On the *cd*-Index for Alternating Descents

Qiongqiong Pan Cheng Qian

Submitted: Oct 5, 2023; Accepted: Sep 30, 2024; Published: Nov 1, 2024

Abstract

In 2008, Chebikin considered the *cd*-index of $\mathfrak{S}_n$ with respect to the alternating descent statistic and asked for a combinatorial interpretation for its coefficients. In this paper, we provide an answer to Chebikin's open problem in terms of permutations without double descents and ending with an ascent, with respect to a new statistic defined on these permutations. Additionally, we demonstrate a *cd*-index approach to proving the gamma-expansions of alternating Eulerian polynomials. Furthermore, we offer a direct combinatorial interpretation for their gamma-coefficients.

Mathematics Subject Classifications: 05A05, 05A15, 05A19

1 Introduction

The *cd*-index was first defined by Fine [2] and it is an encoding of the numbers of chains, specified by ranks, in the poset. It is the most efficient such encoding, incorporating all the affine relations on the flag numbers of Eulerian posets. There are two main issues for research on *cd*-indices. One is the question of the nonnegativity of the coefficients, the other issue is the combinatorial interpretation of the coefficients, either directly in terms of the poset or in terms of other combinatorial objects. In the last thirty-five years, much research has been carried out on these issues. For example, Stanley [21] proved that the *cd*-indices of S-shellable CW spheres are nonnegative. Karu [15] proved the *cd*-index of every Gorenstein* poset is nonnegative. Purtill's early results on nonnegativity of the *cd*-index resulted from studying CL-shellings of polytopes [20]. In this study he showed that the *cd*-index of the simplex is the (noncommutative) André polynomial of Foata and Schützenberger [8]. For further results on *cd*-index and related topics, the reader is referred to the recent survey by Bayer [1].

The objective of this paper is to provide a combinatorial interpretation for the coefficients of an analogue of *cd*-index of symmetric groups, as introduced by Chebikin in [5]. Subsequently, by employing this analogue, we establish the γ-expansions of alternating Eulerian polynomials and present a novel combinatorial interpretation for their γ-coefficients.

College of Mathematics and Physics, Wenzhou University, Wenzhou 325035, PR China (`qpan@wzu.edu.cn`, `cqian@stu.wzu.edu.cn`).

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, we define a *descent* (resp. an *ascent*) to be an index $i \in [n-1]$ such that $\sigma_i > \sigma_{i+1}$ (resp. $\sigma_i < \sigma_{i+1}$). Let $\mathrm{d}(\sigma)$ be the number of descents of σ. Then, the Eulerian polynomials $A_n(x)$ can be described as descent polynomials of $\mathfrak{S}_n$,

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{d}(\sigma)} = \sum_{k=0}^{n-1} A(n,k)x^k.$$

There is a well-known formula for the exponential generating function for $A_n(x)$:

$$\sum_{n>0} A_n(x)\frac{z^n}{n!} = \frac{1 - \exp(z(x-1))}{\exp(z(x-1)) - x}.$$

In [5], Chebikin studied new statistics on permutations that are variations on the descent statistic. In particular, he considered the *alternating descent set* of a permutation σ. We say that $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ has an *alternating descent* at position i if either $\sigma_i > \sigma_{i+1}$ and i is odd, or else if $\sigma_i < \sigma_{i+1}$ and i is even. Let $\hat{D}(\sigma)$ be the set of positions at which σ has an alternating descent, and let $\hat{\mathrm{d}}(\sigma) = |\hat{D}(\sigma)|$. Then the alternating Eulerian polynomials $\hat{A}(x)$ is defined as

$$\hat{A}(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\hat{d}(\sigma)} = \sum_{k=0}^{n-1} \hat{A}(n,k)x^k. \tag{1}$$

And their exponential generating function is,

$$\sum_{n>0} \hat{A}_n(x)\frac{z^n}{n!} = \frac{\sec(1-x)z + \tan(1-x)z - 1}{1 - x(\sec(1-x)z) + \tan(1-x)z)}.$$

Besides the generating polynomials for the alternating descent statistics, another natural generating function to consider is one counting permutations by their alternating descent set. We begin by recalling some well-known facts about the analogous generating function for the classical descent set statistics.

Fix a positive integer n. For a subset $S \subseteq [n-1]$, define the monomial u_S in two non-commuting variable $\mathbf{a}$ and $\mathbf{b}$ by $u_S = u_1u_2 \cdots u_{n-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S\ , \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

Consider the generating function

$$\Psi_n(\mathbf{a}, \mathbf{b}) := \sum_{S \subseteq [n-1]} \beta_n(S)u_S,$$

where $\beta_n(S)$ is the number of permutations in $\mathfrak{S}_n$ with descent set S. The polynomial $\Psi_n(\mathbf{a}, \mathbf{b})$ is known as the *ab-index of* $\mathfrak{S}_n$ *with respect to descent set statistic.* One of the remarkable properties of $\Psi_n(\mathbf{a},\mathbf{b})$ is that it can be expressed in terms of the variables $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$. The polynomial $\Phi_n(\mathbf{c},\mathbf{d})$ defined by $\Psi_n(\mathbf{a},\mathbf{b}) = \Phi_n(\mathbf{a}+\mathbf{b},\mathbf{ab}+\mathbf{ba})$ is called the *cd-index of* $\mathfrak{S}_n$. The polynomial $\Phi_n(\mathbf{c},\mathbf{d})$ has positive integer coefficients, for which several combinatorial interpretations have been found. Here we give one which in terms of so-called *Simsun permutations*. Simsun permutations were named after Rodica Simion and Sheila Sundaram. They were first described in print by Sundaram [23], and were initially used by Simion and Sundaram in the study of homology representations of symmetric group $\mathfrak{S}_n$ [24, 25]. We proceed with a definition of statistics on permutations before giving the definition of Simsum permutations.

Definition 1. For a permutation $\sigma = \sigma_1\cdots\sigma_n$ of $[n]$ with $\sigma_0 = 0$, and for $1 \leqslant i \leqslant n-1$ the entry σ_i is

- a *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a *valley* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$;
- a *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- a *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Let $\mathrm{Pk}(\sigma)$ (resp. $\mathrm{Val}(\sigma)$, $\mathrm{Da}(\sigma)$, $\mathrm{Dd}(\sigma)$) denote the set of peaks (resp. valleys, double descents, double ascents) of σ.

Definition 2. We say permutation $\sigma = \sigma_1\cdots\sigma_n \in \mathfrak{S}_n$ is a *Simsun permutation*, if $\sigma_{[k]}$ has no double descent for all $n \geqslant k \geqslant 1$, where $\sigma_{[k]}$ is the subword of σ consisting of $1,\ldots,k$ in the order they appear in σ.

For instance, $\sigma = 425163 \in \mathfrak{S}_6$ is not a Simsun permutation since $\sigma_{[4]} = 4213$ has a double descent 2, while the permutation $\tau = 31245$ is a Simsun permutation. Let $\mathcal{SS}_n$ be the set of Simsun permutations in $\mathfrak{S}_n$ with last entry n. (Thus $\mathcal{SS}_n$ is essentially the set of Simsun permutations of $[n-1]$ with an n attached at the end.)

For a permutation $\sigma \in \mathcal{SS}_n$, define the $(\mathbf{c},\mathbf{d})$-monomial $\mathrm{cd}(\sigma)$ as follows: write out the descent set of σ as a string of $\mathbf{a}$'s and $\mathbf{b}$'s denoting ascents and descents, respectively, and then replace each occurrence of $\mathbf{ba}$ by $\mathbf{d}$, and each remaining $\mathbf{a}$'s by $\mathbf{c}$. This definition is valid because a Simsun permutation has no double descents. For example, consider the permutation $423516 \in \mathcal{SS}_6$. Its descent set in the above notation is $\mathbf{baaba}$, and thus $\mathrm{cd}(423516) = \mathbf{dcd}$.

Then the Simsun permutations provide a combinatorial interpretation for the *cd*-index of $\mathfrak{S}_n$:

$$\Phi_n(\mathbf{c},\mathbf{d}) = \sum_{\sigma\in\mathcal{SS}_n} \mathrm{cd}(\sigma). \tag{2}$$

In [5], Chebikin defines the analogue of $\Psi_n(\mathbf{a},\mathbf{b})$ for the alternating descent set statistic:

$$\hat{\Psi}_n(\mathbf{a},\mathbf{b}) := \sum_{S\subseteq[n-1]} \hat{\beta}_n(S)u_S, \tag{3}$$

where $\hat{\beta}_n(S)$ is the number of permutations in $\mathfrak{S}_n$ with alternating descent set S. It turns out that the polynomial $\hat{\Psi}_n(\mathbf{a},\mathbf{b})$ satisfies the following proposition.

Proposition 3. *[5] There exists a polynomial $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ such that*

$$\hat{\Phi}_n(\mathbf{a}+\mathbf{b},\mathbf{ab}+\mathbf{ba}) = \hat{\Psi}_n(\mathbf{a},\mathbf{b}),$$

namely, $\hat{\Phi}_n(\mathbf{c},\mathbf{d}) = \Phi_n(\mathbf{c},\mathbf{c}^2-\mathbf{d})$.

Although the polynomial $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ has both positive and negative coefficients, the polynomial $\hat{\Phi}_n(\mathbf{c},-\mathbf{d}) = \Phi_n(\mathbf{c},\mathbf{c}^2+\mathbf{d})$ has only positive coefficients. So in [5], Chebikin asked to give a combinatorial interpretation for these coefficients similar to that of the coefficients of $\Phi_n(\mathbf{c},\mathbf{d})$, so that the coefficients of $\hat{\Phi}_n(\mathbf{c},-\mathbf{d})$ enumerate permutations of a certain kind according to some statistic. Though Chebikin did not find a combinatorial interpretation for the coefficients of $\hat{\Phi}_n(\mathbf{c},-\mathbf{d})$, he proved, using the approach of min-tree representation of permutations introduced by Hetyei and Reiner [14], that the sum of coefficients of $\hat{\Phi}_n(\mathbf{c},-\mathbf{d})$ is equal to the number of permutations without double descents and ending with an ascent. In this paper, Section 2 will provide a combinatorial interpretation for the coefficients of the polynomial $\hat{\Phi}_n(\mathbf{c},-\mathbf{d})$, involving permutations without double descents and ending with an ascent, along with a statistic defined on these permutations. This addresses the open problem posed by Chebikin in [5]. In Section 3, we will explore the γ-expansion of alternating Eulerian polynomials and present a combinatorial interpretation for their γ-coefficients using the *cd*-index of alternating descents.

2 Combinatorial interpretation of $\hat{\Phi}_n(\mathbf{c},-\mathbf{d})$

We first recall some definitions from [9, 10]. Let $w = x_1x_2\cdots x_n$ be a permutation of a finite subset $\{a_1 < a_2 < \cdots < a_n\}$ of $\mathbb{N}$. The word u obtained by concatenation of two words v and w in this order is written $u = vw$. The word v (resp. w) is the left (resp. right) factor of u. Generally, a *factorization of length q* ($q > 0$) of a word w is a sequence $(w_1, w_2, \ldots, w_q)$ of words (some of them possibly empty) such that the concatenation product $w_1w_2\cdots w_q$ is equal to w. For two words $u = u_1u_2\cdots u_n$, $v = v_1v_2\cdots v_m$, and a letter b, we denote $b \in u$, if $b = u_i$ for some $i \in [n]$, and denote $v \subseteq u$, if $v = u_{k+1}u_{k+2}\cdots u_{k+m}$ for some $k \in [n]$. Then, we give the following definition of the x-factorization of a permutation, which was given as a lemma in [10, Lemma 1].

Definition 4. Let $w = x_1x_2\cdots x_n$ ($n > 0$) be a permutation and x be one of the letters x_i ($1 \leqslant i \leqslant n$). Then w has a unique factorization (w_1, w_2, x, w_4, w_5) of length 5, called its *x-factorization*, which is characterized by the tree properties:

(i) w_1 is empty or its last letter is less than x;

(ii) w_2 (resp. w_4) is empty or all its letters are greater than x;

(iii) w_5 is empty or its first letter is less than x.

The André permutations in this paper are called *André permutations of the second kind* in the literature and have several equivalent definitions, see [6, 9]. The following definition of André permutation was given in [9, Definition 4.12].

Definition 5. A permutation σ of $[n]$ is an André permutation, if for all $1 \leqslant k \leqslant n$, $\sigma_{[k]}$ has no double descents and ends with an ascent, where $\sigma_{[k]}$ is the subword of σ consisting of $1, \ldots, k$ in the order they appear in σ.

Let $\mathcal{A}_n$ be the set of André permutations of $[n]$. By definition, an André permutation is always a Simsun permutation, but the reverse is not true. For example, the permutation $\sigma = 24135$ is Simsun but not André, because $\sigma_{[2]} = 21$ ends with a descent. In [13], Hetyei reformulate Foata and Strehl's definition of André permutations in terms of the x-factorization. Pan and Zeng [18] also proved that these two definitions of André permutations are equivalent and presented the Hetyei's definition as a proposition in [18, Proposition 3]. Here, we follow the manner of Pan and Zeng.

Proposition 6. *[18] For $n \geqslant 0$, a permutation $\sigma \in \mathfrak{S}_n$ is an André permutation if and only if it is empty (when $n = 0$) or satisfies the following:*

(i) σ has no double descents;

(ii) $n-1$ is not a descent position, i.e., $\sigma_{n-1} < \sigma_n$;

(iii) If σ_i is a valley of σ with the x-factorization $(w_1, w_2, \sigma_i, w_4, w_5)$, then $\min(w_2) > \min(w_4)$, i.e., the minimum letter of w_2 is larger than the minimum letter of w_4.

Let $\mathcal{DD}_n$ be the set of permutations of $[n]$ without double descents and ending with an ascent. For instance, $\mathcal{DD}_3 = \{123, 213, 312\}$.

Definition 7. If $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{DD}_n$ with $\sigma_0 = 0$, and σ_i is a valley of σ with σ_i-factorization $(w_1, w_2, \sigma_i, w_4, w_5)$, then we say that σ_i is an *André valley of* σ if $\min(w_2) > \min(w_4)$. Otherwise, it is a *non-André valley of* σ.

For $\sigma \in \mathcal{DD}_n$. Let $\operatorname{vala}(\sigma)$ be the number of André valleys of σ. By Proposition 6, it is clear that all the valleys of an André permutation are André valleys.

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{DD}_n$ with $\sigma_0 = 0$, the $\mathbf{a}\bar{\mathbf{a}}\mathbf{b}$-variation monomial $\hat{\mathrm{ab}}(\sigma)$ of σ is a word $v_1 \cdots v_{n-1}$ in non-commuting variables $\mathbf{a}$, $\bar{\mathbf{a}}$, and $\mathbf{b}$, such that for every $i \in \{1, 2, \ldots, n-1\}$ we have

$$v_i = \begin{cases} \mathbf{a} & \text{if } i \text{ is an ascent and } \sigma_i \text{ is a double ascent or an André valley,} \\ \bar{\mathbf{a}} & \text{if } i \text{ is an ascent and } \sigma_i \text{ is a non-André valley,} \\ \mathbf{b} & \text{if } i \text{ is a descent.} \end{cases}$$

The **cd**-variation monomial $\hat{\mathrm{cd}}(\sigma)$ is the word in non-commuting variables **c** and **d** obtained from $\hat{\mathrm{ab}}(\sigma)$ by first replacing every pair **ba** with **d** and $\mathbf{b\bar{a}}$ with $\mathbf{c}^2$, and then replacing the remaining letters with **c**. This definition is valid because the permutations in $\mathcal{DD}_n$ have no double descents. For the empty permutation, we define its $\mathbf{a\bar{a}b}$-variation monomial, as well as its **cd**-variation monomial, to be 1. For instance, take $\sigma = 263415 \in \mathcal{DD}_6$ with $\sigma_0 = 0$. We have that 1 is an ascent and σ_1 is a double descent, so $v_1 = \mathbf{a}$. Both 2 and 4 are descents, hence $v_2 = v_4 = \mathbf{b}$. Since 3 and 1 are valleys, their corresponding factorizations are $(2, 6, 3, 4, 15)$ and $(\epsilon, 2634, 1, 5, \epsilon)$ respectively, where ϵ denotes the empty word. Thus, 3 is an André valley and 1 is a non-André valley; therefore, $v_3 = \mathbf{a}$ and $v_5 = \bar{\mathbf{a}}$. Thus, $\hat{\mathrm{ab}}(\sigma) = \mathbf{abab\bar{a}}$, and $\hat{\mathrm{cd}}(\sigma) = \mathbf{cdc}^2$. The following theorem is our first main result.

Theorem 8. *For $n \geqslant 1$, we have*

$$\hat{\Phi}_n(\mathbf{c}, -\mathbf{d}) = \sum_{\sigma \in \mathcal{DD}_n} \hat{\mathrm{cd}}(\sigma). \tag{4}$$

Example 9. For $n = 3$, we have $\hat{\Phi}_3(\mathbf{c}, -\mathbf{d}) = \Phi_3(\mathbf{c}, \mathbf{c}^2 + \mathbf{d})$, $\mathcal{SS}_3 = \{123, 213\}$, $\mathrm{cd}(123) = \mathbf{c}^2$, and $\mathrm{cd}(213) = \mathbf{d}$. Hence, $\Phi_3(\mathbf{c}, \mathbf{d}) = \mathbf{d} + \mathbf{c}^2$, and $\hat{\Phi}_3(\mathbf{c}, -\mathbf{d}) = \Phi_3(\mathbf{c}, \mathbf{c}^2 + \mathbf{d}) = 2\mathbf{c}^2 + \mathbf{d}$. On the other hand, we also have $\mathcal{DD}_3 = \{123, 213, 312\}$, $\hat{\mathrm{cd}}(123) = \mathbf{c} \cdot \mathbf{c} = \mathbf{c}^2$, $\hat{\mathrm{cd}}(213) = \mathbf{c}^2$, and $\hat{\mathrm{cd}}(312) = \mathbf{d}$. Thus,

$$\begin{aligned}\sum_{\sigma \in \mathcal{DD}_3} \hat{\mathrm{cd}}(\sigma) &= 2\mathbf{c}^2 + \mathbf{d} \\ &= \hat{\Phi}_3(\mathbf{c}, -\mathbf{d}).\end{aligned}$$

To prove Theorem 8, we need to construct a mapping ρ from $\mathcal{A}_n$ to $\mathcal{SS}_n$. Recall that $\mathcal{SS}_n$ is the set of simsun permutations in $\mathfrak{S}_n$ with last entry n. Prior to this, we will first introduce statistics defined on permutations of $\mathfrak{S}_n$. For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, we call the entry σ_i a *right-to-left minimum*, if $\sigma_i < \sigma_j$ for all $j > i$, and i is called a *position of the right-to-left minimum* of σ.

Definition of mapping $\rho : \mathcal{A}_n \to \mathcal{SS}_n$.
Let $i_1 < i_2 < \cdots < i_l$ denote the positions of the right-to-left minima of σ with $\sigma \in \mathcal{A}_n$. It is evident that $\sigma_{i_1} = 1$ and $i_l = n$. Since σ is an André permutation, Proposition 6 (ii) implies that $i_{l-1} = n - 1$.
Let $\tau = \tau_1\tau_2 \cdots \tau_n$, where $\tau_n = n$, and for $1 \leqslant i \leqslant n - 1$, τ_i is defined as follows:

$$\tau_i = \begin{cases} \sigma_i - 1 & \text{if } i \notin \{i_1, \ldots, i_l\}, \\ \sigma_{i_k} - 1 & \text{if } i = i_{k-1} \text{ for } k = 2, \ldots, l. \end{cases} \tag{5}$$

Then, the following two lemmas are the keys to defining the mapping ρ.

Lemma 10. *For $\sigma \in \mathcal{A}_n$, and τ is defined as* (5)*, then the position of each right-to-left minimum of σ is also a position of the right-to-left minimum of τ.*

Proof. Let $i_1 < i_2 < \cdots < i_l$ denote the positions of the right-to-left minima of σ. For $j \in \{i_1, i_2, \ldots, i_l\}$, we now show that j is also a position of the right-to-left minimum of τ. Since $\tau_{n-1} < \tau_n = n$, both $i_{l-1} = n-1$ and $i_l = n$ are positions of the right-to-left minima of τ. For $i_k \in \{i_1, \ldots, i_{l-2}\}$ and $j > i_{k+1}$, considering that i_{k+1} is a position of the right-to-left minimum of σ, we have:

$$\tau_{i_k} = \sigma_{i_{k+1}} - 1 < \sigma_j - 1 = \begin{cases} \tau_j & \text{if } j \notin \{i_{k+1}, \ldots, i_l\}, \\ \tau_{i_{m-1}} & \text{if } j = i_m \text{ for } m = k+2, \ldots, l. \end{cases}$$

That is, $\tau_{i_k} < \tau_j$ for $j \geqslant i_{k+1}$. For $i_k < j < i_{k+1}$, let us take the largest j such that $\sigma_{i_{k+1}} - 1 = \tau_{i_k} > \tau_j = \sigma_j - 1$. This implies j is the largest index where $\sigma_j < \sigma_{i_{k+1}}$, indicating that j is a position of the right-to-left minimum of σ. However, this leads to a contradiction. Therefore, $\tau_{i_k} < \tau_j$ for all $i_k < j < i_{k+1}$. By combining this with the previous discussion, we conclude that $\tau_{i_k} < \tau_j$ for all $i_k < j$, indicating that i_k is a position of the right-to-left minimum in τ. □

Lemma 11. *For $\sigma \in \mathcal{A}_n$, τ is defined as* (5)*; therefore, we have $\tau \in \mathcal{SS}_n$.*

Proof. By Definition 2, we only need to prove that $\tau_{[k]}$ does not have double descents for all $1 \leqslant k \leqslant n$. Suppose, for some $1 \leqslant k \leqslant n$, $\tau_{[k]} = a_1 a_2 \cdots a_k \in \mathfrak{S}_k$ has a double descent, say a_j, that is, $a_{j-1} > a_j > a_{j+1}$. This yields all the letters between a_{j-1} and a_{j+1} in τ, say $b_1, b_2, \ldots, b_q$ are larger than a_{j-1}. These letters and a_{j-1} are not the right-to-left minima of τ. So by (5) and Lemma 10, the position of each right-to-left minimum of σ is also a position of the right-to-left minimum of τ. Therefore, $a_{j-1}+1$ and $b_1+1, b_2+1, \ldots, b_q+1$ are not the right-to-left minima in σ, and they are consecutive letters in σ. Let $m = a_{j-1}+1$; we claim that $\sigma_{[m]}$ has a double descent a_j+1. By (5) and the previous discussions, if $a_{j+1}+1$ is not a right-to-left minimum of σ, then $a_{j-1}+1$, a_j+1, and $a_{j+1}+1$ in $\sigma_{[m]}$ are consecutive letters. Otherwise a_{j-1}, a_j and a_{j+1} are not consecutive letters in $\tau_{[k]}$, so a_j+1 is a double descent of $\sigma_{[m]}$. If $a_{j+1}+1$ is a right-to-left minimum of σ, say $a_{j+1}+1 = \sigma_{i_p}$ for some $p \in \{2, \ldots, l\}$, then $a_{j-1}+1$, a_j+1, and $\sigma_{i_{p-1}}$ in $\sigma_{[m]}$ are consecutive letters. This also means a_j+1 is a double descent of $\sigma_{[m]}$. In conclusion, the existence of double descent in $\sigma_{[m]}$ contradicts the property of σ being an André permutation. Therefore, $\tau_{[k]}$ does not have double descents for all $1 \leqslant k \leqslant n$, with $\tau_n = n$, implying that $\tau \in \mathcal{SS}_n$. □

Now, by Lemma 11, we can define $\rho : \mathcal{A}_n \to \mathcal{SS}_n$ by setting $\rho(\sigma) = \tau$ for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{A}_n$, where τ is obtained from (5).

In what follows, we give two propositions about ρ.

Proposition 12. *The mapping ρ is actually a bijection from $\mathcal{A}_n$ to $\mathcal{SS}_n$.*

Proof. We construct the inverse ρ^{-1} of ρ as follows. For $\tau = \tau_1\tau_2\cdots\tau_n \in \mathcal{SS}_n$ with its positions of the right-to-left minima $\{i_1 < i_2 < \cdots < i_l\}$, it is clear that $\tau_{i_1} = 1$,

$i_{l-1} = n-1$, and $i_l = n$. Then $\rho^{-1}(\tau) = \sigma = \sigma_1\sigma_2\cdots\sigma_n$, where σ_i is given by

$$\sigma_i = \begin{cases} \tau_i + 1 & \text{if } i \notin \{i_1, \ldots, i_l\}, \\ 1 & \text{if } i = i_1, \\ \tau_{i_{k-1}} + 1 & \text{if } i = i_k \text{ for } k = 2, 3, \ldots, l. \end{cases} \tag{6}$$

We first prove that for $j \in \{i_1, i_2, \ldots, i_l\}$, it is also a position of the right-to-left minimum of σ. Since $\sigma_{i_1} = 1$, $\sigma_{n-1} = \sigma_{i_{l-1}} = \tau_{i_{l-2}} + 1$, and $\sigma_n = \sigma_{i_l} = \tau_{i_l - 1} + 1$, we have $\sigma_{n-1} < \sigma_n$. So i_1, i_{l-1}, and i_l are positions of the right-to-left minima of σ. For $i_k \in \{i_2, \ldots, i_{l-2}\}$ and $j > i_{k-1}$, we have

$$\sigma_{i_k} = \tau_{i_{k-1}} + 1 < \tau_j + 1 = \begin{cases} \sigma_j & \text{if } j \notin \{i_k, \ldots, i_l\}, \\ \sigma_{i_{m+1}} & \text{if } j = i_m \text{ for } m = k, \ldots, l. \end{cases}$$

That is, $\sigma_{i_k} < \sigma_j$ for $j > i_{k-1}$, which implies that i_k is a right-to-left minimum of σ. We then show that σ is an André permutation. We assume that there exists some $3 \leqslant k \leqslant n$ such that $\sigma_{[k]} = b_1 b_2 \cdots b_k \in \mathfrak{S}_k$ has a double descent, denoted by b_i, where $b_{i-1} > b_i > b_{i+1}$. This implies that all the letters between b_{i-1} and b_{i+1} in σ, denoted as $a_1, a_2, \ldots, a_q$, are larger than b_{i-1}. In other words, these letters and b_{i-1} are not the right-to-left minima of σ. By using (6) and the fact that each right-to-left minimum of τ is also a position of a right-to-left minimum of σ, we can conclude that $b_{i-1} - 1$ and $a_1 - 1, a_2 - 1, \ldots, a_q - 1$ are not the right-to-left minima of τ, and they are consecutive letters in τ. Let $m = b_{i-1} - 1$. We claim that $\tau_{[m]}$ has a double descent $b_i - 1$. Considering (6) and the previous discussions, if $b_{i+1} - 1$ is not a right-to-left minimum of τ, then $b_{i-1} - 1$, $b_i - 1$, and $b_{i+1} - 1$ in $\tau_{[m]}$ are consecutive letters, which means $b_i - 1$ is a double descent of $\tau_{[m]}$. If $b_{i+1} - 1$ is a right-to-left minimum of τ, denoted as $b_{i+1} - 1 = \tau_{i_p}$ for some $p \in \{1, 2, \ldots, l-1\}$, then $b_{i-1} - 1$, $b_i - 1$, and $\tau_{i_{p+1}}$ in $\tau_{[m]}$ are consecutive letters. This also implies $b_i - 1$ in $\tau_{[m]}$ is a double descent. In conclusion, $\tau_{[m]}$ has a double descent, which contradicts the assumption that τ is a simsun permutation. Therefore, $\sigma_{[k]}$ does not have double descents for all $1 \leqslant k \leqslant n$. We also need to prove that $\sigma_{[k]}$ ends with an ascent for all $1 \leqslant k \leqslant n$. Suppose for some $1 \leqslant k \leqslant n$, $\sigma_{[k]} = b_1 b_2 \cdots b_k$ ends with a descent. It is clear that b_{k-1} is not a right-to-left minimum in σ. Based on our previous discussion, $b_{k-1} - 1$ is not a right-to-left minimum in τ as well. So in τ, there is a τ_{i_q} which is the first right-to-left minimum to the right of $b_{k-1} - 1$ in τ. By using (6), we obtain that in σ, σ_{i_q} and $\sigma_{i_{q+1}}$ are at the right side of b_{k-1} and $\sigma_{i_q} < \sigma_{i_{q+1}} < b_{k-1}$. This means that in $\sigma_{[k]}$, there are at least two letters at the right side of b_{k-1}, which is a contradiction. Therefore, we have $\sigma \in \mathcal{A}_n$. It is not difficult to verify that ρ^{-1} is the inverse of ρ, which yields the result. □

Remark 13. From Lemma 10 and the previous part of the proof of Proposition 12, we see that for $\sigma \in \mathcal{A}_n$, σ and $\rho(\sigma)$ have the same set of positions of the right-to-left minima.

Proposition 14. *The bijection ρ preserves the cd-indices, that is, for $n \geqslant 1$ and $\sigma \in \mathcal{A}_n$, we have*

$$\mathrm{cd}(\sigma) = \mathrm{cd}(\rho(\sigma)). \tag{7}$$

Proof. If $n = 1$, then $\mathrm{cd}(\sigma) = \mathrm{cd}(\rho(\sigma)) = 1$. For $n \geqslant 2$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{A}_n$ with its positions of the right-to-left minima $\{i_1 < i_2 < \cdots < i_l\}$, and $\rho(\sigma) = \tau = \tau_1\tau_2\cdots\tau_n \in \mathcal{SS}_n$. For two consecutive letters $\sigma_i\sigma_{i+1}$ in σ, where $i \in \{1, \ldots, n-1\}$, there are four cases as follows.

(i) If $i,\ i+1 \notin \{i_1, i_2, \ldots, i_l\}$, then $\sigma_i > \sigma_{i+1}$ (resp. $\sigma_i < \sigma_{i+1}$) implies that $\tau_i > \tau_{i+1}$ (resp. $\tau_i < \tau_{i+1}$), since $\tau_i = \sigma_i - 1$ and $\tau_{i+1} = \sigma_{i+1} - 1$.

(ii) If $i = i_k \in \{i_1, i_2, \ldots, i_{l-1}\}$ for some k and $i+1 \notin \{i_1, i_2, \ldots, i_l\}$, then $\sigma_i < \sigma_{i+1}$, as σ and τ have the same set of positions of the right-to-left minima, which yields $\tau_i < \tau_{i+1}$.

(iii) If $i \notin \{i_1, i_2, \ldots, i_{l-1}\}$ and $i+1 = i_k \in \{i_1, i_2, \ldots, i_l\}$ for some $k = 1, 2, \ldots, l$, then $\sigma_i > \sigma_{i+1}$. Since σ and τ have the same set of positions of the right-to-left minima, this implies that $\tau_i > \tau_{i+1}$

(iv) If $i = i_k \in \{i_1, i_2, \ldots, i_{l-1}\}$ and $i+1 = i_{k+1} \in \{i_1, i_2, \ldots, i_l\}$ for some $k = 1, 2, \ldots, l$, then $\sigma_i < \sigma_{i+1}$ implies that $\tau_i < \tau_{i+1}$ for the same reason as (ii) and (iii).

By analyzing these four cases, we can deduce that τ and σ have the same descent set, and therefore, the same ascent set. By the definition of the *cd*-index, we have $\mathrm{cd}(\sigma) = \mathrm{cd}(\tau) = \mathrm{cd}(\rho(\sigma))$. □

Now, let us go back to the x-factorization of permutations. First, let us recall the involutions defined on permutations introduced in [10]. Let (w_1, w_2, x, w_4, w_5) be the x-factorization of permutation w containing the letter x. We define ψ_x on w as

$$\psi_x\, w = w_1w_4xw_2w_5.$$

Clearly (w_1, w_4, x, w_2, w_5) is the x-factorization of $\psi_x\, w$. In [10], Foata and Strehl also proved that for any $x,\ y \in [n]$, ψ_x and ψ_y commute. Since all the ψ_x's commute, for any $X \subseteq [n]$, we can define

$$\psi_X = \prod_{x\in X} \psi_x,$$

when X is empty, we let ψ_X be the identity map of $\mathfrak{S}_n$. Then we have the following proposition.

Proposition 15. *If $\sigma \in \mathcal{DD}_n$ and $X \subseteq \mathrm{Val}(\sigma)$, then $\psi_X\, \sigma \in \mathcal{DD}_n$ and*

$$\mathrm{Val}(\sigma) = \mathrm{Val}(\psi_X\, \sigma). \tag{8}$$

Proof. First, we demonstrate that for any $x \in X$, we have $\psi_x\, \sigma \in \mathcal{DD}_n$ and $\mathrm{Val}(\sigma) = \mathrm{Val}(\psi_x\, \sigma)$. Let the x-factorization of σ be denoted as (w_1, w_2, x, w_4, w_5), and ψ_x acting on $\sigma = w_1w_2xw_4w_5$ is to exchange w_2 and w_4. Consequently, the situation where the stated conclusion does not hold true will only arise at special positions: the last letter of w_1, the first letter of w_2, the last letter of w_2, the first letter of w_4, the last letter of w_4, and the first letter of w_5. Since x is a valley of σ, w_2 and w_4 are not empty words. Let the first letter of w_2 be $\mathrm{first}(w_2)$ and the first letter of w_4 be $\mathrm{first}(w_4)$, where $\mathrm{first}(w_2) > x$ and $\mathrm{first}(w_4) > x$. If w_1 is not empty, then let $\mathrm{last}(w_1)$ be its last letter such that $\mathrm{last}(w_1) < x$. This implies $\mathrm{last}(w_1) < \mathrm{first}(w_4)$ and $\mathrm{last}(w_1) < \mathrm{first}(w_2)$, which means $\mathrm{last}(w_1)$ is a valley of σ if and only if it is a valley of $\psi_x \sigma$ and $\mathrm{last}(w_1)$ is not a double descent of $\psi_x \sigma$. If w_5 is not empty, then $\mathrm{last}(w_4) > \mathrm{first}(w_5)$ and $\mathrm{last}(w_2) > \mathrm{first}(w_5)$ because $\mathrm{first}(w_5) < x$, $\mathrm{last}(w_2) > x$, and $\mathrm{last}(w_4) > x$. This implies that $\mathrm{first}(w_5)$ is a valley of σ if and only if it is a valley of $\psi_x \sigma$, and $\mathrm{first}(w_5)$ is not a double descent of $\psi_x \sigma$.

In σ, $\mathrm{first}(w_2)$, $\mathrm{last}(w_2)$, $\mathrm{first}(w_4)$, and $\mathrm{last}(w_4)$ are not valleys. This is because either $\mathrm{first}(w_2) > \mathrm{last}(w_1)$ or w_1 is empty, and $\mathrm{last}(w_2) > x$, $\mathrm{first}(w_4) > x$, and $\mathrm{last}(w_4) > \mathrm{first}(w_5)$ or w_5 is empty. In $\psi_x \sigma$, either $\mathrm{first}(w_4) > x > \mathrm{last}(w_1)$ or w_1 is empty, and $\mathrm{last}(w_4) > x$, $\mathrm{first}(w_2) > x$, and $\mathrm{last}(w_2) > x > \mathrm{first}(w_5)$ or w_5 is empty. Consequently, $\mathrm{first}(w_2)$, $\mathrm{last}(w_2)$, $\mathrm{first}(w_4)$, and $\mathrm{last}(w_4)$ are not valleys of $\psi_x \sigma$, and $\mathrm{first}(w_2)$ and $\mathrm{first}(w_4)$ are not double descents of $\psi_x \sigma$. Since $\sigma \in \mathcal{DD}_n$, w_2 must end with an ascent or be a single letter. Thus, if w_5 is not empty, then $\mathrm{last}(w_2)$ in $\psi_x \sigma$ is a double ascent or a peak, since $\mathrm{last}(w_2) > \mathrm{first}(w_5)$. If w_5 is not empty, then w_4 must end with an ascent, or it is a single letter; if w_5 is empty, then w_4 must end with an ascent, because σ ends with an ascent. In all cases, $\mathrm{last}(w_4)$ is not a double descent of $\psi_x \sigma$. Now, we demonstrate that $\psi_x \sigma$ ends with an ascent. If w_5 is not empty, then it must have at least two letters. Otherwise, $\mathrm{last}(w_4) > \mathrm{first}(w_5) = w_5$, indicating that σ ends with a descent, which is a contradiction. Thus, in this case, $\psi_x \sigma$ ends with an ascent because w_5 ends with an ascent. If w_5 is empty, then $\psi_x \sigma$ also ends with an ascent, since w_2 ends with an ascent or is a single letter. If it is a single letter, then $w_2 > x$. Hence, $\psi_x \sigma \in \mathcal{DD}_n$, and $\mathrm{Val}(\sigma) = \mathrm{Val}(\psi_x \sigma)$. Since $\psi_X = \prod_{x \in X} \psi_x$, we obtain the desired result. $\square$

Recall that for $\sigma \in \mathcal{DD}_n$, $\mathrm{vala}(\sigma)$ is the number of André valleys of σ and see Definition 7 to recall the definition of André valleys.

Lemma 16. *If $\sigma \in \mathcal{DD}_n$ and $x \in \mathrm{Val}(\sigma)$, then*

$$\mathrm{vala}(\psi_x \sigma) = \begin{cases} \mathrm{vala}(\sigma) + 1 & \text{if } x \text{ is a non-André valley of } \sigma; \\ \mathrm{vala}(\sigma) - 1 & \text{if } x \text{ is an André valley of } \sigma. \end{cases} \tag{9}$$

Proof. Suppose the x-factorization of σ is (w_1, w_2, x, w_4, w_5), so that $\psi_x \sigma = w_1 w_4 x w_2 w_5$ with the x-factorization (w_1, w_4, x, w_2, w_5). In other words, ψ_x switches x from André valley to non-André valley or from non-André valley to André valley. If $|\mathrm{Val}(\sigma)| = 1$, then by Proposition 15, (9) holds true. However, if $|\mathrm{Val}(\sigma)| \geqslant 2$, then for $x, y \in \mathrm{Val}(\sigma)$, without loss of generality, let us assume that $x > y$ and that x precedes y in σ. Suppose that the y-factorization of σ is $(w'_1, w'_2, y, w'_4, w'_5)$. Since x comes before y in σ, it follows that either $x \in w'_2$ or $x \in w'_1$.

- If $x \in w'_2$, then since $w_2 x w_4$ are consecutive letters in σ that are larger than y, we have $w_2 x w_4 \subseteq w'_2$. This implies that the y-factorization of $\psi_x \sigma$ is $(w'_1, w''_2, y, w'_4, w'_5)$, where w''_2 is obtained by swapping w_2 and w_4 in w'_2. Therefore, w'_2 and w''_2 are composed of the same letters, meaning $\min(w'_2) = \min(w''_2)$.

- If $x \in w'_1$, then $w_2 x w_4 \subseteq w'_1$. Otherwise, if there is a letter in $w_2 x w_4$ that is also in w''_2, then $w_2 x w_4 \subseteq w'_2$ due to w'_2 comprising the maximal consecutive letters in σ larger than x. This situation leads to a contradiction since $x \in w'_1$. Since $w_2 x w_4 \subseteq w'_1$, the y-factorization of $\psi_x \sigma$ is $(w''_1, w'_2, y, w'_4, w'_5)$, where w''_1 is obtained by swapping w_2 and w_4 in w'_1.

Both cases indicate that if y is an André valley (resp. a non-André valley) in σ, then y remains an André valley (resp. a non-André valley) in $\psi_x \sigma$ as well. In other words, ψ_x only switches x from André valley to non-André valley or from non-André valley to André valley, which allows us to obtain (9). □

Since all the valleys of André permutations are André valleys and $\psi_X = \prod_{x \in X} \psi_x$, we establish the following lemma according to Proposition 15 and Lemma 16.

Lemma 17. *If $\sigma \in \mathcal{A}_n$ and $X \subseteq \mathrm{Val}(\sigma)$, then the number of André valleys of $\psi_X \sigma$ and the cardinality of X satisfy the following relationship,*

$$\mathrm{vala}(\psi_X \sigma) + |X| = |\mathrm{Val}(\sigma)|. \tag{10}$$

Now, we are in a position to prove Theorem 8.

Proof of Theorem 8. Let $\mathcal{DD}_{n,k} = \{\sigma \in \mathcal{DD}_n : |\mathrm{Val}(\sigma)| = k\}$ and $\mathcal{A}_{n,k} = \{\sigma \in \mathcal{A}_n : |\mathrm{Val}(\sigma)| = k\}$. For $\sigma \in \mathcal{DD}_{n,k}$, let $\mathrm{Orb}(\sigma)$ be the orbit of σ under the action $\{\psi_X : X \subseteq \mathrm{Val}(\sigma)\}$. By Proposition 15 and Lemma 16, we observe that $\mathrm{Orb}(\sigma) \subseteq \mathcal{DD}_{n,k}$, and there is exactly one André permutation in $\mathrm{Orb}(\sigma)$, namely, the permutation with all of its valleys being André valleys. Since $\mathcal{A}_{n,k} \subseteq \mathcal{DD}_{n,k}$, we have

$$\mathcal{DD}_n = \bigcup_{k=1}^{\lfloor n/2 \rfloor} \mathcal{DD}_{n,k} = \bigcup_{k=1}^{\lfloor n/2 \rfloor} \bigcup_{\tau \in \mathcal{A}_{n,k}} \mathrm{Orb}(\tau). \tag{11}$$

For $\sigma \in \mathcal{A}_n$, let $\Phi_{n,\sigma}(\mathbf{c}, \mathbf{d}) = \mathrm{cd}(\sigma)$. By the definition of *cd*-index, we can observe that in $\mathrm{cd}(\sigma)$, the letters $\mathbf{d}$'s correspond to the valley triples of σ. These valley triples, denoted as $(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$, satisfy $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$ for $1 \leqslant i \leqslant n-1$, where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ with $\sigma_0 = 0$. In other words, the occurrence of the letter $\mathbf{d}$ in $\mathrm{cd}(\sigma)$ signifies the presence of an André valley in σ. So by Lemma 16 and Lemma 17, we have

$$\Phi_{n,\sigma}(\mathbf{c}, \mathbf{d} + \mathbf{c}^2) = \sum_{\tau \in \mathrm{Orb}(\sigma)} \hat{\mathrm{cd}}(\tau). \tag{12}$$

By Proposition 14 we derive that

$$\sum_{\sigma \in \mathcal{A}_n} \Phi_{n,\sigma}(\mathbf{c}, \mathbf{d}) = \sum_{\sigma \in \mathcal{A}_n} \mathrm{cd}(\sigma) \tag{13}$$

$$= \sum_{\sigma \in \mathcal{SS}_n} \mathrm{cd}(\sigma) \tag{14}$$

$$= \Phi_n(\mathbf{c}, \mathbf{d}). \tag{15}$$

So, combining (11), (12), and (15), we have

$$\begin{aligned}
\hat{\Phi}_n(\mathbf{c},-\mathbf{d}) &= \Phi_n(\mathbf{c},\mathbf{d}+\mathbf{c}^2)\\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\sigma\in\mathcal{A}_{n,k}} \Phi_{n,\sigma}(\mathbf{c},\mathbf{d}+\mathbf{c}^2)\\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\sigma\in\mathcal{A}_{n,k}} \sum_{\tau\in\mathrm{Orb}(\sigma)} \hat{\mathrm{cd}}(\tau)\\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\sigma\in\mathcal{DD}_{n,k}} \hat{\mathrm{cd}}(\sigma)\\
&= \sum_{\sigma\in\mathcal{DD}_{n}} \hat{\mathrm{cd}}(\sigma),
\end{aligned}$$

which establishes Theorem 8.

3 γ-expansions of alternating Eulerian polynomials

Recall that any polynomial $h(t) = \sum_{i=0}^{n} h_i t_i$ satisfying $h_i = h_{n-i}$ can be expressed uniquely in the form $\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i t^i (1+t)^{n-2i}$. The coefficients γ_i are called the γ-coefficients of $h(t)$. If the γ-coefficients γ_i are all nonnegative, then $h(t)$ is said to be γ-nonnegative. γ-positivity appears surprisingly often in combinatorial and geometric contexts, the notion of γ-positivity appeared first in the work of Foata and Schützenberger [7], and subsequently of Foata and Strehl [10, 11] on the classical Eulerian polynomials. After having implicitly reappeared in the theory of enriched poset partitions of Stembridge [22, Section 2.3]. It was brought again to light independently by Brändén [3, 4] and Gal [12] in the course of their study of poset Eulerian polynomials and face enumeration of flag triangulations of spheres, respectively. These works made it clear that γ-positivity is a concept of independent interest which provides a powerful approach to the problem of unimodality for symmetric polynomials.

In 2022, Lin, Ma, Wang and Wang [16] demonstrated that the alternating Eulerian polynomials (1) have γ-expansions with γ-coefficients alternating in sign. Ma, Fang, Mansour and Yeh [17] presented grammatical interpretations for the alternating Eulerian polynomials of type A and B. And then, they established an interesting connection between alternating Eulerian polynomials of type B and left peak polynomials, which implies that the type B alternating Eulerian polynomials also have γ-expansions with γ-coefficients alternating in sign. Subsequently, Pan and Zeng [19] combined algebraic and combinatorial approaches to provide a combinatorial interpretation for their γ-coefficients in terms of min-max trees. Here, we present a *cd*-index approach to proving the γ-expansions of (1) and directly offer a new combinatorial interpretation for their γ-coefficients. The following theorem represents our result.

Theorem 18. *For $n \geqslant 1$, the alternating Eulerian polynomial $\hat{A}_n(x)$ has the following γ-expansion:*

$$\hat{A}_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-2)^k b_{n,k} x^k (1+x)^{n-1-2k}, \tag{16}$$

where $b_{n,k} = |\{\sigma \in \mathcal{DD}_n : \mathrm{vala}(\sigma) = k\}|$.

Proof. By the definition of ab-index of $\mathfrak{S}_n$ with respect to alternating descent set statistic, let $\mathbf{a} = 1$ and $\mathbf{b} = x$. Then, $\hat{\Psi}_n(1, x) = \hat{A}_n(x)$. As shown by Proposition 3, this yields $\hat{\Phi}_n(1 + x, 2x) = \hat{\Psi}_n(1, x) = \hat{A}_n(x)$. Considering the definition of the cd-index of $\mathcal{DD}_n$, we can derive that for $\sigma \in \mathcal{DD}_n$, the number of the letter $\mathbf{d}$ in $\widehat{\mathrm{cd}}(\sigma)$, denoted as k, and the number of letter $\mathbf{c}$ in $\widehat{\mathrm{cd}}(\sigma)$, denoted as l, satisfy $2k + l = n - 1$. Furthermore, since the number of $\mathbf{d}$'s in $\widehat{\mathrm{cd}}(\sigma)$ corresponds to the number of André valleys of σ, by Theorem 8, we have

$$\hat{\Phi}_n(1 + x, 2x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} b_{n,k} (-1)^k (2x)^k (1+x)^{n-1-2k}, \tag{17}$$

where $b_{n,k} = |\{\sigma \in \mathcal{DD}_n : \mathrm{vala}(\sigma) = k\}|$. This completes our proof. $\square$

Acknowledgements

The authors thank Professor Isabella Novik for providing helpful suggestions. The authors are grateful to the referees for carefully reading of the paper and giving detailed comments which helped improving the readability of our paper. The first author's work was supported by the National Natural Science Foundation of China grant 12201468.

References

[1] M. M. Bayer, The *cd*-index: A survey, in Polytopes and discrete geometry, 1-19, *Contemp. Math.*, 764, *Amer. Math. Soc.*, 2021.

[2] M. M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* 6 (1991), no. 1, 33-47.

[3] P. Brändén, Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture, in Stanley Festschrift (B. E. Sagan, ed.), *Electron. J. Combin.* 11(2) (2004), #R9, 15pp (electronic).

[4] P. Brändén, Actions on permutations and unimodality of descent polynomials, *European J. Combin.* 29 (2008), 514-531.

[5] D. Chebikin, Variations on descents and inversions in permutations, *Electron. J. Combin.*, 15 (1): #R132, 2008.

[6] D. Foata and G.-N. Han, André permutation calculus: a twin Seidel matrix sequence, *Sémin. Lothar. Comb.* 73 (2014-2016) B73e.

[7] D. Foata and M.-P. Schützenberger, Théorie Géometrique des polynmes Eulériens, *Lect. Notes Math.* 138, *Springer-Verlag*, 1970.

[8] D. Foata and M.-P. Schützenberger, Nombres d'Euler et permutations alternantes, *Manuscript, University of Florida, Gainesville, FL*, 1971.

[9] D. Foata and M.-P. Schützenberger, Nombres d'Euler et permutations alternantes, *in: A survey of Combinatorial Theory, Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo.*, 1971, *North-Holland, Amsterdam*, 1973, pp. 173-187.

[10] D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, *Math. Z.* 137 (1974) 257-264.

[11] D. Foata and V. Strehl, Euler numbers and variations of permutations, *in Colloquio Internazionale sulle Teorie Combinatorie* (Roma, 1973), *Tomo, I, Atti dei Convegni Lincei*, No. 17, *Accad. Naz. Lincei, Rome*, 1976, pp. 119-131.

[12] Ś. R. Gal, Real root conjecture fails for five-and higher-dimensional sphere, *Discrete Comput. Geom.* 34 (2005), 269-284.

[13] G. Hetyei, On the *cd*-variation polynomials of André and Simsun permutations, *Discrete Comput. Geom.* 16 (1996), 259-275.

[14] G. Hetyei and E. Reiner, Permutation trees and variation statistics, *European J. Combin.* 19 (1998), pp. 847-866.

[15] K. Karu, The *cd*-index of fans and posets, *Compos. Math.* 142 (2006), no. 3, 701-718.

[16] Z. Lin, S-M. Ma, D. G. L. Wang and L. Wang, Positivity and divisibility of enumerators of alternating descents, *Ramanujan J.* 58, 203-228 (2022).

[17] S-M. Ma, Q. Fang, T. Mansour and Y. Yeh, Alternating Eulerian polynomials and left peak polynomials, *Discrete Math.* 345 (2022) 112714.

[18] Q. Q. Pan and J. Zeng, Brändén's (p, q)-Eulerian polynomials André permutations and continued fractions, *J. Comb. Theory Ser. A* 181 (2021) 105445.

[19] Q. Pan and J. Zeng, Enumeration of permutations by the parity of descent positions, *Discrete Math.* 346 (2023) 113575.

[20] M. Purtill, André permutations, lexicographic shellability and the *cd*-index of a convex polytope, *Trans. Amer. Math. Soc.* 338 (1993), no. 1, 77-104.

[21] R. P. Stanley, Flag f-vectors and the *cd*-index, *Math. Z.* 216 (1994), no. 3, 483-499.

[22] J. R. Stembridge, Enriched P-partitions, *Trans. Amer. Math. Soc.* 349 (1997), 763-788.

[23] S. Sundaram, The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice, *Adv. Math.* 104 (1994), 225-296.

[24] S. Sundaram, The homology of partitions with an even number of blocks, *J. Algebraic Combin.* 4 (1995) 69-92.

[25] S. Sundaram, Plethysm, partitions with an even number of blocks and Euler numbers, *In "Formal Power Series and Algebraic Combinatorics 1994," DIMACS Series in Discrete Math. and Theor. Comput. Sci.* 24 (1996), *Amer. Math. Soc.*.