A new presentation for Specht modules with distinct parts

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Abstract

We obtain a new presentation for Specht modules whose conjugate shapes have strictly decreasing parts by introducing a linear operator on the space generated by column tabloids. The generators of the presentation are column tabloids and the relations form a proper subset of the Garnir relations of Fulton. The results in this paper generalize earlier results of the authors and Stanley on Specht modules of staircase shape.

Mathematics Subject Classifications: 05E10, 20C30

1 Introduction

The Specht modules S^λ, where λ is a partition of n, give a complete set of irreducible representations of the symmetric group $\mathfrak{S}_n$ over a field of characteristic 0, say $\mathbb{C}$. They can be constructed as subspaces of the regular representation $\mathbb{C}\mathfrak{S}_n$ or as presentations given in terms of generators and relations, known as Garnir relations. This paper deals primarily with the latter type of construction.

Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_l)$ be a partition of n. A *Young tableaux* of shape λ is a filling of the Young diagram of shape λ with distinct entries from the set $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{T}_\lambda$ be the set of Young tableaux of shape λ.

To construct the Specht module as a submodule of the regular representation, one can use Young symmetrizers. For $t \in \mathcal{T}_\lambda$, the *Young symmetrizer* is defined by

$$e_t := \sum_{\alpha \in R_t} \alpha \sum_{\beta \in C_t} \operatorname{sgn}(\beta)\beta, \tag{1}$$

where C_t is the column stabilizer of t and R_t is the row stabilizer. The *Specht module* S^λ is the submodule of the regular representation $\mathbb{C}\mathfrak{S}_n$ spanned by $\{\tau e_t : \tau \in \mathfrak{S}_n\}$.

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To construct the Specht module as a presentation, one can use column tabloids and Garnir relations. Let M^λ be the vector space (over $\mathbb{C}$) generated by $\mathcal{T}_\lambda$ subject only to column relations, which are of the form $t+s$, where $s \in \mathcal{T}_\lambda$ is obtained from $t \in \mathcal{T}_\lambda$ by switching two entries in the same column. Given $t \in \mathcal{T}_\lambda$, let $\bar{t}$ denote the coset of t in M^λ. These cosets, which are called *column tabloids*, generate M^λ. A Young tableau is *column strict* if the entries of each of its columns increase from top to bottom. Clearly, $\{\bar{t} : t \text{ is a column strict Young tableau of shape } \lambda\}$ is a basis for M^λ.

Example:

$$\overline{\begin{array}{|c|c|}\hline 3 & 5 \\ \hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}} = -\overline{\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}} = \overline{\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}} = -\overline{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}},$$

There are various different presentations of the Specht module S^λ in the literature that involve column tabloids and Garnir relations, see e.g. [Fu, Sa]. Here we are interested in a presentation of S^λ discussed in Fulton [Fu, Section 7.4]. The generators are the column tabloids $\bar{t}$, where $t \in \mathcal{T}_\lambda$. The Garnir relations are of the form $\bar{t} - \sum \bar{s}$, where the sum is over all $s \in \mathcal{T}_\lambda$ obtained from $t \in \mathcal{T}_\lambda$ by exchanging any k entries of a fixed column with the top k entries of the next column, while maintaining the vertical order of each of the exchanged sets. There is a Garnir relation $g^t_{c,k}$ for every $t \in \mathcal{T}_\lambda$, every column $c \in [\lambda_1 - 1]$, and every k from 1 to the length l_{c+1} of column $c+1$.

Example: For

$$t = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array},$$

we have

$$g^t_{1,1} = \overline{\begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5 & 1 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 5 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}}$$

and

$$g^t_{1,2} = \overline{\begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5 & 1 & 7 \\ \hline 6 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5 & 1 & 7 \\ \hline 2 & 3 \\ \cline{1-2} 6 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5 & 1 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 5 & 3 \\ \cline{1-2} 6 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 5 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}}.$$

Let G^λ be the subspace of M^λ generated by the Garnir relations in

$$\{g^t_{c,k} : c \in [\lambda_1 - 1], k \in [l_{c+1}], t \in \mathcal{T}_\lambda\}. \tag{2}$$

The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{T}_\lambda$ by replacing each entry of a tableau by its image under the permutation in $\mathfrak{S}_n$. This induces a representation of $\mathfrak{S}_n$ on M^λ. Clearly G^λ is invariant under the action of $\mathfrak{S}_n$. The presentation of S^λ obtained in Section 7.4 of [Fu] is given by

$$M^\lambda/G^\lambda \cong_{\mathfrak{S}_n} S^\lambda. \tag{3}$$

On page 102 (before Ex. 16) of [Fu], a presentation of S^λ with a smaller set of relations is given. In this presentation, the index k in $g^t_{c,k}$ of (2) is restricted to a single value: $k = \min[l_{c+1}] = 1$. More precisely, the presentation is

$$M^\lambda/G^{\lambda,\min} \cong_{\mathfrak{S}_n} S^\lambda, \tag{4}$$

where $G^{\lambda,\min}$ is the subspace of G^λ generated by the subset of Garnir relations,

$$\{g^t_{c,1} : c \in [\lambda_1 - 1], t \in \mathcal{T}_\lambda\}.$$

In this paper we obtain an analogous presentation of S^λ, also with a smaller set of relations, when the conjugate λ^* has distinct parts. The index k in $g^t_{c,k}$ of (2) is again restricted to a single value, but now that value is the maximum value: $k = \max[l_{c+1}] = l_{c+1}$. The presentation is given in our main result:

Theorem 1. *Let λ be a partition whose conjugate has distinct parts. Then as $\mathfrak{S}_n$-modules,*

$$M^\lambda/G^{\lambda,\max} \cong S^\lambda, \tag{5}$$

where $G^{\lambda,\max}$ is the subspace of G^λ generated by

$$\{g^t_{c,l_{c+1}} : c \in [\lambda_1 - 1], t \in \mathcal{T}_\lambda\}.$$

Moreover, we can further reduce this set of relations by restricting t to the set of column strict tableaux.

Theorem 1 is obtained as a consequence of Corollary 8 below, which gives all of the eigenspaces of a certain linear operator on M^λ, where λ is any two-column shape 2^a1^b. This eigenspace result improves an earlier result of the authors and Stanley [FHSW2] (see also [FHSW1]), in the special case that $b = 1$. The earlier result was presented in the setting of an n-ary generalization of Lie algebra called a LAnKe or Filippov algebra, where only the $b = 1$ case was relevant. An observation in [FHSW2] that the n-ary Jacobi relations correspond to the restricted class of Garnir relations $G^{\lambda,\max}$ for $\lambda = 2^{n-1}1^1$ provided a special case of Theorem 1 for staircase shapes and motivated the work in the current paper. Our proof in this paper of the eigenspace result corrects an error in the proof for the special case given in [FHSW2].

The method of introducing a linear operator on the space of column tabloids was subsequently used in [BF] to obtain a different presentation of S^λ with a reduced number of relations. This presentation works for *all* shapes, but rather than using a subset of the Garnir relations, it uses a set consisting of sums of the Garnir relations that generate $G^{\lambda,\min}$.

In the next section we prove our eigenspace result and use it to prove Theorem 1. Actually, we obtain a slightly more general version of Theorem 1 along with a converse.

2 The proof of Theorem 1: reduction to the two-column case

Theorem 1 holds more generally when we allow the conjugate λ^* to have multiple parts equal to 1, while still requiring the parts greater than 1 to be distinct. Note that a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_l)$ meets this requirement if and only if $\lambda_i - \lambda_{i+1} \leqslant 1$ for all $i = 2, \ldots l-1$ and $\lambda_l = 1$. For example, $\lambda = (5,4,2,1,1,1)^* = (6,3,2,2,1)$ meets this requirement, but $\lambda = (4,3,2,2)^* = (4,4,2,1)$ does not. The following result implies Theorem 1.

Theorem 2. *Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell)$ be a partition of n. If $\lambda_i - \lambda_{i+1} \leqslant 1$ for each $i = 2, \ldots, l-1$ and $\lambda_l = 1$ then as $\mathfrak{S}_n$-modules,*

$$M^\lambda / G^{\lambda,\max} \cong S^\lambda, \tag{6}$$

where $G^{\lambda,\max}$ is the subspace of G^λ generated by

$$\{g^t_{c,l_{c+1}} : c \in [\lambda_1 - 1], t \in \mathcal{T}_\lambda\}.$$

Moreover, we can further reduce this set of relations by restricting t to the set of column strict tableaux. Conversely, if (6) holds then $\lambda_i - \lambda_{i+1} \leqslant 1$ for each $i = 2, \ldots, l-1$ and $\lambda_l = 1$.

It follows from Fulton's presentation given in (3) that we need only prove Theorem 2 for shapes with just two columns. Indeed, let λ be a partition of n. By (3) $M^\lambda/G^{\lambda,\max} = S^\lambda$ if and only if $G^{\lambda,\max} = G^\lambda$. Since $G^{\lambda,\max}$ equals the direct sum over all columns c of the subspaces spanned by $\{g^t_{c,l_{c+1}} : t \in \mathcal{T}_\lambda\}$, and G^λ equals the direct sum over all columns c of the subspaces spanned by the larger set $\{g^t_{c,k} : k \in [l_{c+1}], t \in \mathcal{T}_\lambda\}$, we have that $G^{\lambda,\max} = G^\lambda$ if and only if

$$\langle g^t_{c,l_{c+1}} : t \in \mathcal{T}_\lambda \rangle = \langle g^t_{c,k} : k \in [l_{c+1}], t \in \mathcal{T}_\lambda \rangle \tag{7}$$

for all columns c. By applying (3) to the conjugate of (l_c, l_{c+1}) we have that (7) holds if and only if

$$M^{(l_c,l_{c+1})^*}/G^{(l_c,l_{c+1})^*,\max} = M^{(l_c,l_{c+1})^*}/\langle g^t_{c,l_{c+1}} : t \in \mathcal{T}_{(l_c,l_{c+1})^*}\rangle = S^{(l_c,l_{c+1})^*}.$$

Putting all this together yields $M^\lambda/G^{\lambda,\max} = S^\lambda$ if and only if

$$M^{(l_c,l_{c+1})^*}/G^{(l_c,l_{c+1})^*,\max} = S^{(l_c,l_{c+1})^*}$$

for all columns c of λ. Hence our claim that we need only consider two-column shapes holds.

For the remainder of this section we consider only two-column shapes $2^m 1^{n-m}$. For $n \geqslant m$, let $V_{n,m} := M^{2^m 1^{n-m}}$. For each n-element subset S of $[n+m]$, let t_S be the column strict Young tableau of shape $2^m 1^{n-m}$ whose first column consists of the elements of S

and second column consists of the elements of $[n+m] \setminus S$, and write v_S for the column tabloid $\bar{t}_S$ indexed by S. Clearly,

$$\left\{ v_S : S \in \binom{[n+m]}{n} \right\} \tag{8}$$

is a basis for $V_{n,m}$. Thus $V_{n,m}$ has dimension $\binom{n+m}{n}$.

For each $S \in \binom{[n+m]}{n}$, let g_S denote the Garnir relation $g^{t_S}_{1,m}$, that is

$$g_S := v_S - \sum \bar{t},$$

where the sum is over all tableaux t obtained from t_S by choosing an m-subset S' of S and for each $i = 1, 2, \ldots, m$, exchanging the ith smallest element of S' (which is in column 1 of t_S) with the ith element of column 2 of t_S. For example, if $n = 3$ and $m = 2$ then

$$\begin{aligned} g_{\{2,4,5\}} &= \overline{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline 5 & \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}}\,. \\ &= v_{\{2,4,5\}} - v_{\{1,3,5\}} + v_{\{1,3,4\}} + v_{\{1,2,3\}}. \end{aligned}$$

Lemma 3. *For all $v \in V_{n,m}$ and $S \in \binom{[n+m]}{n}$, let $\langle v, v_S \rangle$ denote the coefficient of v_S in the expansion of v in the basis given in (8). Also let $p := n - m$. Then for all $S, T \in \binom{[n+m]}{n}$,*

$$\langle g_S, v_T \rangle = \begin{cases} 1 & \text{if } S = T \\ (-1)^{d_1 + \cdots + d_p + \binom{p+1}{2} + 1} & \text{if } S \cap T = \{d_1, \ldots, d_p\} \\ 0 & \text{if } S \neq T \text{ but } |S \cap T| > p. \end{cases}$$

Proof. It is easy to see that the only possible cases are: $S = T$, $|S \cap T| = p$ and $|S \cap T| > p$. It is straightforward that the result for the first and third cases holds. For the second case, let $S = \{a_1, a_2, \ldots, a_n\}$ and $[n+m] \setminus S = \{b_1, b_2, \ldots, b_m\}$, where the a's and b's are listed in increasing order. Then the terms of g_S other than v_S are of the form $-\bar{t}$, where t is a tableau whose first column listed from top to bottom is

$$b_1, \ldots, b_{i_1-1}, a_{i_1}, b_{i_1}, \ldots, b_{i_2-2}, a_{i_2}, b_{i_2-1}, \ldots, b_{i_p-p}, a_{i_p}, b_{i_p-p+1}, \ldots, b_m$$

with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$, and whose second column is $[n+m] \setminus T$ listed in increasing order, where $T = \{a_{i_1}, \ldots, a_{i_p}, b_1, \ldots, b_m\}$. Clearly $\bar{t}$ is equal to $(-1)^{i_1 + \cdots + i_p - \binom{p+1}{2}}$ times the tabloid whose first column from top to bottom is

$$a_{i_1}, \ldots, a_{i_p}, b_1, \ldots, b_m$$

and whose second column is $[n+m] \setminus T$ listed in increasing order. This tabloid is equal to $(-1)^{j_1+j_2+\cdots+j_p} v_T$, where each j_k is the number of b's that are less than a_{i_k}. Since $i_k + j_k = a_{i_k}$ for each k, we have

$$\langle g_S, v_T \rangle = (-1)^{a_{i_1}+a_{i_2}+\cdots+a_{i_p}-\binom{p+1}{2}+1}.$$

Now note that $\{a_{i_1}, a_{i_2}, \ldots, a_{i_p}\} = S \cap T$. □

Let $G_{n,m}$ be the subspace of $V_{n,m}$ generated by $\{g_S : S \in \binom{[n+m]}{n}\}$. It is not difficult to see that $G_{n,m}$ is invariant under the action of $\mathfrak{S}_{n+m}$ on $V_{n,m}$. The two-column case of Theorem 2 can now be stated as follows.

Theorem 4. *Let $m \leqslant n$. Then as $\mathfrak{S}_{n+m}$-modules,*

$$V_{n,m}/G_{n,m} \cong S^{2^m 1^{n-m}}$$

if and only if $m < n$ or $m = n = 1$.

We will introduce a linear operator that will enable us to prove this result (and thereby prove Theorems 2 and 1). First we note that due to the column relations, as an $\mathfrak{S}_{n+m}$–module, $V_{n,m}$ is isomorphic to the representation of $\mathfrak{S}_{n+m}$ induced from the sign representation of the Young subgroup $\mathfrak{S}_n \times \mathfrak{S}_m$:

$$V_{n,m} \cong_{\mathfrak{S}_{n+m}} (\mathrm{sgn}_n \times \mathrm{sgn}_m) \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} .$$

Hence by Pieri's rule,

$$V_{n,m} \cong_{\mathfrak{S}_{n+m}} \bigoplus_{i=0}^{m} S^{2^i 1^{n+m-2i}}. \tag{9}$$

Now consider the linear operator $\varphi : V_{n,m} \to V_{n,m}$ defined on basis elements by

$$\varphi(v_S) = g_S.$$

It is not difficult to see that φ is an $\mathfrak{S}_{n+m}$-module homomorphism whose image is $G_{n,m}$.

Lemma 5. *Let $m \leqslant n$. The operator φ acts as multiplication by a scalar on each irreducible submodule of $V_{n,m}$. Moreover, as $\mathfrak{S}_{n+m}$–modules,*

$$\ker \varphi \cong V_{n,m}/G_{n,m}. \tag{10}$$

Proof. This follows from the fact that $V_{n,m}$ is multiplicity-free, as indicated in (9), and Schur's Lemma. □

It follows from (10) that to prove Theorem 4, we need only show that the kernel of φ is isomorphic to $S^{2^m 1^{n-m}}$ if and only if $m < n$ or $m = n = 1$. This is handled by the following theorem and its corollaries.

Theorem 6. *Let $m \leqslant n$. The operator $\varphi : V_{n,m} \to V_{n,m}$ acts as multiplication by the scalar*

$$w_i := 1 - \binom{n-i}{m-i}(-1)^{m-i}, \tag{11}$$

on the irreducible submodule isomorphic to $S^{2^i 1^{(n+m)-2i}}$ for $i = 0, 1, \ldots, m$.

Remark 7. In the case that $m = n - 1$, Theorem 6 reduces to Theorem 2.4 of [FHSW2], which was given in the setting of n-ary Jacobi relations, where only the $m = n - 1$ case arises.

Corollary 8. *For $m < n$, the operator φ has $m + 1$ distinct eigenvalues $w_0, w_1, \ldots, w_m$. Moreover, if E_i is the eigenspace corresponding to w_i then as $\mathfrak{S}_{n+m}$-modules,*

$$E_i \cong S^{2^i 1^{(n+m)-2i}}$$

for each $i = 0, 1, \ldots, m$. Consequently,

$$\ker \varphi \ \cong_{\mathfrak{S}_{n+m}} \ S^{2^m 1^{n-m}}. \tag{12}$$

Corollary 9. *For $m = n$ the operator φ has the eigenvalues 0 and 2, with*

$$w_i = \begin{cases} 0 & \text{if } n-i \text{ is even,} \\ 2 & \text{if } n-i \text{ is odd,} \end{cases}$$

for each $i = 0, \ldots, n$. Consequently,

$$\ker \varphi \ \cong_{\mathfrak{S}_{2n}} \ \bigoplus_{\substack{i=0 \\ n-i \text{ even}}}^{n} S^{2^i 1^{2n-2i}}, \tag{13}$$

and the eigenspace with eigenvalue 2 is given by

$$\bigoplus_{\substack{i=0 \\ n-i \text{ odd}}}^{n} S^{2^i 1^{2n-2i}}.$$

Proof of Theorem 6. The proof follows (and at the same time corrects) the argument for the $m = n - 1$ case given in [FHSW2, Theorem 2.4]. Indeed, if we set $m = n - 1$ in the following proof, we get a corrected version of the proof in [FHSW2], which was given in the language of n-ary Jacobi relations, where only the $m = n - 1$ case was relevant. On occasion, we will refer back to the arguments in the proof of [FHSW2, Theorem 2.4].

By Lemma 5, φ acts as a scalar on each irreducible submodule. To compute the scalar, we start by letting t be the standard Young tableau of shape $2^i 1^{n+m-2i}$ whose first column is the concatenation of the sequences $1, 2, \ldots, n$ and $n + i + 1, n + i + 2, \ldots, n + m$ and whose second column is the sequence $n + 1, n + 2, \ldots, n + i$. Now set $r_t := \sum_{\alpha \in R_t} \alpha$ and $c_t := \sum_{\beta \in C_t} \operatorname{sgn}(\beta)\beta$, where R_t and C_t denote the row and column stabilizer of t, respectively. Recall from (1) that $e_t = r_t c_t$ and that the Specht module $S^{2^i 1^{n+m-2i}}$ is the submodule of the regular representation $\mathbb{C}\mathfrak{S}_{n+m}$ spanned by $\{\tau e_t : \tau \in \mathfrak{S}_{n+m}\}$.

Now let F_t be the set of permutations σ in C_t that fix the elements of $\{n+1, n+2, \ldots, n+i\}$ and satisfy

$$\sigma(1) < \cdots < \sigma(n), \quad \sigma(n+i+1) < \cdots < \sigma(n+m),$$

and let

$$f_t := \sum_{\sigma \in F_t} \operatorname{sgn}(\sigma)\sigma.$$

Also let $T := \{1, 2, \ldots, n\}$. We claim[1] that

$$c_t v_T = n!(m-i)!i! f_t v_T. \tag{14}$$

Indeed,

$$c_t = \sum_{\sigma \in F_t} \operatorname{sgn}(\sigma)\sigma \sum_{\tau \in \mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C} \operatorname{sgn}(\tau)\tau,$$

where $A = \{1, \ldots, n\}$, $B = \{n+i+1, \ldots, n+m\}$, and $C = \{n+1, \ldots, n+i\}$. It follows from the antisymmetry of the columns of τv_T that

$$c_t v_T = n!(m-i)!i! \sum_{\sigma \in F_t} \operatorname{sgn}(\sigma)\sigma v_T = n!(m-i)!i! f_t v_T,$$

as claimed.

It follows from (14) that $r_t f_t v_T$ is a scalar multiple of $e_t v_T$. Since the coefficient of v_T in the expansion of $r_t f_t v_T$ is 1, we have $e_t v_T \neq 0$. Now following the proof of [FHSW2, Theorem 2.4], we can show that the subspace spanned by $\{\pi e_t v_T : \pi \in \mathfrak{S}_{n+m}\}$ is the unique submodule of $V_{n,m}$ isomorphic to $S^{2^i 1^{(n+m)-2i}}$. This allows us to abuse notation by letting $S^{2^i 1^{(n+m)-2i}}$ denote this submodule of $V_{n,m}$.

Since $r_t f_t v_T$ is a scalar multiple of $e_t v_T$, it is in $S^{2^i 1^{(n+m)-2i}}$. It follows that

$$\varphi(r_t f_t v_T) = c r_t f_t v_T,$$

for some scalar c, which we want to show equals w_i. Similarly to the proof of [FHSW2, Theorem 2.4], we can use Lemma 3 to show that

$$c = \langle \varphi(r_t f_t v_T), v_T \rangle = 1 + \sum_{\substack{S \in \binom{[n+m]}{n} \setminus \{T\} \\ S \cap T = p}} \langle r_t f_t v_T, v_S \rangle \langle \varphi(v_S), v_T \rangle.$$

Let $S \cap T = \{d_1, d_2, \ldots, d_p\}$, in which case $\langle \varphi(v_S), v_T \rangle = (-1)^{d_1 + \cdots + d_p + \binom{p+1}{2} + 1}$. Hence

$$c = 1 + \sum_{D \in \binom{[n]}{p}} (-1)^{\left(\sum_{d \in D} d\right) + \binom{p+1}{2} + 1} \langle r_t f_t v_T, v_{S(D)} \rangle, \tag{15}$$

[1] In the proof of [FHSW2, Theorem 2.4], the same claim was made *erroneously* for $f_t := \sum_{\sigma \in F_t} \operatorname{sgn}(\sigma)\sigma^{-1}$ in the case $m = n-1$. Here we provide a proof of the correct claim and then proceed with an argument analogous to what is given in [FHSW2].

where

$$S(D) = D \cup \{n+1, n+2, \ldots, n+m\}.$$

To compute $\langle r_t f_t v_T, v_{S(D)} \rangle$, we must consider how we get $v_{S(D)}$ from the action on v_T of a permutation $\alpha\sigma$, where $\alpha \in R_t$ and $\sigma \in F_t$. In order to get $S(D)$ for some $D := \{d_1 < d_2 < \cdots < d_p\} \in \binom{[n]}{p}$, we must have that $D \in \binom{\{i+1,\ldots,n\}}{p}$ and that σ fixes the elements of $\{n+1, \ldots, n+i\}$ and interchanges $\{n+i+1, \ldots, n+m\}$ with $\{i+1, \ldots, n\} \setminus D$ putting D in positions $i+1, \ldots, i+p$. More precisely, in one line notation, σ restricted to the first column of t is the concatenation of the sequences

$$1, \ldots, i$$

$$d_1, \ldots, d_p$$

$$n+i+1, \ldots, n+m$$

$$i+1, \ldots, d_1 - 1, d_1 + 1, \ldots, d_2 - 1, d_2 + 1, \ldots, d_p - 1, d_p + 1, \ldots, n.$$

In one line notation, σ restricted to the second column is $n+1, \ldots, n+i$. We must also have

$$\alpha = (1, n+1)(2, n+2) \cdots (i, n+i),$$

giving us $S(D)$ in the first n entries of the first column of $\alpha\sigma t$.

The number of inversions in σ restricted to the first column of t is

$$\sum_{k=1}^{p} (d_k - k - i) + (m-i)(n-i-p) = \sum_{k=1}^{p} d_k - \binom{p+1}{2} - ip + (m-i)^2,$$

and σ restricted to the second column of t has no inversions. It follows that

$$\operatorname{sgn}(\sigma) = (-1)^{\left(\sum_{k=1}^{p} d_k\right) - \binom{p+1}{2} - ip + (m-i)}. \tag{16}$$

In terms of our basis, the first column of $\alpha\sigma v_T$ (written horizontally) is

$$n+1, \ldots, n+i, d_1, \ldots, d_p, n+i+1, \ldots, n+m$$

and the second column is

$$1, \ldots, i, i+1, \ldots, d_1 - 1, d_1 + 1, \ldots, d_p - 1, d_p + 1, \ldots, n.$$

To put this basis in canonical form, we need to put the first column in increasing order by moving all the d_k's all the way to the top of the column. That requires i transpositions for each d_k, resulting in the sign $(-1)^{ip}$. The second column is already in increasing order. This yields $\alpha\sigma v_T = (-1)^{ip} v_{S(d_1, d_2, \ldots, d_p)}$. Hence by (16),

$$\operatorname{sgn}(\sigma)\alpha\sigma v_T = (-1)^{\left(\sum_{k=1}^{p} d_k\right) - \binom{p+1}{2} + (m-i)} v_{S(d_1, d_2, \ldots, d_p)}.$$

We can now conclude that for all $D \in \binom{[n]}{p}$,

$$\langle r_t f_t v_T, v_{S(D)} \rangle = \begin{cases} (-1)^{\left(\sum_{d\in D} d\right) - \binom{p+1}{2} + (m-i)} & \text{if } D \in \binom{\{i+1,\ldots,n\}}{p} \\ 0 & \text{otherwise.} \end{cases}$$

By substituting this into (15), we get the eigenvalue,

$$c = 1 - \sum_{D \in \binom{\{i+1,\ldots,n\}}{p}} (-1)^{m-i} = 1 - \binom{n-i}{n-m} (-1)^{m-i} = w_i,$$

which is all that is needed to complete the proof of the theorem. □

Proof of Theorem 4. We can now use (10), (12), and the $n = 1$ case of (13) to conclude that the "if" direction of Theorem 4 holds. The "only if" direction also follows from (10), (12), and (13). □

Final Remarks. Corollary 9 has another interesting consequence. Indeed, the decomposition of $\ker \varphi$ given in (13) yields a decomposition of $V_{n,n}/G_{n,n}$ into irreducibles. Since $V_{n,n}/G_{n,n}$ is clearly isomorphic to the composition product (see [Mac, (6.2) on p. 158]) of the trivial representation of $\mathfrak{S}_2$ with the sign representation of $\mathfrak{S}_n$, we recover a well-known result on the decomposition of this composition product into irreducibles; see e.g. [Mac, p. 140].

One may ask what happens if the k in the Garnir relations (2) is restricted to a single value other than the minimum 1 and the maximum l_{c+1}. For which partitions λ do such Garnir relations generate the entire set of Garnir relations? Is there a generalization of Theorem 2 that answers this question? We leave these questions open.[2]

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[2]Motivated by an earlier version of our paper, Maliakas, Metzaki, and Stergiopoulou address these questions in [MMS]. One of us has also addressed them in [F].

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