, \{\mathtt{a}, \mathtt{t}\}).$$

This is also the case for lattices with known (or well-approximated) connective constants.

Proposition 74. *The geodesic connective constants of the square grid, ladder graph and hexagonal grid are as follows:*

- $\mu^g(\mathbb{Z}^2) = 2$,
- $\mu^g(\mathbb{L}) = 1$,
- $\mu^g(\mathbb{H}) = \sqrt{2}$.

Proof.

- For the square lattice, we know that $\Gamma_{\mathbb{Z}^2,\{\mathtt{a},\mathtt{b}\}}(n) \leqslant 2^{n+3}$ which implies $h(\mathbb{X}^g_{\mathbb{Z}^2,\{\mathtt{a},\mathtt{b}\}}) = \log(2)$, as $\mathbb{X}^g_{G,S}$ contains the full-shift $\{\mathtt{a}, \mathtt{b}\}^{\mathbb{Z}}$.
- Recall that the ladder graph $\mathbb{L}$ is the Cayley graph of $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generating set $\{\mathtt{t}, \mathtt{s}\}$, where $\mathtt{s}^2 =_G \epsilon$ and $\mathtt{t}$ is the generator for $\mathbb{Z}$. In this case, the geodesic growth is given by $\Gamma(n) = n^2 + 3n$ when $n \geqslant 2$. Thus, the geodesic connective constant is 1.
- Also recall that the hexagonal grid $\mathbb{H}$ is the Cayley graph of the Coxeter group $\tilde{A}_2$ with generating set $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ (see Example 7). From [5] we know that the generating function for the geodesic growth of $\tilde{A}_2$ in this case is given by

  $$f(z) = \frac{2z^3 + z^2 + z + 1}{(1-z)(1-2z^2)}.$$

  Thus, the geodesic connective constant is given by the reciprocal of the smallest zero of the denominator, which is $\sqrt{2}$. □

On the other hand, if we take the infinite dihedral group $\mathcal{D}_\infty$ with the generating set $S = \{\mathtt{a}, \mathtt{b}\}$ as seen in Example 6, we have that $\mu(\mathcal{D}_\infty, S) = \mu^g(\mathcal{D}_\infty, S) = 1$.

Question 75. Under which conditions $\mu(G, S) = \mu^g(G, S)$? Under which conditions is the inequality strict?

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