Dual Bipartite Q-Polynomial Distance-Regular Graphs and Dual Uniform Structures

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Submitted: May 27, 2024; Accepted: Oct 3, 2024; Published: Nov 1, 2024

Abstract

Let Γ denote a dual bipartite Q-polynomial distance-regular graph with vertex set X and diameter $D \geqslant 3$. Fix $x \in X$, and let L^* and R^* denote the corresponding *dual lowering* and *dual raising matrix*, respectively. We show that a certain linear dependency among $R^*L^{*2}, L^*R^*L^*, L^{*2}R^*, L^*$ holds, and determine whether this linear dependency endow Γ with a *dual uniform* or *dual strongly uniform structure*. Precisely, except for two special cases a dual uniform structure is always attained, and except for four special cases a dual strongly uniform structure is always attained.

Mathematics Subject Classifications: 05E99, 05C50

1 Introduction

In his thesis [5], Delsarte introduced the Q-polynomial property for a distance-regular graph Γ (see Section 2 for formal definitions). Since then, the Q-polynomial property has been investigated by many authors; see for example [1, 2, 3]. A survey about the Q-polynomial property can be found in [4].

Assume Γ is Q-polynomial. In [13], Terwilliger introduced the subconstituent algebra of Γ. For each vertex x of Γ, the corresponding subconstituent algebra $T = T(x)$ is generated by the adjacency matrix A and a certain diagonal matrix $A^* = A^*(x)$. The eigenspaces of A^* are the subconstituents of Γ with respect to x. The matrices A and A^* satisfy two relations called the tridiagonal relations [14, Lemma 5.4], [15]. The first (resp., second) tridiagonal relation is of degree 3 in A (resp., A^*) and of degree 1 in A^* (resp., A). In [14], the tridiagonal relations are used to describe the combinatorics of Γ.

Assume for a moment that Γ is bipartite. For any fixed vertex x of Γ, define two $(0,1)$-matrices, $L = L(x)$ and $R = R(x)$ (indexed on the set of vertices of Γ) as follows. For the vertices y, z of Γ, the (z, y)-entry of L is 1 if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) - 1$, and 0 otherwise; here ∂ denotes the path-length distance of Γ. The matrix R is the transpose of

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L. Then the adjacency matrix A of Γ naturally decomposes as $A = L+R$. The matrices L and R are respectively called the lowering matrix and raising matrix of Γ with respect to x [10, 12]. In the first tridiagonal relation, if one eliminates A using $A = L+R$, one finds that on each x-subconstituent of Γ the matrices $RL^2,\ LRL,\ L^2R,\ L$ are linearly dependent. The coefficients in this linear dependence depend on the subconstituent. The collection of these dependencies is called an R/L dependency structure. Motivated by these R/L dependency structures, in [12] Terwilliger introduced the uniform property for a partially ordered set. In that work, he described the algebraic structure of the uniform posets and displayed eleven infinite families of examples. A careful study of the known connection between the Q-polynomial property and uniform posets was completed in [10], introducing a variation on the uniform property, called strongly uniform. Strongly uniform implies uniform. For each Q-polynomial structure on Γ (bipartite distance-regular), the authors determined precisely when the corresponding R/L dependency structure is uniform or strongly uniform.

Now, assume that our distance-regular graph Γ has a dual bipartite Q-polynomial structure (see Section 4 for formal definitions). Then the diagonal matrix A^* of Γ can be decomposed as $A^* = L^* + R^*$, where $L^* = L^*(x)$ (resp., $R^* = R^*(x)$) is the dual lowering matrix (resp., dual raising matrix) of Γ with respect to x (see Section 3). The matrix R^* is the transpose of L^*. In the second tridiagonal relation, replacing A^* by $L^* + R^*$, one finds that on each eigenspace of A the matrices

$$R^*L^{*2}, \quad L^*R^*L^*, \quad L^{*2}R^*, \quad L^*$$

are linearly dependent. The coefficients in this linear dependence depend on the eigenspace. We call this collection of dependencies an R^*/L^* dependency structure (see Section 5). In light of the analogy with the arguments from the previous paragraph, one can naturally introduce the concept of a dual (strongly) uniform structure and study the connection with the dual bipartite Q-polynomial structure. The purpose of this paper is to determine for each (dual bipartite) Q-polynomial structure on Γ when the corresponding R^*/L^* dependency structure is dual uniform or dual strongly uniform. Throughout the paper, we constantly refer to the classification of P- and Q-polynomial orderings contained in [1, Theorem 5.1]. Precisely, when we say that Γ is of *type (roman numeral)*, we mean that its parameters are those listed in the corresponding *case (roman numeral)* of [1, Theorem 5.1].

To describe our main result, assume that our distance-regular graph Γ has vertex set X and diameter $D \geqslant 3$, and that it admits a dual bipartite Q-polynomial structure. Fix $x \in X$, and consider the corresponding dual lowering matrix and dual raising matrix, i.e., $L^* = L^*(x)$ and $R^* = R^*(x)$, respectively. Consider the following cases.

(i) Γ is the ordinary $2D$-gon.

(ii) Γ is the cube $H(D, 2)$, D even, type [1, Theorem 5.1(III)].

(iii) Γ is the halved cube $\frac{1}{2}H(2D, 2)$.

(iv) Γ is as in Lemma 8 with $s \in \{q^{-1}, q^{-2D-1}\}$, $q^{2D} \neq 1$ (see Sections 4,7 for more details).

We show that: in Cases (i) and (ii), the corresponding R^*/L^* dependency structure is not dual uniform; in Cases (iii) and (iv), this structure is dual uniform but not dual strongly uniform; in all other cases, this structure is dual strongly uniform.

The paper is organized as follows. In Sections 2 and 3, we discuss the Bose-Mesner algebra and the dual Bose-Mesner algebra of a distance-regular graph. In Section 4, we consider the dual Q-polynomial bipartite case, and for it, in Sections 5 and 6, we introduce R^*/L^* dependency structures and dual (strongly) uniform structures. In Sections 7-10, for each given (dual bipartite) Q-polynomial structure on Γ, we determine precisely when the corresponding R^*/L^* dependency structure is dual uniform or dual strongly uniform. Our main result is Theorem 29.

2 Preliminaries

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by X. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with entries in $\mathbb{R}$ and rows indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. We refer to V as the *standard module* of $\mathrm{Mat}_X(\mathbb{R})$. We endow V with the bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ that satisfies $\langle u, v\rangle = u^t v$ for $u, v \in V$, where t denotes transpose. For $y \in X$, let $\hat{y}$ denote the vector in V that has y-coordinate 1 and all other coordinates 0. Observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Throughout the paper, let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set $\mathcal{R}$, path-length distance function ∂, and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and an integer i, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) := \Gamma_1(x)$. For an integer $k \geqslant 0$, we say Γ is *regular* with valency k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers $0 \leqslant h, i, j \leqslant D$ and all $x, y \in X$ with $\partial(x, y) = h$ the number $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1\,i-1}$ $(1 \leqslant i \leqslant D)$, $a_i := p^i_{1i}$ $(0 \leqslant i \leqslant D)$, $b_i := p^i_{1\,i+1}$ $(0 \leqslant i \leqslant D-1)$, $k_i := p^0_{ii}$ $(0 \leqslant i \leqslant D)$, and $c_0 := 0, b_D := 0$. For the rest of this paper, assume Γ is distance-regular with diameter $D \geqslant 3$. By the triangle inequality, for $0 \leqslant h, i, j \leqslant D$ we have $p^h_{ij} = 0$ (resp., $p^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp., equal to) the sum of the other two. In particular, $c_i \neq 0$ for $1 \leqslant i \leqslant D$ and $b_i \neq 0$ for $0 \leqslant i \leqslant D-1$. Observe that Γ is regular with valency $k = b_0 = k_1$ and that $c_i + a_i + b_i = k$ for $0 \leqslant i \leqslant D$.

We recall the Bose-Mesner algebra of Γ. For $0 \leqslant i \leqslant D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, z)-entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y, z) = i, \\ 0 & \text{if } \partial(y, z) \neq i \end{cases} \qquad (y, z \in X).$$

The matrix A_i is called the *ith distance matrix* of Γ. We abbreviate $A := A_1$ and call it the *adjacency matrix* of Γ. Observe that (ai) $A_0 = I$; (aii) $J = \sum_{i=0}^{D} A_i$; (aiii) $A_i^t = A_i$ $(0 \leqslant i \leqslant D)$; (aiv) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \leqslant i, j \leqslant D)$, where I (resp., J) denotes the identity matrix (resp., all 1's matrix) in $\mathrm{Mat}_X(\mathbb{R})$. Using these properties, we find $\{A_i\}_{i=0}^D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$, known as the *Bose-Mesner algebra* of Γ. By [1, p. 190], A generates M. Furthermore, by [3, p. 45], M has a basis $\{E_i\}_{i=0}^D$ such that (ei) $E_0 = |X|^{-1}J$; (eii) $I = \sum_{i=0}^{D} E_i$; (eiii) $E_i^t = E_i$ $(0 \leqslant i \leqslant D)$; (eiv) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant D)$. We call $\{E_i\}_{i=0}^D$ the *primitive idempotents* of Γ. The primitive idempotent E_0 is said to be *trivial*.

Since $\{E_i\}_{i=0}^D$ form a basis for M, there exist scalars $\{\theta_i\}_{i=0}^D$ in $\mathbb{R}$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Combining this with (eiv), we find

$$AE_i = E_i A = \theta_i E_i \quad (0 \leqslant i \leqslant D). \tag{1}$$

We call θ_i the *eigenvalue of* Γ *associated with* E_i. The $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M. By (ei) we have $\theta_0 = k$. By (eii)-(eiv),

$$V = E_0V + E_1V + \cdots + E_DV \text{ (orthogonal direct sum).}$$

For $0 \leqslant i \leqslant D$, the space E_iV is the eigenspace of A associated with θ_i. Let m_i denote the rank of E_i, and note that m_i is the dimension of E_iV. We call m_i the *multiplicity* of θ_i.

We recall the Krein parameters of Γ. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{R})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant D$, so M is closed under $\circ$. Thus, there exist scalars $q_{ij}^h \in \mathbb{R}$ $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leqslant i, j \leqslant D).$$

The parameters q_{ij}^h are called the *Krein parameters* of Γ. By [3, Proposition 4.1.5], these parameters are nonnegative. The given ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents is said to be *Q-polynomial* if for $0 \leqslant h, i, j \leqslant D$ the Krein parameter $q_{ij}^h = 0$ (resp., $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp., equal to) the sum of the other two. Let E denote a nontrivial primitive idempotent of Γ and let θ denote the corresponding eigenvalue. We say that Γ is *Q-polynomial with respect to* E (or θ) whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ such that $E_1 = E$.

3 The dual Bose-Mesner algebra

We continue to discuss the distance-regular graph Γ from the previous sections. In this section, we recall the dual Bose-Mesner algebra of Γ. Fix $x \in X$. For $0 \leqslant i \leqslant D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the *ith dual idempotent* of Γ with respect to x. For convenience, set $E_i^* = 0$ for $i < 0$ or $i > D$. It is known that (esi) $I = \sum_{i=0}^{D} E_i^*$; (esii) $E_i^{*t} = E_i^*$ $(0 \leqslant i \leqslant D)$; (esiii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leqslant i, j \leqslant D)$. By these properties, $\{E_i^*\}_{i=0}^{D}$ forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{R})$, known as the *dual Bose-Mesner algebra* of Γ with respect to x; also,

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \text{ (orthogonal direct sum).}$$

The algebras M and M^* are related as follows. By [13, Lemma 3.2],

$$E_i^* A_j E_h^* = 0 \text{ if and only if } p_{ij}^h = 0 \quad (0 \leqslant h, i, j \leqslant D).$$

Let E denote a nontrivial primitive idempotent of Γ. For the rest of the section, assume Γ is Q-polynomial with respect to E. Let $A^* = A^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$A_{yy}^* = |X| E_{xy} \quad (y \in X).$$

The matrix A^* is called the *dual adjacency matrix* of Γ corresponding to E and x. By [13, Lemma 3.11(ii)], A^* generates M^*. Since $\{E_i^*\}_{i=0}^{D}$ forms a basis for M^*, there exist scalars $\{\theta_i^*\}_{i=0}^{D}$ in $\mathbb{R}$ such that $A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Combining this with (esiii), we obtain

$$A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \quad (0 \leqslant i \leqslant D).$$

We call $\{\theta_i^*\}_{i=0}^{D}$ the *dual eigenvalue sequence* for the given Q-polynomial structure. Note that the $\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct since A^* generates M^*. For $0 \leqslant i \leqslant D$, the space $E_i^* V$ is the eigenspace of A^* associated with θ_i^*. By [1, Proposition 3.4.(iv)], we have that $\theta_0^* = \mathrm{rank}(E) = \dim(EV)$.

Lemma 1. *([14, Lemma 5.4]) Let $\{E_i\}_{i=0}^{D}$ denote a Q-polynomial ordering of the primitive idempotents of Γ and, for $0 \leqslant i \leqslant D$, let θ_i denote the eigenvalue of Γ for E_i. Let $\{\theta_i^*\}_{i=0}^{D}$ denote the dual eigenvalue sequence for the given Q-polynomial structure. Then the following (i)-(iii) hold.*

(i) There exists $\beta \in \mathbb{R}$ such that

$$\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

for $2 \leqslant i \leqslant D - 1$.

(ii) There exist $\gamma, \gamma^* \in \mathbb{R}$ such that

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \quad \gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$$

for $1 \leqslant i \leqslant D - 1$.

(iii) There exist $\varrho, \varrho^* \in \mathbb{R}$ such that

$$\begin{aligned}\varrho &= \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma\left(\theta_{i-1} + \theta_i\right),\\ \varrho^* &= \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*\left(\theta_{i-1}^* + \theta_i^*\right)\end{aligned}$$

for $1 \leqslant i \leqslant D$.

Lemma 2. *([14, Lemma 5.4]) Let E denote a Q-polynomial primitive idempotent of Γ and let $A^* = A^*(x)$ denote the corresponding dual adjacency matrix. Then*

$$\begin{aligned}\left[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma\left(AA^* + A^*A\right) - \varrho A^*\right] = 0,\\ \left[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*\left(A^*A + AA^*\right) - \varrho^* A\right] = 0,\end{aligned}$$

where $[r,s] = rs - sr$ and $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ are from Lemma 1.

We now recall some special matrices for the graph Γ. Define the matrices $R^* = R^*(x), F^* = F^*(x), L^* = L^*(x)$ by

$$R^* = \sum_{i=0}^{D} E_{i+1}A^*E_i, \quad F^* = \sum_{i=0}^{D} E_iA^*E_i, \quad L^* = \sum_{i=0}^{D} E_{i-1}A^*E_i, \tag{2}$$

where $E_{D+1} = E_{-1} = 0$. They are known as the *dual raising*, *dual flat*, and *dual lowering matrices* of Γ with respect to x, respectively. Note that R^*, F^*, and L^* have real entries. Also, observe that F^* is symmetric and $R^* = (L^*)^t$. Moreover,

$$A^* = R^* + F^* + L^*. \tag{3}$$

Using Equation (2) and the convention that $E_{-1} = 0, E_{D+1} = 0$, we find

$$\begin{aligned}&R^*E_i = E_{i+1}R^* \ (-1 \leqslant i \leqslant D),\\ &F^*E_i = E_iF^* \ (0 \leqslant i \leqslant D),\\ &L^*E_i = E_{i-1}L^* \ (0 \leqslant i \leqslant D+1).\end{aligned} \tag{4}$$

Furthermore, one can easily derive the following lemma from the above considerations.

Lemma 3. *Let L^*, R^* be as in (2). Then the following (i), (ii) hold.*

(i) $R^*E_iV \subseteq E_{i+1}V$ for $0 \leqslant i \leqslant D-1$, and $R^*E_DV = 0$;

(ii) $L^*E_iV \subseteq E_{i-1}V$ for $1 \leqslant i \leqslant D$, and $L^*E_0V = 0$.

4 Dual bipartite Q-polynomial distance-regular graphs

We continue to refer to the distance-regular graph Γ from the previous sections. Assume that Γ admits a Q-polynomial structure. Such a Q-polynomial structure is *dual bipartite* whenever $a_i^* := q_{1i}^i = 0$ for $0 \leqslant i \leqslant D$. In this case, using [13, Lemma 3.12], we have

$$E_iA^*E_h = 0 \quad \text{if } |h-i| \neq 1 \quad (0 \leqslant h, i \leqslant D). \tag{5}$$

Moreover, it follows from (2), (3), and (5) that $F^* = 0$ and $A^* = R^* + L^*$.

For Γ admitting a dual bipartite Q-polynomial structure, the following classification results are known.

Theorem 4. *([6, Theorem 1.1]) Let Γ denote a distance-regular graph with diameter $D \geqslant 3$ and valency k, and suppose Γ is not bipartite. Then Γ has a dual bipartite Q-polynomial structure if and only if $b_i = c_{D-i}$ for $0 \leqslant i \leqslant D-1$ and the array $\{c_1, \ldots, c_D\}$ for Γ is one of the following:*

(i) $\{1, 4, 9, \ldots, D^2\}$*;*

(ii) $\{1, 6, 15, \ldots, 2D^2 - D\}$*;*

(iii) $\{1, k - a_1 - 1, k\}$*; or*

(iv) $\{1, \delta\eta, (\delta^2 - 1)(2\eta - \delta + 1), \delta(2\eta + 2\eta\delta - \delta^2)\}$*, where* $\delta \geqslant 3$*,* $\eta \geqslant 3\delta/4$ *are integers and* η *divides* $\delta^2(\delta^2 - 1)/2$*.*

The arrays (i) and (ii) are uniquely realized by the Johnson graph $J(D, 2D)$ and the halved cube $\frac{1}{2}H(2D, 2)$, respectively. The graphs with array (iii) are the Taylor graphs.

Remark 5. With reference to Theorem 4(iii), the Gosset graph is a Taylor graph with array $\{1, 10, 27\}$ [4, Section 5.1].

Remark 6. An example with array (iv) in Theorem 4 is the Meixner double cover ($\delta = 4, \eta = 6$) [4, Section 3.2.4]. However, the array (iv) with δ, η odd has been ruled out by Jurišić and Koolen [9, Corollary. 3.2].

Theorem 7. *([6, Theorem 1.2]) Let Γ denote a distance-regular graph with diameter $D \geqslant 3$ and valency k, and suppose Γ is bipartite. Then Γ has a dual bipartite Q-polynomial structure if and only if $b_i = c_{D-i}$ for $0 \leqslant i \leqslant D-1$ and the array $\{c_1, \ldots, c_D\}$ for Γ is one of:*

(i) $\{1, 1, \ldots, 1, 2\}$*;*

(ii) $\{1, 2, 3, \ldots, k\}$*;*

(iii) $\{1, k-1, k\}$*;*

(iv) $\{1, 2\xi, 4\xi - 1, 4\xi\}$*, where* $\xi \geqslant 1$ *is an integer; or*

(v) $\{1, c, k-c, k-1, k\}$*, where* $k = \xi(\xi^2 + 3\xi + 1)$*,* $c = \xi(\xi+1)$*, and* $\xi \geqslant 2$ *is an integer.*

The arrays (i), (ii), and (iii) are uniquely realized by the ordinary $2D$-gon, the cube $H(D, 2)$, and the complement of $K_{k+1} \times K_2$, respectively. The graphs with array (iv) are the Hadamard graphs of order 4ξ. The array (v) is uniquely realized for $\xi = 2$ by the double cover of the Higman-Sims graph. We know of no examples with $\xi \geqslant 3$.

The lemma below will play a crucial role in our analysis.

Lemma 8. *([11]) Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$, and suppose Γ has a dual bipartite Q-polynomial structure with dual eigenvalues $\theta_0^*, \ldots, \theta_D^*$. Furthermore, suppose Γ is not the cube $H(D, 2)$, the halved cube $\frac{1}{2}H(2D, 2)$, the Johnson graph $J(D, 2D)$, or the Gosset graph. Then $b_i = c_{D-i}$ for $0 \leqslant i \leqslant D-1$, and there exist complex numbers $q \neq 0$ and s with*

$$\begin{aligned} q^j &\neq 1, \quad 1 \leqslant j \leqslant D, \\ q^j &\neq -1, \quad 0 \leqslant j \leqslant D-1, \\ sq^j &\neq 1, \quad 2 \leqslant j \leqslant 2D, \end{aligned}$$

such that

$$a_i = \frac{\left(q^{2i}-1\right)\left(q^{2D}-q^{2i}\right)\left(q^D+q^2\right)\left(1+sq^{D+1}\right)}{(q-1)\left(q^{2i}+q^{D+1}\right)\left(q^{2i+1}+q^D\right)\left(1-sq^{2D}\right)}, \quad 1 \leqslant i \leqslant D-1,$$

$$c_i = \frac{q^{2i-1}\left(q^D+q\right)\left(q^D+q^2\right)\left(q^{2i}-1\right)\left(1-sq^{2D+2-2i}\right)}{\left(q^2-1\right)\left(q^{2i}+q^D\right)\left(q^{2i}+q^{D+1}\right)\left(1-sq^{2D}\right)}, \quad 1 \leqslant i \leqslant D-1,$$

$$c_D = \frac{\left(q^D-1\right)\left(q^D+q^2\right)\left(1-sq^2\right)}{q\left(q^2-1\right)\left(1-sq^{2D}\right)},$$

$$\theta_i^* = \frac{q^D\left(q^{-i}-q^{i-D}\right)\left(1-sq^3\right)}{(q-1)\left(1-sq^{D+2}\right)}, \quad 0 \leqslant i \leqslant D.$$

Proposition 9. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$, and suppose Γ has a dual bipartite Q-polynomial structure. Then the statements (ii) and (iii) from Lemma 1 hold with $\gamma^* = 0$, and*

$$A^{*3}A - AA^{*3} - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) = \varrho^*(A^*A - AA^*). \tag{6}$$

Proof. If Γ is $H(D, 2)$, $\frac{1}{2}H(2D, 2)$, $J(D, 2D)$, or the Gosset graph, then Γ has *classical parameters*[1] and so admits a dual bipartite Q-polynomial structure with dual eigenvalues $\theta_i^* = D - i$, $0 \leqslant i \leqslant D$ [3, Corollary 8.4.2] (see also [1, Theorem 5.1], type (IIC) for $H(D, 2)$, and type (IIA) for the others). Thus, from Lemma 1, we have $\beta = 2$ and $\gamma^* = 0$. Note that for the cube $H(D, 2)$, with D even, there exists another (dual bipartite) Q-polynomial structure with dual eigenvalues $\theta_i^* = (-1)^i(D - 2i)$, $0 \leqslant i \leqslant D$ [1, p. 305 – type (III)]. Here, Lemma 1 gives $\beta = -2$ and $\gamma^* = 0$. When Γ is none of the previous graphs, we use the expression of θ_i^* from Lemma 8, so obtaining $\beta = q + q^{-1}$ and $\gamma^* = 0$ by Lemma 1.
Equation (6) follows from Lemma 2 with $\gamma^* = 0$. □

[1] For the sake of completeness, we address the reader to [3, Sections 6.1, 8.4, 8.5, Chapter 9] for all basic definitions and results regarding graphs with classical parameters. However, all these details, except those explicitly mentioned, are not relevant in the context of this article.

Remark 10. The proof of Proposition 9 also reveals that the parameter β from Lemma 1 takes on different values depending on the (dual bipartite) Q-polynomial structure we consider for our graph Γ. Precisely,

(a) if Γ is $H(D,2)$– type (IIC), $\frac{1}{2}H(2D,2)$, $J(D,2D)$, or the Gosset graph, then $\beta = 2$;

(b) if Γ is $H(D,2)$– type (III), with D even, then $\beta = -2$;

(c) otherwise, $\beta = q + q^{-1} \neq \pm 2$ (see Lemma 8).

From now on, we refer to the following notational convention.

Notation 11. Assume that our distance-regular graph Γ admits a dual bipartite Q-polynomial structure with diameter $D \geqslant 3$. Let $\{E_i\}_{i=0}^D$ denote the corresponding Q-polynomial ordering of the primitive idempotents of Γ, and let $\{\theta_i\}_{i=0}^D$ denote the corresponding eigenvalues. Abbreviate $E = E_1$, and consider our fixed vertex $x \in X$. For $0 \leqslant i \leqslant D$, let $E_i^* = E_i^*(x)$ denote the ith dual idempotent of Γ with respect to x. Let $A^* = A^*(x)$ denote the dual adjacency matrix of Γ that corresponds to E and x. Let $\{\theta_i^*\}_{i=0}^D$ denote the dual eigenvalue sequence for the given Q-polynomial structure. Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ be as in Lemma 1. Let the matrices $L^* = L^*(x)$ and $R^* = R^*(x)$ be defined as in (2).

5 A R^*/L^* dependency relation

In this section, we display a certain linear dependency among $R^*L^{*2}, L^*R^*L^*, L^{*2}R^*, L^*$.

Proposition 12. *With reference to Notation 11, for* $1 \leqslant i \leqslant D$,

$$\begin{aligned} &\frac{\theta_i - \theta_{i-1} + (\beta+1)\left(\theta_{i-2} - \theta_{i-1}\right)}{\theta_i - \theta_{i-1}} R^*L^{*2} + (\beta+2)L^*R^*L^* \\ &+ \frac{\theta_i - \theta_{i-1} + (\beta+1)\left(\theta_i - \theta_{i+1}\right)}{\theta_i - \theta_{i-1}} L^{*2}R^* = \varrho^* L^* \end{aligned} \tag{7}$$

holds on E_iV*, where* θ_{-1} *and* θ_{D+1} *are inderteminants.*

Proof. Multiplying (6) by E_{i-1} on the left and by E_i on the right, it follows from (1) that

$$(\theta_i - \theta_{i-1}) E_{i-1}A^{*3}E_i + (\beta+1)E_{i-1}\left(A^*AA^{*2} - A^{*2}AA^*\right)E_i = \varrho^*\left(\theta_i - \theta_{i-1}\right)E_{i-1}A^*E_i.$$

Using $A^* = R^* + L^*$ along with (1), (4) and $E_iE_j = \delta_{ij}E_i$ $(1 \leqslant i \leqslant D)$, we find

$$\begin{aligned} &(\theta_i - \theta_{i-1})\left(R^*L^{*2} + L^*R^*L^* + L^{*2}R^*\right)E_i + (\beta+1)\left(\left(\theta_{i-2} - \theta_{i-1}\right)R^*L^{*2} + \left(\theta_i - \theta_{i-1}\right)L^*R^*L^*\right. \\ &\left. + \left(\theta_i - \theta_{i+1}\right)L^{*2}R^*\right)E_i = \varrho^*\left(\theta_i - \theta_{i-1}\right)L^*E_i. \end{aligned}$$

Equation (7) is obtained once we factor out the above equation with respect to R^*L^{*2}, $L^*R^*L^*$, $L^{*2}R^*$, L^* and divide the result by $\theta_i - \theta_{i-1}(\neq 0)$. In the end, observe that (7) still holds at the extremes $(i = 1, D)$ as $L^{*2}E_1V = 0$ and $R^*E_DV = 0$ from Lemma 3. ☐

We call Equation (7) the R^*/L^* *dependency structure* which corresponds to the given (dual bipartite) Q-polynomial structure of Γ.

6 Dual uniform structures

In this section, with reference to Notation 11, we introduce the notion of a *dual uniform structure* for Γ, which somehow *dualizes* that of a *uniform structure* for bipartite Q-polynomial distance-regular graphs; see [12], [10], [8], and [7] for further details. We first need to define what a parameter matrix is.

Definition 13. A *parameter matrix* $U = (e^*_{ij})_{1\leqslant i,j\leqslant D}$ is a tridiagonal matrix with entries in $\mathbb{C}$ satisfying the following conditions:

(i) $e^*_{ii} = 1$ for $1 \leqslant i \leqslant D$;

(ii) $e^*_{i\,i-1} \neq 0$ for $2 \leqslant i \leqslant D$ or $e^*_{i-1\,i} \neq 0$ for $2 \leqslant i \leqslant D$;

(iii) the principal submatrix $U(p,t) = (e^*_{ij})_{p\leqslant i,\,j\leqslant t}$ is nonsingular for $1 \leqslant p \leqslant t \leqslant D$.

For convenience we write $e^{*-}_i := e^*_{i\,i-1}$ for $2 \leqslant i \leqslant D$ and $e^{*+}_i := e^*_{i\,i+1}$ for $1 \leqslant i \leqslant D-1$. We also set $e^{*-}_1 := 0$ and $e^{*+}_D := 0$.

Definition 14. A *dual uniform structure* of Γ is a pair (U, f), where U is a parameter matrix and $f = \{f^*_i\}_{i=1}^D$ is a vector in $\mathbb{C}^D$, such that

$$e^{*-}_i R^* L^{*2} + L^* R^* L^* + e^{*+}_i L^{*2} R^* = f^*_i L^* \tag{8}$$

is satisfied on E_iV $(1 \leqslant i \leqslant D)$, where E_i are the idempotents of Γ. In addition, a *dual strongly uniform structure* of Γ is a dual uniform structure (U, f) for which $e^{*-}_i \neq 0$ for $2 \leqslant i \leqslant D$ and $e^{*+}_i \neq 0$ for $1 \leqslant i \leqslant D-1$.

Consider our Q-polynomial structure from Notation 11. Our next goal is to determine in which cases the corresponding R^*/L^* dependency structure is dual uniform or dual strongly uniform. According to Lemma 8, there are five distinct graphs to deal with. We start from the case where Γ is different from $H(D,2)$, $\frac{1}{2}H(2D,2)$, $J(D,2D)$, and the Gosset graph. Recall that in this case $\beta = q + q^{-1} \neq \pm 2$ (see Remark 10).

7 Case $\beta = q + q^{-1}$

In this section, assume that the graph Γ from Notation 11 is different from $H(D,2)$, $\frac{1}{2}H(2D,2)$, $J(D,2D)$, and the Gosset graph. We determine when the corresponding R^*/L^* dependency structure is dual uniform or dual strongly uniform.

Lemma 15. *With reference to Notation 11, assume that Γ satisfies the conditions of Lemma 8. Then the corresponding R^*/L^* dependency structure is given by*

$$-\frac{q^2(1-q^{2i-3}s)}{(q+1)(1-q^{2i}s)} R^* L^{*2} + L^* R^* L^* - \frac{1-q^{2i+3}s}{q(q+1)(1-q^{2i}s)} L^{*2} R^* = \frac{q^{D-1}(1-q^3 s)^2}{(1-q^{D+2}s)^2} L^* \tag{9}$$

on E_iV for $1 \leqslant i \leqslant D$. Here, the parameters q, s are as in Lemma 8.

Proof. According to the notation of [1, Theorem 5.1], Γ is of type (I) with $r_2 = -r_1, r_3 = q^{-1-D}, s = r_1^2$, and $s^* = -q^{-1-D}$. In this case, we find the equations in Lemma 8 as well as

$$\theta_i = \theta_0 + hq^{-i}(1-q^i)(1-sq^{i+1}) \qquad (0 \leqslant i \leqslant D).$$

Then, combining this with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), i.e.,

$$\begin{aligned}&-\frac{q(q+1)(1-q^{2i-3}s)}{1-q^{2i}s}R^*L^{*2} + (q+q^{-1}+2)L^*R^*L^*\\ &-\frac{(q+1)(1-q^{2i+3}s)}{q^2(1-q^{2i}s)}L^{*2}R^* = \frac{q^{D-2}(q+1)^2(1-q^3s)^2}{(1-q^{D+2}s)^2}L^*,\end{aligned}$$

which holds on E_iV for $1 \leqslant i \leqslant D$. The result is obtained once we divide the previous equation by $\beta + 2 = (q+q^{-1}+2) \neq 0$. $\square$

Proposition 16. *With reference to Notation 11, asssume that Γ is not $H(D,2)$, $\frac{1}{2}H(2D,2)$, $J(D,2D)$, or the Gosset graph. The corresponding R^*/L^* dependency structure is dual uniform if and only if Γ is not the ordinary $2D$-gon. Furthermore, it is dual strongly uniform if and only if Γ is bipartite or $s \notin \{q^{-1}, q^{-2D-1}\}$.*

Proof. We need to check whether the coefficients

$$e_i^{*-} := -\frac{q^2(1-q^{2i-3}s)}{(q+1)(1-q^{2i}s)} \ \ (2 \leqslant i \leqslant D), \qquad e_i^{*+} := -\frac{1-q^{2i+3}s}{q(q+1)(1-q^{2i}s)} \ \ (1 \leqslant i \leqslant D-1)$$

of Equation (9) satisfy the conditions (ii), (iii) from Definition 13. Here, we set $f_i^* := \frac{q^{D-1}(1-q^3s)^2}{(1-q^{D+2}s)^2}$ $(1 \leqslant i \leqslant D)$. By Lemma 8, $e_i^{*-} \neq 0$ for $3 \leqslant i \leqslant D$, and $e_i^{*+} \neq 0$ for $1 \leqslant i \leqslant D-2$. For $1 \leqslant p \leqslant t \leqslant D$, let $U_{p,t} = (e_{ij}^*)_{p \leqslant i,j \leqslant t}$ denote the principal submatrix of U as in Definition 13. It is easy to see that if $p = t$ then $\det(U_{p,t}) = 1$. Similarly, if $t = p+1$, then

$$\det(U_{p,t}) = 1 - e_p^{*+}e_{p+1}^{*-} = \frac{(q^2+q+1)(1-q^{2p+1}s)^2}{(q+1)^2(1-q^{2p}s)(1-q^{2(p+1)}s)}.$$

If $t \geqslant p+2$, expanding $\det(U_{p,t})$ by the first row and then by the first column, we obtain that

$$\det(U_{p,t}) = \det(U_{p+1,t}) - e_p^{*+}e_{p+1}^{*-}\det(U_{p+2,t}).$$

A simple induction argument shows that

$$\det(U_{p,t}) = \frac{(q^{t-p+2}-1)(1-q^{p+t}s)\displaystyle\prod_{i=0}^{t-p-1}(1-q^{2p+2i+1}s)^2}{(q-1)(q+1)^{t-p+1}\displaystyle\prod_{i=0}^{2(t-p)}(1-q^{2p+i}s)}. \tag{10}$$

It follows from Lemma 8 that all factors in the numerator (and denominator) of (10) are nonzero, implying that $U_{p,t}$ is nonsingular. Here, we have to distinguish some different cases. If $s = q^{-1}$ and $q^{2D} = 1$, then $e_2^{*-} = e_{D-1}^{*+} = 0$, and the corresponding R^*/L^* dependency structure is not dual uniform. This is exactly the case when Γ is the ordinary $2D$-gon from Theorem 7(i).

If $s \in \{q^{-1}, q^{-2D-1}\}$, with $q^{2D} \neq 1$, the conditions (ii), (iii) from Definition 13 are both attained; so Γ has a dual uniform R^*/L^* dependency structure, which is not dual strongly uniform since $e_2^{*-} = 0$ or $e_{D-1}^{*+} = 0$. In this case, Γ cannot be bipartite (otherwise, $s = -q^{-D-1}$ [6, Lemma 3.1]), and it is of type (iii) or (iv) from Theorem 7, depending on whether $D = 3$ or $D = 4$, respectively.

If $s \notin \{q^{-1}, q^{-2D-1}\}$, the conditions (ii), (iii) from Definition 13 are satisfied with $e_i^{*-} \neq 0$ $(2 \leqslant i \leqslant D)$, $e_i^{*+} \neq 0$ $(1 \leqslant i \leqslant D-1)$, and hence Γ has a dual strongly uniform R^*/L^* dependency structure. □

8 Case $\beta = 2$

In this section, we assume that our graph Γ from Notation 11 has *classical parameters* (see [3, Sections 6.1, 8.4, 8.5, Chapter 9] for a detailed description). For such a graph, there is a dual bipartite Q-polynomial structure with dual eigenvalues $\theta_i^* = D - i$ $(0 \leqslant i \leqslant D)$ [3, Corollary 8.4.2][2]; and so, as computed in the proof of Proposition 9, we have $\beta = 2$. Our analysis will be then split into four subsections, one for each of the possible cases among $H(D,2)$, $\frac{1}{2}H(2D,2)$, $J(D,2D)$, and the Gosset graph; we will show that the corresponding R^*/L^* dependency structure is always dual strongly uniform, except for the case of the halved cube, whose R^*/L^* dependency structure is dual uniform but not dual strongly uniform.

8.1 The cube $H(D,2)$

With reference to [1, Theorem 5.1], let Γ be the cube $H(D,2)$ of type (IIC), where $r = 2$, $r_1 = -D-1$, and $s = s^* = -2$. Then

$$\theta_i - \theta_0 = \theta_i^* - \theta_0^* = -2i \quad (0 \leqslant i \leqslant D). \tag{11}$$

Lemma 17. *With reference to Notation 11, if Γ is the cube $H(D,2)$ [1, Theorem 5.1(IIC)], then the corresponding R^*/L^* dependency structure is given by*

$$-\frac{1}{2}R^*L^{*2} + L^*R^*L^* - \frac{1}{2}L^{*2}R^* = L^* \tag{12}$$

on E_iV for $1 \leqslant i \leqslant D$.

Proof. Using (11) together with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), i.e.,

$$-2R^*L^{*2} + 4L^*R^*L^* - 2L^{*2}R^* = 4L^*$$

[2]The Q-sequence $\theta_i^* = \theta_0^* - s^*i$ $(0 \leqslant i \leqslant D)$ from [1, Theorem 5.1(IIC),(IIA)] is here normalized in such a way that $\theta_{D-1}^* = 1, \theta_D^* = 0$.

holds on E_iV for $1 \leqslant i \leqslant D$. The result is obtained once we divide the previous equation by $\beta + 2 = 4$. $\square$

Proposition 18. *With reference to Notation 11, asssume that* Γ *is the cube* $H(D, 2)$ *[1, Theorem 5.1(IIC)]. Then the corresponding* R^*/L^* *dependency structure is dual strongly uniform.*

Proof. We need to check whether the coefficients

$$e_i^{*-} := -\frac{1}{2} \quad (2 \leqslant i \leqslant D), \qquad e_i^{*+} := -\frac{1}{2} \quad (1 \leqslant i \leqslant D-1)$$

of Equation (12) satisfy the conditions (ii), (iii) from Definition 13. Here, we set $f_i^* := 1$ $(1 \leqslant i \leqslant D)$. It is obvious that $e_i^{*-} \neq 0$ for $2 \leqslant i \leqslant D$, and $e_i^{*+} \neq 0$ for $1 \leqslant i \leqslant D-1$. For $1 \leqslant p \leqslant t \leqslant D$, let $U_{p,t} = (e_{ij}^*)_{p \leqslant i,j \leqslant t}$ denote the principal submatrix of U as in Defintion 13. A simple linear recurrence argument shows that

$$\det(U_{p,t}) = \frac{t-p+2}{2^{t-p+1}},$$

implying that $U_{p,t}$ is nonsingular. Since the conditions (ii),(iii) from Definition 13 are satisfied with $e_i^{*-} \neq 0$ $(2 \leqslant i \leqslant D)$, $e_i^{*+} \neq 0$ $(1 \leqslant i \leqslant D-1)$, Γ supports a dual strongly uniform R^*/L^* dependency structure. $\square$

Remark 19. It is known the cube $H(D, 2)$ [1, Theorem 5.1(IIC),(III)] gives rise to a P- and Q-polynomial association scheme such that $P = Q$, i.e., a self-dual scheme [1, p. 310]. Thus, the corresponding R/L and R^*/L^* dependency structures have the same coefficients; also, the first is uniform if and only if the second is dual uniform. In [10] it was shown that the cube $H(D, 2)$ of type (IIC) has a strongly uniform R/L dependency structure. In light of the arguments above, this means that the corresponding R^*/L^* dependency structure is dual strongly uniform, and so Proposition 18 is proved.

8.2 The halved cube $\frac{1}{2}H(2D, 2)$

With reference to [1, Theorem 5.1], let Γ be the halved cube $\frac{1}{2}H(2D, 2)$, which is of type (IIA) with $r_1 = -D-1$, $r_2 = -D-\frac{1}{2}$, $s = -2D-1$, and $s^* = -4$. Then

$$\theta_i = \theta_0 + hi(i - 2D), \quad \theta_i^* = \theta_0^* - 4i \quad (0 \leqslant i \leqslant D). \tag{13}$$

Lemma 20. *With reference to Notation 11, if* Γ *is the halved cube* $\frac{1}{2}H(2D, 2)$*, then the corresponding* R^*/L^* *dependency structure is given by*

$$-\frac{D-i+2}{2D-2i+1} R^* L^{*2} + L^* R^* L^* - \frac{D-i-1}{2D-2i+1} L^{*2} R^* = 4L^* \tag{14}$$

on E_iV *for* $1 \leqslant i \leqslant D$.

Proof. Using (13) together with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), i.e.,

$$-\frac{4D-4i+8}{2D-2i+1}R^*L^{*2}+4L^*R^*L^*-\frac{4D-4i-4}{2D-2i+1}L^{*2}R^*=16L^*$$

holds on E_iV for $1\leqslant i\leqslant D$. The result is obtained once we divide the previous equation by $\beta+2=4$. □

Proposition 21. *With reference to Notation 11, assume that* Γ *is the halved cube* $\frac{1}{2}H(2D,2)$. *Then the corresponding* R^*/L^* *dependency structure is dual uniform, but not dual strongly uniform.*

Proof. We need to check whether the coefficients

$$e_i^{*-}:=-\frac{D-i+2}{2D-2i+1}\ \ (2\leqslant i\leqslant D),\qquad e_i^{*+}:=-\frac{D-i-1}{2D-2i+1}\ \ (1\leqslant i\leqslant D-1)$$

of Equation (14) satisfy the conditions (ii), (iii) from Definition 13. Here, we set $f_i^*:=4$ $(1\leqslant i\leqslant D)$. It is easy to see that $e_i^{*-}\neq 0$ for $2\leqslant i\leqslant D$, and $e_i^{*+}\neq 0$ for $1\leqslant i\leqslant D-2$. For $1\leqslant p\leqslant t\leqslant D$, let $U_{p,t}=(e^*_{ij})_{p\leqslant i,j\leqslant t}$ denote the principal submatrix of U as in Defintion 13. Note that if $p=t$ then $\det(U_{p,t})=1$. Similarly, if $t=p+1$, then

$$\det(U_{p,t})=1-e_p^{*+}e_{p+1}^{*-}=\frac{3(D-p)^2}{(2D-2p-1)(2D-2p+1)}.$$

If $t\geqslant p+2$, expanding $\det(U_{p,t})$ by the first row and then by the first column, we obtain that

$$\det(U_{p,t})=\det(U_{p+1,t})-e_p^{*+}e_{p+1}^{*-}\det(U_{p+2,t}).$$

A simple induction argument shows that

$$\det(U_{p,t})=\frac{2^{t-p-1}(2D-t+1)(t-p+2)\displaystyle\prod_{i=0}^{t-p-1}(D-t+i+1)^2}{\displaystyle\prod_{i=0}^{2(t-p)}(2D-2t+i+1)}. \tag{15}$$

Observe that all factors in the numerator (and denominator) of (15) are nonzero, implying that $U_{p,t}$ is nonsingular. Since both the conditions (ii) and (iii) from Definition 13 are satisfied, Γ supports a dual uniform R^*/L^* dependency structure, which is not dual strongly uniform as $e_{D-1}^{*+}=0$. □

8.3 The Johnson graph $J(D,2D)$

With reference to [1, Theorem 5.1], let Γ be the Johnson graph $J(D,2D)$, which is of type (IIA) with $r_1=-D-1$, $r_2=-D+1$, $s=-2(D+1)$, and $s^*=-\frac{2}{D}(2D-1)$. Then

$$\theta_i=\theta_0+hi(i-2D-1),\quad \theta_i^*=\theta_0^*-\frac{2i(2D-1)}{D}\quad (0\leqslant i\leqslant D). \tag{16}$$

Lemma 22. *With reference to Notation 11, if* Γ *is the Johnson graph* $J(D,2D)$*, then the corresponding* R^*/L^* *dependency structure is given by*

$$-\frac{2D-2i+5}{4(D-i+1)}R^*L^{*2}+L^*R^*L^*-\frac{2D-2i-1}{4(D-i+1)}L^{*2}R^*=\frac{(2D-1)^2}{D^2}L^* \tag{17}$$

on E_iV *for* $1 \leqslant i \leqslant D$.

Proof. Using (16) together with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), i.e.,

$$-\frac{2D-2i+5}{D-i+1}R^*L^{*2}+4L^*R^*L^*-\frac{2D-2i-1}{D-i+1}L^{*2}R^*=\frac{4(2D-1)^2}{D^2}L^*$$

holds on E_iV for $1 \leqslant i \leqslant D$. The result is obtained once we divide the previous equation by $\beta+2=4$. □

Proposition 23. *With reference to Notation 11, asssume that* Γ *is the Johnson graph* $J(D,2D)$*. Then the corresponding* R^*/L^* *dependency structure is dual strongly uniform.*

Proof. We need to check whether the coefficients

$$e_i^{*-} := -\frac{2D-2i+5}{4(D-i+1)} \;\; (2 \leqslant i \leqslant D), \qquad e_i^{*+} := -\frac{2D-2i-1}{4(D-i+1)} \;\; (1 \leqslant i \leqslant D-1)$$

of Equation (17) satisfy the conditions (ii), (iii) from Definition 13. Here, we set $f_i^* := \frac{(2D-1)^2}{D^2}$ $(1 \leqslant i \leqslant D)$. It is easy to see that $e_i^{*-} \neq 0$ for $2 \leqslant i \leqslant D$, and $e_i^{*+} \neq 0$ for $1 \leqslant i \leqslant D-1$. For $1 \leqslant p \leqslant t \leqslant D$, let $U_{p,t} = (e_{ij}^*)_{p \leqslant i,j \leqslant t}$ denote the principal submatrix of U as in Defintion 13. Note that if $p=t$ then $\det(U_{p,t})=1$. Similarly, if $t=p+1$, then

$$\det(U_{p,t}) = 1 - e_p^{*+}e_{p+1}^{*-} = \frac{3(2D-2p+1)^2}{16(D-p)(D-p+1)}.$$

If $t \geqslant p+2$, expanding $\det(U_{p,t})$ by the first row and then by the first column, we obtain that

$$\det(U_{p,t}) = \det(U_{p+1,t}) - e_p^{*+}e_{p+1}^{*-}\det(U_{p+2,t}).$$

A simple induction argument shows that

$$\det(U_{p,t}) = \frac{(2D-t-p+2)(t-p+2)\displaystyle\prod_{i=0}^{2(t-p)}(2D-2t+i+2)}{2^{3(t-p+1)}\displaystyle\prod_{i=0}^{t-p}(D-t+i+1)^2}. \tag{18}$$

Observe that all factors in the numerator (and denominator) of (18) are nonzero, implying that $U_{p,t}$ is nonsingular. Since both the conditions (ii) and (iii) from Definition 13 are satisfied with $e_i^{*-} \neq 0$ $(2 \leqslant i \leqslant D)$, $e_i^{*+} \neq 0$ $(1 \leqslant i \leqslant D-1)$, Γ supports a dual strongly uniform R^*/L^* dependency structure. □

8.4 The Gosset graph

With reference to [1, Theorem 5.1], let Γ be the Gosset graph, which is of type (IIA) with $r_1 = -4$, $r_2 = -\frac{13}{4}$, $s = -\frac{13}{2}$, and $s^* = -\frac{14}{3}$. Then

$$\theta_i = (9-4i)(3-i) - i, \quad \theta_i^* = 7 - \frac{14}{3}i \quad (0 \leqslant i \leqslant 3). \tag{19}$$

Lemma 24. *With reference to Notation 11, if Γ is the Gosset graph, then the corresponding R^*/L^* dependency structure is given by*

$$-\frac{4i-19}{2(4i-13)}R^*L^{*2} + L^*R^*L^* - \frac{4i-7}{2(4i-13)}L^{*2}R^* = \frac{49}{9}L^* \tag{20}$$

on E_iV for $1 \leqslant i \leqslant D$.

Proof. Using (19) together with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), i.e.,

$$-\frac{8i-38}{4i-13}R^*L^{*2} + 4L^*R^*L^* - \frac{8i-14}{4i-13}L^{*2}R^* = \frac{196}{9}L^*$$

holds on E_iV for $1 \leqslant i \leqslant D$. The result is obtained once we divide the previous equation by $\beta + 2 = 4$. □

Proposition 25. *With reference to Notation 11, assume that Γ is the Gosset graph. Then the corresponding R^*/L^* dependency structure is dual strongly uniform.*

Proof. We need to check whether the coefficients

$$e_i^{*-} := -\frac{4i-19}{2(4i-13)} \quad (i = 2, 3), \qquad e_i^{*+} := -\frac{4i-7}{2(4i-13)} \quad (i = 1, 2)$$

of Equation (20) satisfy the conditions (ii), (iii) from Definition 13. Here, we set $f_i^* := \frac{49}{9}$ $(1 \leqslant i \leqslant 3)$. It is clear that $e_i^{*-} \neq 0$ for $i = 2, 3$, and $e_i^{*+} \neq 0$ for $i = 1, 2$. For $1 \leqslant p \leqslant t \leqslant 3$, let $U_{p,t} = (e_{ij}^*)_{p \leqslant i,j \leqslant t}$ denote the principal submatrix of U as in Defintion 13. Note that $\det(U_{p,p}) = 1$, and

$$\det(U_{1,2}) = \frac{49}{60}, \qquad \det(U_{2,3}) = \frac{27}{20}, \qquad \det(U) = \det(U_{1,3}) = \frac{7}{6}, \tag{21}$$

which are all nonzero. Since both the conditions (ii) and (iii) from Definition 13 are satisfied with $e_i^{*-} \neq 0$ $(i = 2, 3)$, $e_i^{*+} \neq 0$ $(i = 1, 2)$, Γ supports a dual strongly uniform R^*/L^* dependency structure. □

9 Case $\beta = -2$

As mentioned in the proof of Proposition 9, the cube $H(D, 2)$ admits a further (dual bipartite) Q-polynomial structure with respect to the original one, which yields the restrictions $r_1 = -D - 1$, $r_2 = -\frac{D+1}{2}$, $r_3 = -r_2$, and $s = s^* = D + 1$ [1, p. 305 – type (III)]. Then

$$\theta_i = \theta_i^* = (-1)^i (D - 2i) \quad (0 \leqslant i \leqslant D), \tag{22}$$

and hence Lemma 1 gives $\beta = -2$. We will show that in this case the corresponding R^*/L^* dependency structure is not dual uniform.

Lemma 26. *With reference to Notation 11, if Γ is the cube $H(D, 2)$ [1, Theorem 5.1(III)], then the corresponding R^*/L^* dependency structure is given by*

$$-\frac{2}{D - 2i + 1} R^* L^{*2} + \frac{2}{D - 2i + 1} L^{*2} R^* = 4L^* \tag{23}$$

on $E_i V$ for $1 \leqslant i \leqslant D$.

Proof. Using (22) together with Lemma 1, we can easily compute the corresponding R^*/L^* dependency structure from Equation (7), so obtaining (23). ☐

Proposition 27. *With reference to Notation 11, assume that Γ is the cube $H(D, 2)$ [1, Theorem 5.1(III)]. Then the corresponding R^*/L^* dependency structure is not dual uniform.*

Proof. Equation (23) does not match the form of (8). ☐

Remark 28. In [10] it was shown that the cube $H(D, 2)$ of type (III) has a R/L dependency structure which is not uniform. In light of the same arguments as in Remark 19, this means that the corresponding R^*/L^* dependency structure is not dual uniform, and so Proposition 27 is proved.

10 The main result

In this section, we simply collect together all the results obtained in the previous sections, thus providing our main theorem.

Theorem 29. *Let Γ be a distance-regular graph with vertex set X and diameter $D \geqslant 3$. Assume Γ has a dual bipartite Q-polynomial structure. Fix $x \in X$, and let $L^* = L^*(x)$, $R^* = R^*(x)$ respectively denote the corresponding dual lowering matrix and dual raising matrix as in (2). Then the corresponding R^*/L^* dependency structure is dual strongly uniform with the following exceptions:*

(i) Γ *is the ordinary $2D$-gon:*

(ii) Γ *is the cube $H(D, 2)$, D even, type [1, Theorem 5.1(III)];*

(iii) Γ *is the halved cube* $\frac{1}{2}H(2D, 2)$*;*

(iv) Γ *is as in Lemma 8 with* $s \in \{q^{-1}, q^{-2D-1}\}$, $q^{2D} \neq 1$.

Precisely, in Cases (i) and (ii), the corresponding R^*/L^* *dependency structure is not dual uniform; in Cases (iii) and (iv), this structure is dual uniform but not dual strongly uniform.*

Proof. Immediate from Propositions 16, 18, 21, 23, 25, and 27. □

Acknowledgments

Giusy Monzillo's research is partially supported by the Ministry of Education, Science and Sports of the Republic of Slovenia (University of Primorska Developmental funding pillar).

Conflicts of Interest

None declared.

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