Improved lower bounds for Queen's Domination via a relaxation

Archit Karandikar[a] Akashnil Dutta[b]

Submitted: Apr 13, 2023; Accepted: Aug 20, 2024; Published: Nov 15, 2024

Abstract

The Queen's Domination problem, studied for over 160 years, poses the following question: What is the least number of queens that can be arranged on a $m \times n$ chessboard so that they either attack or occupy every cell?

We propose a relaxation of the Queen's Domination problem and solve it exactly for rectangular chessboards. As a consequence, we improve on the best known lower bound for rectangular chessboards for one-eighth of the nontrivial cases. As another consequence, we generalize and provide a new interpretation of the best known lower bounds for Queen's Domination of square $n \times n$ chessboards for $n \not\equiv 3 \bmod 4$.

Finally, we show some results and make some conjectures towards the goal of simplifying the long complicated proof for the best known lower bound for square boards when $n \equiv 3 \bmod 4$ (and $n > 11$). These simply stated conjectures may also be of independent interest.

Mathematics Subject Classifications: 05C69, 05B99

1 Introduction

The *Queen's graph* $Q_{m\times n}$ has the squares of an $m\times n$ chessboard as its vertex set $V(Q_{m\times n})$, with two vertices connected by an edge if they are in the same row, column or diagonal. A subset $\mathcal{D}$ of the vertex set is said to *dominate* $Q_{m\times n}$ if every vertex is either contained in $\mathcal{D}$ or is adjacent to an element in $\mathcal{D}$. The problem of finding the smallest dominating set of $Q_{m\times n}$ - the size of which is denoted by $\gamma(Q_{m\times n})$ - is the *Queen's Domination problem.*

We assume that the chessboard is placed on a Cartesian plane with edges parallel to the coordinate axes. The x-coordinate increases from left to right and the y-coordinate increases from bottom to top. The origin of the Cartesian plane is the center of the bottom-left square of the chessboard. The squares of the chessboard have unit length. We identify the mn squares of the $m \times n$ chessboard by their coordinates (x, y) where $x \in \{0, \dots, m-1\}$ denotes column number (increases from left to right) and $y \in \{0, \dots, n-1\}$

[a] CEO of CoInvent AI, Inc. (archit@coinvent.ai).
[b] LinkedIn Inc. (akashnil07@gmail.com).

 https://doi.org/10.37236/12003

denotes row number (increases from bottom to top). The center of square (x, y) is the point (x, y) on the Cartesian plane. Diagonals which go left as they go up are called *sum diagonals* since the value $x + y$ is fixed along them. Diagonals which go right as they go up are called *difference diagonals* and the value $x - y$ is fixed along them. Rows, columns, sum diagonals and difference diagonals are collectively referred to as *lines* and numerically identified by their fixed y, x, $x + y$ and $x - y$ quantities respectively.

The *Queen's Domination problem* is a special case of the set cover problem, which is itself a discrete constrained optimization problem. It can be stated as follows:

Definition 1.1 (Queen's Domination problem)**.**

$$\begin{aligned}\gamma(Q_{m\times n}) = \text{minimum of } p \\ &\text{such that } \exists \text{ rows } R_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ columns } C_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ sum diagonals } S_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ difference diagonals } D_i,\ 0 \leqslant i \leqslant p-1 \\ &\text{subject to } V(Q_{m\times n}) = (\bigcup_{i=0}^{p-1} R_i) \cup (\bigcup_{i=0}^{p-1} C_i) \cup (\bigcup_{i=0}^{p-1} S_i) \cup (\bigcup_{i=0}^{p-1} D_i) \\ &\qquad \text{and } R_i \cap C_i \cap S_i \cap D_i \neq \emptyset,\ 0 \leqslant i \leqslant p-1.\end{aligned}$$

We ignore the slight misuse of notation in these definitions which take the graph as argument for the γ function but define the problem in terms of the chessboard that underlies the graph. The row R_i, column C_i and diagonals S_i, D_i are the ones occupied by the i^{th} queen. Note that these rows, columns and diagonals can repeat. The version where they cannot repeat is called the *Independent Queen's Domination problem.*

Definition 1.2 (Independent Queen's Domination problem)**.**

$$\begin{aligned}i(Q_{m\times n}) = \text{minimum of } p \\ &\text{such that } \exists \text{ distinct rows } R_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ distinct columns } C_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ distinct sum diagonals } S_i,\ 0 \leqslant i \leqslant p-1 \\ &\qquad\qquad \exists \text{ distinct difference diagonals } D_i,\ 0 \leqslant i \leqslant p-1 \\ &\text{subject to } V(Q_{m\times n}) = (\bigcup_{i=0}^{p-1} R_i) \cup (\bigcup_{i=0}^{p-1} C_i) \cup (\bigcup_{i=0}^{p-1} S_i) \cup (\bigcup_{i=0}^{p-1} D_i) \\ &\qquad \text{and } R_i \cap C_i \cap S_i \cap D_i \neq \emptyset,\ 0 \leqslant i \leqslant p-1.\end{aligned}$$

The last constraint in both the above formulations, which requires that the i^{th} row, i^{th} column, i^{th} sum diagonal and i^{th} difference diagonal all intersect at a square on the chessboard, corresponds to queen placement and seems a little hard to tackle. We relax this constraint and refer to the resulting problems as the Relaxed Queen's Domination problem and the Relaxed Independent Queen's Domination problem. The *Relaxed Queen's Domination problem* - whose solution we denote by $\gamma^*(Q_{m\times n})$ - asks for the least number p such that one can choose p rows, p columns, p sum diagonals and p difference diagonals which together suffice to cover each square of a $m \times n$ chessboard. Note that some of these p chosen lines of each kind may be repeated. The *Relaxed Independent Queen's Domination problem* - whose solution we denote by $i^*(Q_{m\times n})$ - is identical apart from imposing the additional constraint that the p chosen lines of each kind must be distinct.

Definition 1.3 (Relaxed Queen's Domination problem)**.**

$$\begin{aligned}\gamma^*(Q_{m\times n}) = \text{minimum of } p & \\ \text{such that } &\exists \text{ rows } R_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ columns } C_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ sum diagonals } S_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ difference diagonals } D_i,\ 0 \leqslant i \leqslant p-1\\ \text{subject to } V(Q_{m\times n}) &= (\bigcup_{i=0}^{p-1} R_i) \cup (\bigcup_{i=0}^{p-1} C_i) \cup (\bigcup_{i=0}^{p-1} S_i) \cup (\bigcup_{i=0}^{p-1} D_i).\end{aligned}$$

Definition 1.4 (Relaxed Independent Queen's Domination problem)**.**

$$\begin{aligned}i^*(Q_{m\times n}) = \text{minimum of } p & \\ \text{such that } &\exists \text{ distinct rows } R_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ distinct columns } C_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ distinct sum diagonals } S_i,\ 0 \leqslant i \leqslant p-1\\ &\exists \text{ distinct difference diagonals } D_i,\ 0 \leqslant i \leqslant p-1\\ \text{subject to } V(Q_{m\times n}) &= (\bigcup_{i=0}^{p-1} R_i) \cup (\bigcup_{i=0}^{p-1} C_i) \cup (\bigcup_{i=0}^{p-1} S_i) \cup (\bigcup_{i=0}^{p-1} D_i). \qquad (1)\end{aligned}$$

The Relaxed Queen's Domination problem stated in Definition 1.3 is the crux of our work. The central result of our work is that this relaxation is both (a) loose enough to be solvable exactly using simple constructions and impossibility proofs and (b) tight enough to improve the best known lower bound on the Queen's Domination problem in a few cases and match it in most cases. We will solve this relaxation exactly for all rectangular boards and in the process we will improve on the best known lower bound for Queen's Domination of rectangular boards. As an intermediate result, we will also define and solve the Relaxed Bishop's Domination problem exactly. We will only approach the Queen's Domination problem indirectly via the Relaxed Queen's Domination problem. As a consequence of this approach, throughout this work, we will deal with choosing queen-occupied rows, columns and diagonals rather than placing queens.

2 Related Work

The Queen's Domination problem on square boards was included (Problem C18) in a collection of unsolved problems [11] in number theory in 1994. It was first considered by de Jaenisch [9] in 1862, when he gave minimum dominating sets of $Q_{n\times n}$ for $n \leqslant 8$. Ball [3] considered several new questions about Queen's Domination. Ahrens [1] and von Szily [15, 16] gave minimum dominating sets of $Q_{n\times n}$ for $9 \leqslant n \leqslant 13$ and $n = 17$. Many of these were shown to be minimum by recent works on lower bounds. These works also made efforts to list all minimum dominating sets for each n, producing lists modulo symmetries. Ahrens [2] summarized these results in his 1910 book.

The first nontrivial lower bound for square boards was obtained by Raghavan and Venkatesan in 1987 [14] when they showed that $\gamma(Q_{n\times n}) \geqslant (n-1)/2$. Another proof of this bound was given by Spencer in 1990 (as cited and stated in [5, 7, 18]). Weakley [18] improved the bound by showing that $\gamma(Q_{n\times n}) \geqslant (n+1)/2$ for the case $n \equiv 1 \bmod 4$. Watkins [17] collected these results in his treatise of mathematical chessboard problems. Östergård and Weakley [13] showed that these established bounds were all very good (either exact or off-by-1) for $n \leqslant 120$ by adapting an algorithm from Knuth [12] originally developed for the exact cover problem. Weakley and Finozhenok [10, 18, 19] improved the bound for the case $n \equiv 3 \bmod 4$ (when $n > 11$) to $(n+1)/2$ with a long proof spanning 3 papers and over 35 pages. For even n there has been no direct improvement over the very first $(n-1)/2$ lower bound, but Weakley [22] showed several results and made some conjectures towards this goal, especially for the case $n \equiv 0 \bmod 4$.

The progression of the best known lower bound for square boards over the years is shown in Table 1. For square boards of side n where $n \not\equiv 3 \bmod 4$, our results match the best known lower bounds. In fact, we generalize and provide a new interpretation of them by showing that these are in fact the solutions to the Relaxed Queen's Domination problem. The case $n \equiv 3 \bmod 4$ (when $n > 11$), which has the aforementioned long complex proof [10, 18, 19], is the only case where the solution to the Relaxed Queen's Domination problem does not match the best known lower bound for the Queen's Domination problem. In section 5 we make some conjectures towards the goal of simplifying the proof of the best known lower bound for this case. These may be of independent interest.

Case	Raghavan et al. [14], Spencer [7]	Weakley [18]	Weakley et al. [10, 18, 19]	Our Results
$n = 4k$	$\gamma(Q_n) \geqslant 2k$	$\gamma(Q_n) \geqslant 2k$	$\gamma(Q_n) \geqslant 2k$	$\gamma^*(Q_n) = 2k$
$n = 4k+1$	$\gamma(Q_n) \geqslant 2k$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma^*(Q_n) = 2k+1$
$n = 4k+2$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma^*(Q_n) = 2k+1$
$n = 4k+3$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma(Q_n) \geqslant 2k+1$	$\gamma(Q_n) \geqslant 2k+2$ for $n > 11$	$\gamma^*(Q_n) = 2k+1$
Year	1987	1995	2007	2024

Table 1: Progression of the lower bound for square boards over the years. We have used Q_n here as a shorthand for $Q_{n\times n}$. Note that $\gamma(Q_n) \geqslant \gamma^*(Q_n)$ as shown in Theorem 3.1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	-1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
4	-1	-1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
5	-1	-1	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
6	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
7	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
8	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1
9	-1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1
10	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
11	-1	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
12	-1	-1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
13	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
14	-1	-1	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
15	-1	-1	-1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
16	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
17	-1	-1	-1	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
18	-1	-1	-1	-1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
19	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
20	-1	-1	-1	-1	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
21	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
22	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
23	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0

	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
4	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
7	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
8	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
9	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
10	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
11	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
12	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
13	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
14	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
15	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
16	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1
17	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1
18	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
19	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
20	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
21	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
22	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
23	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

Figure 1: Summary of our improvements to the best known [4] lower bound on $\gamma(Q_{m\times n})$. Row and column indices denote m and n respectively. A value of -1 (red) indicates that $\gamma(Q_{m\times n})$ is trivially equal to $\min\{m,n\}$. A value of 1 (blue) indicates that we improve upon the $(m+n-2)/4$ lower bound by 1. We improve the lower bound for one-eighth of the cases. The improvement for square boards (green) was already established by Weakley [18]. A value of 0 (yellow) indicates that we match the $(m+n-2)/4$ lower bound.

The first lower bound for the general case of rectangular boards was obtained by Raghavan and Venkatesan in 1987 [14] when they showed that $\gamma(Q_{m\times n}) \geqslant \min\{m-1, n-1, (m+n-2)/4\}$. We refer to the boards where $\max\{m,n\} \geqslant 3\min\{m,n\}-2$ as trivial boards. Bozoki et al. [4] marginally improved the lower bound to $\min\{m,n\}$ on the trivial case and $(m+n-2)/4$ on the nontrivial case. We denote this best known lower bound by $\gamma_{\text{lb}}(Q_{m\times n})$. They showed several results for rectangular boards and posed the question of whether this bound can be improved which we answer in the affirmative.

In this work we improve (Theorem 4.5) the best known lower bound for the Queen's Domination problem on rectangular boards as summarized in Figure 1. This can be interpreted as a generalization of Weakley's improvement [18] to rectangular boards. Specifically we improve the $(m+n-2)/4$ lower bound on the nontrivial boards with m rows and n columns by 1 where (a) m,n are even and $m+n \equiv 6 \bmod 8$ and where (b) m,n are odd and $m+n \equiv 2 \bmod 8$. Cases (a) and (b) together account for one-eighth of all nontrivial boards since the blue/green cells in every row and every column in Figure 1 are separated by exactly 7 yellow cells. On some of these specific small values of (m,n), a better computational lower bound than the generally applicable $\gamma_{\text{lb}}(Q_{m\times n})$ lower bound is known. Figure 1 uses $\gamma_{\text{lb}}(Q_{m\times n})$ as the reference for its cell colouring. The dark outline highlights that exact values of $\gamma(Q_{m\times n})$ are known for $1 \leqslant m,n \leqslant 18$ [4]. We find an exact solution for $\gamma^*(Q_{m\times n})$ in Theorem 4.7 which implies that we also generalize and provide a new interpretation for the known lower bounds. The essential idea for the proof of Theorem 4.5 and more broadly of this work is captured in Lemma 4.3.

3 Preliminaries

In this section we state the definitions and results which are used several times in the following sections. We start with the *Fundamental Relaxation Inequality* for the Queen's Domination and Independent Queen's Domination problems.

Theorem 3.1 (Fundamental Relaxation Inequality)**.** *For all natural numbers m, n,*

$$\gamma^*(Q_{m\times n}) = i^*(Q_{m\times n}) \leqslant \gamma(Q_{m\times n}) \leqslant i(Q_{m\times n}).$$

Proof. Relaxing a constraint leads to minimization over a superset of the original solution space, and so

$$\begin{aligned} \gamma^*(Q_{m\times n}) &\leqslant \gamma(Q_{m\times n}) \\ i^*(Q_{m\times n}) &\leqslant i(Q_{m\times n}). \end{aligned}$$

The Queen's Domination problem is a relaxation of the Independent Queen's Domination problem, with the distinctness constraint of the chosen lines being relaxed. An analogous relationship holds between the relaxed versions of these two problems. Hence

$$\begin{aligned} \gamma(Q_{m\times n}) &\leqslant i(Q_{m\times n}) \\ \gamma^*(Q_{m\times n}) &\leqslant i^*(Q_{m\times n}). \end{aligned}$$

An optimal solution to the Relaxed Queen's Domination problem with $\gamma^*(Q_{m\times n})$ chosen lines of each kind can be transformed to a solution to the Relaxed Independent Queen's Domination problem by repeatedly replacing duplicate rows, columns and diagonals with unchosen ones. There are always unchosen lines available since $\gamma^*(Q_{m\times n}) \leqslant \min(m, n)$. We conclude that

$$\begin{aligned} i^*(Q_{m\times n}) &\leqslant \gamma^*(Q_{m\times n}) \\ \gamma^*(Q_{m\times n}) = i^*(Q_{m\times n}) &\leqslant \gamma(Q_{m\times n}) \leqslant i(Q_{m\times n}). \end{aligned} \qquad \square$$

We now define two constructs which are at the heart of all our lower bound proofs. For the purpose of Definition 3.2, think of our process of selecting p distinct lines of each type in two steps. In the first step we select p distinct rows and p distinct columns. In the second step we choose p distinct diagonals of each kind. The sets $\{C'_i\}_{i=0}^{m'-1}$ and $\{R'_i\}_{i=0}^{n'-1}$ are the $m' = m - p$ unchosen columns and $n' = n - p$ unchosen rows after the first step. The spaced grid is the set of cells not covered after the first step.

Definition 3.2 (Spaced grid)**.** Given a set of m' columns $\{C'_i\}_{i=0}^{m'-1}$ and a set of n' rows $\{R'_i\}_{i=0}^{n'-1}$, we define the *spaced grid* with these rows and columns as

$$G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1}) = (\bigcup_{i=0}^{m'-1} C'_i) \cap (\bigcup_{i=0}^{n'-1} R'_i).$$

Definition 3.3 (Cellular perimeter of a spaced grid)**.** Consider a spaced grid denoted by $G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1})$ which consists of m' columns $\{C'_i\}_{i=0}^{m'-1}$ and n' rows $\{R'_i\}_{i=0}^{n'-1}$. Assume without loss of generality that its columns are indexed from left to right and that the rows are indexed from bottom to top - both in increasing order of their fixed coordinate. We define its *cellular perimeter* as its set of outermost cells, or formally as

$$\begin{aligned}\mathrm{CP}\Big[G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1})\Big] &= (C'_0 \cup C'_{m-1} \cup R'_0 \cup R'_{n-1}) \cap G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1}) \\ &= \Big((C'_0 \cup C'_{m-1}) \cap (\bigcup_{i=0}^{n'-1} R'_i)\Big) \cup \Big((R'_0 \cup R'_{n-1}) \cap (\bigcup_{i=0}^{m'-1} C'_i)\Big).\end{aligned}$$

The last definition in this section is the *Relaxed Bishop's Domination problem*, which we will also solve exactly for square and rectangular boards in Section 4. Note that the graph $B_{m\times n}$ has the squares of an $m \times n$ chessboard as its vertex set $V(B_{m\times n})$, with two vertices connected by an edge if they are in the same diagonal.

Definition 3.4 (Relaxed Bishop's Domination Problem)**.** .

$$\begin{aligned} b^*(B_{m\times n}) = \text{minimum of } p & \\ \text{such that } \exists \text{ sum diagonals } S_i, 0 \leqslant i \leqslant p-1 & \\ \exists \text{ difference diagonals } D_i, 0 \leqslant i \leqslant p-1 & \\ \text{subject to } V(B_{m\times n}) = (\bigcup_{i=0}^{p-1} S_i) \cup (\bigcup_{i=0}^{p-1} D_i). & \end{aligned} \tag{2}$$

Finally we show that each of these relaxations are monotonic in m and n. Note that monotonicity does not hold for the Queen's Domination problem since $\gamma(Q_{8\times 11}) = 6 > 5 = \gamma(Q_{9\times 11}) = \gamma(Q_{10\times 11}) = \gamma(Q_{11\times 11})$ as shown in [4].

Lemma 3.5 (Monotonicity of Relaxation)**.** *If $m' \leqslant m$, $n' \leqslant n$ then*

$$\begin{aligned}\gamma^*(Q_{m'\times n'}) &\leqslant \gamma^*(Q_{m\times n}) \\ i^*(Q_{m'\times n'}) &\leqslant i^*(Q_{m\times n}) \\ b^*(B_{m'\times n'}) &\leqslant b^*(B_{m\times n}).\end{aligned}$$

Proof. We will prove only the first statement. The others can be proven identically. Consider a covering of a $m \times n$ board using $\gamma^*(Q_{m\times n})$ chosen lines of each of the four types. Consider the bottom-left $m' \times n'$ sub-board of the larger board. The chosen lines which intersect this sub-board along with arbitrary replacements for those which do not intersect it form a solution to $\gamma^*(Q_{m'\times n'})$ implying that $\gamma^*(Q_{m'\times n'}) \leqslant \gamma^*(Q_{m\times n})$. ☐

In Section 4, we will use the Fundamental Relaxation Inequality to keep our proofs for lower bounds on the Relaxed Queen's Domination problem simple by showing the same lower bound instead on the more constrained Relaxed Independent Queen's Domination problem. The latter is easier to tackle because of the distinctness constraint of the chosen lines. We will use the cellular perimeter and spaced grid constructs often in these proofs.

4 Relaxed Queen's Domination on Rectangular Boards

In this section we will completely solve the Relaxed Queen's Domination problem on all rectangular boards (Theorem 4.7) and hence improve the lower bound for the Queen's Domination problem on one-eighth of the nontrivial rectangular boards as shown in Figure 1. We generalize and provide a new interpretation of the currently known lower bound [4] for Queen's Domination of rectangular boards in Theorem 4.2. Theorem 4.5 generalizes Weakley's improvement [18] for the lower bound on $\gamma(Q_{(4k+1)\times(4k+1)})$ to rectangular boards. We also completely solve the Relaxed Bishop's Domination problem on rectangular boards in Lemma 4.6.

We start by proving a Lemma which we will use as a building block multiple times in the rest of the paper. This is the idea that we use to match the lower bound for square and rectangular boards obtained by Raghavan and Venkatesan [14] and Spencer [7]. Lemma 4.3 which improves on this for a special case is the essential idea underlying the improvement in the lower bound for the Queen's Domination problem that we establish in this work.

Lemma 4.1. *Given natural numbers m', n', a set of m' distinct columns $\{C'_i\}_{i=0}^{m'-1}$ and a set of n' distinct rows $\{R'_i\}_{i=0}^{n'-1}$, let p be a natural number such that p distinct sum diagonals and p distinct difference diagonals suffice to cover the spaced grid $G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1})$. Then*

$$p \geqslant (m' + n' - 2)/2.$$

Proof. The result is trivial if $\min\{m', n'\} = 1$. Otherwise, we consider the cellular perimeter K of the spaced grid G. Note that $|K| = 2m' + 2n' - 4$ and each of the chosen $2p$ diagonals can cover at most 2 squares in K. Since all squares in the cellular perimeter are covered, we infer that

$$\begin{aligned} 2 \times 2p &\geqslant 2m' + 2n' - 4 && (3) \\ p &\geqslant (m' + n' - 2)/2. && \square \end{aligned}$$

Theorem 4.2. *For all natural numbers m, n,*

$$\gamma(Q_{m\times n}) \geqslant \gamma^*(Q_{m\times n}) \geqslant \begin{cases} \min\{m, n\} & \textit{if } \max\{m, n\} \geqslant 3\min\{m, n\} - 2 \\ (m + n - 2)/4 & \textit{otherwise.} \end{cases}$$

Proof. We assume that $m \geqslant n$ without loss of generality. We will show the claimed bound on $i^*(Q_{m\times n})$, which equals $\gamma^*(Q_{m\times n})$ by Theorem 3.1. Let $i^*(Q_{m\times n}) = p$ so that there is a choice of p distinct rows, p distinct columns, p distinct sum diagonals and p distinct difference diagonals which satisfy (1) to cover $V(Q_{m\times n})$.

Trivial case: When $m \geqslant 3n - 2$, we need to show that $p \geqslant n$. Suppose that $p < n$. Consider an unchosen row of m squares. Each of the $3p$ chosen columns and diagonals can cover at most one square in it. The number of covered squares in this row is at most $3p \leqslant 3(n - 1) < 3n - 2 \leqslant m$ and so this row is not completely covered, a contradiction.

Nontrivial case: Note first that $m < 3n-2$ implies $(m+n-2)/4 < n-1$. If $p \geqslant n$ then $p > (m+n-2)/4$. If $p < n$, consider the spaced grid of cells that are uncovered by the chosen columns and rows. This grid has $m-p$ columns and $n-p$ rows and since it must be covered by the chosen diagonals, Lemma 4.1 implies that

$$\begin{aligned} p &\geqslant ((m-p)+(n-p)-2)/2 \\ p &\geqslant (m+n-2)/4. \end{aligned}$$

□

This proof of Theorem 4.2 (illustrated in Figure 2) generalizes the proofs by Raghavan and Venkatesan [14] and Bozoki et al. [4]. It has similarities with Theorem 1 in [22] which is the only other work we know that considers the cellular perimeter K. We will now prove Theorem 4.5 which improves the best known lower bound for the Queen's Domination problem on rectangular boards. The central idea for its proof and more broadly of this work is contained in the following Lemma 4.3 which improves on Lemma 4.1.

Lemma 4.3. *Given odd natural numbers m', n', a set of m' distinct columns $\{C'_i\}_{i=0}^{m'-1}$ and a set of n' distinct rows $\{R'_i\}_{i=0}^{n'-1}$, let p be a natural number such that p distinct sum diagonals and p distinct difference diagonals suffice to cover $G(\{C'_i\}_{i=0}^{m'-1}, \{R'_i\}_{i=0}^{n'-1})$, the spaced grid formed by these rows and columns. Then*

$$p \geqslant (m'+n'-2)/2+1.$$

Proof. The result is trivial if $\min\{m', n'\} = 1$. Otherwise we know that $p \geqslant (m'+n'-2)/2$ from Lemma 4.1 and it suffices to show that equality cannot hold in this case. Assume for the sake of contradiction that $p = (m'+n'-2)/2$, so that each of the $2p$ chosen diagonals covers exactly 2 squares in the cellular perimeter K of G. We assume that columns and rows are indexed in increasing order. Let $l = C'_0$, $r = C'_{m'-1}$ denote the leftmost, rightmost columns and $b = R'_0$, $t = R'_{n'-1}$ denote the bottommost, topmost rows.

Construct the rectangle $\mathcal{T}$ which has its vertices at the centers of the squares (b, l), (b, r), (t, r) and (t, l). We construct a purple segment corresponding to each chosen sum diagonal S'_i which joins the centers of squares in it and is confined to $\mathcal{T}$ (has both endpoints on $\mathcal{T}$). Similarly, we construct an analogously defined yellow segment corresponding to each chosen difference diagonal D'_i. Figure 2 illustrates these constructions.

Consider a vertical cross-section v (i.e. vertical line segment spanning both horizontal edges) of $\mathcal{T}$ which intersects $\mathcal{T}$ and does not pass through the center of any square. Let $q_V(v)$ denote the difference between the number of purple segments and the number of yellow segments it intersects. We observe that $q_V(v)$ is invariant as we slide v from left to right. The only places that v stops or starts intersecting some segment is when it crosses over the center of any column. In such transitions, there can be at most two segments that it stops or starts intersecting, as illustrated in the four cases shown at the bottom of Figure 2. In all cases, $q_V(v)$ does not change. The quantity $q_H(h)$ defined analogously is invariant as we slide from bottom to top a horizontal cross-section h which intersects $\mathcal{T}$.

We focus on the difference between the total length l_P of purple segments and the total length l_Y of yellow segments. Since $q_V(v)$ is invariant, every (δx)-width vertical

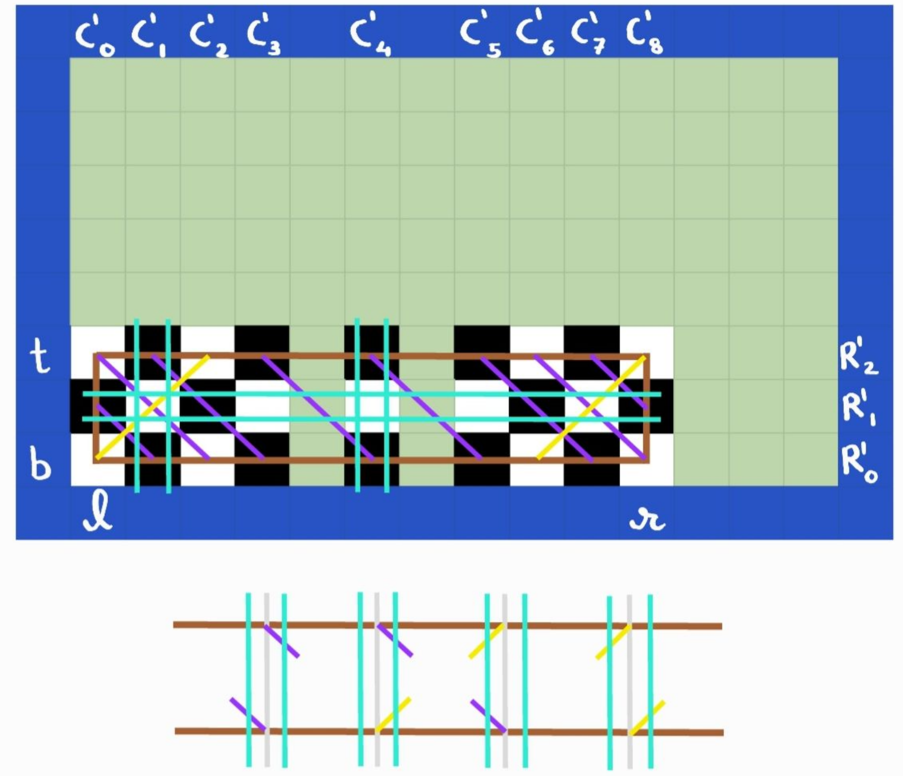

Figure 2: Illustration of the proofs of Lemmas 4.1, 4.3 (for $m' = 9, n' = 3$) and Theorems 4.2, 4.5 (for $m = 14, n = 8$). The black and white squares form the spaced grid G. The green squares are those covered by the chosen rows and columns. Rectangle $\mathcal{T}$ is shown in brown. Purple and yellow segments correspond to chosen sum and difference diagonals respectively. Vertical (v) and horizontal (h) cross-sections are cyan coloured. We see that $q_V(v) = 1$ and $q_H(h) = 5$ are invariant. As per (4) we have $q_V(v)(r-l) = q_H(h)(t-b) = 10$. The 4 cases for the transition of the vertical cross-section are shown at the bottom.

(analogously, horizontal) cross-section contributes $\sqrt{2}q_V(v)(\delta x)$ to this quantity and hence

$$l_P - l_Y = \sqrt{2}q_V(v)(r-l) = \sqrt{2}q_H(h)(t-b). \tag{4}$$

Another way to see this is that the sum of q_V over column boundaries $\sum_{i=l}^{r-1} q_V(i+1/2)$ equals $(l_P - l_Y)/(\sqrt{2})$. This is because every purple or yellow segment when partitioned into subsegments of length $\sqrt{2}$ intersects a column boundary at the midpoint of each subsegment. This expression also equals $q_V(v)(r-l)$ because of the invariance of $q_V(v)$.

Say there are k_P purple and k_Y yellow diagonals crossing the line $x = l + 1/2$. Then $q_V(v) = q_V(l + 1/2) = k_P - k_Y$ and the number of diagonals intersecting $x = l$ is $k_P + k_Y = n'$. Hence $q_V(v) = n' - 2k_Y$ is odd and similarly $q_H(h) = m' - 2k_Y$ is odd. Either both these must be positive or both must be negative because of (4). We will show that $Q = 2q_V(v) + 2q_H(h) = 0$ which contradicts the previous statement. This quantity Q equals $q_V(l) + q_V(r) + q_H(b) + q_H(t)$ which is the difference between the number of purple and yellow segments intersecting each side of $\mathcal{T}$, summed up over all 4 sides. Since each of the p sum diagonals (purple segments) intersects $\mathcal{T}$ exactly twice it contributes 2 to Q. Similarly each of the p difference diagonals (yellow segments) contributes -2 to Q. The only thing left is to account for the double counting at the corners in the quantity Q. Each corner covered by a purple or yellow segment contributes an additional 1 or -1 to Q respectively, but these additional corner contributions sum to zero since two corners each must be covered by purple and yellow segments. We infer that $Q = 2p - 2p = 0$, which as noted above is a contradiction. Hence $p \geqslant (m' + n' - 2)/2 + 1$ if m', n' are odd. ☐

Definition 4.4. We now introduce the notation $f(m,n)$. For all natural numbers m, n,

$$f(m,n) = \begin{cases} \min\{m,n\} & \text{if } \max\{m,n\} \geqslant 3\min\{m,n\} - 2 \\ (m+n-2)/4+1 & \text{else if } m,n \text{ are even and } m+n \equiv 6 \bmod 8 \\ (m+n-2)/4+1 & \text{else if } m,n \text{ are odd and } m+n \equiv 2 \bmod 8 \\ \lceil (m+n-2)/4 \rceil & \text{otherwise.} \end{cases}$$

Theorem 4.5. *For all natural numbers m, n,*

$$\gamma(Q_{m\times n}) \geqslant \gamma^*(Q_{m\times n}) \geqslant f(m,n).$$

Proof. We need to improve the lower bound from Theorem 4.2 by 1 (a) when m, n are even and $m + n \equiv 6 \bmod 8$ and (b) when m, n are odd and $m + n \equiv 2 \bmod 8$. Consider such a case and assume for the sake of contradiction that $\gamma^*(Q_{m\times n}) = i^*(Q_{m\times n}) = p = (m+n-2)/4$ so that there is a choice of p distinct rows, p distinct columns, p distinct sum diagonals and p distinct difference diagonals which satisfy (1) to cover $V(Q_{m\times n})$. This integer p is odd in case (a) and even in case (b). In both cases, the number of unchosen columns $m' = m - p$ and number of unchosen rows $n' = n - p$ are both odd.

Consider the $m' \times n'$ spaced grid G of squares not covered by the chosen rows and columns. The chosen p distinct sum diagonals and p distinct difference diagonals cover G. Since m', n' are odd, Lemma 4.3 implies that

$$\begin{aligned} p &\geqslant (m' + n' - 2)/2 + 1 \\ &= (m - p + n - p - 2)/2 + 1 \\ &= (4p - 2p)/2 + 1 \\ &= p + 1. \end{aligned}$$

This contradiction completes the proof. ☐

Lemma 4.6. *For all natural numbers m, n*

$$b^*(B_{m\times n}) = \begin{cases} (m+n-2)/2+1 & \textit{if } m,n \textit{ are odd} \\ \lceil (m+n-2)/2 \rceil & \textit{otherwise.} \end{cases}$$

Proof. The $V(B_{m\times n})$ board is a special case of a spaced grid with consecutive rows and columns. An argument analogous to Theorem 3.1 implies that we can assume that the

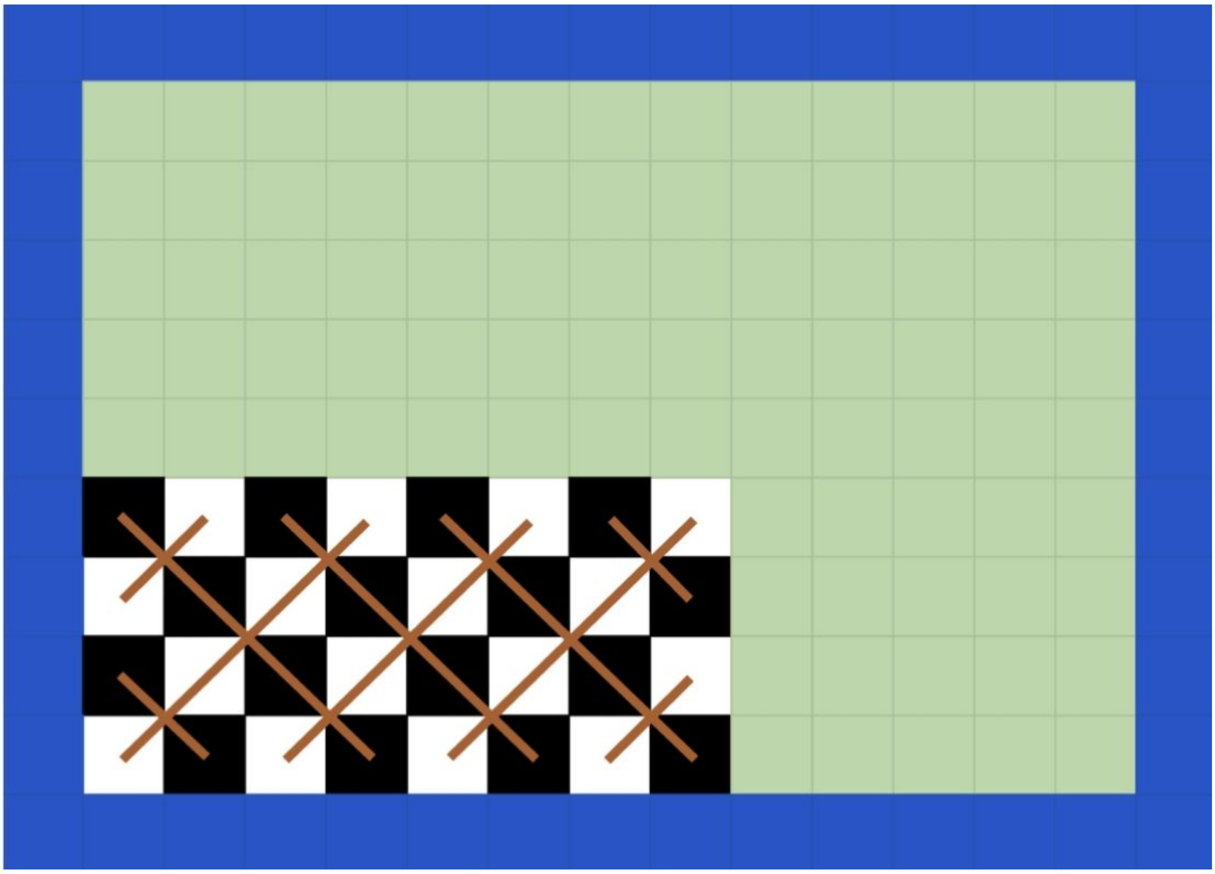

Figure 3: Illustration of the proof of Theorem 4.7 for $m_1 = 13, n_1 = 9$. Blue squares are not part of the board. The squares in the $f(m_1, n_1)$ chosen rows and $f(m_1, n_1)$ columns are coloured green. The $f(m_1, n_1)$ chosen sum diagonals and $f(m_1, n_1)$ chosen difference diagonals are shown in brown. The black-and-white section of the boards also illustrates the chosen diagonals in Lemma 4.6 for $m_2 = 8, n_2 = 4$.

chosen diagonals are all distinct without loss of generality. Under this assumption, Lemma 4.3 implies that $b^*(B_{m\times n}) \geqslant g(m,n)$, where $g(m,n) = (m+n-2)/2 + 1$ if m, n are odd and $\lceil (m+n-2)/2 \rceil$ otherwise.

We will now show that $b^*(B_{m\times n}) \leqslant g(m,n)$. Lemma 3.5 implies that it suffices to show this in the case where m, n are both even, since all other cases can be reduced to this case by increasing m or n or both without increasing $f(m,n)$. The set of possible sums $\hat{S}$ equals $\{x + y : (x,y) \in V(B_{m\times n})\} = \{0, \ldots m+n-2\}$ and the set of possible differences $\hat{D}$ equals $\{x - y : (x,y) \in V(B_{m\times n})\} = \{-(n-1), \ldots m-1\}$. We choose the set of sum diagonals with odd sums $\{s \in \hat{S} : s \text{ is odd}\}$ and the set of difference diagonals with even differences $\{d \in \hat{D} : d \text{ is even}\}$. Each of these is of size $(m+n-2)/2$ since m, n are both even. Together these cover $V(B_{m\times n})$ since for each (x,y), the parity of $x+y$ is the same as the parity of $x-y$. Hence $b^*(B_{m\times n}) \leqslant (m+n-2)/2 = g(m,n)$. □

Theorem 4.7. *For all natural numbers m, n*

$$\gamma^*(Q_{m\times n}) = f(m,n).$$

Proof. Theorem 4.5 states that $\gamma^*(Q_{m\times n}) \geqslant f(m,n)$, where $f(m,n)$ is the notation introduced in Definition 4.4. We will now show that $\gamma^*(Q_{m\times n}) \leqslant f(m,n)$ in all cases. We assume that $m \geqslant n$ without loss of generality. In the trivial case where $m \geqslant 3n-2$, simply choosing all n rows suffices to cover the board. Hereafter we consider the nontrivial case.

It suffices to show that $\gamma^*(Q_{m\times n}) \leqslant f(m,n)$ for the cases (a) m, n are even and $m+n \equiv 2 \bmod 8$ and (b) m, n are odd and $m+n \equiv 6 \bmod 8$. All other cases can be reduced using Lemma 3.5 to one of these two cases by increasing m or n or both without increasing $f(m,n)$. Let us now consider such a case. We know that $(m+n-2)/4 < n-1$ since this is equivalent to $m < 3n-2$. Let $p = f(m,n) = (m+n-2)/4$ and choose the rightmost p columns and the topmost p rows. This integer p is even in case (a) and odd in case (b). The spaced grid of squares not covered yet forms a $m' \times n'$ board $V(B_{m'\times n'})$ where $m' = m-p$ and $n' = n-p$ are both even. Lemma 4.6 implies that p' sum diagonals and p' difference diagonals can cover $V(B_{m'\times n'})$, where

$$\begin{aligned} p' &= (m'+n'-2)/2 \\ &= (m-p+n-p-2)/2 \\ &= (4p-2p)/2 = p \end{aligned}$$

which implies that $\gamma^*(Q_{m\times n}) \leqslant p = f(m,n)$. □

We have fully solved the Relaxed Queen's Domination problem on rectangular boards.

5 Queen's Domination of (4k+3)-Square Boards

In this section we will show some results and make some conjectures towards the goal of simplifying the long complex proof [10, 18, 19] that $\gamma(Q_{n\times n}) \geqslant (n+1)/2$ when $n \equiv 3 \bmod 4$ and $n > 11$. This is the only case for square boards in which Theorem 4.7 does not generalize the proof of the best known lower bound, as summarized in Table 1. For this, we will continue choosing rows, columns and diagonals rather than placing queens. Having solved the Relaxed Queen's Domination problem completely for square (and rectangular) boards, we now investigate the properties of its optimal solutions.

Assume for the remainder of this section that $n = 4k+3$ where k is a non-negative integer. Theorem 4.7 implies that $\gamma(Q_{n\times n}) \geqslant (n-1)/2$ and what remains is to improve the lower bound by 1 by showing that equality cannot occur. The aforementioned proof spanning 3 papers and over 35 pages did this by proving a sequence of results, stated below. In stating all these results, we assume further for the sake of contradiction that $\gamma(Q_{n\times n}) = (n-1)/2 = 2k+1$ and that $\mathcal{D}$ denotes the corresponding dominating set of queens. We also consider the sub-board U bounded by the leftmost and rightmost unoccupied columns and the bottommost and topmost unoccupied rows.

Result 5.1. *$U, \mathcal{D}$ have the following properties:*

(a) $\mathcal{D}$ is independent.

(b) U is a square sub-board of size $j \times j$ where j is odd and $3(n+1)/4 \leqslant j \leqslant n$.

(c) Each edge square of U is attacked exactly once.

(d) For each queen in $\mathcal{D}$, at least one of the row and the column through it intersects U. Both diagonals through it intersect the edge squares of U in two squares each.

(e) The sum diagonal joining the top-left and bottom-right corner of U contains exactly one queen in $\mathcal{D}$. Of the remaining $2k$ queens, half are above it and half below it. A similar result holds for the difference diagonal joining the two other corners of U.

We use Result 5.1 to define the center of U as the new origin for our coordinate system, similar to [19]. Results 5.2, 5.3 and Theorem 5.13 use this new origin, in contrast to previous sections which used the center of the bottom-left square as the origin. The column and row numbers still increase from left-to-right and bottom-to-top respectively. The square U is bounded by the rows and columns numbered $-(j-1)/2$ and $(j-1)/2$.

Result 5.2. *$U = V(Q_{n\times n})$. Each queen in $\mathcal{D}$ has both coordinates even.*

Result 5.3. *There exists some integer w, $0 \leqslant w \leqslant k$ such that the $2k+1$ occupied sum diagonal numbers and difference diagonal numbers are both*

$$0, \pm 2, \pm 4, \dots \pm 2w, \pm(2w+4), \pm(2w+8), \dots \pm (4k-2w).$$

Result 5.4. $n \in \{3, 11\}$.

Result 5.1 was proven in [18] and is equivalently stated as Theorem 2 in [19]. We will generalize most of it in the Relaxed Queen's Domination setting. Result 5.2 is the hardest part of the proof. It spans 19 pages [19] and uses 8 lemmas. We propose open questions and make conjectures towards simplifying it. Result 5.3 is the easiest of the four results. It was proven as Theorem 3 in [19]. We generalize it and provide an alternative visual proof. The proof of result 5.4 is based on the Pell's equation [10, 19]. This cannot be simplified by just choosing occupied rows, columns and diagonals since every solution from Result 5.3 is valid for the Relaxed Queen's Domination problem for all $n \equiv 3 \bmod 4$.

For the rest of this section, let $p = \gamma^*(Q_{n\times n}) = (n-1)/2 = 2k+1$ and let rows $(R_i)_{i=0}^{p-1}$, columns $(C_i)_{i=0}^{p-1}$, sum diagonals $(S_i)_{i=0}^{p-1}$ and difference diagonals $(D_i)_{i=0}^{p-1}$ constitute a solution to the Relaxed Queen's Domination Problem for an $n \times n$ board. The cellular perimeter of the spaced grid G of cells not covered by the chosen rows and columns is denoted by K. Parts (a) through (e) of the following Theorem 5.5 almost fully generalize parts (a) through (e) of Result 5.1 respectively.

Theorem 5.5. *Consider the sub-board U bounded by the leftmost (l) and rightmost (r) unchosen columns and the bottommost (b) and topmost (t) unchosen rows. Then*

(a) *The rows* $(R_i)_{i=0}^{p-1}$ *are all distinct. Similarly, the columns* $(C_i)_{i=0}^{p-1}$*, sum diagonals* $(S_i)_{i=0}^{p-1}$ *and difference diagonals* $(D_i)_{i=0}^{p-1}$ *are each also all distinct.*

(b) *U is a square sub-board.*

(c) *Each edge square of U is covered by exactly one of the $4p$ chosen lines.*

(d) *Each chosen diagonal intersects the edge squares of U in two points.*

(e) *The sum diagonal joining the top-left and bottom-right corner of U is chosen. Of the remaining $2k$ chosen sum diagonals, half are above it and half are below it. An analogous result holds for the difference diagonals.*

Proof. *(a)* The spaced grid G is defined by at least $2k+2$ columns and $2k+2$ rows which implies $|K| \geqslant 4(2k+2)-4 = 8k+4$. Each of the chosen $2k+1$ sum diagonals and $2k+1$ difference diagonals can cover at most two squares in K, and so the number of covered squares in K is at most $8k+4$. Since all squares in K must be covered, both of these inequalities must be equalities. This implies that all chosen lines must be distinct.

(c),(d) We also infer that each square in K must be covered exactly once since the number of such squares covered by the $4k+2$ diagonals is $8k+4$. If U_e denotes the edge squares (i.e. the cellular perimeter) of U, then squares in $U_e \setminus K$ by definition are covered exactly once by the chosen rows and columns. Every diagonal can cover at most 2 squares in U_e which are both accounted for by K, and hence no diagonal can intersect $U_e \setminus K$.

(b) Assume for the sake of contradiction that $r-l \neq t-b$. Without loss of generality, let $r-l < t-b$. We colour the top and bottom rows of K (besides corners) yellow and the left and right columns of K (besides corners) purple. Formally, squares $K_Y = \{(x,y) \in K : y \in \{b,t\}, l < x < r\}$ are coloured yellow and squares $K_P = \{(x,y) \in K : x \in \{l,r\}, b < y < t\}$ are coloured purple, as depicted in Figure 4. Note that $|K_Y| = |K_P| = 4k$. The four diagonals which cover the four corners must each also cover an additional purple square since $r-l < t-b$. The remaining $4k-2$ diagonals must then cover four more yellow squares ($4k$) than purple squares ($4k-4$). Hence at least one diagonal must cover two yellow squares, which contradicts $r-l < t-b$.

(e) The squares (r,b) and (l,t) must be connected by a sum diagonal $S^* = r+b = l+t$ since otherwise the two difference diagonals covering them each cover only one square in K, an impossibility. We now colour the squares in K above S^* yellow and those below S^* purple, as depicted in Figure 4. Formally, squares $K_Y = \{(x,y) \in K \ : x+y > S^*\}$ are coloured yellow and squares $K_P = \{(x,y) \in K \ : x+y < S^*\}$ are coloured purple. Each sum diagonal above S^* covers two yellow squares and each one below S^* covers two purple squares. Each difference diagonal covers one yellow and one purple square. Since $|K_Y| = |K_P| = 4k+1$ we infer that among the $2k$ sum diagonals besides S^*, half are above it and half are below it. An analogous result holds for the difference diagonals. ☐

Most of the subresults (a) through (e) of Result 5.1 are immediately implied by subresults (a) through (e) of Theorem 5.5 respectively. The latter results generalize the former results to the Relaxed Queen's Domination setting. Result 5.1 (b) is the only result that

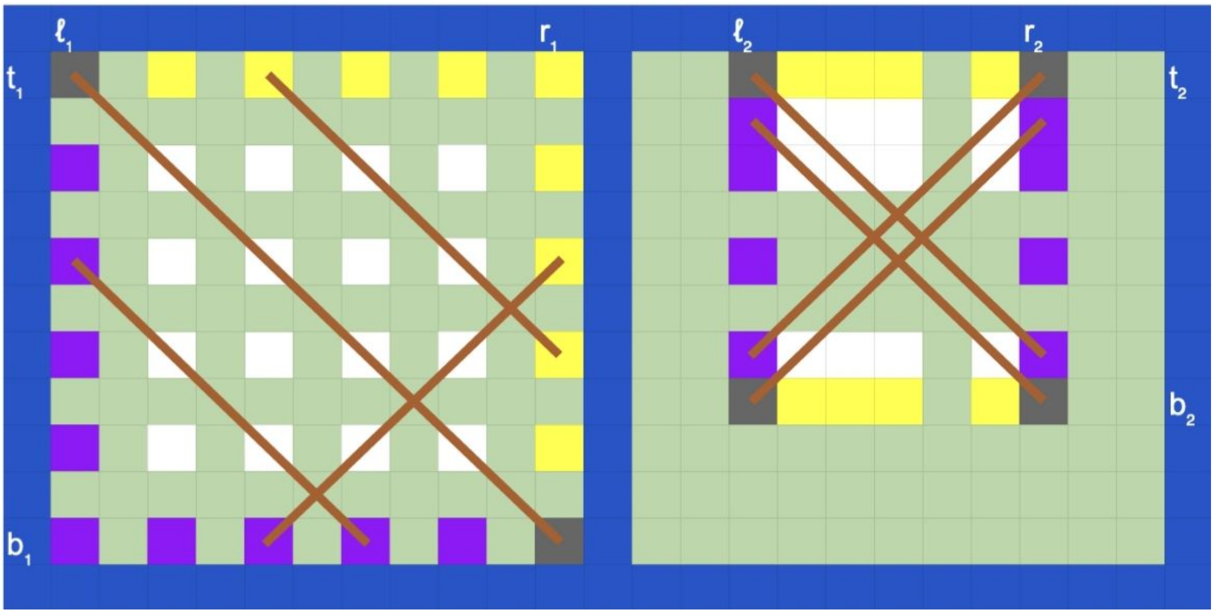

Figure 4: Illustration of the proof of Theorem 5.5, specifically part (e) on the left and part (b) on the right. In both boards, we have $n = 4k + 3 = 11$, $k = 2$ and $p = 2k = 4$. The blue boundary is not part of either board. The chosen rows and columns are coloured green. Non-green squares make up the spaced grid G. Squares coloured grey, yellow(K_Y) and purple(K_P) constitute its cellular perimeter K. Diagonals are coloured brown.

is only partially generalized. Specifically, the parts that are not generalized are two properties of j, the length of square U: (i) j is odd (ii) $3(n+1)/4 \leqslant j \leqslant n$. Note that both of these properties are not true at all in the Relaxed Queen's Domination setting. Simply choose the rightmost $2k+1$ rows and the topmost $2k+1$ columns. Lemma 4.6 implies that the remaining uncovered $(2k+2) \times (2k+2)$ rectangle can be covered with $2k+1$ diagonals of each kind, and this is a counterexample to both the statements above.

Having generalized Result 5.1, we now propose open questions and conjectures towards simplifying Result 5.2. The first open question asks what values the set of rows $(R_i)_{i=0}^{p-1}$ and the set of columns $(C_i)_{i=0}^{p-1}$ can take. Question 5.10 is particularly simple to state independent of the Queen's Domination problem and may be of independent interest.

To state the first open question, we will focus not on the set of chosen rows and columns but the unchosen ones. Let $(R'_i)_{i=0}^{p}$ denote the $p+1$ unchosen rows and $(C'_i)_{i=0}^{p}$ denote the $p+1$ unchosen columns. Without loss of generality, let both of these be sorted so that $0 \leqslant R'_0 < \ldots < R'_p < n = 4k+3$ and $0 \leqslant C'_0 < \ldots < C'_p < n = 4k+3$. The set of squares not covered by the chosen rows and columns is the spaced grid $G(\{C'_i\}_{i=0}^{p}, \{R'_i\}_{i=0}^{p})$ - shorthand G - as defined in Definition 3.2. Theorem 4.5 (b) implies that $C'_p - C'_0 = R'_p - R'_0 = e$ (say). Without loss of generality, we translate the rows and columns so that $0 = R'_0 < \ldots < R'_p = e$ and $0 = C'_0 < \ldots < C'_p = e$. Recall that $p = 2k+1$ and that p sum diagonals $(S_i)_{i=0}^{p-1}$ and p difference diagonals $(D_i)_{i=0}^{p-1}$ fully cover the $(p+1) \times (p+1)$ spaced grid G. Lemma 4.3 implies that p is in fact the least such number that can suffice.

Definition 5.6 (Perfect Diagonal Cover of a spaced grid)**.** A *perfect diagonal cover* for a spaced grid with m' rows and n' columns is a set of exactly $(m'+n'-2)/2$ sum diagonals and $(m'+n'-2)/2$ difference diagonals that covers every square in it.

Let us now state the first open question, generalizing the $e < 4k+3$ constraint: *Which are the spaced grids with an equal number of rows and columns that have a perfect diagonal cover?* Lemma 4.3 implies that this equal number of rows and columns must be even.

Question 5.7. Let $p = 2k+1$ and $e \geqslant p$. What are the values $0 = R'_0 < \ldots < R'_p = e$ and $0 = C'_0 < \ldots < C'_p = e$ such that $G(\{C'_i\}_{i=0}^{p}, \{R'_i\}_{i=0}^{p})$ can be fully covered using p sum diagonals and p difference diagonals?

Each of the $2p$ chosen diagonals must cover 2 points in K, the cellular perimeter of G. This is because equality in Lemma 4.1 implies equality in (3). So far we have only considered the squares in $K \subseteq G$ but we will now consider all squares in G. We denote $\{R'_0, \ldots R'_{p-1}\}$ by R' and $\{C'_0, \ldots C'_{p-1}\}$ by C'. We also denote $-R' = \{-r \bmod e : r \in R'\}$. Each square (C'_i, R'_j) in G must be covered by either a sum diagonal or a difference diagonal which must also intersect U_{edge} in two squares contained in K. This means that either (i) $C'_i + R'_j \bmod e \in C' \cap R'$ or (ii) $C'_i - R'_j \bmod e \in C' \cap -R'$. If neither of these is true, then the square (C'_i, R'_j) is clearly uncoverable - we call such squares uncoverable mod e. This leads to our second open question: *Which are the spaced grids with equal number of rows and columns that have no squares that are uncoverable mod e?*

Question 5.8. Let $e \geqslant p \geqslant 2$. What are the sets $C' = \{C'_0, \ldots C'_{p-1}\} \subseteq \{0, \ldots e-1\}$ and $R' = \{R'_0, \ldots R'_{p-1}\} \subseteq \{0, \ldots e-1\}$ such that $0 \in C' \cap R'$ and for every (i, j) such that $0 \leqslant i, j \leqslant p-1$ one of the following holds:

(i) $C'_i + R'_j \bmod e \in C' \cap R'$

(ii) $C'_i - R'_j \bmod e \in C' \cap (-R')$.

For $(4k+3) \times (4k+3)$ boards, we are only interested in odd p. We wrote a computer program[1] to solve this for small p and e and found that all such spaced grids are symmetric about both their diagonals for odd p (see Figure 5). At least one of these two symmetries holds for even p. This leads us to Conjecture 5.9 which states some necessary but not sufficient properties for Question 5.8. The first part of this conjecture states the following: *Spaced grids with an equal and odd number of rows and columns that have no squares uncoverable mod e are symmetric about both diagonals.*

Conjecture 5.9. All solutions to Question 5.8 for odd p satisfy:

(i) $C' = R'$

(ii) $C' = -R'$.

For even p, at least one of (i) or (ii) is satisfied.

[1] https://github.com/architkarandikar/queens-domination/tree/main/qstn_5p7_cjtr_5p8

Conjecture 5.9 says that solutions to Question 5.8 for odd p are symmetric about both diagonals. This leads us to ask a simpler question: does one symmetry imply the other? We already know that the answer is no for even p. Without loss of generality, we assume symmetry along the sum diagonal, so that $C' = R'$. The two conditions of Question 5.8 now simplify further: for all $C'_i, C'_j \in C'$, one of the following should be true (i) $C'_i + C'_j \bmod e \in C'$ (ii) $C'_i - C'_j \bmod e \in C'$. The question now becomes: which are such subsets C'? Since we have translated so that $C'_0 = 0$, we are only interested in those C' for which $0 \in C'$. For $(4k+3)\times(4k+3)$ boards we are interested in odd-sized subsets. However, we generalize both of these constraints to make the question even more natural. The resulting Question 5.10 is now so simple to state that it is potentially of independent interest as a problem. Although it is stated independently of the Queen's Domination problem, it essentially asks: *Which are the spaced grids with equal number of rows and columns symmetric about one diagonal that have no squares uncoverable mod e?*

Question 5.10. What are the subsets S of $\{0, 1, \dots e-1\}$ such that for all $x, y \in S$, either $x + y \bmod e$ or $x - y \bmod e$ is in S?

We also wrote a computer program[2] to solve this for small values for of e (see Figure 5). For odd-sized subsets we found the symmetry properties we expected from Conjecture 5.9. For even-sized subsets these do not hold. This leads us to Conjecture 5.11, which states necessary but not sufficient properties for Question 5.10: *Odd-sized subsets which are solutions to Question 5.10 always contain* 0 *and are closed under negation mod* e.

Conjecture 5.11. All odd-sized solutions S to Question 5.10 satisfy

(i) $0 \in S$

(ii) $s \in S \Rightarrow (-s) \bmod e \in S$.

In the last part of this section we generalize Result 5.3 by characterizing all perfect diagonal covers for *uniformly spaced grids* with an equal number of rows and columns.

Definition 5.12 (Uniformly spaced grid)**.** A spaced grid $G(\{C'_i\}_{i=0}^{p'-1}, \{R'_i\}_{i=0}^{p'-1})$ consisting of p' rows $R'_0 < \dots < R'_{p'-1}$ and p' columns $C'_0 < \dots < C'_{p'-1}$ is called a *uniformly spaced grid* if and only if $C'_i - C'_{i-1} = R'_i - R'_{i-1} = d$ for some fixed positive integer d for all i such that $1 \leqslant i \leqslant p' - 1$.

Lemma 4.3 implies that a perfect diagonal cover can exist only when m', n' are both even. Theorem 5.13 which follows generalizes Result 5.3 by characterizing all perfect diagonal covers for uniformly spaced grids with p' rows and p' columns when $p' = 2k + 2$ is even. We parameterize p' as $2k + 2$ because the spaced grid G of squares not covered by the chosen rows and columns for $n = 4k + 3$ has $2k + 2$ rows and columns. With $m' = n' = p' = 2k + 2$, perfect diagonal covers have $(m' + n' - 2)/2 = 2k + 1$ diagonals of each kind. We note that $d = 2$ is the only case relevant to the Queen's Domination

[2] https://github.com/architkarandikar/queens-domination/tree/main/qstn_5p9_cjtr_5p10

```
C':0 1 2 6 7   R':0 1 2 6 7
C':0 2 3 5 6   R':0 2 3 5 6
```

(a) Solutions to Qn. 5.8 for $p = 5, e = 8$.

```
C':0 1 2 5   R':0 1 2 5
C':0 1 2 5   R':0 1 4 5
C':0 1 3 4   R':0 1 3 4
C':0 1 3 4   R':0 2 3 5
C':0 1 4 5   R':0 1 2 5
C':0 1 4 5   R':0 1 4 5
C':0 2 3 5   R':0 1 3 4
C':0 2 3 5   R':0 2 3 5
```

(b) Solutions to Qn. 5.8 for $p = 4, e = 6$.

```
S u {e}: 0 6
S u {e}: 0 2 4 6
S u {e}: 0 1 5 6
S u {e}: 0 1 2 4 5 6
```

(c) Odd-sized solutions S to Qn. 5.10 for $e = 6$.

```
S u {e}: 6
S u {e}: 0 1 6
S u {e}: 0 2 6
S u {e}: 0 3 6
S u {e}: 0 4 6
S u {e}: 2 4 6
S u {e}: 0 1 3 4 6
S u {e}: 0 5 6
S u {e}: 0 1 2 5 6
S u {e}: 0 2 3 5 6
S u {e}: 0 1 4 5 6
S u {e}: 1 2 4 5 6
S u {e}: 0 1 2 3 4 5 6
```

(d) Even-sized solutions S to Qn. 5.10 for $e = 6$.

Figure 5: Explanatory examples for Questions 5.8, 5.10 and Conjectures 5.9, 5.11.

problem since approximately $n(d-1)/d$ queens are needed for the spaced grid of squares not covered by the chosen rows and columns to be a uniformly spaced grid with spacing d. The special case of this result for $d = 2$ was proven as Theorem 1 in [20] as a generalization of Theorem 3 of [6]. We generalize it further and provide an alternative visual proof.

Theorem 5.13. *Consider a uniformly spaced grid $G(\{C'_i\}_{i=0}^{p'-1}, \{R'_i\}_{i=0}^{p'-1})$ with $p' = 2k+2$ columns $\{C'_i\}_{i=0}^{p'-1}$, p' rows $\{R'_i\}_{i=0}^{p'-1}$ and spacing d between consecutive rows and columns. We identify columns by their x-coordinate and rows by their y-coordinate. Without loss of generality, translate the columns and rows to be symmetric about both axes, so that $C'_i = (i-k-1/2)d$ and $R'_i = (i-k-1/2)d$. A set of $2k+1$ sum diagonals $S = \{S_i\}_{i=0}^{2k}$ and $2k+1$ difference diagonals $D = \{D_i\}_{i=0}^{2k}$ - identified by the invariant sum and difference of their square coordinates respectively - is a perfect cover if and only if*

$$S = D = Q_w \text{ for some } w \text{ such that } 0 \leqslant w \leqslant k \text{ where}$$
$$Q_w = \{0, \underbrace{\pm d, \pm 2d, \dots \pm wd}_{\text{count: } 2w}, \underbrace{\pm(w+2)d, \pm(w+4)d, \dots \pm (2k-w)d}_{\text{count: } 2(k-w)}\}.$$

Proof. Without loss of generality, we prove this for $d = 2$. Note that all R_i and C_i values are odd integers and all S_i and D_i values are even integers for $d = 2$. Our proof is based on the following *central observation*: all G-squares in difference diagonal $2f$ (for $f \in [1, 2k+1]$) are covered if and only if either difference diagonal $2f$ is chosen or if sum diagonals $H_f = \{2f - 4k - 2, 2f - 4k + 2, \dots 4k + 2 - 2f\}$ are all chosen. This is because the G-squares in difference diagonal $2f$ are $\{(2f - (2k+1), -(2k+1)), (2f - (2k-1), -(2k-1)), \dots (2k+1, (2k+1) - 2f)\}$. Figure 6 explains this argument visually. Similar results also hold for difference and sum diagonals $\pm 2f$ for all $f \in [1, 2k+1]$.

Let (S, D) be a perfect diagonal cover of G and consider the largest integer $w \geqslant 0$ such that $\{0, \pm 2, \pm 4 \dots \pm 2w\} \subseteq S \cap D$. If $w = k$, the claim trivially holds since $|S \cap D| = 2k + 1 = |S| = |D|$ implies that $S = D = Q_k$. Otherwise one of $\pm 2(w+1)$ does not belong to either S or D. Without loss of generality, let $2(w+1) \notin D$. The central observation now implies that $H_{w+1} = \{2w - 4k, 2w - 4k + 4, \dots 4k - 2w\} \subseteq S$. Hence $\{\pm 2(w+2), \pm 2(w+4), \dots \pm 2(2k-w)\} \subseteq S$ implying that $S = Q_w$, since all $2k+1$ elements of S are accounted for. Hence $2(w+1) \notin S$, which similarly implies that $D = Q_w$.

Conversely let $S = D = Q_w$ for some integer w such that $0 \leqslant w \leqslant k$. We will show that $S \cup D$ covers all squares in G which trivially implies that (S, D) is a perfect cover of G. Each square in G belongs to some difference diagonal $2f$ where $-2k+1 \leqslant f \leqslant 2k+1$. Difference diagonals in Q_w are trivially covered since they are chosen. Those not in Q_w are $A_e \cup B_e$ where $A_e = \{\pm 2(w+1), \pm 2(w+3), \dots \pm 2(2k-w-1)\}$ and $B_e = \{\pm 2(2k-w+1), \pm 2(2k-w+2), \dots \pm 2(2k+1)\}$. We infer from the central observation that all squares in difference diagonal $\pm 2(w+1)$ are covered since $H_{w+1} = \{2w-4k, 2w-4k+4, \dots 4k-2w\} \subseteq S$. The central observation further implies that all squares in each difference diagonal in A_e are covered since $H_{2k-w-1} \subset \dots \subset H_{w+3} \subset H_{w+1}$. All squares in each difference diagonal in B_e are covered since $H_i \subseteq \{0, \pm 2, \pm 4 \dots \pm 2w\} \subseteq S$ for all i such that $2k-w+1 \leqslant i \leqslant 2k+1$. Hence all squares in G are covered by $S \cup D$. ☐

This proof of Theorem 5.13 illustrated in Figure 6 characterizes all perfect covers of a uniformly spaced grid. This concludes the last section of this work.

6 Conclusion

We have improved the lower bound for the Queen's Domination problem on rectangular boards in one-eighth of the nontrivial cases by proposing and completely solving the Relaxed Queen's Domination problem. By doing so, we have answered an open question posed in [4] in the affirmative. We have generalized and provided a new interpretation of lower bounds on rectangular boards, including most of the known bounds for square boards. We have also generalized Weakley's improvement [18] to the lower bound for $(4k+1)$-square boards to rectangular boards. We have generalized some results for the only case for square boards for which the known lower bound goes beyond the one implied by the Relaxed Queen's Domination problem. We have formulated open questions and conjectures relevant to this problem that may be of broader interest.

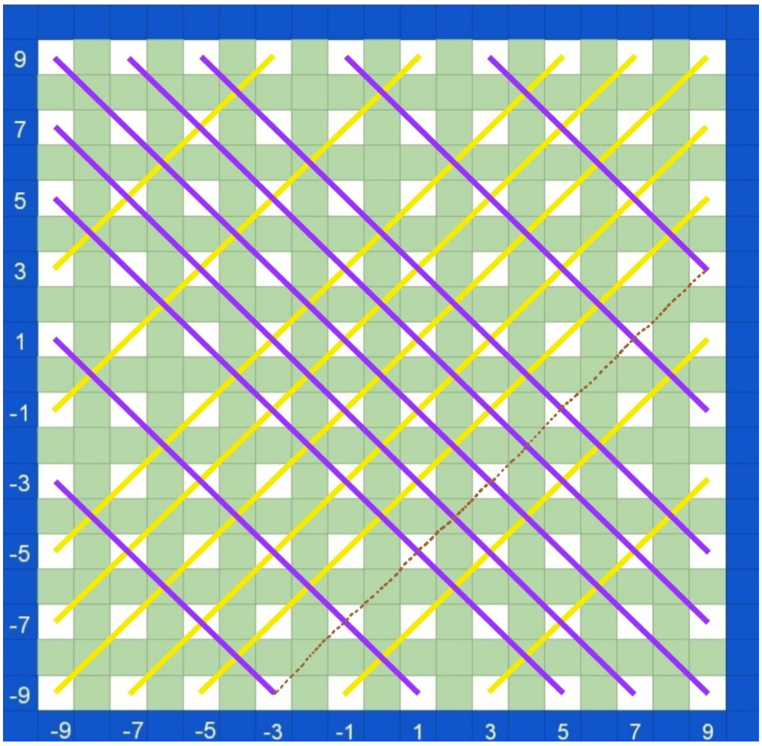

Figure 6: A perfect cover for a uniformly spaced grid with 10 rows and columns, which illustrates the proof of Theorem 5.13 for $k = 4$, $d = 2$, $w = 2$ and $S = D = Q_w = \{0, \pm 2, \pm 4, \pm 8, \pm 12\}$. Chosen sum and difference diagonals are shown in purple and yellow respectively. The dotted brown difference diagonal $2f$ ($f = 3$) illustrates the central observation. Since it is not chosen, the sum diagonals $H_f = \{-12, -8, -4, 0, 4, 8, 12\}$ must all be chosen.

Acknowledgements

Thanks to our good friend Varun Dwarkanathan for introducing this problem to us.

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