Permutations that Separate Close Elements, and Rectangle Packings in the Torus

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Abstract

Let n, s and k be positive integers. For distinct $i, j \in \mathbb{Z}_n$, define $||i, j||_n$ to be the distance between i and j when the elements of $\mathbb{Z}_n$ are written in a circle. So

$$||i, j||_n = \min\{(i - j) \bmod n, (j - i) \bmod n\}.$$

A permutation $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ is (s, k)-*clash-free* if $||\pi(i), \pi(j)||_n \geqslant k$ whenever $||i, j||_n < s$. So an (s, k)-clash-free permutation moves every pair of close elements (at distance less than s) to a pair of elements at large distance (at distance at least k). The notion of an (s, k)-clash-free permutation can be reformulated in terms of certain packings of $s \times k$ rectangles on an $n \times n$ torus.

For integers n and k with $1 \leqslant k < n$, let $\sigma(n, k)$ be the largest value of s such that an (s, k)-clash-free permutation of $\mathbb{Z}_n$ exists. Strengthening a recent paper of Blackburn, which proved a conjecture of Mammoliti and Simpson, we determine the value of $\sigma(n, k)$ in all cases.

Mathematics Subject Classifications: 05B40, 05A05

1 Introduction

Let n, s and k be positive integers. As in the abstract, for $i, j \in \mathbb{Z}_n$ we define $||i, j||_n$ to be the distance between i and j when the elements of $\mathbb{Z}_n$ are written in a circle, so $||i, j||_n = \min\{(i - j) \bmod n, (j - i) \bmod n\}$. Let $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ be a permutation of $\mathbb{Z}_n$. An (s, k)-*clash* is a pair of distinct elements $i, j \in \mathbb{Z}_n$ such that $||i, j||_n < s$ and $||\pi(i), \pi(j)||_n < k$. (So i and j, and their image under π are both close.) The permutation π is (s, k)-*clash-free* if there are no (s, k)-clashes.

For integers n and k with $1 \leqslant k < n$, we define $\sigma(n, k)$ be the largest value of s such that an (s, k)-clash-free permutation π of $\mathbb{Z}_n$ exists. What can be said about

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$\sigma(n, k)$? This question was first considered by Mammoliti and Simpson [9], who showed that $\sigma(n, k) \leqslant \lfloor (n-1)/k \rfloor$ and conjectured that

$$\sigma(n, k) \in \{\lfloor (n-1)/k \rfloor - 1, \lfloor (n-1)/k \rfloor\} \tag{1}$$

for all n and k. This conjecture was proved by Blackburn [3], who gave an explicit construction of an (s, k)-clash-free permutation with $s = \lfloor (n-1)/k \rfloor - 1$ for all n and k.

As pointed out in [9], we can think of the problem of constructing (s, k)-clash-free permutations as a problem of packing rectangles on a torus as follows. Consider an $n \times n$ array consisting of n^2 cells, each cell indexed by a pair of elements in $\mathbb{Z}_n$. We index the cells in cartesian fashion, so for example cell $(0, 0)$ is at the lower left corner of the array, and cell $(n-1, 0)$ is at the lower right corner of the array. Given a permutation π of $\mathbb{Z}_m$, we add n dots to the array in cells $(i, \pi(i))$ for $i \in \mathbb{Z}_n$. (This is sometimes known as the *plot* or the *graph* of the permutation.) We add $s \times k$ rectangles (so rectangles that are s cells wide and k cells high) to the array, by shading cells $(i + x, \pi(i) + y)$ for $0 \leqslant x < s$ and $0 \leqslant y < k$. Then π is (s, k)-clash-free if and only if these rectangles do not overlap. See Figure 1 for an example. So we may rephrase the problem of constructing an (s, k)-clash-free permutation as finding a (non-overlapping) packing of n rectangles of size $s \times k$ in an $n \times n$ discrete torus, such that no two rectangles are aligned horizontally or aligned vertically.

The case $k = 2$ of this problem has been considered [1, 5, 8] under the name of cyclic matching sequencibility of graphs. Non-cyclic versions of the problem (in which we work over the integers $\{0, 1, \ldots, n-1\}$ rather than modulo n) have also been studied [9]. (The special case of the non-cyclic problem when $k = s = 2$ can be rephrased as the problem of placing n non-attacking kings on an $n \times n$ chessboard, one in each row and each column. This is the problem of Hertzsprung [6] from 1887, rediscovered by Kaplansky [7] in 1944.) Packings of diamonds rather than rectangles (called permuted packings) have applications to permutation patterns [2, 4]; here we are looking for permutations whose dots have large minimum distance in the Manhattan metric (the ℓ_1 metric).

Given (1), a very natural question to ask is: When does $\sigma(n, k) = \lfloor (n-1)/k \rfloor$? Mammoliti and Simpson found many parameters where this is the case. For example, they showed [9, Theorem 3.7] that, setting $s = \lfloor (n-1)/k \rfloor$, we have $\sigma(n, k) = s$ when

$$k|n,\ s|n,\ \gcd(n, k) = 1,\ \text{or}\ \gcd(n, s) = 1. \tag{2}$$

Indeed, using this result and the fact that an (s, k)-clash-free permutation is (s', k')-clash-free when $s' \leqslant s$ and $k' \leqslant k$, they were able to show that $\sigma(n, k) = \lfloor (n-1)/k \rfloor$ for all choices of n and k with $n \leqslant 30$, apart from $(n, k) \in \{(18, 4), (26, 4), (26, 6)\}$. In these last three cases, they showed by computer search that $\sigma(n, k) = \lfloor (n-1)/k \rfloor - 1$. As pointed out in [3], the first open cases with $n > 30$ after applying the techniques in [9] are $(n, k) \in \{(34, 4), (34, 8), (38, 6), (39, 6), (40, 6)\}$.

The main theorem in this paper completely resolves the question of the value of $\sigma(n, k)$, for all values of n and k. Indeed, we prove the following theorem:

Theorem 1. *Let n and k be fixed positive integers, with $k < n$. Write $s = \lfloor (n-1)/k \rfloor$, so $n = sk + r$ where $1 \leqslant r \leqslant k$. Define $d_k = \gcd(n, k)$ and $d_s = \gcd(n, s)$.*

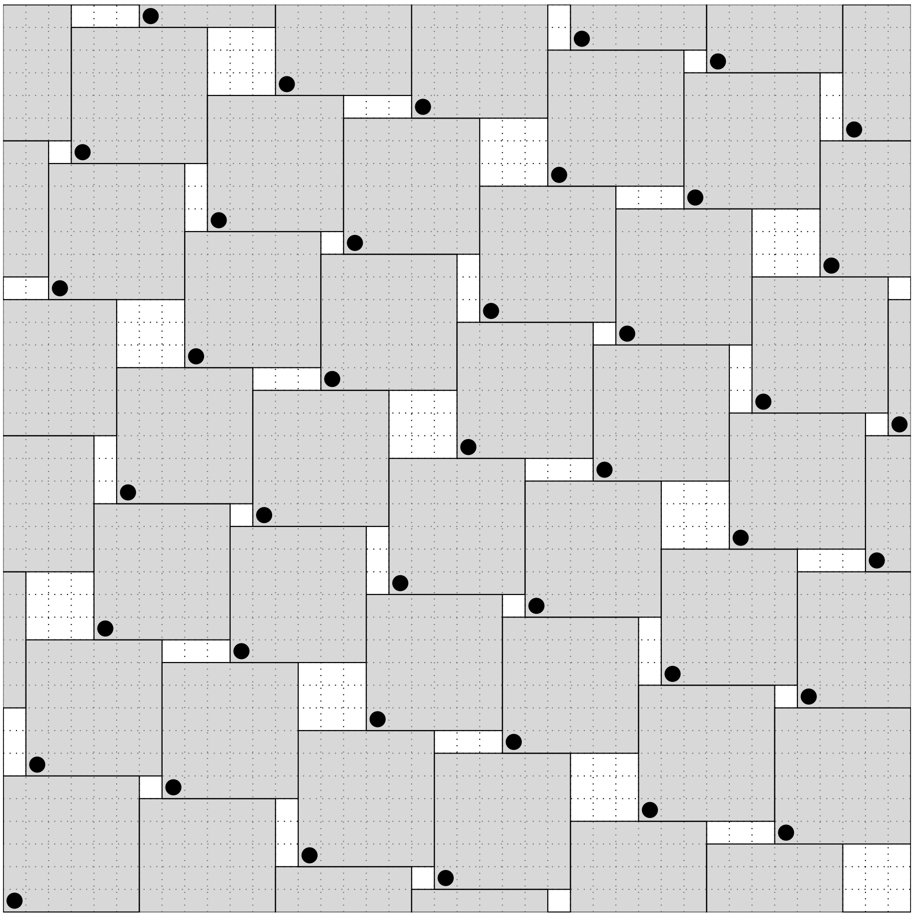

Figure 1: When $n = 40$, $k = s = 6$ and $\pi : \mathbb{Z}_{40} \to \mathbb{Z}_{40}$ is the permutation mapping $0, 1, 2, 3, 4, \ldots$ to $0, 6, 27, 33, 12, \ldots$ respectively.

- *If $r \geqslant s$ or $k = r$, then $\sigma(n,k) = \lfloor (n-1)/k \rfloor$.*
- *If $r < s$ and $r < k$ and $d_s d_k$ divides n, then $\sigma(n,k) = \lfloor (n-1)/k \rfloor$.*
- *If $r < s$ and $r < k$ and $d_s d_k$ does not divide n, then $\sigma(n,k) = \lfloor (n-1)/k \rfloor - 1$.*

Given (1), this theorem is equivalent to the assertion that an (s,k)-clash-free permutation with $s = \lfloor (n-1)/k \rfloor$ exists if and only if $s \leqslant r$, $k = r$, or $d_s d_k$ divides n.

Considering the small open cases listed above, Theorem 1 shows in particular (see Figure 1) that $\sigma(n,k) = \lfloor (n-1)/k \rfloor$ when $(n,k) = (40,6)$, and $\sigma(n,k) = \lfloor (n-1)/k \rfloor - 1$ when $(n,k) \in \{(34,4),(34,8),(38,6),(39,6)\}$.

We remark that the constructions in [3] and [9], together with Theorem 9 below, provide explicit constructions for permutations meeting the bound $\sigma(n,k)$ in all cases.

We observe that the first statement of the theorem follows from known results. To show this, first observe that if $k = r$ then k divides n, and so $\sigma(n,k) = \lfloor (n-1)/k \rfloor$ by [9, Theorem 3.7] and the theorem holds in this case. So we may assume without loss of generality that $1 \leqslant r < k$. Secondly, we observe that if $r > s$ then

$$\lfloor (n-1)/s \rfloor = \lfloor (ks + (r-1))/s \rfloor \leqslant k+1$$

and so $\lfloor (n-1)/s \rfloor - 1 \leqslant k$. In this case the construction from Blackburn [3, Section 2] with the roles of r and s swapped gives a (k,s)-clash-free permutation of $\mathbb{Z}_n$. Since the existence of an (s,k)-clash-free permutation is equivalent to the existence of a (k,s)-clash-free permutation, we find that the theorem holds in this case also. Hence it suffices to assume that $r \leqslant s$. Finally, if $r = s$ then s divides r, and the theorem follows by [9, Theorem 3.7]. Hence the first assertion of the theorem holds.

So it suffices to prove the final two assertions of the theorem. To this end, we will assume from now on that $n = sk + r$ where r is positive, $r < s$ and $r < k$.

In Section 2, we will provide some structural information on permutations π of $\mathbb{Z}_n$ that are $(\lfloor (n-1)/k \rfloor, k)$-clash-free. In particular we show (see Theorem 7 below) that if an (s,k)-clash-free permutation exists then $d_s d_k$ must divide n. So if $d_s d_k$ does not divide n then no (k,s)-clash-free permutation exists and hence the third statement of the Theorem 1 will follow.

In Section 3 we assume that $d_s d_k$ divides n and provide an explicit construction for an (s,k)-clash-free permutation; see Theorem 9 below. This will prove the second statement of the theorem, and completes the proof of Theorem 1.

In fact, the techniques of Sections 2 and 3 can be extended to provide a comprehensive classification of (s,k)-clash-free permutations with $s = \lfloor (n-1)/k \rfloor$ when $r < k$, $s < k$ and $d_s d_k$ divides n. Section 4 sketches a proof of this classification.

2 The structure of extremal clash-free permutations

Let n and k be such that $k < n$. Define $s = \lfloor (n-1)/k \rfloor$ and write $n = sk + r$ with $1 \leqslant r$. Assume that $r < s$ and $r < k$. Let π be an (s,k)-clash-free permutation. This section explores the structure of π. We will argue from the point-of-view of a rectangle packing for the sake of clarity. So we represent the torus $\mathbb{Z}_n^2$ as an $n \times n$ grid of 1×1 cells, indexed in the usual cartesian fashion with the origin at the lower left-hand corner. We place dots in the cells with coordinates $(i, \pi(i))$. We place n rectangles in the torus, all s cells wide and k cells high, with the ith rectangle R_i having cell $(i, \pi(i))$ at its lower left corner. The fact that π is (s, k)-clash-free implies that these rectangles do not overlap.

Lemma 2. *Every column of the grid contains exactly r cells that are not covered by a rectangle. Every row contains exactly r cells that are not covered by a rectangle.*

Proof. Let $x \in \mathbb{Z}_n$, and consider the column of all cells with first co-ordinate x. The ith rectangle R_i intersects our column if and only if $i \in \{x, x-1, \ldots, x-(s-1)\}$. If the rectangle intersects our column, it covers exactly k cells. Since the s rectangles that intersect our column are disjoint, exactly sk cells in our column are covered. So there are $n - sk = r$ cells in our column not covered by a rectangle. The argument for the row of cells with second co-ordinate $y \in \mathbb{Z}_n$ is essentially the same: there are k rectangles intersecting our row (the rectangles R_i where $\pi(i) \in \{y, y-1, \ldots, y-(k-1)\}$) and each rectangle covers s cells in the row. □

A cell is called *free* if it is not contained in a rectangle, and is otherwise *covered*. So the lemma above states that every row and every column contains exactly r free cells.

Lemma 3. *Let R be a fixed rectangle. Then exactly one rectangle touches the top edge of R. The same is true for the left, right and bottom edges of R.*

Proof. Consider the s cells that lie just above R. All these cells lie in the same row y, and this row contains r free cells by Lemma 2. Since $r < s$, not all of the s cells are free. So at least one is occupied, and there is a rectangle touching the top edge of R. There cannot be two rectangles touching the top edge of R, since then these rectangles would both have dots with second coordinate y, contradicting the fact that π is injective. So there is exactly one rectangle touching the top edge of y, and the first statement of the theorem follows. The same argument (applied three times) establishes the second statement of the theorem. □

Let Γ be the graph with vertex set $\mathbb{Z}_n$, with vertices i and j joined by an edge if and only if the rectangles R_i and R_j touch on their left or right sides. Let Δ be the graph with the same vertex set $\mathbb{Z}_n$, but with vertices i and j joined by an edge if and only if the rectangles R_i and R_j touch on their top or bottom sides. We call the connected components of Γ *warp threads*. We call the connected components of Δ *weft threads*. (The terminology warp and weft comes from weaving. Parallel warp threads are set up in a loom, and the shuttle with a weft thread attached weaves through the warp threads, thus forming the transverse threads in the woven textile.)

Lemma 4. *Each warp thread U is a cycle containing n/d_s vertices. The vertices in U correspond to rectangles R_i where i lies in a fixed congruence class modulo d_s. Each weft thread V is a cycle containing n/d_k vertices. The vertices in V correspond to rectangles R_i where $\pi(i)$ lies in a fixed congruence class modulo d_k.*

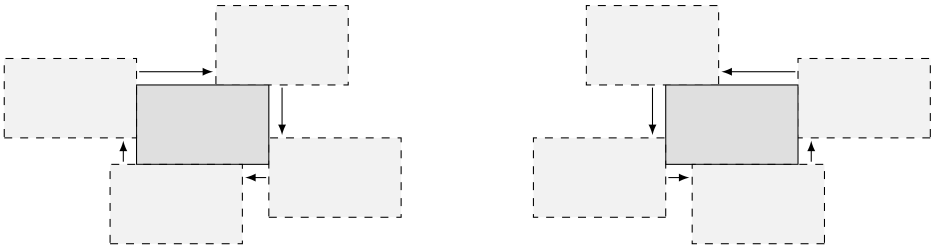

Figure 2: Clockwise (left) and anticlockwise (right) rectangles.

Proof. Lemma 3 implies that warp threads are all cycles, since every vertex of the thread has degree 2. The first co-ordinates of rectangles in a warp thread U increases by s as we move right along the cycle. So we return to the start of the cycle after ℓ steps, where ℓ is the additive order of s modulo n. Since $\ell = n/d_s$, the first statement of the theorem follows. The second statement follows from this observation, together with the fact that the number of elements $i \in \mathbb{Z}_n$ in a congruence class modulo d_s is also n/d_s. The final statement of the lemma is proved similarly, using the fact that two touching rectangles in a weft thread V have second co-ordinates that differ by k. □

Let R be a rectangle in our packing. There are four rectangles touching R, none of which is completely aligned horizontally or vertically with R (because π is a permutation). As can be seen in Figure 2 (where the arrows span a positive number of cells), the fact that rectangles do not overlap shows that the rectangles touching R can be arranged in one of two ways: we call R a *clockwise rectangle* or *anticlockwise rectangle* respectively. This figure also shows that touching rectangles are clockwise when R is clockwise, and anticlockwise otherwise. So all the rectangles in our packing are either clockwise or anticlockwise: we call π a clockwise or anticlockwise permutation depending on which case occurs. Reflecting in a horizontal line (or replacing π by $-\pi$) shows that a clockwise (s, k)-clash-free permutation exists if and only if an anticlockwise (s, k)-clash-free permutation exists.

We define the *gap* of a rectangle R to be the (edge) connected region of free cells that includes free cells just above R.

Lemma 5. *Fix a warp thread U. The set of rectangles touching the upper edges of rectangles in U is another warp thread U'. The gap of any rectangle $R \in U$ is rectangular in shape. The width of the gap of $R \in U$ depends only on U (not R). The set of rectangles touching the lower edges of rectangles in U form another warp thread. Fix a weft thread V. The set of rectangles touching the left edges of rectangles in V forms a weft thread, as does the set of rectangles touching the right edges of rectangles in V. The gap of $R \in V$ is a rectangle, and its height depends only on V (not R).*

Proof. We may assume that π is a clockwise permutation. (The proof for an anticlockwise permutation is identical to the clockwise case.) We will prove the statements for warp threads: the corresponding proofs for weft threads are similar.

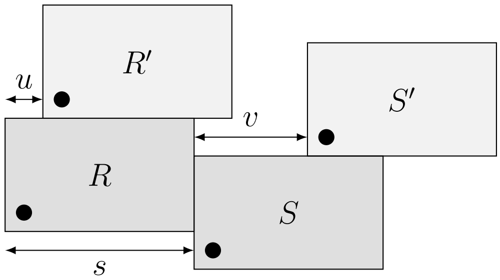

Figure 3: Can R' and S' fail to touch?

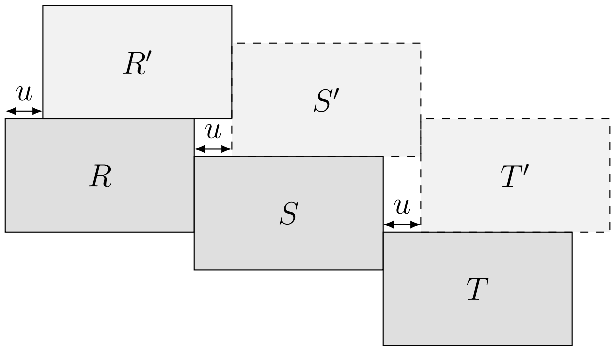

Figure 4: Gaps above a warp thread have constant width.

Let R and S be adjacent rectangles in a warp thread U. Let R' and S' be the rectangles touching the upper sides of R and S respectively; see Figure 3. To prove the first statement of the lemma, it suffices to prove that R' and S' lie on the same warp thread. The first coordinates a of the dot in R and b of the dot in S satisfy the equality $b - a = s$, since R and S touch. The first coordinate of the dot in R' is of the form $a' = a + u$ where $1 \leqslant u < s$, and the first coordinate of the dot in S' is $b' = b + v$ where $1 \leqslant v < s$. The number of columns separating R' and S' is $b' - a' - s$, where

$$b' - a' - s = (b + v) - (a + u) - s = v - u < v < s.$$

So there can be no rectangle between R' and S', as R and S are separated by fewer than s columns. But this means that R' and S' touch, since otherwise the k cells immediately to the left of S are all free. This implies that R' and S' lie on the same warp thread, as required.

The same argument implies the fourth statement of the lemma, that the rectangles touching the lower sides of a warp thread form another warp thread.

In Figure 4, R, S and T lie in the same warp thread. Let R', S' and T' be the rectangles touching the upper sides of R, S and T respectively. We note that once the position of the rectangles R, S, T and R' are known, the position of S' is determined since S' touches S and R', and similarly the position of T' are determined since it touches T and S'. The figure shows that the gaps of S and T both rectangular, of the same width.

This suffices to establish the second and third statements of the lemma. □

Note that, in particular, Lemma 4 shows that there are d_s warp threads and d_k weft threads. We may use the results of Lemma 5 to define two maps on the set of warp threads, and two maps on the set of all weft threads, as follows. For a warp thread U, define $\tau(U)$ to be the warp thread consisting of all rectangles (not in U) that touch the upper edges of U. Define $\delta(U)$ to be the warp thread consisting of all rectangles (not in U) that touch the lower edges of U. For a weft thread V, define $\rho(V)$ to be the weft thread consisting of all rectangles (not in V) that touch the right edges of rectangles in V. Finally, define $\lambda(V)$ to be the weft thread consisting of all rectangles (not in V) that touch the left edges of rectangles in V.

Lemma 6. *The function τ defined above is a permutation of the set of all d_s warp threads; indeed it is a d_s-cycle. The function ρ defined above is a cyclic permutation of the set of weft threads; indeed it is an d_k-cycle.*

Proof. The functions τ and ρ are permutations, as they have inverses δ and λ respectively. Let U be a fixed warp thread. Then the region of the torus consisting of the rectangles in $\bigcup_{i\in\mathbb{Z}}\tau^i(U)$ together with their gaps has no boundary and so is the whole of the torus. Hence every warp thread lies in $\bigcup_{i\in\mathbb{Z}}\tau^i(U)$, and so τ is an n/d_s-cycle. Similarly, ρ is an n/d_k-cycle. □

Theorem 7. *Let n and k be fixed positive integers, with $k < n$. Write $s = \lfloor (n-1)/k \rfloor$, so $n = sk + r$ where $1 \leqslant r \leqslant k$. Define $d_k = \gcd(n, k)$ and $d_s = \gcd(n, s)$. Suppose that $r < k$ and $r < s$. If an (s, k)-clash-free permutation of $\mathbb{Z}_n$ exists, then $d_s d_k$ divides n.*

Proof. Fix a rectangle R. The rectangle R lies on a warp thread U_0 and a weft thread V. Journeying upwards along our weft thread V, we obtain a sequence $U_0, U_1, \ldots$ of warp threads, where the ith rectangle in our journey lies in warp thread U_i. The definition of τ shows that $\tau(U_i) = U_{i+1}$ for $i \geqslant 0$. By Lemma 6, τ is a d_s-cycle and so the sequence $U_0, U_1, \ldots$ has exact period d_s. But our weft thread contains n/d_k rectangles and so after n/d_k steps our journey reaches R and begins to repeat. So the sequence $U_0, U_1, \ldots$ has period dividing n/d_k. Hence d_s divides n/d_k and so $d_s d_k$ divides n, as required. □

Previously, the cases when $\sigma(n, k) = \lfloor (n-1)/k \rfloor - 1$ were established by computer search (and so only finitely many parameters were known). Theorem 7 allows us to construct infinitely many parameters with this property:

Corollary 8. *There are infinitely many choices of parameters n and k such that $\sigma(n, k) = \lfloor (n-1)/k \rfloor - 1$.*

Proof. Let g be any integer with $g > 1$, and define $n = 6g^2 + g$ and $k = 2g$. In the notation of the theorem, we have $s = \lfloor (n-1)/k \rfloor = 3g$, and so $r = n - sk = g$. Moreover, $d_k = d_s = g$. Since n is not divisible by $d_k d_s$ in this situation, Theorem 7 implies that no (s, k)-clash-free permutation exists and so $\sigma(n, k) \neq \lfloor (n-1)/k \rfloor$. Hence $\sigma(n, k) = \lfloor (n-1)/k \rfloor - 1$, by (1). □

3 An explicit construction of (s, k)-clash-free permutations

Theorem 9. *Let n and k be fixed positive integers, with $k < n$. Write $s = \lfloor (n-1)/k \rfloor$, so $n = sk + r$ where $1 \leqslant r \leqslant k$. Define $d_k = \gcd(n, k)$ and $d_s = \gcd(n, s)$. Suppose that $r < k$ and $r < s$. If $d_s d_k$ divides n, then there exists an (s, k)-clash-free permutation.*

Proof. Suppose that $d_s d_k$ divides n. We define a function $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ as follows. Let $x \in \mathbb{Z}_n$. We claim there exist unique integers $j \in \{0, 1, \ldots, d_s - 1\}$, $i \in \{0, 1, \ldots, d_k - 1\}$ and $\alpha \in \{0, 1, \ldots, n/(d_s d_k) - 1\}$ such that $x = (\alpha d_k + i)s + j \pmod n$. To see this, we first use the quotient and remainder theorem to write $x = q d_s + j \pmod n$ for unique integers $q \in \{0, 1, \ldots, n/d_s - 1\}$ and $j \in \{0, 1, \ldots, d_s - 1\}$. Since $\{q d_s \bmod n : q \in \mathbb{Z}\} = \{fs \bmod n : f \in \mathbb{Z}\}$, we see that $q d_s = fs \pmod n$ for a unique integer $f \in \{0, 1, \ldots, n/d_s - 1\}$. We write $f = \alpha d_k + i$, where $0 \leqslant i < d_k$ and $0 \leqslant \alpha < n/(d_s d_k)$. Then $x = (\alpha d_k + i)s + j \pmod n$, and our claim follows. We define

$$\pi(x) = \pi((\alpha d_k + i)s + j) = jk - i - \alpha d_s d_k \in \mathbb{Z}_n.$$

We first show that π is a permutation. It suffices to prove that π is injective. Suppose

$$x = (\alpha d_k + i)s + j \text{ and } x' = (\alpha' d_k + i')s + j' \tag{3}$$

are elements of $\mathbb{Z}_n$ such that $\pi(x) = \pi(x')$. We may look at $\pi(x) = jk - i - \alpha d_s d_k$ and $\pi(x') = j'k - i' - \alpha' d_s d_k$ modulo d_k, since d_k divides n. Using the fact that d_k also divides k, we see that $i = i'$ modulo d_k and so $i = i'$. Hence

$$j(k/d_k) - \alpha d_s = j'(k/d_k) - \alpha' d_s \pmod{n/d_k}. \tag{4}$$

Since d_s divides n/d_k, we may deduce that

$$j(k/d_k) = j'(k/d_k) \pmod{d_s}. \tag{5}$$

Since $d_k = \gcd(n, k)$, we have that k/d_k and n/d_k are coprime. But d_s divides n/d_k, and so k/d_k and d_s are coprime. Hence (5) implies that $j = j' \pmod{d_s}$ and so $j = j'$. Now (4) implies that $\alpha = \alpha' \pmod{n/(d_s d_k)}$ and so $\alpha = \alpha'$. We have shown that $i = i'$, $j = j'$ and $\alpha = \alpha'$. Hence $x = x'$ by (3). Hence π is injective, as required.

We now show that π is (s, k)-clash-free. Define integers a and b by $a = r - (d_s - 1)$ and $b = d_s d_k - (d_k - 1)$. We claim that $0 < a < s$ and $0 < b < k$. Clearly b is positive. To prove the other inequalities in our claim, we first note that $r = n - sk$ and $d_s d_k$ divides n and so r is a (positive) multiple of $d_s d_k$. In particular, we see that $a \leqslant r$ and $b \leqslant r$. Since $r < s$ and $r < k$, we see that $a < s$ and $b < k$. Finally, since $r \geqslant d_s d_k$ we see that a is positive, and our claim follows.

Consider the non-overlapping collection of $s \times k$ rectangles in the infinite grid $\mathbb{Z}^2$, as depicted in Figure 5. This packing of rectangles is divided into $d_k \times d_s$ *blocks* of rectangles (with different shadings in the figure). There are d_s rows and d_k columns of rectangles in each block. The gaps above the rectangles are all of width and height 1, apart from the gaps above the rectangles in the top row of each block which have width a, and the gaps

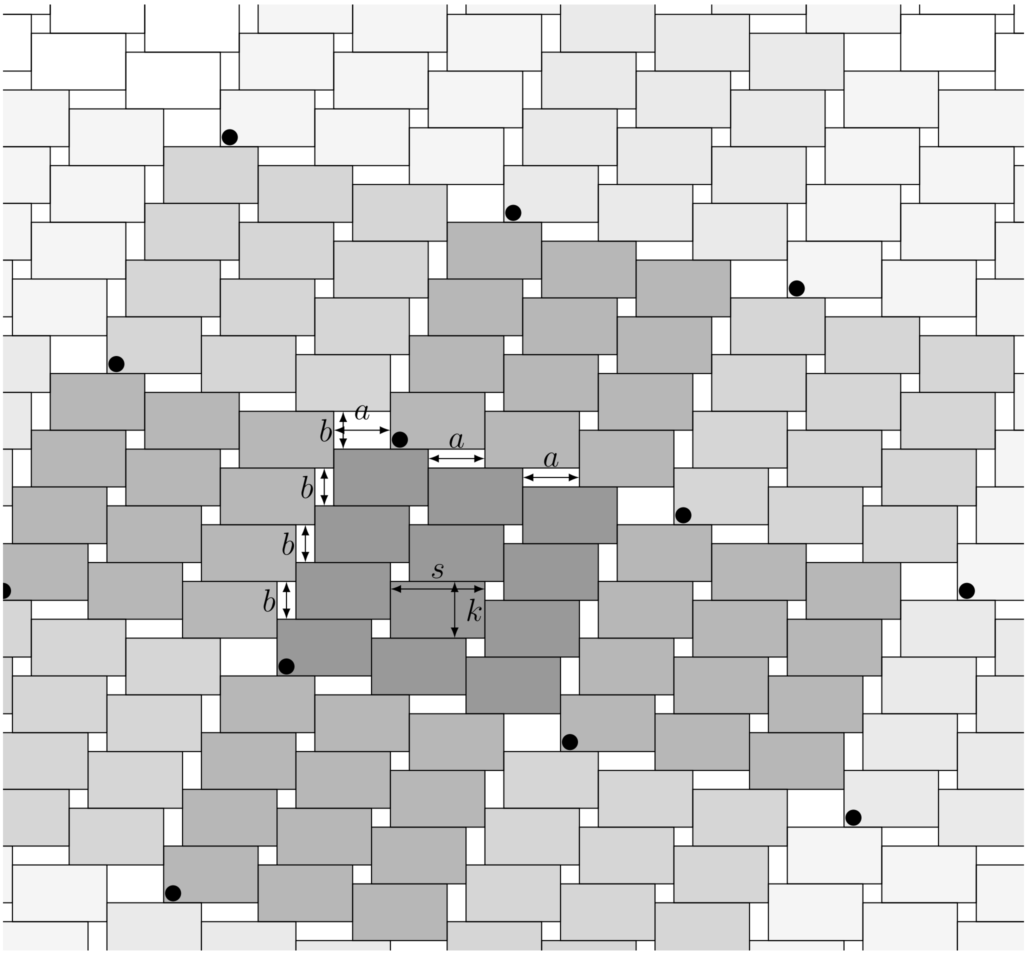

Figure 5: A planar rectangle packing, with some blocks of rectangles highlighted.

above the the left column of each block, which have height b. (This is a special case of a packing obtained from two bi-infinite sequences (a_i) and (b_j) of positive integers, with $a_i \leqslant s$ and $b_i \leqslant k$, by defining the gap above the (i,j)th rectangle to have a gap of width a_i and height b_j.)

Fix one of the blocks of rectangles (coloured more darkly in Figure 5), and choose coordinates so that the lower left corner of the lower left rectangle in the block is $(0,0)$. (This is marked as a dot in the most darkly shaded block.) The set of lower left dots in all blocks (the dots in the figure) form a lattice L, where

$$\begin{aligned}&L = \{\alpha u + \beta v : \alpha, \beta \in \mathbb{Z}\}, \text{ with}\\ &u = (d_k s, -(d_k - 1) - b) = (d_k s, -d_s d_k) \text{ and}\\ &v = ((d_s - 1) + a, d_s k) = (r, d_s k).\end{aligned}$$

(If we take a tile consisting of the rectangles in a fixed block and their associated gaps, the images of this tile under L form a lattice tiling of the plane.)

Let $\mathcal{P}$ be the set of coordinates of the lower left corners of all rectangles (not all blocks) in the packing. So

$$\mathcal{P} = \{w + (is + j, jk - i) : 0 \leqslant i \leqslant d_k - 1, 0 \leqslant j \leqslant d_s - 1 \text{ and } w \in L\}.$$

Rectangles in the same block are associated with the same element $w \in L$. It is clear that the transformation of the plane defined by $x \mapsto x + w$ is a symmetry of our packing for any $w \in L$. We note that $(n,0), (0,n) \in L$, since

$$\begin{aligned}(n,0) &= \left(\frac{k}{d_k}\right) u + v, \text{ and}\\ (0,n) &= -\left(\frac{r}{d_s d_k}\right) u + \left(\frac{s}{d_s}\right) v.\end{aligned}$$

So our rectangle packing is periodic of period dividing n in both horizontal and vertical directions, and therefore gives a well-defined packing of the torus $\mathbb{Z}_n^2$ with $s \times k$ rectangles. But the permutation π gives rise to rectangles in this packing, since the $s \times k$ rectangles associated with π are at coordinates

$$(x, \pi(x)) = \big((\alpha d_k + i)s + j, jk - i - \alpha d_s d_k\big) = \alpha u + (is + j, jk - i)$$

modulo n, which all lie in $\mathcal{P}$. Since π is a permutation, these rectangles are distinct modulo n and hence non-overlapping. Hence π is (s,k)-clash-free, as required. □

4 More structure theory

As in previous sections, we assume that n and k are fixed integers with $k < n$, and we set $s = \lfloor (n-1)/k \rfloor$ with $n = sk + r$ for $1 \leqslant r \leqslant k$. We define d_s and d_k as before, and assume that $r < k$ and $r < s$. Futhermore, we suppose that $d_s d_k$ divides n, so (s,k)-clash-free permutations exist. In this section, we sketch how the structural information in Section 2 may be extended, and the construction in Section 3 may be generalised, to provide a tighter characterisation of (s,k)-clash-free permutations.

The orbits of the map taking a permutation π to the permutation $-\pi$ partition the set of (s,k)-clash-free permutations into pairs, one clockwise and one anticlockwise. So, without loss of generality, we may assume that our (s,k)-clash-free permutation is clockwise. Furthermore, replacing π by the permutation π' defined by $\pi'(x)=\pi(x)-\pi(0)$ for all $x\in\mathbb{Z}_n$, we may assume that $\pi(0)=0$. We aim to classify clockwise (s,k)-clash-free permutations with $\pi(0)=0$ by bringing them into bijection with the set of objects we call (s,k,n)-jumpers. It turns out that the rectangles of these clash-free permutations may be divided up into blocks, generalising the approach in Section 3, but the gaps associated with the rectangles in a block have a more complicated structure that is determined by an (s,k,n)-jumper. We use the term jumper to keep within our weaving theme, and because a jumper determines how rectangles are arranged, jumping from a completely regular lattice. We leave the reader to think of a brand name that the jumpers might belong to.

An (s,k,n)-*jumper* is a pair $\big((a_i),(b_i)\big)$ of sequences of integers with the following properties:

1. (a_i) has period dividing d_s, and (b_i) has period dividing d_k.
2. We have $1\leqslant a_i<s$ and $1\leqslant b_i<k$ for $i\geqslant 0$.
3. The d_k partial sums $\sum_{i=0}^{\ell-1} b_i$ where $0\leqslant \ell<d_k$ are distinct modulo d_k. Moreover, d_sd_k divides σ_b where $\sigma_b=\sum_{i=0}^{d_k-1} b_i$.
4. The d_s partial sums $\sum_{i=0}^{m-1} a_i$ where $0\leqslant m<d_s$ are distinct modulo d_s. Moreover, d_sd_k divides σ_a where $\sigma_a=\sum_{i=0}^{d_s-1} a_i$.
5. Defining σ_a and σ_b as above, $\sigma_a\sigma_b=d_sd_kr$.

We define $J(s,k,n)$ to be the set of (s,k,n)-jumpers.

For example, suppose $n=216$, $s=10$ and $k=21$. So $n=sk+r$ with $r=6$, and we have $d_s=2$ and $d_k=3$. We can check that $\big((a_i),(b_i)\big)$ is a jumper, Let $(a_i)=(a_0,a_1,\ldots)$ be the sequence of period 2 with $a_0=1$ and $a_1=5$. Let (a'_i) be the constant sequence of period 1 where $a'_i=3$ for all i. Let $(b_i)=(b_0,b_1,\ldots)$ be the sequence of period 3 where $b_0=b_1=1$ and $b_2=4$. Let $(b'_i)=(b'_0,b'_1,\ldots)$ be the sequence of period 3 where $b'_0=1$, $b'_1=4$ and $b'_2=1$. Then is not hard to check that when $(c_i)\in\{(a_i),(a'_i)\}$ and $(d_i)\in\{(b_i),(b'_i)\}$ we have that $\big((c_i),(d_i)\big)$ is an (s,k,n)-jumper.

Under our assumption that d_sd_k divides n, (s,k,n)-jumpers always exist. Indeed, a (s,k,n)-jumper can be constructed as follows. Since $n=sk+r$, we see that d_sd_k divides r. Write $r/(d_sd_k)=r_ar_b$ for some positive integers r_a and r_b. (This factorisation can be trivial.) Define $\sigma_a=d_sd_kr_a$ and $\sigma_b=d_sd_kr_b$. Let $(a_0,a_1,\ldots)$ be the sequence of period dividing d_s with $a_1=\cdots a_{d_s-1}=1$ and $a_0=\sigma_a-(d_s-1)$. Similarly, let $(b_0,b_1,\ldots)$ be the sequence of period dividing d_k with $b_1=\cdots b_{d_k-1}=1$ and $b_0=\sigma_b-(d_k-1)$. It is easy to check (using our assumption that $r<s$ and $r<k$) that $\big((a_i),(b_i)\big)$ is an (s,k,n)-jumper.

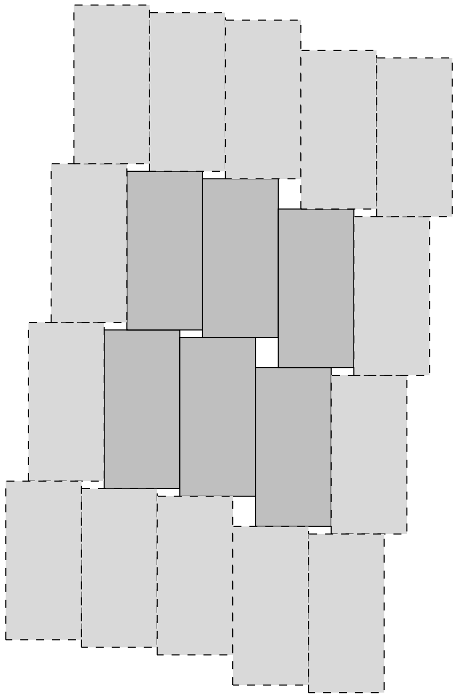

Figure 6: A block in a permutation with jumper $\big((a'_i),(b'_i)\big)$. All gaps have width 3, and height either 1 or 4.

We will see below that a jumper $\big((a_i),(b_i)\big)$ determines the size of gaps associated with rectangles in a block of a permutation: the (u,v)th rectangle in a block has a gap that is rectangular of width a_v and height b_u to its north-east. For example, a block of a permutation associated with the jumper $\big((a'_i),(b'_i)\big)$ is depicted in Figure 6.

Lemma 10. *Let n and k be fixed integers with $k < n$. Set $s = \lfloor (n-1)/k \rfloor$, and define r by $n = sk + r$ for $1 \leqslant r \leqslant k$. Define $d_s = \gcd(n,s)$ and $d_k = \gcd(n,k)$. Assume that $d_s d_k$ divides n. Let $\big((a_i),(b_i)\big)$ be an (s,k,n)-jumper, and define $\sigma_a = \sum_{i=0}^{d_s-1} a_i$ and $\sigma_b = \sum_{i=0}^{d_k-1} b_i$. Then*

$$\gcd\big(n/(d_s d_k), \sigma_b/(d_s d_k)\big) = 1.$$

Proof. The third and fourth conditions for being a jumper imply that we be write $\sigma_a = d_s d_k r_a$ and $\sigma_b = d_s d_k r_b$ for some integers r_a and r_b. The final condition of being a jumper implies that $r = d_s d_k r_a r_b$ and so in particular $r/(d_s d_k)$ is a multiple of $\sigma_b/(d_s d_k)$. Thus it suffices to show that

$$\gcd\big(n/(d_s d_k), r/(d_s d_k)\big) = 1.$$

Suppose, for a contradiction, that there exists a prime p dividing both $n/(d_s d_k)$ and $r/(d_s d_k)$. Then p divides

$$(n-r)/(d_s d_k) = sk/(d_s d_k) = (s/d_s)(k/d_k),$$

and so p divides s/d_s or k/d_k. Suppose that p divides s/d_s. Since p divides $n/(d_s d_k)$, we see that p divides n/d_s and so $\gcd(n/d_s, s/d_s) \geqslant p$. But this contradicts the definition of d_s. Similarly, if we assume that p divides k/d_k we derive a contradiction using the definition of d_k. So $\gcd\big(n/(d_s d_k), r/(d_s d_k)\big) = 1$ and the lemma follows. □

Theorem 11. *Let n and k be fixed integers with $k < n$. Set $s = \lfloor (n-1)/k \rfloor$, and define r by $n = sk + r$ for $1 \leqslant r \leqslant k$. Define $d_s = \gcd(n, s)$ and $d_k = \gcd(n, k)$. Assume that $r < k$ and $r < s$. Futhermore, suppose that $d_s d_k$ divides n. There is a bijection between the set of clockwise (s,k)-clash-free permutations with $\pi(0) = 0$ and the set $J(s,k,n)$ of (s,k,n)-jumpers.*

Proof. Let π be a clockwise (s,k)-clash-free permutation with $\pi(0) = 0$. We produce an (s,k,n)-jumper as follows.

Consider the rectangle R whose lower left cell is at $(0,0)$. As in the proof of Theorem 7, we may define a sequence $U_0, U_1, \ldots$ of warp threads, where U_0 is the warp thread containing R and where $U_{i+1} = \tau(U_i)$ for $i > 0$. The proof of Theorem 7 shows that this sequence has period exactly d_s. Let a_i be the width of a gap above a rectangle in U_i. (This width does not depend on the rectangle in U_i we choose, by Lemma 5.) The sequence (a_i) has period dividing d_s, and $1 \leqslant a_i < s$ for $i \geqslant 0$. Arguing similarly, by travelling along the warp thread through R, we may produce a sequence $V_0, V_1, \ldots$ of weft threads, where V_0 is the weft thread containing R and $V_{i+1} = \rho(V_i)$ for $i > 0$. This sequence has period exactly d_k. Let b_i be the height of a gap above a rectangle in V_i (which is well-defined by Lemma 5). The sequence (b_i) has period dividing d_k, and $1 \leqslant b_i < k$ for $i \geqslant 0$. We claim that the pair $\big((a_i), (b_i)\big)$ of sequences is an (s,k,n)-jumper.

We have already established the first two conditions required for being a jumper. To show the third and fourth conditions are also satisfied, we argue as follows. Moving along the warp thread containing R, we find a sequence $R = R_{(0)}, R_{(1)}, \ldots$ of rectangles. The lower left cell of rectangle $R_{(\ell)}$ is at position $(\ell s, -\sum_{i=0}^{\ell-1} b_i)$. The rectangles $R_{(0)}, R_{(1)}, \ldots, R_{(d_k-1)}$ lie in distinct weft threads $V_0, V_1, \ldots, V_{d_k-1}$, and so the elements $-\sum_{i=0}^{\ell-1} b_i$ are distinct modulo d_k, by the last statement of Lemma 4. Hence the d_k partial sums $\sum_{i=0}^{x-1} b_i$ are distinct modulo d_k. The sequence $R = R_{(0)}, R_{(1)}, \ldots$ of rectangles has period n/d_s, again by Lemma 4. Comparing the second coordinates of $R_{(0)}$ and $R_{(n/d_k)}$ we find that

$$0 \equiv -\sum_{i=0}^{n/d_s - 1} b_i \equiv -(n/(d_s d_k))\sigma_b \pmod{n},$$

and so $(n/(d_s d_k))\sigma_b \equiv 0 \pmod{d_s d_k}$. Thus the third condition holds. The fourth condition similarly holds, by arguing that the rectangles encountered when moving along the weft thread containing R have lower left corners at positions $(\sum_{i=0}^{m-1} a_i, mk)$ and using the second statement of Lemma 4.

It remains to check that the final condition for being a jumper is satisfied. Since the sequence (b_j) has period d_k, the gaps of rectangles in warp thread U_i have total area

$$\sum_{j=0}^{n/d_s-1}(a_ib_j) = a_i\left(\sum_{j=0}^{n/d_s-1} b_j\right) = a_i(n/d_sd_k)\sigma_b.$$

So the total area of all gaps can be written as

$$(n/d_sd_k)\sum_{i=0}^{d_s-1} a_i\sigma_b = (n/d_sd_k)\sigma_a\sigma_b.$$

But Lemma 2 shows that every row of our torus contains r free cells, and so there are nr free cells in total. Hence $nr = (n/d_sd_k)\sigma_a\sigma_b$ and so $\sigma_a\sigma_b = rd_sd_k$. This establishes our claim that $((a_i),(b_i))$ is an (s,k,n)-jumper, as required.

Let $\big((a_i),(b_i)\big)$ be an (s,k,n)-jumper. We construct a permutation $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ as follows. Let $x \in \mathbb{Z}_n$. By the fourth condition of being an (s,k,n)-jumper, and by considering x modulo d_s, there exist unique integers $q \in \{0,1,\dots,n/d_s-1\}$ and $m \in \{0,1,\dots,d_s-1\}$ such that $x = qd_s + \sum_{i=0}^{m-1} a_i \pmod{n}$. As in the proof of Theorem 9, we may write $qd_s = fs \pmod{n}$ for a unique integer $f \in \{0,1,\dots,n/d_s-1\}$ and we may write $f = \alpha d_k + \ell$ where $0 \leqslant \ell < d_k$ and $0 \leqslant \alpha < n/(d_sd_k)$. We define $\pi(x) = y$ where

$$\pi(x) = \pi\left((\alpha d_k+\ell)s + \sum_{i=0}^{m-1} a_i\right) = mk - \sum_{i=0}^{\ell-1} b_i - \alpha\sigma_b.$$

In order to check that π is a permutation, it suffices to show that π is injective. Suppose that

$$mk - \sum_{i=0}^{\ell-1} b_i - \alpha\sigma_b = m'k - \sum_{i=0}^{\ell'-1} b_i - \alpha'\sigma_b \pmod{n}.$$

Considering this equality modulo d_k, and using the third condition of being an (s,k,n)-jumper, we see that $\ell = \ell'$. So $mk - \alpha\sigma_b = m'k - \alpha'\sigma_b \pmod{n}$ and hence (using the fact that d_k divides σ_b)

$$m(k/d_k) - \alpha(\sigma_b/d_k) = m'(k/d_k) - \alpha'(\sigma_b/d_k) \pmod{n/d_k}.$$

Since d_s divides both n/d_k and σ_b/d_k, we deduce that $m(k/d_k) = m'(k/d_k) \pmod{d_s}$. But k/d_k is coprime to n/d_k and so k/d_k is coprime to d_s. Hence $m = m' \pmod{d_s}$ and so $m = m'$. We deduce that $\alpha\sigma_b = \alpha'\sigma_b \pmod{n}$, and so $\alpha\sigma_b/(d_sd_k) = \alpha'\sigma_b/(d_sd_k) \pmod{n/(d_sd_k)}$. Now $\sigma_b/(d_sd_k)$ is coprime to $n/(d_sd_k)$, by Lemma 10, and so $\alpha = \alpha'$.

We have shown that π is a permutation, it is clear that $\pi(0) = 0$, and it is not hard to see that π is clockwise. Consider the generalisation of the packing of $s \times k$ rectangles in the plane $\mathbb{Z}^2$ mentioned in Section 3. So the (ℓ,m)th rectangle in the packing has lower left corner at the cell with position

$$\left(\ell s + \sum_{i=0}^{m-1} a_i, mk - \sum_{i=0}^{\ell-1} b_i\right).$$

This rectangle packing is invariant with respect to the integer lattice $\mathcal{L}$ generated by u and v, where

$$u = (d_k s, -\sum_{i=0}^{d_k-1} b_i) = (d_k s, -\sigma_b) \text{ and}$$
$$v = (\sum_{i=0}^{d_s-1} a_i, d_s k) = (\sigma_a, d_s k).$$

We note that

$$\frac{k}{d_k}u + \frac{\sigma_b}{d_s d_k}v = (ks + r, 0) = (n, 0) \text{ and}$$
$$-\frac{\sigma_a}{d_s d_k}u + \frac{s}{d_s}v = (0, r + ks) = (0, n).$$

Now $d_s d_k$ divides σ_a and σ_b, since we have an (n, s, k)-jumper. So $(n, 0)$ and $(0, n)$ are $\mathbb{Z}$-linear combinations of u and v, and therefore lie in $\mathcal{L}$. Thus our packing induces a well-defined packing of the torus $\mathbb{Z}_n^2$. The argument in Theorem 9 now shows that π is (s, k)-clash-free.

The process of producing a jumper from a permutation, and the process of producing a permutation from a jumper, are inverse to each other. So we have a bijections between (s, k, n)-jumpers and clockwise (s, k)-clash free permutations of $\mathbb{Z}_n$ with $\pi(0) = 0$, as required. □

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