Modified Ascent Sequences and Bell Numbers

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Submitted: May 17, 2023; Accepted: Oct 2, 2024; Published: Nov 1, 2024

Abstract

In 2011, Duncan and Steingrímsson conjectured that modified ascent sequences avoiding any of the patterns 212, 1212, 2132, 2213, 2231 and 2321 are counted by the Bell numbers. Furthermore, the distribution of the number of ascents is the reverse of the distribution of blocks on set partitions. We solve the conjecture for all the patterns except 2321. We describe the corresponding sets of Fishburn permutations by pattern avoidance, and leave some open questions for future work.

Mathematics Subject Classifications: 05A19, 05A05

1 Introduction

In recent years, several papers have been devoted to the study of combinatorial structures enumerated by the Fishburn numbers. The sequence of Fishburn numbers is recorded as A022493 in the OEIS [21] and has the elegant generating function [22]

$$\sum_{n\geqslant 0}\prod_{k=1}^{n}\big(1-(1-t)^k\big) = 1+t+2t^2+5t^3+15t^4+53t^5+217t^6+\cdots$$

The milestone paper [2] by Bousquet-Mélou, Claesson, Dukes and Kitaev introduced bijections between four families of Fishburn-type objects. Among them, we find ascent sequences, Fishburn permutations and unlabeled (**2**+**2**)-free posets. The set of Fishburn permutations consists of permutations avoiding a certain bivincular pattern. They can be constructed inductively by successive insertion of a new maximum value; by recording the positions where the new maximum is inserted at each step, we obtain ascent sequences. Later, Dukes and Parviainen [15] showed a recursive bijection between ascent sequences and upper triangular matrices with nonnegative integer entries and no null rows or columns, the Fishburn matrices. Modified ascent sequences were introduced to better clarify the relation between ascent sequences and (**2**+**2**)-free posets. Originally [2], they were defined as the bijective image of ascent sequences under a certain hat mapping. More recently, Claesson and the current author [7] characterized them independently as Cayley

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 https://doi.org/10.37236/12080

permutations where ascent tops and leftmost copies coincide. The same authors [8] introduced Fishburn trees to simplify the bijections relating all the Fishburn-type structures mentioned so far. They act as central objects from which modified ascent sequences, Fishburn matrices and (**2**+**2**)-free posets are obtained transparently. Claesson and the current author [7] also initiated the development of a theory of transport of patterns between Fishburn permutations and modified ascent sequences. Their framework is based on the fact that the map relating these two structures can be described by Burge transpose of Burge words, an operation that behaves well with respect to pattern containment. Pattern avoidance on (modified) ascent sequences and Fishburn permutations have been discussed recently by other authors [13, 16, 17, 18].

The current paper, which fits in the same line of research, is mostly devoted to the proof of the following conjecture proposed by Duncan and Steingrímsson [16] in 2011:

Conjecture 1. On modified ascent sequences, the patterns

$$212,\ 1212,\ 2132,\ 2213,\ 2231,\ 2321$$

are all Wilf-equivalent and the enumeration of modified ascent sequences avoiding any of these patterns is given by the Bell numbers. Moreover, the distribution of the number of ascents is the reverse of the distribution of blocks on set partitions.

The Bell numbers appear as sequence A000110 in the OEIS [21]:

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975.$$

Note that 12132 is the only modified ascent sequence of length 5 that contains any of the patterns listed in Conjecture 1; hence there are 52 such sequences instead of 53. It is well known that the nth Bell number is equal to the number of set partitions over $\{1, 2, \ldots, n\}$, and the distribution of blocks is given by the Stirling numbers of the second kind.

In Section 2, we provide the necessary tools and definitions. We define restricted growth functions and show that restricted growth functions of length n with maximum value k encode bijectively set partitions of $\{1, 2, \ldots, n\}$ with k blocks. We define modified ascent sequences and Fishburn permutations, as well as the operation of Burge transpose that relates them. We define classical patterns and mesh patterns, both on classical permutations and on permutations with repeated entries, i.e. Cayley permutations. We end the section with some general results concerning the enumerative aspects of modified ascent sequences.

In Section 3, we solve Conjecture 1 for the first three patterns 212, 1212 and 2132. To start, we show that in fact the avoidance of any pattern in $\{212, 1212, 2132, 12132\}$ determines the same set of modified ascent sequences, adding 12132 to the conjecture in the process. Then we find a recursive construction of 212-avoiding modified ascent sequences by successive insertion of new maxima. This construction embodies well known equations defining the Stirling numbers of the second kind, and a proof of Conjecture 1 follows immediately. We also show that restricted growth functions encode active sites of 212-avoiding modified ascent sequences in a way that is similar to how ascent sequences encode active sites of Fishburn permutations. En passant, we obtain a bijection between these two structures. An alternative bijection is then obtained by slightly tweaking the Burge transpose. Quite surprisingly, the same construction seems to be working for the pattern 2213 as well, a fact that we were not able to prove.

In Section 4, we solve the patterns 2213 and 2231. We provide a recursive construction of 2213- and 2231-avoiding modified ascent sequences in terms of the number of copies of 1 they contain. The resulting equations are different from the one obtained previously for the pattern 212, but they still lead to the Bell numbers. A bijection with the set of restricted growth functions is once again an immediate outcome of our approach.

In Section 5, we describe by pattern avoidance the sets of Fishburn permutations corresponding to modified ascent sequences avoiding 212, 2213 and 2321, respectively. A description of the set corresponding to 2231-avoiding modified ascent sequences remains to be determined. More suggestions for future work can be found in Section 6.

2 Preliminaries

Let n be a natural number. An *endofunction* of size n is a map $x : [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$. We often identify the endofunction x with the word $x = x_1 \ldots x_n$, where $x_i = x(i)$ for each $i \in [n]$. If $n = 0$, we allow the empty endofunction ϵ. A *Cayley permutation* [4, 20] is an endofunction $x : [n] \to [n]$ such that $\operatorname{Im}(x) = [k]$, for some $k \leqslant n$. Alternatively, a nonempty endofunction $x = x_1 \cdots x_n$ is a Cayley permutation if it contains at least a copy of every integer between 1 and its maximum value. In this paper, given a set A whose elements are equipped with a notion of size, we will denote by A_n the set of elements in A that have size n. Conversely, given a definition of A_n (of elements of size n) we let $A = \cup_{n\geqslant 0} A_n$. For instance, we let Cay_n be the set of Cayley permutations of size n and $\mathrm{Cay} = \cup_{n\geqslant 0}\mathrm{Cay}_n$ be the set of all Cayley permutations. For $n \leqslant 3$, we have $\mathrm{Cay}_0 = \{\epsilon\}$, $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$ and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

It is well known that a Cayley permutation x encodes the ballot (ordered set partition) $B_1 \ldots B_k$, where $i \in B_{x_i}$ and $k = \max(x)$. For instance,

$$x = 311241334 \quad \text{encodes the ballot} \quad \{2, 3, 6\}\{4\}\{1, 7, 8\}\{5, 9\}.$$

This encoding of ballots by Cayley permutations is bijective, and thus Cayley permutations are counted by the Fubini numbers (sequence A000670 in the OEIS [21]). If we apply the same encoding to set partitions, listed with minima of blocks in increasing order, we obtain *restricted growth functions*

$$\mathrm{RGF}_n = \{x_1 \cdots x_n :\ x_1 = 1,\ x_{i+1} \leqslant \max(x_1 \cdots x_i) + 1 \ \text{ for each } i < n\}.$$

As a result, $|\mathrm{RGF}_n|$ is equal to the nth Bell number; furthermore, the number of restricted growth functions of size n whose maximum value is k equals the number of set partitions of $[n]$ with k blocks. These are counted by the (n,k)th Stirling number of the second kind, denoted here by $S(n,k)$. For $n = 3$, we obtain

$$\mathrm{RGF}_3 = \{111, 112, 121, 122, 123\}.$$

The shortest restricted growth function that is not a modified ascent sequence (defined in Section 2.1) is 1212; on the other hand, 1312 is a modified ascent sequence, but not a restricted growth function.

Let $x \in \mathrm{Cay}_n$ and $y \in \mathrm{Cay}_k$ be two Cayley permutations, with $k \leqslant n$. Then x *contains* y if x contains a subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$, with $i_1 < i_2 < \cdots < i_k$, which is *order isomorphic* to y; that is, $x_{i_s} < x_{i_t}$ if and only if $y_s < y_t$ and $x_{i_s} = x_{i_t}$ if and only if $y_s = y_t$. In this case, we write $y \leqslant x$ and $x_{i_1}x_{i_2}\cdots x_{i_k} \simeq y$; the subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$ is called an *occurrence* of the *pattern* y in x. Otherwise, we say that x *avoids* y. We denote by $\mathrm{Cay}(y)$ the set of Cayley permutations that avoid y. A notable example is the set of *permutations* $\mathrm{Sym} = \mathrm{Cay}(11)$; equivalently, Sym is the set of bijective endofunctions. More generally, when B is a set of patterns, $\mathrm{Cay}(B)$ shall denote the set of Cayley permutations avoiding every pattern in B. We use analogous notations for subsets of Cay. For instance, $\hat{\mathcal{A}}(212)$ denotes the set of modified ascent sequences (defined in Section 2.1) avoiding the pattern 212. For a detailed introduction to permutation patterns we refer to Bevan's note [1].

A more general notion of containment is obtained via *mesh patterns* [12] and *Cayley-mesh patterns* [5]. A mesh pattern is a pair (y, R), where $y \in \mathrm{Sym}_k$ is a permutation (classical pattern) and $R \subseteq [0,k] \times [0,k]$ is a set of pairs of integers. The pairs in R identify the lower left corners of unit squares in the plot of x which specify forbidden regions. An occurrence of the mesh pattern (y, R) in the permutation x is an occurrence of the classical pattern y such that no other points of x occur in the forbidden regions specified by R. Cayley-mesh patterns, i.e. mesh patterns on Cayley permutations, are defined analogously, but with additional regions that allow the possibility of having repeated entries. In this paper, we will often define mesh patterns (both on permutations and Cayley permutations) by plotting the underlying classical pattern, with the forbidden regions shaded. For instance, the mesh patterns that characterize Fishburn permutations and modified ascent sequences (see Section 2.1) are illustrated in Figure 1; here, $\mathfrak{f}$ would be more extensively defined as

$$\mathfrak{f} = (231, R), \quad \text{with } R = \{(1,0), (1,1), (1,2), (1,3), (0,1), (2,1), (3,1)\}.$$

Note that the shaded regions of $\mathfrak{f}$ consist of rows or columns only; that is, if a box is shaded, then its whole row or column is shaded as well. A mesh pattern that satisfies this property is called a *bivincular pattern* [2]. Roughly speaking, shaded columns impose a constraint of adjacency on the positions, while shaded rows impose a constraint of adjacency on the values. An occurrence of $\mathfrak{f}$ is an occurrence of 231 where the (entries playing the role of) 2 and 3 are in consecutive positions, and the 2 and 1 are consecutive in value. As an example, the permutation 31524 avoids $\mathfrak{f}$ but contains 231, while the permutation 32541 contains an occurrence of $\mathfrak{f}$ realized by the entries 251.

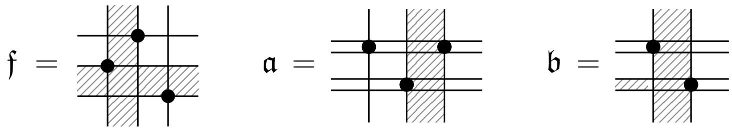


Figure 1: Mesh patterns such that $F = \mathrm{Sym}(\mathfrak{f})$ and $\hat{\mathcal{A}} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b})$.

2.1 Modified ascent sequences and Fishburn permutations

Given a Cayley permutation x, let

$$\mathrm{asctops}(x) = \{(1, x_1)\} \cup \{(i, x_i) : 1 < i \leqslant n,\ x_{i-1} < x_i\}$$

be the set of *ascent tops* and their indices, including the first element; let also

$$\mathrm{nub}(x) = \{(\min x^{-1}(j), j) : 1 \leqslant j \leqslant \max(x)\}$$

be the set of *leftmost copies* and their indices[1]. The set nub has recently played a central role in the context of modified (difference) ascent sequences [6, 10, 11]. As an example, we have

$$\mathrm{asctops}(1212) = \{(1,1),(2,2),(4,2)\} \quad \text{while} \quad \mathrm{nub}(1212) = \{(1,1),(2,2)\}.$$

On the other hand,

$$\mathrm{asctops}(1312) = \mathrm{nub}(1312) = \{(1,1),(2,3),(4,2)\}.$$

When there is no ambiguity, we will sometimes abuse notation and simply write $x_i \in \mathrm{nub}(x)$ or $x_i \in \mathrm{asctops}(x)$. If $x_i \in \mathrm{nub}(x)$ and $x_i = a$, we say that x_i is the leftmost copy of a in x; or, that x_i is a leftmost copy in x. Claesson and the current author [7] proved the following characterization of the set $\hat{\mathcal{A}}$ of *modified ascent sequences*[2], and we shall use it as the definition of $\hat{\mathcal{A}}$.

Proposition 2. *[7, Lemma 2.1] We have*

$$\hat{\mathcal{A}} = \{x \in \mathrm{Cay} : \mathrm{asctops}(x) = \mathrm{nub}(x)\}.$$

In particular, all the ascent tops have distinct values and $\max(x) = |\mathrm{asctops}(x)|$.

The equation $\mathrm{asctops}(x) = \mathrm{nub}(x)$ can be equivalently expressed with Cayley-mesh patterns as

$$\hat{\mathcal{A}} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b}),$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are depicted in Figure 1. Indeed, for any Cayley permutation x, it is easy to see that

$$x \text{ avoids } \mathfrak{a} \quad \text{if and only if} \quad \mathrm{asctops}(x) \subseteq \mathrm{nub}(x)$$

[1] The name "nub" comes from an Haskell function that removes duplicate elements from a list, keeping only the first occurrence of each element. One may also think of nub as a short for "not used before".

[2] We depart slightly from the original paper [2] where modified ascent sequences are zero-based.

and

$$x \text{ avoids } \mathfrak{b} \quad \text{if and only if} \quad \text{asctops}(x) \supseteq \text{nub}(x).$$

Indeed, the rightmost entry in any occurrence of $\mathfrak{a}$ is an ascent top which is not a leftmost copy; and, conversely, the rightmost entry in any occurrence of $\mathfrak{b}$ is a leftmost copy that is not an ascent top. The set $\hat{\mathcal{A}}$ can be alternatively defined in a recursive fashion as follows [7]. There is exactly one modified ascent sequence of length zero, the empty word, and of length one, the single letter word 1. For $n \geqslant 1$, every $y \in \hat{\mathcal{A}}_{n+1}$ is of one of two forms depending on whether the last letter forms an ascent with the penultimate letter:

- $y = x_1 \cdots x_n\, x_{n+1}$, with $1 \leqslant x_{n+1} \leqslant x_n$, or
- $y = \tilde{x}_1 \cdots \tilde{x}_n\, x_{n+1}$, with $x_n < x_{n+1} \leqslant 1 + \max(x_1 \cdots x_n)$,

where $x_1 \cdots x_n \in \hat{\mathcal{A}}_n$ and, for $i \in [n]$,

$$\tilde{x}_i = \begin{cases} x_i & \text{if } x_i < x_{n+1} \\ x_i + 1 & \text{if } x_i \geqslant x_{n+1}. \end{cases}$$

In other words, each modified ascent sequence x gives rise to $\max(x) + 1$ modified ascent sequences of length one more, obtained by

- adding a new rightmost entry less than or equal to $\max(x) + 1$;
- and, if the newly added element is an ascent top, increasing by one all the entries of x that are greater than or equal to the new entry.

The set of Fishburn permutations is defined as $F = \text{Sym}(\mathfrak{f})$, where $\mathfrak{f}$ is the bivincular pattern depicted in Figure 1. Claesson and the current author [7] reformulated the original bijection [2] relating $\hat{\mathcal{A}}$ and F in terms of Burge transpose of Burge words. Let I_n be the subset of Cay_n consisting of the weakly increasing Cayley permutations

$$I_n = \{u \in \text{Cay}_n :\ u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n\}.$$

Define the set of biwords

$$\text{Bur}_n = \{(u, v) \in I_n \times \text{Cay}_n :\ \text{Des}(u) \subseteq \text{Des}(v)\},$$

where $\text{Des}(v) = \{i : v_i \geqslant v_{i+1}\}$ is the set of weak descents of v. Biwords in Bur_n are called *Burge words* due to their connection with the Burge variant of the RSK correspondence [3]. Here we will write a Burge word either as a pair (u, v), or, more extensively, as a biword (i.e. two-line array)

$$\binom{u}{v} = \begin{pmatrix} u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}.$$

The *Burge transpose* of the Burge word (u, v) is the biword $(u, v)^T$ obtained by turning each column of (u, v) upside down, and sorting the columns of the resulting biword in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. The reason why this operation is called a transposition lies in the fact that Burge words are in bijection with *Burge matrices*, i.e. matrices with nonnegative integer entries whose every row and column has at least one nonzero entry. Under this bijection, the operation of transposing a Burge word corresponds to the usual matrix transposition; that is, if M is the matrix associated with (u, v), then the transpose of M is associated with $(u, v)^T$. Now, it is possible to show [7] that Bur_n is closed under transpose and $w = (u, v)$ is a Burge word if and only if $(w^T)^T = w$. Furthermore, if x is a modified ascent sequence of length n and $\mathrm{id} = 12 \cdots n$ denotes the identity permutation (of the same length), then the Fishburn permutation p corresponding to x can be computed as

$$\binom{\mathrm{id}}{x}^T = \binom{\mathrm{sort}(x)}{p},$$

where $\mathrm{sort}(x)$ is obtained by sorting the entries of x in weakly incrasing order. Define the map $\gamma : \mathrm{Cay} \to \mathrm{Sym}$ accordingly by letting, for $x \in \mathrm{Cay}$,

$$\binom{\mathrm{id}}{x}^T = \binom{\mathrm{sort}(x)}{\gamma(x)}.$$

Then, the restriction of γ to $\hat{\mathcal{A}}$ is a size-preserving bijection from $\hat{\mathcal{A}}$ to F. As an example to illustrate this construction, let $x = 141233551$. Note that x is a modified ascent sequence, e.g. by Proposition 2, since

$$\mathrm{asctops}(x) = \{(1, 1), (2, 4), (4, 2), (5, 3), (7, 5)\} = \mathrm{nub}(x).$$

Finally, we have

$$\binom{\mathrm{id}}{x}^T = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,4\,1\,2\,3\,3\,5\,1 \end{pmatrix}^T = \begin{pmatrix} 1\,1\,1\,2\,3\,3\,4\,5 \\ 8\,3\,1\,4\,6\,5\,2\,7 \end{pmatrix}$$

and the Fishburn permutation corresponding to x is $\gamma(x) = 83146527$. It is easy to check that $\gamma(x)$ is in fact a Fishburn permutation, that is, it avoids $\mathfrak{f}$.

The following theorem describes how pattern avoidance is transported from Fishburn permutations to modified ascent sequences.

Theorem 3. *[7, Theorem 5.1] For every permutation p, we have*

$$F(p) = \gamma\big(\hat{\mathcal{A}}(B_p)\big),$$

where $B_p = \{x \in \mathrm{Cay} : \gamma(x) = p\}$ is the Fishburn basis *of p.*

Theorem 3 shows that the set $F(p)$ of Fishburn permutations avoiding p is associated via the bijection γ with the set $\hat{\mathcal{A}}(B_p)$ of modified ascent sequences avoiding every pattern in B_p. A method to construct B_p explicitly can be found in the same paper [7].

2.2 General results on $\hat{\mathcal{A}}$

In this section we prove some properties of the set $\hat{\mathcal{A}}$ that will be useful later.

Proposition 4. *In a modified ascent sequence, all the copies of its maximum value are in consecutive positions.*

Proof. Let $x \in \hat{\mathcal{A}}$ and let $m = \max(x)$. We show that all the entries of x that are equal to m are in consecutive positions. Equivalently, we show that, if $x_i = x_j = m$ for some $i < j$, then it must be $x_\ell = m$ for each $i < \ell < j$. For a contradiction, suppose that there is at least one entry between x_i and x_j that is smaller than m, and let

$$\ell = \max\{k : \ i < k < j \text{ and } x_k < m\}$$

be the index of the rightmost such entry. Then $x_{\ell+1} = m$, $x_\ell < x_{\ell+1}$ and $x_i x_\ell x_{\ell+1}$ is an occurrence of $\mathfrak{a}$ in x, which is impossible due to Proposition 2. □

Given a natural number n, let $f_n = |\hat{\mathcal{A}}_n|$ denote the nth Fishburn number. For $1 \leqslant k \leqslant n$, let

$$\hat{\mathcal{A}}_n(k) = \{x \in \hat{\mathcal{A}}_n : \ x \text{ contains } k \text{ copies of } \max(x)\}$$

and let $f_n(k) = |\hat{\mathcal{A}}_n(k)|$. Note that $f_n = \sum_{k=1}^{n} f_n(k)$.

Proposition 5. *For $n \geqslant 1$, we have*

$$f_{n+1}(1) = f_{n+1} - f_n.$$

Proof. By Proposition 4, in a modified ascent sequence all the entries that are equal to its maximum value are in consecutive positions. As a result, for $k \geqslant 2$, the removal of the rightmost such entry yields a bijection from $\hat{\mathcal{A}}_{n+1}(k)$ to $\hat{\mathcal{A}}_n(k-1)$. Thus $f_{n+1}(k) = f_n(k-1)$ and also

$$\begin{aligned} f_{n+1} &= \sum_{k=1}^{n+1} f_{n+1}(k) \\ &= f_{n+1}(1) + \sum_{k=2}^{n+1} f_{n+1}(k) && [f_{n+1}(k) = f_n(k-1)] \\ &= f_{n+1}(1) + \sum_{k=2}^{n+1} f_n(k-1) && [i = k-1] \\ &= f_{n+1}(1) + \sum_{i=1}^{n} f_n(i) \\ &= f_{n+1}(1) + f_n, \end{aligned}$$

from which $f_{n+1}(1) = f_{n+1} - f_n$ follows immediately. □

Corollary 6. *For $n \geqslant 1$, we have*

$$f_n = \sum_{i=1}^{n} f_i(1);$$

that is, Fishburn numbers are the partial sum of the sequence $\{f_n(1)\}_{n \geqslant 1}$.

Proof. The case $n = 1$ holds since $f_1 = f_1(1)$. For $n \geqslant 2$, we repeatedly iterate Proposition 5 to obtain

$$f_n = f_n(1) + f_{n-1} = f_n(1) + f_{n-1}(1) + f_{n-2} = \cdots = \sum_{i=1}^{n} f_i(1). \qquad \square$$

The first ten terms of the sequence $\{f_n(1)\}_{n \geqslant 1}$ are

$$1, 1, 3, 10, 38, 164, 797, 4321, 25905, 170368.$$

We call these the 2-*Fishburn numbers*. The 2-Fishburn numbers can be alternatively expressed in terms of

$$f_{n,m} = |\{x \in \hat{\mathcal{A}}_n : \max(x) = m\}| \quad \text{as} \quad f_{n+1}(1) = \sum_{m=1}^{n} m f_{n,m}. \tag{1}$$

Indeed, we have $f_n = \sum_{m=1}^{n} f_{n,m}$. Furthermore, referring to the recursive construction of $\hat{\mathcal{A}}$ described in Section 2.1, if $x \in \hat{\mathcal{A}}_n$ has maximum value $\max(x) = m$, then x gives rise (by insertion of a new rightmost entry) to $m + 1$ modified ascent sequences of length $n + 1$. Therefore,

$$f_{n+1} = \sum_{m=1}^{n} (m + 1) f_{n,m}$$

and

$$f_{n+1}(1) = f_{n+1} - f_n = \sum_{m=1}^{n} (m + 1) f_{n,m} - \sum_{m=1}^{n} f_{n,m} = \sum_{m=1}^{n} m f_{n,m}.$$

The triangle $f_{n,m}$ is sequence A137251 in the OEIS [21]. Note that we were not able to find a combinatorial construction that embodies Equation (1) directly.

Next we show that the insertion of a new strict maximum between two consecutive entries of a modified ascent sequence yields another modified ascent sequence if and only if the two entries form a weak descent.

Proposition 7. *Let $x \in \hat{\mathcal{A}}_n$ and let $m = \max(x)$. For $i \in [n]$, denote by $x^{(i)}$ the sequence*

$$x^{(i)} = x_1 \cdots x_i \ (m + 1) \ x_{i+1} \cdots x_n$$

obtained by inserting $m + 1$ immediately after x_i. Then

$$x^{(i)} \in \hat{\mathcal{A}}_{n+1} \quad \textit{if and only if} \quad x_i \geqslant x_{i+1} \textit{ or } i = n.$$

Proof. It is a direct consequence of the equality $\mathrm{nub}(x) = \mathrm{asctops}(x)$ that defines $\hat{\mathcal{A}}$. Indeed, if $x_i < x_{i+1}$, then $x_{i+1} \in \mathrm{asctops}(x) = \mathrm{nub}(x)$. Now, due to the insertion of $m+1$, the entry x_{i+1} is no longer an ascent top in $x^{(i)}$; however, it is still a leftmost copy. Since the other pairs of consecutive entries are not affected by the insertion of $m+1$, the set $\mathrm{nub}(x^{(i)})$ is strictly contained in $\mathrm{asctops}(x^{(i)})$, and thus $x^{(i)}$ is not a modified ascent sequence. On the other hand, if $x_i \geqslant x_{i+1}$ or $i = n$, all the ascent tops and all the leftmost copies of x are preserved in $x^{(i)}$. Also the new entry $m+1$ is both an ascent top and a leftmost copy. Hence $\mathrm{asctops}(x^{(i)}) = \mathrm{nub}(x^{(i)})$ and $x^{(i)}$ is a modified ascent sequence. □

3 Patterns 212,1212,2132,12132

In this section we solve Conjecture 1 for every pattern $y \in \{212, 1212, 2132\}$. In fact we show that

$$\hat{\mathcal{A}}(212) = \hat{\mathcal{A}}(1212) = \hat{\mathcal{A}}(2132) = \hat{\mathcal{A}}(12132)$$

and add 12132 to the list of patterns.

Lemma 8. *Let $y = y_1 y_2 \cdots y_k$ be a Cayley permutation. Suppose that $y_2 = 1$ is the only copy of 1 in y. Then*

$$\hat{\mathcal{A}}(y) = \hat{\mathcal{A}}(1y).$$

Proof. Let $x \in \hat{\mathcal{A}}$. If x avoids y, then it avoids $1y$. Hence the inclusion $\hat{\mathcal{A}}(y) \subseteq \hat{\mathcal{A}}(1y)$ holds. To prove the opposite inclusion, we show that if x contains y, then it contains $1y$ too. Let $x_{i_1} x_{i_2} \cdots x_{i_k}$ be an occurrence of y in x, with $i_1 < i_2 < \cdots < i_k$. Let x_j be the leftmost smallest entry between x_{i_1} and x_{i_3}; more formally, let

$$j = \min\{j : \ i_1 < j < i_3, \ x_j = \min(x_{i_1} x_{i_1+1} \cdots x_{i_3})\}.$$

Note that $x_{i_1} x_j x_{i_3} \cdots x_{i_k} \simeq y$ since $x_j \leqslant x_{i_2}$ and $y_2 = 1$ is the only copy of 1 in y. Let j' be the index of the leftmost copy of x_j in x. Due to our choice of j, we have that $(j, x_j) \notin \mathrm{asctops}(x) = \mathrm{nub}(x)$. In particular, it must be $j' < i_1$. Hence $x_{j'} x_{i_1} x_j x_{i_3} \cdots x_{i_k}$ is an occurrence of $1y$ in x, as desired. □

Corollary 9. *We have*

$$\hat{\mathcal{A}}(212) = \hat{\mathcal{A}}(1212) \quad \textit{and} \quad \hat{\mathcal{A}}(2132) = \hat{\mathcal{A}}(12132).$$

Proof. Both equalities follow immediately from Lemma 8 by letting $y = 212$ and $y = 2132$, respectively. □

Proposition 10. *We have*

$$\hat{\mathcal{A}}(212) = \hat{\mathcal{A}}(2132).$$

Proof. The inclusion $\hat{\mathcal{A}}(212) \subseteq \hat{\mathcal{A}}(2132)$ is trivial. To prove the opposite inclusion, suppose that x contains an occurrence $x_i x_j x_k$ of 212. We show that x contains 2132. Note that $(k, x_k) \notin \mathrm{nub}(x)$ since $x_i = x_k$ and $i < k$. Thus, due to the equality $\mathrm{nub}(x) = \mathrm{asctops}(x)$ defining $\hat{\mathcal{A}}$, it must be $x_{k-1} \geqslant x_k$. If $x_{k-1} > x_k$, then $x_i x_j x_{k-1} x_k$ is an occurrence of 2132 in x, as wanted. Finally, if $x_{k-1} = x_k$, then we can repeat the same argument on the occurrence $x_i x_j x_{k-1}$ of 212 until we eventually fall back in the previous case. □

By Corollary 9 and Proposition 10, we have

$$\hat{\mathcal{A}}(212) = \hat{\mathcal{A}}(1212) = \hat{\mathcal{A}}(2132) = \hat{\mathcal{A}}(12132).$$

For the rest of this section, let $B = \hat{\mathcal{A}}(212)$. For $n \geqslant 1$ and $1 \leqslant k, m \leqslant n$, in analogy with the definitions of $f_n(k)$ and $f_{n,m}$ given in Section 2.2, define the sets

$$\begin{array}{lcl} B_n(k) & = & \{x \in B_n :\ x \text{ contains } k \text{ copies of } \max(x)\}; \\ B_{n,m} & = & \{x \in B_n :\ \max(x) = m\}; \\ B_{n,m}(k) & = & B_n(k) \cap B_{n,m}. \end{array}$$

Denote their cardinalities by

$$b_n = |B_n|, \quad b_n(k) = |B_n(k)|, \quad b_{n,m} = |B_{n,m}|, \quad b_{n,m}(k) = |B_{n,m}(k)|.$$

We wish to prove that $b_{n,m} = S(n, n-m+1)$, where $S(n,j)$ is the (n,j)th Stirling number of the second kind. Since in any modified ascent sequence the number of ascent tops equals the maximum value, this settles Conjecture 1 for $y \in \{212, 1212, 2132, 12132\}$.

Let us start by showing that if $x \in B_{n+1}(1)$ and $m = \max(x)$, then the sequence obtained by removing the only copy of m from x is a modified ascent sequence. Together with Proposition 7, this implies that every such $x \in B_{n+1}(1)$ is obtained uniquely from an element of B_n by inserting a new strict maximum between two consecutive entries that form a weak descent. Building upon this property, in combination with the fact that all the copies of $\max(x)$ are in consecutive positions (by Proposition 4), we will obtain a recursive construction for the set B_n.

Proposition 11. *Let $x \in B_{n+1}(1)$. Let x_i be the only copy of $\max(x)$ in x. Denote by $\tilde{x}$ the sequence*

$$\tilde{x} = x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}$$

obtained from x by removing x_i. Then $\tilde{x} \in B_n$.

Proof. It is easy to see that $\tilde{x}$ avoids 212 since x does so. We shall prove that $\tilde{x}$ is a modified ascent sequence by showing that $\operatorname{asctops}(\tilde{x}) = \operatorname{nub}(\tilde{x})$. Recall that $\operatorname{asctops}(x) = \operatorname{nub}(x)$ since $x \in \hat{\mathcal{A}}$. In particular, we have $\max(x) = x_i > x_{i+1}$ and thus $x_{i+1} \notin \operatorname{nub}(x)$. Let j be the index of the leftmost copy of x_{i+1} in x, so that $x_j = x_{i+1}$ and $(j, x_j) \in \operatorname{nub}(x)$. Note that it must be $x_{i-1} \geqslant x_{i+1}$, or else we would have an occurrence $x_j x_{i-1} x_{i+1}$ of 212 in x, which is impossible. Therefore, $x_{i-1} \geqslant x_{i+1}$, i.e. $x_{i+1} \notin \operatorname{asctops}(\tilde{x})$, and also $x_{i+1} \notin \operatorname{nub}(\tilde{x})$. Since the other pairs of consecutive elements are not affected by the removal of x_i, we have $\operatorname{asctops}(\tilde{x}) = \operatorname{nub}(\tilde{x})$, which concludes the proof. □

In general, the previous proposition does not hold if x contains 212. For instance, $12132 \in \hat{\mathcal{A}}_5(1)$, but 1212 is not a modified ascent sequence.

It is easy to see that, if x avoids 212 and $x^{(i)}$ is defined as in Proposition 7, i.e. by inserting a new strict maximum immediately after x_i, then $x^{(i)}$ avoids 212 as well. Indeed the new entry $m+1$, where $m = \max(x)$, can not be part of an occurrence of 212 in

$x^{(i)}$. Therefore, by Proposition 7 and Proposition 11, each modified ascent sequence x in $B_{n+1}(1)$ is obtained uniquely from some $\tilde{x} \in B_n$ by inserting a new entry equal to $\max(x)+1$, either between two consecutive entries $x_i \geqslant x_{i+1}$ or after x_n. Recall also that $\max(x) = |\mathrm{asctops}(x)|$. Therefore, if $x \in B_{n,m}$, then there are exactly $n-m+1$ positions where the insertion of a new strict maximum yields an element of $B_{n+1}(1)$. Thus

$$b_{n+1,m+1}(1) = (n-m+1)b_{n,m} \quad \text{and} \quad b_{n+1}(1) = \sum_{m=1}^{n}(n-m+1)b_{n,m}.$$

Furthermore, by Proposition 4, all the copies of $\max(x)$ are in consecutive positions, hence

$$b_{n+1,m}(k+1) = b_{n,m}(k).$$

Putting everything together, we obtain

$$\begin{aligned} b_{n,m} &= \sum_{i=1}^{n} b_{n,m}(i) \\ &= b_{n,m}(1) + \sum_{i=2}^{n} b_{n,m}(i) && [b_{n,m}(1) = (n-m+1)b_{n-1,m-1}] \\ &= (n-m+1)b_{n-1,m-1} + \sum_{i=2}^{n} b_{n-1,m}(i-1) && [j = i-1] \\ &= (n-m+1)b_{n-1,m-1} + \sum_{j=1}^{n-1} b_{n-1,m}(j) \\ &= (n-m+1)b_{n-1,m-1} + b_{n-1,m}. \end{aligned}$$

Thus, the coefficients $b_{n,m}$ obey the recurrence

$$b_{n,m} = (n-m+1)b_{n-1,m-1} + b_{n-1,m}. \tag{2}$$

The Stirling numbers of the second kind are defined by

$$\begin{cases} S(n,n) = 1 & n \geqslant 0; \\ S(n,0) = S(0,n) = 0 & n > 0; \\ S(n,k) = kS(n-1,k) + S(n-1,k-1) & 0 < k < n. \end{cases}$$

We prove that $b_{n,m} = S(n, n-m+1)$. It is easy to see that the initial conditions are satisfied. For instance, $b_{n,1} = S(n,n) = 1$ since the only $x \in B_{n,1}$ is $x = 11\cdots 1$. Let $n \geqslant 2$ and $m \in [n]$; using induction on n and Equation (2):

$$\begin{aligned} b_{n,m} &= (n-m+1)b_{n-1,m-1} + b_{n-1,m} \\ &= (n-m+1)S(n-1,n-m+1) + S(n-1,n-m) && [k = n-m+1] \\ &= kS(n-1,k) + S(n-1,k-1) \\ &= S(n,k) \\ &= S(n,n-m+1). \end{aligned}$$

Theorem 12. *Let $y \in \{212, 1212, 2132, 12132\}$. Then $|\hat{\mathcal{A}}_n(y)|$ is equal to the nth Bell number. Furthermore, the number of modified ascent sequences in $\hat{\mathcal{A}}_n(y)$ whose maximum value is equal to m is given by the $(n, n-m+1)$th Stirling number of the second kind.*

By Proposition 5, modified ascent sequences that contain exactly one copy of their maximum value are counted by the 2-Fishburn numbers. In other words, $f_{n+1}(1) = f_{n+1} - f_n$ and Fishburn numbers are the partial sums of the 2-Fishburn numbers. Similarly, since $|B_n|$ is equal to the nth Bell number, the sequence $\{b_n(1)\}_{n \geqslant 1}$ is given by the 2-*Bell numbers* (A005493 [21]). This can be proved with a completely analogous argument by observing that $b_{n+1}(k+1) = b_n(k)$. We leave the details to the reader.

Next we define a bijection ϕ from RGF_n to B_n. As suggested by Theorem 12, we shall in fact map $\mathrm{RGF}_{n,n-m+1}$ to $B_{n,m}$, where

$$\mathrm{RGF}_{n,m} = \{x \in \mathrm{RGF}_n :\ \max(x) = m\}.$$

Indeed, under the usual encoding of set partitions by restricted growth functions described in Section 2, the set $\mathrm{RGF}_{n,m}$ corresponds to set partitions of $[n]$ with m blocks. We proceed as follows. First we describe more directly the recursive construction of $B_{n,m}$ embodied by Equation (2). Then we introduce the related notion of *active site*, and show that restricted growth functions encode this construction in a way that is similar to how ascent sequences encode active sites of Fishburn permutations.

Let $x \in B_{n,m}$. As observed in the paragraph leading to Proposition 11, if x contains one copy of $m = \max(x)$, then x is obtained uniquely from some $\tilde{x} \in B_{n-1,m-1}$ by inserting m either between two consecutive elements that form a weak descent or after the last entry. Note that there are $n-m+1$ such positions in $\tilde{x}$ since $\max(\tilde{x}) = |\mathrm{asctops}(\tilde{x})|$. On the other hand, suppose that x contains at least two copies of m. By Proposition 4, all the copies of m are in consecutive positions, hence the removal of the rightmost copy of m from x determines (uniquely) an element of $B_{n-1,m}$. In other words, every $x \in B_{n,m}$ gives rise to $(n-m+1)+1$ sequences in B_{n+1}:

(*i*) The $n-m+1$ sequences in $B_{n+1,m+1}$ obtained by inserting a new strict maximum between two entries $x_i \geqslant x_{i+1}$, including the last spot after x_n; we call these positions the *active sites* of x. We also label the active sites with the integers $1, 2, \dots, n-m+1$, going from left to right.

(*ii*) The (only) sequence in $B_{n+1,m}$ obtained by inserting a new weak maximum at the end of the string of consecutive maxima of x.

Similarly, given a restricted growth function $r \in \mathrm{RGF}_{n,m}$, by inserting a new rightmost entry $r_{n+1} = i$ one obtains

(*i*) m sequences in $\mathrm{RGF}_{n+1,m}$, one for each $i = 1, 2, \dots, m$;

(*ii*) One sequence in $\mathrm{RGF}_{n+1,m+1}$, when $i = m+1$ is a new strict maximum.

Now, given a restricted growth function $r = r_1 \cdots r_n$, define $x = \phi(r)$ inductively as follows.

- Map the only restricted growth function of length one, $r = 1$, to the only modified ascent sequence of length one, $x = 1$. Note that the sum $\max(r) + \max(x)$ is equal to one plus the length of r (or x).
- Let $n \geqslant 1$ and let $\phi(r_1 \cdots r_n) = x_1 \cdots x_n$ be defined inductively, with $\max(r) + \max(x) = n + 1$. Let $\bar{r} = r_1 \cdots r_n r_{n+1}$. We define $\bar{x} = \phi(\bar{r})$ as follows, according to whether $r_{n+1} \leqslant \max(r)$ or $r_{n+1} = \max(r) + 1$.
    1. If $r_{n+1} = i \leqslant \max(r)$, then we let $\bar{x}$ be obtained from x by inserting a new strict maximum $\max(x) + 1$ in the ith active spot of x. Note that $\max(r) = n - \max(x) + 1$, thus the number of possible choices for i equals the number of active sites of x.
    2. If $r_{n+1} = \max(r) + 1$, then we let $\bar{x}$ be obtained from x by inserting a new weak maximum (at the end of the string of consecutive maxima).

Note that in each case we have $\max(\bar{r}) + \max(\bar{x}) = n + 2$. In general, if $r \in \mathrm{RGF}_{n,j}$ and $x = \phi(r)$, then $x \in B_{n,n+1-j}$. As a result, the map ϕ defined this way is a bijection between $\mathrm{RGF}_{n,n-m+1}$ and $B_{n,m}$[3]. We have thus obtained an alternative proof of the fact that the statistic $|\mathrm{asctops}(x)| = \max(x)$ on B_n is equidistributed with the reverse of the number of blocks, i.e. $\max(r)$, on set partitions of $[n]$. Below we illustrate the step-by-step computation of $\phi(123224135) = 141233551$. Here superscripts denote labels of active sites, while positions between consecutive elements that have no superscript are forbidden. At each step, we underline the newly added element in the modified ascent sequence (on the right).

$$\begin{array}{rcl}
1 & \longmapsto & \underline{1}^1 \\
12 & \longmapsto & 1^1\underline{1}^2 \\
123 & \longmapsto & 1^1 1^2 \underline{1}^3 \\
1232 & \longmapsto & 1^1 1\underline{2}^2 1^3 \\
12322 & \longmapsto & 1^1 12\underline{3}^2 1^3 \\
123224 & \longmapsto & 1^1 123^2 \underline{3}^3 1^4 \\
1232241 & \longmapsto & 1\underline{4}^1 123^2 3^3 1^4 \\
12322413 & \longmapsto & 14^1 123^2 3\underline{5}^3 1^4 \\
r = 123224135 & \longmapsto & 14^1 123^2 35^3 \underline{5}^4 1^5 = \phi(r).
\end{array}$$

3.1 Bijection via transposition of biwords

In the previous section, two recursive constructions of $\hat{\mathcal{A}}(212)$ and RGF lead to the definition of the bijection ϕ. Here we slightly tweak the Burge transpose to obtain a new

[3] It is in fact an isomorphism between the generating trees of RGF and B induced by the generating rules (i) and (ii).

bijection ψ. Compared to ϕ, the construction of ψ is more straightforward, and arguably more elegant. Despite that, the proof that ψ is a bijection come with a considerable amount of technical details. In light of that, and also due to the fact that what we show in this section is not strictly necessary for the rest of the paper, most of them will be omitted.

Let $x = x_1 \cdots x_n$ be a modified ascent sequence and let $m = \max(x)$. The definition of the restricted growth function $\psi(x)$ goes as follows. Initially, we use a step-by-step procedure to label each entry x_j of x with a positive integer u_j. For $i = 1, \dots, m$, at the ith step we label all the copies of i in x.

- For $i = 1$, we label the copies of 1 with increasing integers $1, 2, 3, \dots$, starting from the leftmost copy and going from left to right.
- Let $i \geqslant 2$. Let t be the maximum label assigned at the previous steps. Let $x_j = i$ be the leftmost copy of i in x. Since $\operatorname{asctops}(x) = \operatorname{nub}(x)$, we have $x_{j-1} < x_j$. In particular, the entry x_{j-1} has been labeled with u_{j-1} at a previous step of the procedure. Then we let $u_j = u_{j-1}$ and assign labels $t+1, t+2, \dots$ to the remaining copies of i (going from left to right).

Finally, we arrange x and the resulting labels $u = u_1 \cdots u_n$ in the biword

$$\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

To sum up this procedure in the most succint way, at the ith step we give to the leftmost copy of i the same label as the entry immediately to its left, and give new (increasing) labels to the other copies of i. An example that illustrates this construction is the following. Let $x = 141233551$. The five steps needed to determine u are illustrated below, where labels defined at each step are underlined.

$$\text{Step 1:} \quad \begin{pmatrix} \underline{1} & & \underline{2} & & & & & & \underline{3} \\ 1 & 4 & 1 & 2 & 3 & 3 & 5 & 5 & 1 \end{pmatrix}$$

$$\text{Step 2:} \quad \begin{pmatrix} 1 & & 2 & \underline{2} & & & & & 3 \\ 1 & 4 & 1 & 2 & 3 & 3 & 5 & 5 & 1 \end{pmatrix}$$

$$\text{Step 3:} \quad \begin{pmatrix} 1 & & 2 & 2 & \underline{2} & \underline{4} & & & 3 \\ 1 & 4 & 1 & 2 & 3 & 3 & 5 & 5 & 1 \end{pmatrix}$$

$$\text{Step 4:} \quad \begin{pmatrix} 1 & \underline{1} & 2 & 2 & 2 & 4 & & & 3 \\ 1 & 4 & 1 & 2 & 3 & 3 & 5 & 5 & 1 \end{pmatrix}$$

$$\text{Step 5:} \quad \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 4 & \underline{4} & \underline{5} & 3 \\ 1 & 4 & 1 & 2 & 3 & 3 & 5 & 5 & 1 \end{pmatrix}$$

In the end, we obtain

$$\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} 1\,1\,2\,2\,2\,4\,4\,5\,3 \\ 1\,4\,1\,2\,3\,3\,5\,5\,1 \end{pmatrix}.$$

Now, define the biword $(u, x)^{T'}$ by

- Flipping $\binom{u}{x} \mapsto \binom{x}{u}$;
- Sorting the top row in weakly increasing order, breaking ties by sorting the bottom entries in *increasing* order.

The definition of T' is analogous to the Burge transpose T, the only difference being that in case of ties we sort the bottom entries in increasing order. Biwords where columns are sorted this way play a central role in the RSK correspondence, and are often called *generalized permutations* [19]. For instance, the T'-transpose of the biword $(u, x) = (112224453, 141233551)$ is

$$\begin{pmatrix} 1\,1\,2\,2\,2\,4\,4\,5\,3 \\ 1\,4\,1\,2\,3\,3\,5\,5\,1 \end{pmatrix} \xmapsto{\text{flip}} \begin{pmatrix} 1\,4\,1\,2\,3\,3\,5\,5\,1 \\ 1\,1\,2\,2\,2\,4\,4\,5\,3 \end{pmatrix} \xmapsto{\text{sort}} \begin{pmatrix} 1\,1\,1\,2\,3\,3\,4\,5\,5 \\ 1\,2\,3\,2\,2\,4\,1\,4\,5 \end{pmatrix}.$$

Finally, we let $\psi(x)$ be the bottom row of the biword $(u, x)^{T'}$. In our example, we have obtained $\psi(x) = 123224145$. Note that the bijection ϕ defined in the previous section maps the same restricted growth function 123224135 to the modified ascent sequence 141233551, and not to x. That is, $\psi \neq \phi^{-1}$.

Next we prove that $\psi(x)$ is a restricted growth function and ψ is a bijection from $\hat{\mathcal{A}}(212)$ to RGF. Let us prove that $\psi(x) \in$ RGF first. Consider the decomposition of $(u, x)^{T'}$ obtained by splitting the biword according to the value of the top entry; more explicitly, we have

$$\begin{pmatrix} u \\ x \end{pmatrix}^{T'} = \left(\begin{array}{cccc|cccc|cccc|c|cccc} 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 & 3 & 3 & \dots & 3 & \dots & m & m & \dots & m \\ t_1 & 2 & \dots & \ell_1 & t_2 & \ell_1+1 & \dots & \ell_2 & t_3 & \ell_2+1 & \dots & \ell_3 & \dots & t_m & \ell_{m-1}+1 & \dots & \ell_m \end{array}\right),$$

where t_i is the label assigned under ψ to the leftmost copy of i in x, $m = \max(x)$, and $\ell_1, \dots, \ell_m$ are nonnegative integers. To prove that $x \in$ RGF, it suffices to show that $t_1 = 1$ and $t_i \leqslant \ell_{i-1}$ for each $i \geqslant 2$. By definition of ψ, we have $t_1 = 1$. Furthermore, if j is the index of the leftmost copy of i in x, then t_i is equal to the label u_{j-1} of x_{j-1}. For $i = 2$, we have $x_{j-1} < x_j = 2$, thus $x_{j-1} = 1$ and $t_2 \in \{1, 2, \dots, \ell_1\}$, i.e. $t_2 \leqslant \ell_1$. A completely analogous argument shows that $t_i \leqslant \ell_{i-1}$ for each i. As a result, $\psi(x)$ is a restricted growth function.

Finally, we show that ψ is bijective by defining its inverse map ψ^{-1} from RGF to $\hat{\mathcal{A}}(212)$. Let $r = r_1 \cdots r_n \in$ RGF. For $i \in [n]$, we define the label y_i of r_i as hinted by the decomposition of $(u, x)^{T'}$ considered above. That is, we let $y_1 = 1$ and, for $i \geqslant 2$,

$$y_i = \begin{cases} y_{i-1} & \text{if } r_i = \max(r_1 \cdots r_{i-1}) + 1; \\ y_{i-1} + 1 & \text{if } r_i \leqslant \max(r_1 \cdots r_i). \end{cases}$$

In other words, scanning r from left to right, we repeat the same label if the bottom entry is a new integer in r; otherwise, we use a new label (equal to one plus the label used before). We will now describe an iterative procedure to arrange the columns (r_i, y_i) in a biword, and the desired modified ascent sequence $x = \psi^{-1}(r)$ will be obtained as the bottom row of such biword. For $j = 1, 2, \dots, \max(y)$, at the jth step of the procedure we shall arrange all the columns where $y_i = j$ as a consecutive block, sorting them in increasing order with respect to the top entry r_i. Their position in the biword will be determined by the smallest top label of such entries; more precisely, if the smallest top label of the entries $y_i = j$ is equal to ℓ, then we insert all such columns immediately after the rightmost column in the current biword where the top entry is equal to ℓ. At the first step, $j = 1$, we simply arrange all the columns with bottom entry 1 (sorting them in increasing order with respect to the top entry). The fact that r is a restricted growth function guarantees that such ℓ is well defined at each next step of the procedure.

Instead of proving that the map defined this way is the inverse map of ψ, we wish to better clarify this construction with a concrete example. Consider the restricted growth function $r = 123224145$ obtained previously as $r = \psi(x)$, for $x = 141233551$. Letting $y = y_1 \cdots y_n$, we have

$$\begin{array}{rl} r = & 1\,2\,3 \,\big|\, 2 \,\big|\, 2\,4 \,\big|\, 1 \,\big|\, 4\,5; \\ y = & 1\,1\,1 \,\big|\, 2 \,\big|\, 3\,3 \,\big|\, 4 \,\big|\, 5\,5. \end{array}$$

We arrange the columns (r_i, y_i) in a biword following the iterative procedure described above. At the jth step, the newly inserted columns are highlighted and ℓ_j denotes the smallest top label of the entries $y_i = j$.

$$\begin{array}{lll} \text{Step 1:} & & \begin{pmatrix} \mathbf{123} \\ \mathbf{111} \end{pmatrix}; \\ \text{Step 2:} & \ell_2 = 2 \longrightarrow & \begin{pmatrix} 12\,\mathbf{2}\,3 \\ 11\,\mathbf{2}\,1 \end{pmatrix}; \\ \text{Step 3:} & \ell_3 = 2 \longrightarrow & \begin{pmatrix} 122\,\mathbf{24}\,3 \\ 112\,\mathbf{33}\,1 \end{pmatrix}; \\ \text{Step 4:} & \ell_4 = 1 \longrightarrow & \begin{pmatrix} 1\,\mathbf{1}\,22243 \\ 1\,\mathbf{4}\,12331 \end{pmatrix}; \\ \text{Step 5:} & \ell_5 = 4 \longrightarrow & \begin{pmatrix} 112224\,\mathbf{45}\,3 \\ 141233\,\mathbf{55}\,1 \end{pmatrix}. \end{array}$$

In the end, the bottom row of the resulting biword is $\psi^{-1}(r) = 141233551 = x$, as expected.

Remark 13. The map ψ is in fact defined on the set $\hat{\mathcal{A}}$ of all modified ascent sequences. Quite remarkably, a numerical investigation suggests that the restriction of ψ to $\hat{\mathcal{A}}(2213)$ yields a bijection to RGF too. A proof of this fact remains to be found. When ψ is extended to $\hat{\mathcal{A}}$, the smallest example of a collision is given by

$$\psi(12132) = \psi(12213) = 12132.$$

This is the only collision for sequences of length five (and indeed $|\mathrm{RGF}_5| = |\hat{\mathcal{A}}_5| - 1$). Note that 12132 and 12213 are the shortest modified ascent sequences containing 212 and 2213, respectively. On the other hand, for instance due to the same example of collision, the restriction of ψ to $\hat{\mathcal{A}}(2231)$ and $\hat{\mathcal{A}}(2321)$ is not bijective.

4 Patterns 2213 and 2231

In Section 3, we showed a recursive construction of $\hat{\mathcal{A}}(212)$ that embodies Equation (2) defining the Stirling numbers of the second kind. In this section, we settle Conjecture 1 for the patterns 2213 and 2231. Our methods are similar in that we will once again construct $\hat{\mathcal{A}}(y)$ recursively, for $y \in \{2213, 2231\}$. This construction, however, leads to a different equation

$$g_{n+1}(k) = k \sum_{j=k}^{n} g_n(j),$$

where $g_n(k)$ is the number of modified ascent sequences in $\hat{\mathcal{A}}_n(y)$ that contain k copies of 1. To prove that the Bell numbers satisfy the same equation, we find an analogous construction for RGF. A bijection between the two sets is obtained as a byproduct.

Let us start from RGF. Throughout this section, we denote by $\mathrm{RGF}_n(k)$ the set of restricted growth functions whose length of the maximal, strictly increasing prefix is equal to k. That is, for $k \leqslant n-1$ we let

$$\mathrm{RGF}_n(k) = \{r \in \mathrm{RGF}_n :\ r_1 \cdots r_k = 12 \cdots k,\ r_{k+1} \leqslant k\},$$

while $\mathrm{RGF}_n(n)$ is the singleton containing the sequence $12 \cdots n$. In terms of set partitions, sequences in $\mathrm{RGF}_n(k)$ correspond to set partitions of $[n]$ where exactly k blocks contain their own index as an element; equivalently, where k is the maximum integer such that $1, 2, \ldots, k$ are contained in distinct blocks. Let $h_n(k) = |\mathrm{RGF}_n(k)|$. We show that $h_{n+1}(n+1) = 1$ and, for $k = 1, \ldots, n$,

$$h_{n+1}(k) = k \sum_{j=k}^{n} h_n(j). \tag{3}$$

Indeed, we just noted that $\mathrm{RGF}_{n+1}(n+1)$ is a singleton. Furthermore, let $k \in [n]$ and let $r \in \mathrm{RGF}_n(j)$, for some $j \geqslant k$. Then we have

$$r = r_1 \cdots r_j r_{j+1} \cdots r_n \quad \text{with } r_1 \cdots r_j = 1 \cdots j \text{ and } r_{j+1} \leqslant j.$$

For $i \in [k]$, let $r(k, i)$ be the sequence obtained from r by inserting a new entry equal to i between r_k and r_{k+1}; that is, let

$$\begin{aligned} r(k,i) &= r_1 \cdots r_k \quad i \quad r_{k+1} \cdots r_j \quad r_{j+1} \cdots r_n \\ &= 1 \cdots k \quad i \quad k+1 \cdots j \quad r_{j+1} \cdots r_n. \end{aligned}$$

Clearly, $r(k,i) \in \mathrm{RGF}_{n+1}(k)$. On the other hand, let $y \in \mathrm{RGF}_{n+1}(k)$, with $k < n+1$. Then $y = r(k,i)$, where $i = y_{k+1}$ and r is obtained by removing y_{k+1} from y. Note that $r \in \mathrm{RGF}_n(j)$, for some $j \geqslant k$. As a result, for $k < n+1$, we have the bijection

$$\begin{aligned} [k] \times \bigcup_{j=k}^{n} \mathrm{RGF}_n(j) &\longrightarrow \mathrm{RGF}_{n+1}(k) \\ (i, r) &\longmapsto r(k, i) \end{aligned}$$

and thus

$$k\sum_{j=k}^{n} h_n(j) = h_{n+1}(k).$$

The triangle $h_n(k)$ is recorded as A259691 in the OEIS [21]. To the best of our knowledge, the combinatorial interpretation given here appears to be new.

Next we turn our attention to $\hat{\mathcal{A}}(2213)$. For $n \geqslant 1$ and $k \in [n]$, let

$$G_n(k) = \{x \in \hat{\mathcal{A}}_n(2213) : \ x \text{ contains } k \text{ copies of } 1\}$$

and let $g_n(k) = |G_n(k)|$. Note that $g_n(n) = 1$ since the only sequence in $G_n(n)$ is $x = 1 \cdots 1$. For $k < n+1$, we mimic what we did before and define a bijection

$$\begin{aligned}[k] \times \bigcup_{j=k}^{n} G_n(j) &\longrightarrow G_{n+1}(k)\\ (i, x) &\longmapsto x(k, i),\end{aligned}$$

proving that the coefficients $g_{n+1}(k)$ satisfy the same recurrence as $h_{n+1}(k)$. Since the initial conditions are the same, we have that $g_n(k) = h_n(k)$.

Let $x \in G_n(j)$, for some $j \geqslant k$, and let $i \in [k]$. Define the sequence $x(k, i)$ as follows:

(1) Increase by one all the entries of x, except for the k leftmost copies of 1.

(2) Insert a new entry equal to 2 immediately to the right of the ith copy of 1 (counting from left to right).

(3) In the special case where $i < k$ and the kth and $(k+1)$th copies of 1 are consecutive in position, move to the end of the sequence the maximal string of consecutive 1s that end with the kth copy.

Next we show in Proposition 14 that $x(k, i) \in G_{n+1}(k)$. Roughly speaking, the reason is that the map $(i, x) \mapsto x(k, i)$ preserves the equality $\text{asctops}(x) = \text{nub}(x)$ and does not create any new occurrence of 2213. The role of (3) is to address the case where the $(k+1)$th copy of 1 would become an ascent top that is not a leftmost copy. The proof consists of a case by case analysis that requires a certain amount of technicalities, some of which will be omitted for the sake of readability. The shortest example where (3) plays a role is given by $x = 1112 \in G_4(3)$. In this case, for $k = 2$ and $i = 1$, we have

$$1\,1\,1\,2 \xmapsto{(1)} 1\,1\,2\,3 \xmapsto{(2)} 1\,\underline{2}\,\boxed{1}\,2\,3 \xmapsto{(3)} 1\,\underline{2}\,2\,3\,\boxed{1} \ = x(2,1),$$

where the underlined entry is the newly inserted 2. Note that, without moving the boxed 1 (i.e. the maximal string of consecutive 1s that end with the kth copy of 1) at the end of the sequence, the $(k+1)$th copy of 1 would be an ascent top in $x(2, 1)$, but not a leftmost copy; consequently, the resulting sequence 12123 would not be a modified ascent sequence. Two more instances of this construction will be illustrated later in Example 15 and Example 16.

Proposition 14. *Let $x \in G_n(j)$, with $k \leqslant j \leqslant n$, and let $i \in [k]$. Then $x(k,i) \in G_{n+1}(k)$.*

Proof. Let $y = x(k,i)$. By definition, y contains k copies of 1. We have to show that $y \in \hat{\mathcal{A}}$ and y avoids 2213. We distinguish two cases depending on whether or not the definition of $y = x(k,i)$ falls under the special case (3).

- Suppose that we are not in the special case (3); that is, y is obtained from x by increasing by one all its entries, except for the leftmost k copies of 1, and inserting a 2 immediately after the ith copy of 1. More explicitly, in this case we have

$$\begin{aligned} x = &\quad 1^{(1)}B_1 \quad \cdots \quad 1^{(i)}B_i \quad \cdots \quad 1^{(k)}B_k \; 1^{(k+1)}B_{k+1} \quad \cdots \quad 1^{(j)}B_j, \\ y = &\quad 1^{(1)}\bar{B}_1 \quad \cdots \quad 1^{(i)}2\bar{B}_i \quad \cdots \quad 1^{(k)}\bar{B}_k \; \bar{1}^{(k+1)}\bar{B}_{k+1} \quad \cdots \quad \bar{1}^{(j)}\bar{B}_j, \end{aligned}$$

  where, for $\ell = 1, \ldots, j$, $1^{(\ell)}$ denotes the ℓth copy of 1, B_ℓ denotes the block of entries between $1^{(\ell)}$ and $1^{(\ell+1)}$, and a bar marks entries and blocks that are increased by one in y. We prove that $y \in \hat{\mathcal{A}}$ by showing that $\text{asctops}(y) = \text{nub}(y)$. Note that $\text{asctops}(x) = \text{nub}(x)$ since $x \in \hat{\mathcal{A}}$. The relative order of the entries of y is almost the same as in x, with the exception of the newly inserted 2 and the increased copies of 1. In particular, the newly inserted 2 is both an ascent top and a leftmost copy in y. Furthermore, if the block B_i is not empty, then its leftmost element is an ascent top both in x and in y, since all the elements in $\bar{B}_i$ are greater than 2. Similarly, it belongs to $\text{nub}(x)$, since $\text{nub}(x) = \text{asctops}(x)$, and to $\text{nub}(y)$ as well, once again due to the fact that the relative order of the entries of y that are greater than 2 is the same as it was in x. The case when B_i is empty can be addressed similarly. Finally, the only other entry that could potentially become a new ascent top in y is $\bar{1}^{(k+1)}$. This happens if and only if B_k is empty, that is, if $1^{(k)}$ and $1^{(k+1)}$ are consecutive in positions. Now, if $i = k$, then the entries

$$1^{(k)}1^{(k+1)} \quad \text{are mapped to} \quad 1^{(k)}2\bar{1}^{(k+1)} = 122$$

  and once again the newly inserted 2 belongs to both $\text{asctops}(y)$ and $\text{nub}(y)$. If instead $i < k$, then we fall under the special case (3), which we consider below. As a result of the above discussion, the equality $\text{asctops}(x) = \text{nub}(x)$ is preserved by the construction $(i, x) \mapsto y$; hence $\text{asctops}(y) = \text{nub}(y)$ and $y \in \hat{\mathcal{A}}_{n+1}$, as wanted.
  Let us show that y avoids 2213 next. Note that x avoids 2213 by our assumptions. Since the relative order of elements of y that are greater than 2 is the same as it was in x, an eventual occurrence $y_{\ell_1}y_{\ell_2}y_{\ell_3}y_{\ell_4}$ of 2213 in y should satisfy $y_{\ell_3} = 1$ and $y_{\ell_1} = y_{\ell_2} = 2$; otherwise, we would have $y_{\ell_4} \geqslant y_{\ell_1} = y_{\ell_2} \geqslant 3$ and the same four entries would form an occurrence of 2213 in x, which is impossible. On the other hand, there is only one entry equal to 2 that precedes the rightmost copy of 1 in y, and thus the case $y_{\ell_3} = 1$ and $y_{\ell_1} = y_{\ell_2} = 2$ is impossible too.

- Let us now take care of the special case (3) where $i < k$ and $1^{(k)}$ and $1^{(k+1)}$ are consecutive in position. Let us write

$$x = \quad 1^{(1)}B_1 \quad \cdots \quad 1^{(i)}B_i \quad \cdots \quad \boxed{1^{(\ell)}1^{(\ell+1)} \cdots 1^{(k)}} \quad 1^{(k+1)}B_{k+1} \quad \cdots \quad 1^{(j)}B_j,$$

where the box contains the maximal string of consecutive 1s that end with $1^{(k)}$ (for some $\ell \geqslant 1$) and the rest is defined as in the previous case. Then

$$\begin{aligned} x = &\quad 1^{(1)}B_1 \quad \cdots \quad 1^{(i)}B_i \quad \cdots \quad \boxed{1^{(\ell)}\cdots 1^{(k)}} \quad 1^{(k+1)}B_{k+1}\cdots 1^{(j)}B_j, \\ y = &\quad 1^{(1)}\bar{B}_1 \quad \cdots \quad 1^{(i)}2\bar{B}_i \quad \cdots \quad \bar{1}^{(k+1)}\bar{B}_{k+1}\cdots \bar{1}^{(j)}\bar{B}_j \quad \boxed{1^{(\ell)}\cdots 1^{(k)}}, \end{aligned}$$

where in y the boxed 1s have been moved at the end under the effect of (3). It is now easy to verify that, similarly to the previous case, the equality $\operatorname{asctops}(y) = \operatorname{nub}(y)$ holds, and we omit the details. We just note that, by moving the boxed 1s at the end of the sequence, we have removed the newly created ascent $1^{(k)}\bar{1}^{(k+1)}$. Otherwise, the entry $\bar{1}^{(k+1)}$ would be an ascent top, but not a leftmost copy since $\bar{1}^{(k+1)} = 2$ and we inserted a new entry equal to 2 immediately after $1^{(i)}$, with $i < k$. The proof that y avoids 2213 is identical to the previous case, with the addition that moving the boxed 1s at the end of the sequence cannot create an occurrence of 2213. □

To see that the map $(i, x) \mapsto x(k, i)$ is a bijection from $[k] \times \bigcup_{j=k}^{n} G_n(j)$ to $G_{n+1}(k)$, let us define its inverse map. Let $y = y_1 \cdots y_{n+1} \in G_{n+1}(k)$. We determine i, j, k and x such that $x \in G_n(j)$ and $y = x(k, i)$. Let

$$\begin{aligned} i &= |\{\ell : \ y_\ell = 1 \text{ and } y_s \neq 2 \ \forall s < \ell\}|, \\ j &= |\{\ell : y_\ell \leqslant 2\}| - 1, \\ k &= |\{\ell : y_\ell = 1\}\}|; \end{aligned}$$

that is, i is equal to the number of 1s preceding the leftmost 2; j is equal to the number of copies of 1 and 2, minus one; and k is equal to number of copies of 1s. Let x be the string obtained from y as follows:

(a) Remove the leftmost copy of 2; note that the leftmost copy of 2 is an ascent top preceded by the ith copy of 1.

(b) In the special case where $y_{n+1} = 1$ and there is at least one copy of 2 left (after (a) has been applied), move the maximal string of consecutive 1s containing y_{n+1} immediately before the leftmost copy of 2.

(c) Decrease by one each entry that is strictly greater than 1.

Clearly, steps (a), (b) and (c) mirror, respectively, steps (2), (3) and (1) in the definition of the map $(x, i) \mapsto x(k, i)$. Note that (b) is applied if and only $x \mapsto x(k, i) = y$ falls under the special case (3). Indeed, if (3) is not applied, then in y all the copies of 1 precede the second copy of 2, i.e. $\bar{1}^{(k+1)}$. Thus, when the leftmost copy of 2 is removed from y at step (a), the last entry $y_{n+1} = 1$ is preceded by another copy of 2 if and only if the final string of consecutive 1s was moved under the effect of (3). In the end, we have $y = x(k, i)$, and the map defined above is the inverse map of $(x, i) \mapsto x(k, i)$.

Two examples of this construction are illustrated below.

Example 15. Let $y = 12613224532$. Note that $y \in G_{11}(2)$. Here we have

$$i = 1, \quad j = 5, \quad k = 2$$

and x is obtained as

$$y = 1\underline{2}613224532 \xmapsto{(a)} 1613224532 \xmapsto{(c)} 1512113421 = x,$$

where the leftmost copy of 2 in y is underlined. Note that (b) is not applied since y does not end with 1. It is now easy to check that $y = x(2, 1)$:

$$x = 1512113421 \xmapsto{(1)} 1613224532 \xmapsto{(2)} 1\underline{2}613224532 = y.$$

Example 16. Let $y = 131551242111$. We have $y \in G_{12}(6)$ and

$$i = 3, \quad j = 7, \quad k = 6.$$

Here we apply (a), (b) and (c) to obtain

$$\begin{aligned} y = 131551\underline{2}42111 &\xmapsto{(a)} 13155142\,\boxed{111} \\ &\xmapsto{(b)} 1315514\,\boxed{111}\,2 \\ &\xmapsto{(c)} 12144131111 = x. \end{aligned}$$

In this case, (c) is applied since y ends with 1 and there is at least one copy of 2 in the sequence resulting from (b). Finally, we have $y = x(6, 3)$:

$$\begin{aligned} x = 12144131111 &\xmapsto{(1)} 13155141112 \\ &\xmapsto{(2)} 131551\underline{2}4\,\boxed{111}\,2 \\ &\xmapsto{(3)} 131551242\,\boxed{111} \; = y. \end{aligned}$$

In the previous part of this section, we have proved Equation (3) by showing a recursive construction where each object of $\mathrm{RGF}_n(j)$, $j = k, k+1, \ldots, n$, gives rise to k objects in $\mathrm{RGF}_{n+1}(k)$. In a similar fashion, we described a recursive construction of $G_n(k) = \{x \in \hat{\mathcal{A}}_n(2213) : x \text{ contains } k \text{ copies of } 1\}$ which leads to the analogous equation

$$g_{n+1}(k) = k \sum_{j=k}^{n} g_n(j),$$

where $g_n(j) = |G_n(j)|$. The next corollary follows immediately from these two results.

Corollary 17. *For $n \geqslant 0$, the number of 2213-avoiding modified ascent sequences of length n is equal to the nth Bell number.*

Proof. Let $k \in [n]$. Due to what proved so far in this section, the number of restricted growth functions in $\mathrm{RGF}_n(k)$ equals the number of 2213-avoiding modified ascent sequences of length n that contain k copies of 1. By summing over k, we obtain

$$|\hat{\mathcal{A}}_n(2213)| = \sum_{k=1}^{n} |G_n(k)| = \sum_{k=1}^{n} |\mathrm{RGF}_n(k)| = |\mathrm{RGF}_n|,$$

which is equal to the nth Bell number. □

Two methods to construct inductively RGF and $\hat{\mathcal{A}}(2213)$ are provided, respectively, by the maps

$$r \mapsto r(k,i) \quad \text{and} \quad x \mapsto x(k,i).$$

A bijection ϕ from RGF_n to $\hat{\mathcal{A}}_n(2213)$ is obtained accordingly by letting

$$\phi(12\cdots n) = 11\cdots 1 \quad \text{and} \quad \phi\big(r(k,i)\big) = \phi(r)(k,i).$$

In other words, we let ϕ map the only sequence $r = 12\cdots n$ in $\mathrm{RGF}_n(n)$ to the only sequence $x = 11\cdots 1$ in $G_n(n)$; and, if $\phi(r) = x$ is defined inductively and $s = r(k,i)$, then we let $\phi(s) = x(k,i)$. Note that $\max(12\cdots n) = n + 1 - \max(11\cdots 1)$; and, if we assume that $\max(r) = n + 1 - \max(x)$, then

$$\max\big(r(k,i)\big) = \max(r) = n + 1 - \max(x) = n + 2 - \max\big(x(k,i)\big),$$

where we used the fact that

$$\max\big(r(k,i)\big) = \max(r) \quad \text{and} \quad \max\big(x(k,i)\big) = \max(x) + 1.$$

Therefore, for each $r \in \mathrm{RGF}_n$, we have

$$\max(r) + \max\big(\phi(r)\big) = n + 1.$$

As observed at the end of Section 3 for the pattern 212, a consequence of this fact is that the distribution of $|\mathrm{asctops}(x)| = \max(x)$ on $\hat{\mathcal{A}}(2213)$ is given by the reverse of the Stirling numbers of the second kind. We have thus settled Conjecture 1 for the pattern 2213.

The pattern 2231 can be solved by slightly tweaking the argument used for 2213. The same recursive construction works, except for the fact that the special rule (3) in the definition of the map $x \mapsto x(k,i)$ must be replaced with:

(3′) In the special case where $i < k$ and the kth and $(k+1)$th copies of 1 are in consecutive positions in x, move the maximal string of consecutive 1s that end with the kth copy *immediately after the maximal string of consecutive* 2*s that starts with* $\bar{1}^{(k+1)}$.

Recall that by applying (3) to the sequence 1112 we obtained

$$1\,1\,1\,2 \xmapsto{(1)} 1\,1\,2\,3 \xmapsto{(2)} 1\,\underline{2}\,\boxed{1}\,2\,3 \xmapsto{(3)} 1\,\underline{2}\,2\,3\,\boxed{1},$$

which avoids 2213 (but contains 2231). If instead we apply (3′), we obtain

$$1\,1\,1\,2 \xmapsto{(1)} 1\,1\,2\,3 \xmapsto{(2)} 1\,\underline{2}\,\boxed{1}\,2\,3 \xmapsto{(3)} 1\,\underline{2}\,2\,\boxed{1}\,3,$$

which avoids 2231 (but contains 2213). Roughly speaking, both the constructions used for 2231 and 2213 preserve the equality $\operatorname{asctops}(x) = \operatorname{nub}(x)$. The special rules (3) and (3′) are necessary in order to address the case where $1^{(k)}$ and $1^{(k+1)}$ are in consecutive positions, and this would result in a strict ascent $1^{(k)}\bar{1}^{(k+1)}$ where $\bar{1}^{(k+1)}$ is not a leftmost copy. More specifically, this is fixed by moving the box of consecutive 1s ending with $1^{(k)}$ somewhere else (to the right) in the sequence. In order to avoid 2213, we move the box as far as possible at the end of the sequence; if instead we want to avoid 2231, we move the box as close as possible to the original spot.

Theorem 18. *Let $y \in \{2213, 2231\}$. Then $|\hat{\mathcal{A}}_n(y)|$ is equal to the nth Bell number. Furthermore, the number of modified ascent sequences in $\hat{\mathcal{A}}_n(y)$ whose maximum value is equal to m is given by the $(n, n-m+1)$th Stirling number of the second kind.*

5 Fishburn permutations

Recall from Section 2.1 the bijection $\gamma : \hat{\mathcal{A}} \to F$ defined by

$$\begin{pmatrix} \mathrm{id} \\ x \end{pmatrix}^T = \begin{pmatrix} \operatorname{sort}(x) \\ \gamma(x) \end{pmatrix},$$

where id is the identity permutation, $x \in \hat{\mathcal{A}}$, $\operatorname{sort}(x)$ is obtained by sorting the entries of x in weakly increasing order, and T denotes the Burge transpose. Recall also that the biwords (id, x) and $\big(\operatorname{sort}(x), \gamma(x)\big)$ are Burge words; that is, the descent set of the top row is a subset of the descent set of the bottom row, and this property is preserved by T.

In this section, we use bivincular patterns to characterize the set $\gamma\big(\hat{\mathcal{A}}(y)\big)$ of Fishburn permutations corresponding to $\hat{\mathcal{A}}(y)$, for $y \in \{212, 2213, 2321\}$. Namely, we show that

$$\gamma\big(\hat{\mathcal{A}}(212)\big) = F(\alpha), \quad \gamma\big(\hat{\mathcal{A}}(2213)\big) = F(\beta_1, \beta_2), \quad \gamma\big(\hat{\mathcal{A}}(2321)\big) = F(\delta_1, \delta_2),$$

where the bivincular patterns α, β_1, β_2, δ_1, δ_2 are depicted—as mesh patterns—in Figure 2. As usual, we shall assume the same picture as their definition. For instance, a more extensive definition of α would be

$$\alpha = \big(2413, \{(2,k) : k = 0, \ldots, 4\} \cup \{(k,3) : k = 0, \ldots, 4\}\big).$$

We could not find a suitable description of $\gamma\big(\hat{\mathcal{A}}(2231)\big)$ in terms of pattern avoidance.

Let us start from the pattern 2321.

Proposition 19. *We have*

$$\gamma\big(\hat{\mathcal{A}}(2321)\big) = F(\delta_1, \delta_2).$$

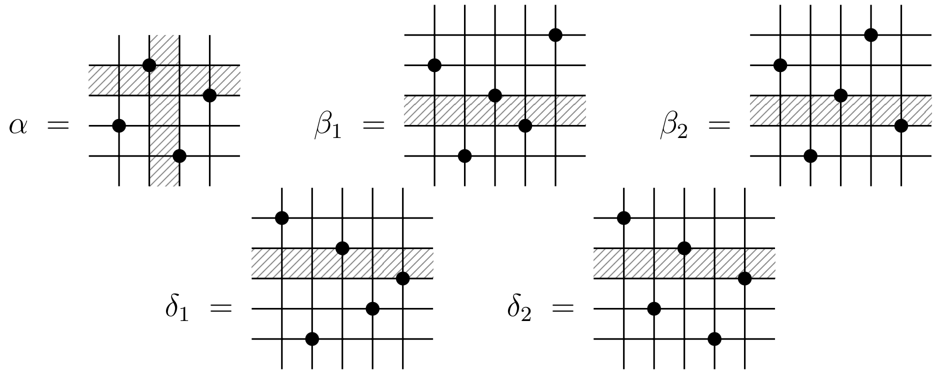


Figure 2: Mesh (bivincular) patterns such that $\gamma\big((\hat{\mathcal{A}}(212)\big) = F(\alpha)$, $\gamma\big((\hat{\mathcal{A}}(2213)\big) = F(\beta_1, \beta_2)$, and $\gamma\big((\hat{\mathcal{A}}(2321)\big) = F(\delta_1, \delta_2)$.

Proof. Let $x \in \hat{\mathcal{A}}_n(2321)$ and let $p = \gamma(x)$ be the corresponding Fishburn permutation. We show that

$$x \text{ contains } 2321 \quad \text{if and only if} \quad p \text{ contains } \delta_1 \text{ or } \delta_2.$$

Suppose that x contains an occurrence $x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ of 2321. Let j be the index of the leftmost copy of x_{i_4} in x, i.e. such that $(j, x_j) \in \mathrm{nub}(x)$ and $x_j = x_{i_4}$. We start by showing that we can assume $i_3 = i_2 + 1$ without losing generality. Indeed, we have $x_{i_1} = x_{i_3}$ and thus, since $\mathrm{asctops}(x) = \mathrm{nub}(x)$, x_{i_3} is not an ascent top, i.e. $x_{i_3-1} \geqslant x_{i_3}$. Now, if $x_{i_3-1} > x_{i_3}$, then $x_{i_1}x_{i_3-1}x_{i_3}x_{i_4}$ is an occurrence of 2321 where the second and third element are in consecutive position, as wanted. Otherwise, if $x_{i_3-1} = x_{i_3}$, then we can replace x_{i_3} with x_{i_3-1} and repeat the same argument until we fall back in the previous case. Similarly, we can assume $j < i_2$. Indeed, we have $x_{j-1} < x_j$ since $(j, x_j) \in \mathrm{nub}(x) = \mathrm{asctops}(x)$. If $j > i_2$ (and thus also $j > i_3 = i_2 + 1$), then $x_{i_1}x_{i_2}x_{i_3}x_{j-1}$ is an occurrence of 2321 and we can once again go on until we eventually fall in the case $j < i_2$. In the end, due to the assumptions $i_3 = i_2 + 1$ and $j < i_2$, we have either

$$x = \cdots x_j \cdots x_{i_1} \cdots x_{i_2}x_{i_3} \cdots x_{i_4} \quad \text{or} \quad x = \cdots x_{i_1} \cdots x_j \cdots x_{i_2}x_{i_3} \cdots x_{i_4},$$

depending on whether $j < i_1$ or $j > i_1$. Now, let us consider more in details the equation $(\mathrm{id}_n, x)^T = (\mathrm{sort}(x), p)$. If $j < i_1$, then we have

$$\begin{pmatrix} \cdots\ j\ \cdots\ i_1\ \cdots\ \ i_2 i_3\ \ \cdots\ i_4\ \cdots \\ \cdots x_j \cdots x_{i_1} \cdots x_{i_2}x_{i_3} \cdots x_{i_4} \cdots \end{pmatrix}^T = \begin{pmatrix} \cdots x_{i_4} \cdots x_j \cdots x_{i_3} \cdots x_{i_1} \cdots x_{i_2} \cdots \\ \cdots\ i_4\ \cdots\ j\ \cdots\ i_3\ \cdots\ i_1\ \cdots\ i_2\ \cdots \end{pmatrix}$$

and i_4, j, i_3, i_1, i_2 is an occurrence of δ_1 in p. Indeed, the underlying pattern is 51423 and $i_3 = i_2 + 1$. On the other hand, suppose that $j > i_1$. Then

$$\begin{pmatrix} \cdots\ i_1\ \cdots\ j\ \cdots\ \ i_2 i_3\ \ \cdots\ i_4\ \cdots \\ \cdots x_{i_1} \cdots x_j \cdots x_{i_2}x_{i_3} \cdots x_{i_4} \cdots \end{pmatrix}^T = \begin{pmatrix} \cdots x_{i_4} \cdots x_j \cdots x_{i_3} \cdots x_{i_1} \cdots x_{i_2} \cdots \\ \cdots\ i_4\ \cdots\ j\ \cdots\ i_3\ \cdots\ i_1\ \cdots\ i_2\ \cdots \end{pmatrix}.$$

In this case, the pattern underlying i_4, j, i_3, i_1, i_2 is 52413 and, since $i_3 = i_2+1$, i_4, j, i_3, i_1, i_2 is an occurrence of δ_2. We have thus proved that if x contains 2321, then p contains δ_1 or δ_2, as desired.

To prove the opposite direction, we show that, if p contains δ_1 or δ_2, then x contains 2321. Suppose initially that p contains δ_1. Let $p_{i_1}p_{i_2}p_{i_3}p_{i_4}p_{i_5}$ be an occurrence of δ_1 in p; that is, an occurrence of 51423 where $p_{i_3} = p_{i_5} + 1$. Let

$$\binom{\mathrm{sort}(x)}{p} = \begin{pmatrix} \cdots \ \ell_1 \ \cdots \ \ell_2 \ \cdots \ \ell_3 \ \cdots \ \ell_4 \ \cdots \ \ell_5 \ \cdots \\ \cdots \ p_{i_1} \ \cdots \ p_{i_2} \ \cdots \ p_{i_3} \ \cdots \ p_{i_4} \ \cdots \ p_{i_5} \ \cdots \end{pmatrix}.$$

Since $(\mathrm{sort}(x), p)$ is a Burge word, we have $\mathrm{Des}\big(\mathrm{sort}(x)\big) \subseteq \mathrm{Des}(p)$. Hence it must be

$$\ell_2 < \ell_3 \ \text{ since } \ p_{i_2} < p_{i_3} \quad \text{and} \quad \ell_4 < \ell_5 \ \text{ since } \ p_{i_4} < p_{i_5}.$$

Furthermore, the Burge transpose acts as

$$\begin{aligned} \binom{\mathrm{sort}(x)}{p}^T &= \begin{pmatrix} \cdots & \ell_1 & \cdots & \ell_2 & \cdots & \ell_3 & \cdots & \ell_4 & \cdots & \ell_5 & \cdots \\ \cdots & p_{i_1} & \cdots & p_{i_2} & \cdots & p_{i_3} & \cdots & p_{i_4} & \cdots & p_{i_5} & \cdots \end{pmatrix}^T \\ &= \begin{pmatrix} \cdots & p_{i_2} & \cdots & p_{i_4} & \cdots & p_{i_5}\, p_{i_3} & \cdots & p_{i_1} & \cdots \\ \cdots & \ell_2 & \cdots & \ell_4 & \cdots & \ell_5\, \ell_3 & \cdots & \ell_1 & \cdots \end{pmatrix} = \binom{\mathrm{id}_n}{x}. \end{aligned}$$

Note that $\ell_5 > \ell_3 > \ell_1$ and ℓ_5 and ℓ_3 are in consecutive positions since $p_{i_3} = p_{i_5} + 1$. In particular, such entry ℓ_3 is not an ascent top in x. Thus the leftmost copy, say x_j, of ℓ_3 precedes ℓ_5 in x (more precisely, it precedes the column (p_{i_5}, ℓ_5)). Therefore, the entry x_j form an occurrence of 2321 together with the bottom entries ℓ_5, ℓ_3 and ℓ_1 in the columns (p_{i_5}, ℓ_5), (p_{i_3}, ℓ_3) and (p_{i_1}, ℓ_1). We have thus proved that, if p contains δ_1, then x contains 2321. Finally, suppose that p contains an occurrence $p_{i_1}p_{i_2}p_{i_3}p_{i_4}p_{i_5}$ of δ_2. The proof that x contains 2321 is identical to the previous case, the only difference being that the positions of the columns (p_{i_2}, i_2) and (p_{i_4}, i_4) is exchanged since the classical pattern underlying δ_2 is 52413 (instead of 51423). □

The two remaining patterns 212 and 2213 can be solved in a similar fashion. Below we just sketch the corresponding proofs, leaving the details to the reader.

Proposition 20. *We have*

$$\gamma\big(\hat{\mathcal{A}}(2213)\big) = F(\beta_1, \beta_2).$$

Proof. Let $x \in \hat{\mathcal{A}}_n(2213)$ and let $p = \gamma(x)$. We show that

$$x \text{ contains } 2213 \quad \text{if and only if} \quad p \text{ contains } \beta_1 \text{ or } \beta_2.$$

Let $x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ be an occurrence of 2213 in x. Let x_j be the leftmost copy of x_{i_3} in x. Note that

$$x_j = x_{i_3} < x_{i_1} = x_{i_2} < x_{i_4}.$$

It is not hard to show that one can assume $j < i_2$ without losing generality. If $i_2 = i_1 + 1$, then i_3, j, i_2, i_1, i_4 is an occurrence of β_1 in p. Otherwise, consider the entry x_{i_2-1}

immediately to the left of x_{i_2}. Since $x_{i_2} \notin \mathrm{nub}(x) = \mathrm{asctops}(x)$, it must be $x_{i_2-1} \geqslant x_{i_2}$ (and $j \neq i_2 - 1$). If $x_{i_2-1} = x_{i_2}$, replace x_{i_2} with x_{i_2-1} and repeat the same argument. Otherwise, suppose that $x_{i_2-1} > x_{i_2}$. If $x_{i_2-1} < x_{i_4}$, then

$$\begin{aligned}\begin{pmatrix}\mathrm{id}\\ x\end{pmatrix}^T &= \begin{pmatrix}\cdots & j & \cdots & i_2-1 & i_2 & \cdots & i_3 & \cdots & i_4 & \cdots\\ \cdots & x_j & \cdots & x_{i_2-1} & x_{i_2} & \cdots & x_{i_3} & \cdots & x_{i_4} & \cdots\end{pmatrix}^T\\ &= \begin{pmatrix}\cdots & x_{i_3} & \cdots & x_j & \cdots & x_{i_2} & \cdots & x_{i_2-1} & \cdots & x_{i_4} & \cdots\\ \cdots & i_3 & \cdots & j & \cdots & i_2 & \cdots & i_2-1 & \cdots & i_4 & \cdots\end{pmatrix} = \begin{pmatrix}\mathrm{sort}(x)\\ p\end{pmatrix}\end{aligned}$$

and i_3, j, i_2, i_2-1, i_4 is an occurrence of β_1 (note that $i_3, j, i_2, i_2-1, i_4 \simeq 41325$). Similarly, if $x_{i_2-1} \geqslant x_{i_4}$, then

$$\begin{aligned}\begin{pmatrix}\mathrm{id}\\ x\end{pmatrix}^T &= \begin{pmatrix}\cdots & j & \cdots & i_2-1 & i_2 & \cdots & i_3 & \cdots & i_4 & \cdots\\ \cdots & x_j & \cdots & x_{i_2-1} & x_{i_2} & \cdots & x_{i_3} & \cdots & x_{i_4} & \cdots\end{pmatrix}^T\\ &= \begin{pmatrix}\cdots & x_{i_3} & \cdots & x_j & \cdots & x_{i_2} & \cdots & x_{i_4} & \cdots & x_{i_2-1} & \cdots\\ \cdots & i_3 & \cdots & j & \cdots & i_2 & \cdots & i_4 & \cdots & i_2-1 & \cdots\end{pmatrix} = \begin{pmatrix}\mathrm{sort}(x)\\ p\end{pmatrix}.\end{aligned}$$

and i_3, j, i_2, i_4, i_2-1 is an occurrence of β_2 (here $i_3, j, i_2, i_2-1, i_4 \simeq 41352$).

Let us now take care of the other direction. Suppose that p contains an occurrence $p_{i_1}p_{i_2}p_{i_3}p_{i_4}p_{i_5}$ of β_1; that is, $p_{i_1}p_{i_2}p_{i_3}p_{i_4}p_{i_5} \simeq 41325$ and $p_{i_3} = p_{i_4}+1$. The Burge transpose acts on $(\mathrm{sort}(x), p)^T = (\mathrm{id}, x)$ by mapping the columns

$$\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5\\ p_{i_1} & p_{i_2} & p_{i_3} & p_{i_4} & p_{i_5}\end{pmatrix} \quad \text{to} \quad \begin{pmatrix}p_{i_2} & p_{i_4} & p_{i_3} & p_{i_1} & p_{i_5}\\ \ell_2 & \ell_4 & \ell_3 & \ell_1 & \ell_5\end{pmatrix},$$

where ℓ_4 and ℓ_3 are in consecutive positions in p since $p_{i_3} = p_{i_4} + 1$. Due to the equality $\mathrm{Des}\big(\mathrm{sort}(x)\big) \subseteq \mathrm{Des}(p)$, we also have

$$\ell_2 < \ell_3 \;\text{ since }\; p_{i_2} < p_{i_3} \quad\text{and}\quad \ell_4 < \ell_5 \;\text{ since }\; p_{i_4} < p_{i_5}.$$

Now, if $\ell_4 = \ell_3$, then $\ell_4, \ell_3, \ell_1, \ell_5$ is an occurrence of 2213 in x. Otherwise, let $\ell_4 > \ell_3$. Then ℓ_3 is not an ascent top in x and $t, \ell_3, \ell_1, \ell_5$ is an occurrence of 2213, where t is the leftmost copy of ℓ_3 in x. In a similar fashion, it is easy to see that, given an occurrence $p_{i_1}p_{i_2}p_{i_3}p_{i_4}p_{i_5}$ of β_2, the Burge transpose maps the entries $p_{i_1}, p_{i_3}, p_{i_4}$ to an occurrence of 213 in x where the entry that plays the role of 2 is not an ascent top—and thus not a leftmost copy. The desired occurrence of 2213 is obtained immediately by adding the leftmost copy of the 2 to these three entries. □

Proposition 21. *We have*

$$\gamma\big(\hat{\mathcal{A}}(212)\big) = F(\alpha).$$

Proof. Let $x \in \hat{\mathcal{A}}(212)$ and let $p = \gamma(x)$. Let $x_{i_1}x_{i_2}x_{i_3}$ be an occurrence of 212 in x. Without losing generality, we can assume that

(*i*) $x_j \neq x_{i_1}$ for each $i_1 < j < i_3$;

(*ii*) $x_{i_3-1} > x_{i_3}$.

Indeed, (*i*) is simply achieved by taking the largest i_1 and the smallest i_3 such that $x_{i_1}x_{i_2}x_{i_3} \simeq 212$. Furthermore, since $x_{i_1} = x_{i_3}$, we have $x_{i_3} \notin \mathrm{nub}(x) = \mathrm{asctops}(x)$. Hence $x_{i_3-1} \geqslant x_{i_3}$ and, due to (*i*), $x_{i_3-1} > x_{i_3}$. Finally, the Burge transpose maps the entries $x_{i_1}x_{i_2}x_{i_3-1}x_{i_3}$ to an occurrence of α in p. Indeed $x_{i_1}x_{i_2}x_{i_3-1}x_{i_3} \simeq 2132$ is mapped by T to an occurrence of 2413. The entries playing the role of 4 and 1 are in consecutive positions due to (*i*); and the 4 and the 3 are consecutive in value due to (*ii*).

Next suppose that p contains α. We show that x contains 212. The classical pattern underlying α is 2413. Due to Theorem 3, since the Fishburn basis of 2413 is

$$B_{2413} = \{x \in \mathrm{Cay} : \gamma(x) = 2413\} = \{2132, 3142\},$$

we have

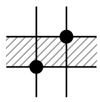
$$F(2413) = \gamma\big(\hat{\mathcal{A}}(2132, 3142)\big).$$

Hence, since p contains 2413, x contains 2132 or 3142. If x contains 2132, then we are done since $\hat{\mathcal{A}}(212) = \hat{\mathcal{A}}(2132)$ by Proposition 10. On the other hand, if x contains 3142, then an occurrence of 212 can be obtained by taking the leftmost copy of the entry that plays the role of 2, which must precede the 1 due to the shaded regions defining α. $\square$

Corollary 22. *The sets of Fishburn permutations*

$$F(\alpha), \quad F(\beta_1, \beta_2),$$

are counted by the Bell numbers. Furthermore, the distribution of the number of occurrences of the pattern

$\mathfrak{g} =$ 

is the reverse of the distribution of blocks on set partitions.

Proof. The first part of the statement follows immediately by Theorem 12, Theorem 18, Proposition 20 and Proposition 21. The second part follows from the same results since γ maps each strict ascent in a modified ascent sequence x to an occurrence of $\mathfrak{g}$ in the corresponding Fishburn permutation $\gamma(x)$. $\square$

The current author [6] has recently solved Conjecture 1 for the remaining pattern 2321. By Proposition 19, the set $F(\delta_1, \delta_2)$ is also counted by the Bell numbers.

6 Final remarks

In this paper, we proved that modified ascent sequences avoiding any of the patterns in $\{212, 1212, 2132, 12132, 2213, 2231\}$ are counted by the Bell numbers; we also showed that the distribution of strict ascents (or, equivalently, of the maximum value) is the reverse of the number of blocks on set partitions. The pattern 2321 was solved recently by the current author [6], thus fully answering Conjecture 1.

Claesson and the current author [8] have recently introduced Fishburn trees to clarify the bijections relating modified ascent sequences, Fishburn matrices and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets. Under these maps, the sets $\hat{\mathcal{A}}(212)$, $\hat{\mathcal{A}}(2213)$, $\hat{\mathcal{A}}(2231)$ and $\hat{\mathcal{A}}(2321)$ determine sets of matrices, trees and posets that are counted by the Bell numbers: can we find independent and interesting description of these sets, e.g. in terms of properties defined directly on each of these structures?

Pattern avoiding modified ascent sequences are related to several other combinatorial structures. One notable instance is given by the set $\hat{\mathcal{A}}(2312, 3412)$. The set of Fishburn permutations corresponding to $\hat{\mathcal{A}}(2312, 3412)$ is $F(3412)$ [7]. Furthermore, the pair of statistics right-to-left maxima and right-to-left minima on $F(3412)$ seems to have the same distribution as the pair left-to-right maxima and right-to-left maxima over the set of 312-sortable permutations [9]. The enumeration of all these sets (see also A202062 [21]) is still unknown.

Sequences in context:
A000670, A000110, A022493, A137251, A005493, A259691, A202062.

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