Super Multiset RSK and a Mixed Multiset Partition Algebra

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Abstract

Through dualities on representations on tensor powers and symmetric powers respectively, the partition algebra and multiset partition algebra have been used to study long-standing questions in the representation theory of the symmetric group. In this paper we extend this story to exterior powers, introducing the mixed multiset partition algebra as well as a generalization of the Robinson-Schensted-Knuth algorithm to two-row arrays of multisets with elements from two alphabets. From this algorithm, we obtain enumerative results that reflect representation-theoretic decompositions of this algebra. Furthermore, we use the generalized RSK algorithm to describe the decomposition of a polynomial ring in sets of commuting and anti-commuting variables as a module over both the general linear group and the symmetric group.

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1 Introduction

Suppose an $A \times B$-module V has a decomposition of the form

$$V \cong \bigoplus_{\lambda \in \Lambda} W_A^\lambda \otimes W_B^\lambda. \tag{1}$$

Such decompositions appear naturally in contexts like mutually centralizing actions or decomposing an algebra A as an $A \times A$-module over itself. Comparing dimensions, this decomposition manifests enumeratively as

$$\dim(V) = \sum_{\lambda \in \Lambda} \dim(W_A^\lambda) \cdot \dim(W_B^\lambda).$$

This equality implies that there exists a bijection

$$\mathcal{V} \xleftarrow{\sim} \biguplus_{\lambda \in \Lambda} \mathcal{A}_\lambda \times \mathcal{B}_\lambda \tag{2}$$

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where $\mathcal{V}$, $\mathcal{A}_\lambda$, and $\mathcal{B}_\lambda$ are indexing sets for bases of V, W_A^λ, and W_B^λ respectively.

In [25] Robinson (and later Schensted in [27]) described a bijection between permutations in S_n and pairs of standard Young tableaux, which explicitly provides the above bijection when $V = \mathbb{C}S_n$ is decomposed as an $S_n \times S_n$ module. In [11] Knuth generalized this process to one taking in a biword and returning a pair of semistandard Young tableaux of the same shape, and this generalization is usually called the RSK algorithm. The RSK algorithm and its further generalizations provide bijections like Equation (2) explicitly in several contexts.

Generalizations of RSK also appear as bijections between pairs of bases of some associative algebras and Lie algebras. Super RSK describes a correspondence between objects called restricted superbiwords and semistandard supertableaux (see [3, 5, 8, 13, 15] for some examples of these combinatorial objects arising in the study of superalgebras and their representations). Although there are multiple bijections between these objects (see [1, 28] for example), the one that produces the semistandard multiset partition tableaux used in Section 5 turns out to be that introduced in [17].

A variant of RSK that inserts arrays of multisets was introduced in [4], in which the authors used this algorithm to describe bijections like Equation (2) for decompositions of diagram algebras such as the partition algebra (introduced in [10, 16]) and the multiset partition algebra (introduced in [18, 23]).

Diagram algebras have been used to address long-standing questions about the representation theory of the symmetric group. For example:

- In [2], the authors use the partition algebra to compute a positive formula for some Kronecker coefficients.

- In [21, 22] the authors use the partition algebra to introduce a basis of symmetric functions in order to study reduced Kronecker coefficients and restriction coefficients.

- In [19], the authors study the uniform block permutation algebra and the connections between its characters and the plethysm operation on symmetric functions.

This paper is organized as follows:

In Section 3, we adapt the insertion algorithm for multiset partitions introduced in [4] by replacing the underlying RSK algorithm with the super RSK algorithm of [17].

In Section 4, we introduce the mixed multiset partition algebra $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$, which generalizes the multiset partition algebras of [18] and [23]. We introduce a basis for the algebra and describe a diagrammatic product on this basis, which generalizes a basis introduced for the multiset partition algebra in [29]. When x is specialized to an integer $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$, the algebra $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ is isomorphic to the centralizer $\mathrm{End}_{S_n}(\mathrm{Sym}^{\boldsymbol{a}}(V_n) \otimes \bigwedge^{\boldsymbol{b}} V_n)$ where V_n is an n-dimension complex vector space.

In Section 5 we construct the simple modules $W_{\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}}^\lambda$ of $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ for $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$. The super multiset partition insertion algorithm of Section 3 is an enumerative manifestation of the decomposition

$$\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n) \cong \bigoplus_{\lambda \in \Lambda^{\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}} (W_{\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}}^\lambda)^* \otimes W_{\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}}^\lambda$$

and as such is employed in this section to prove that a spanning set for $W^\lambda_{\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}}$ is in fact a basis.

Finally, in Section 6 we use these results to recover a combinatorial interpretation for the decomposition of a multivariate polynomial ring as an S_n-module given in [20] as well as provide an interpretation for its decomposition as a GL_n-module using analogous tableaux.

2 Preliminaries and Definitions

2.1 Set Partitions and Multiset Partitions

A *set partition* ρ of a set S is a set of nonempty subsets of S called *blocks* whose disjoint union is S. We write $\ell(\rho)$ for the number of blocks of ρ. A *multiset* of size r from a set S is a collection of r unordered elements of S that can be repeated. We will write multisets in $\{\!\{\,,\}\!\}$ to differentiate them from sets and we will usually denote them by a capital letter with a tilde. We may write multisets using exponential notation $\tilde{S} = \{\!\{{s_1}^{m_1}, \ldots, {s_k}^{m_k}\}\!\}$ where the multiplicity of the element s_i is given by the exponent m_i. We write $m_{s_i}(\tilde{S}) = m_i$ for this multiplicity. Given multisets $\tilde{S} = \{\!\{{s_1}^{m_1}, \ldots, {s_k}^{m_k}\}\!\}$ and $\tilde{R} = \{\!\{{s_1}^{n_1}, \ldots, {s_k}^{n_k}\}\!\}$, write $\tilde{S} \uplus \tilde{R} = \{\!\{{s_1}^{m_1+n_1}, \ldots, {s_k}^{m_k+n_k}\}\!\}$ for the union of the two multisets. A *multiset partition* $\tilde{\rho}$ of a multiset $\tilde{S}$ is a multiset of multisets called blocks whose union is $\tilde{S}$. We write $\ell(\tilde{\rho})$ for the number of blocks of $\tilde{\rho}$.

We will be interested in set and multiset partitions with elements from the following four alphabets of positive integers:

$$[r] = \{1, \ldots, r\} \qquad\qquad [\overline{r}] = \{\overline{1}, \ldots, \overline{r}\}$$
$$[\underline{r}] = \{\underline{1}, \ldots, \underline{r}\} \qquad\qquad [\overline{\underline{r}}] = \{\overline{\underline{1}}, \ldots, \overline{\underline{r}}\}$$

We call the numbers $[\overline{r}]$ and $[\overline{\underline{r}}]$ *barred*, and we call the numbers $[\underline{r}]$ and $[\overline{\underline{r}}]$ *underlined*. Within each alphabet we order the numbers as usual, and between alphabets we adopt the convention that
$$i < \overline{j} < \underline{k} < \overline{\underline{m}}$$
for all $i, j, k,$ and m.

A *weak composition* of an integer r of length k is a sequence of k non-negative integers that sum to r. Write $\mathcal{W}_{r,k}$ for the set of weak compositions of r of length k. For $\boldsymbol{a} \in \mathcal{W}_{r,k}$, write $\boldsymbol{a}_i$ for the ith number in the sequence. We say that $\boldsymbol{a}$ is less than or equal to $\boldsymbol{b}$ in the *dominance order*, written $\boldsymbol{a} \trianglelefteq \boldsymbol{b}$, if

$$\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_i \leqslant \boldsymbol{b}_1 + \cdots + \boldsymbol{b}_i$$

for all $1 \leqslant i \leqslant k$. For $\boldsymbol{a} \in \mathcal{W}_{r,k}$, write $[k]^{\boldsymbol{a}} = \{\!\{1^{\boldsymbol{a}_1}, \ldots, k^{\boldsymbol{a}_k}\}\!\}$.

For $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m}$, we define the following sets of set and multiset partitions where in each set we assume that no barred entry is ever repeated in a block of a multiset partition. Let

- $\Pi_{2(r)}$ be the set of set partitions of $[r] \cup [\underline{r}]$,

- $\Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ be the set of multiset partitions of $[k]^{\boldsymbol{a}} \uplus [\overline{m}]^{\boldsymbol{b}} \uplus [\underline{k}]^{\boldsymbol{a}} \uplus [\underline{\overline{m}}]^{\boldsymbol{b}}$, and

- $\hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$ be the set of multiset partitions in $\Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ where repeated blocks have an even number of barred entries. We call these *restricted multiset partitions*.

Example 1.

$$\{\{1,\underline{1}\},\{2,3,\underline{2},\underline{3}\}\} \in \Pi_{2(3)}$$
$$\{\{\{1,\overline{1},\underline{1},\underline{2}\}\},\{\{2,\overline{1}\}\},\{\{\underline{\overline{1}}\}\},\{\{\underline{\overline{1}}\}\}\} \in \Pi_{2((1,1),(2,0))}$$
$$\{\{\{1,\underline{1},\underline{2}\}\},\{\{2\}\},\{\{\overline{1},\underline{\overline{1}}\}\},\{\{\overline{1},\underline{\overline{1}}\}\}\} \in \hat{\Pi}_{2((1,1),(2,0))}$$

A block of a set or multiset partition is called *propagating* if it contains both an underlined element and a non-underlined element.

Let $\tilde{S}$ and $\tilde{R}$ be multisets from $[k]$. We say that $\tilde{S}$ is less than $\tilde{R}$ in the *last-letter order*, written $\tilde{S} < \tilde{R}$, if one of the following conditions hold: (i) $\tilde{S}$ is empty and $\tilde{R}$ is not, (ii) $\max(\tilde{S}) < \max(\tilde{R})$, or (iii) $\max(\tilde{S}) = \max(\tilde{R}) = u$ and $\tilde{S} \smallsetminus \{u\} < \tilde{R} \smallsetminus \{u\}$.

For $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m}$, define the *coloring map* $\kappa_{\boldsymbol{a},\boldsymbol{b}} : [r+s] \to [k] \cup [\overline{m}]$ by

$$\kappa_{\boldsymbol{a},\boldsymbol{b}}(i) = \begin{cases} 1 & 1 \leqslant i \leqslant \boldsymbol{a}_1 \\ 2 & \boldsymbol{a}_1 < i \leqslant \boldsymbol{a}_1 + \boldsymbol{a}_2 \\ \vdots & \\ k & \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_{k-1} < i \leqslant r \\ \overline{1} & r < i \leqslant r + \boldsymbol{b}_1 \\ \vdots & \\ \overline{m} & r + \boldsymbol{b}_1 + \cdots + \boldsymbol{b}_{m-1} < i \leqslant r + s \end{cases}.$$

We can extend this map to a map from subsets of $[r+s]$ to multisets with elements in $[k] \cup [\overline{m}]$ where $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\{i_1,\ldots,i_\ell\}) = \{\kappa_{\boldsymbol{a},\boldsymbol{b}}(i_1),\ldots,\kappa_{\boldsymbol{a},\boldsymbol{b}}(i_\ell)\}$. Finally, we can extend $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ to $\Pi_{2(r+s)}$ by applying $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ to each block where we set $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\underline{i}) = \underline{\kappa_{\boldsymbol{a},\boldsymbol{b}}(i)}$. Note that each $\tilde{\pi} \in \Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ is in the image $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\Pi_{2(r)})$, but this image also contains multiset partitions with blocks having repeated barred elements.

Example 2. Here we give an example of a set partition that does map to an element of $\Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ under the coloring map.

$$\kappa_{(2,1),(0,2)}\left(\{\{1,2,5\},\{3,4,\underline{3}\},\{\underline{2},\underline{4}\},\{\underline{1},\underline{5}\}\}\right)$$
$$= \{\{\{1,1,\overline{2}\}\},\{\{2,\overline{2},\underline{2}\}\},\{\{\underline{1},\underline{\overline{2}}\}\},\{\{\underline{1},\underline{\overline{2}}\}\}\}$$

Let $\tilde{\pi} \in \Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ and let $\pi \in \kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\tilde{\pi})$ be such that if $\kappa_{\boldsymbol{a},\boldsymbol{b}}(i) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(j)$ and $i < j$, then the block containing i in π is weakly before the block containing j in the last-letter order. The unique such element is called the *standardization* of $\tilde{\pi}$.

Example 3. The standardization of a multiset partition can be computed by first putting its blocks in last-letter order, then for each value, replacing it with its pre-image under $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ increasing left-to-right.

$$\left\{\left\{\left\{1,1,\overline{2}\right\}\right\},\left\{\left\{2,\overline{2},\underline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{\left\{1,2,\overline{2}\right\}\right\},\left\{\left\{2,\overline{2},\underline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{\left\{1,2,\overline{2}\right\}\right\},\left\{\left\{3,\overline{2},\underline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{\left\{1,2,4\right\}\right\},\left\{\left\{3,5,\underline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{\left\{1,2,4\right\}\right\},\left\{\left\{3,5,\underline{2}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{2},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{\left\{1,2,4\right\}\right\},\left\{\left\{3,\underline{3}\right\}\right\},\left\{\left\{\underline{1},\overline{2}\right\}\right\},\left\{\left\{\underline{2},\overline{2}\right\}\right\}\right\}$$
$$\left\{\left\{1,2,4\right\},\left\{3,5,\underline{3}\right\},\left\{\underline{1},\underline{4}\right\},\left\{\underline{2},\underline{5}\right\}\right\}$$

Remark 4. If $\pi = \{A_1,\ldots,A_\ell\}$ is the standardization of $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$, then its blocks satisfy

$$A_i < A_j \implies \kappa_{\boldsymbol{a},\boldsymbol{b}}(A_i) \leqslant \kappa_{\boldsymbol{a},\boldsymbol{b}}(A_j).$$

2.2 Diagrams

For a set partition $\pi \in \Pi_{2(r)}$, there is a classical graph-theoretic representation of π on two rows of vertices with the top row being labeled 1 through r and the bottom being labeled $\underline{1}$ through $\underline{r}$. Two vertices of this graph are connected if and only if their labels are in the same block of π.

Example 5. The set partition $\pi = \{\{1,\underline{1},2,\underline{3}\},\{2,3\},\{\underline{4}\},\{4,5,\underline{5}\}\}$ could be represented by either of the following two graphs.

Because there are many graphs that represent the set partition π, we consider two graphs equivalent if their connected components give rise to the same set partition. The *diagram* of π is the equivalence class of graphs with the same connected components.

We can similarly consider a graph-theoretic representation of any multiset partition $\tilde{\pi} \in \Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ with $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m}$. This time we place $r + s$ vertices on the top labeled by $[k]^{\boldsymbol{a}} \uplus [\overline{m}]^{\boldsymbol{b}}$ in weakly increasing order and place $r + s$ vertices on the bottom labeled by $[\underline{k}]^{\boldsymbol{a}} \uplus [\overline{\underline{m}}]^{\boldsymbol{b}}$ in weakly increasing order. We then connect the vertices in any way so that the labeled connected components taken together are $\tilde{\pi}$.

Example 6. The multiset partition

$$\tilde{\pi} = \left\{\left\{\left\{1,1\right\}\right\},\left\{\left\{1,\underline{1},\underline{1},\underline{1}\right\}\right\},\left\{\left\{\overline{1}\right\}\right\},\left\{\left\{2,\overline{\underline{1}},\underline{1}\right\}\right\},\left\{\left\{\underline{2},\overline{\underline{1}}\right\}\right\}\right\}$$

could be represented by either of the following graphs:

Again the *diagram* of $\tilde{\pi}$ is the equivalence class of graphs whose labeled connected components give $\tilde{\pi}$.

We will often drop the labels on these graphs. A set partition diagram will be distinguished by the black color of its vertices, and a multiset partition diagram will be distinguished by its colored vertices. Its vertices are understood to be labeled with **blue**, **orange**, **green**, and **purple** representing 1, 2, $\bar{1}$, and $\bar{2}$ respectively, where the vertices labeled with barred elements are further distinguished by being drawn as open circles.

The multiset partition diagram for $\tilde{\pi} \in \Pi_{2(\boldsymbol{a},\boldsymbol{b})}$ drawn in *standard form* is the diagram for the standardization π of $\tilde{\pi}$ with vertices labeled i colored with the color $\kappa_{\boldsymbol{a},\boldsymbol{b}}(i)$. The right-most diagram in Example 6 is drawn in standard form.

2.3 Tableaux

A *partition* of n is a weakly-decreasing sequence λ of positive integers called *parts* summing to n. We write $\lambda \vdash n$ to mean that λ is a partition of n. We write λ_i for the ith element of the sequence λ, called the ith part of λ. Given a partition λ, its *Young diagram* is an array of left-justified boxes where the ith row from the bottom has λ_i boxes. For example, the Young diagram of $(5, 3, 3, 1) \vdash 12$ is
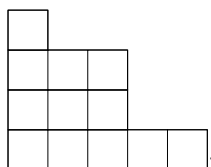
When we refer to the ith row of a Young diagram, we mean the ith row from the bottom, which corresponds to the ith part of λ.

A *standard Young tableau* t of shape $\lambda \vdash n$ is a filling of these boxes with the numbers $[n]$ whose rows increase left-to-right and columns increase bottom-to-top. Write $\mathcal{SYT}(\lambda)$ for the set of standard Young tableaux of shape λ.

A *semistandard Young tableau* is a filling of the Young diagram of λ with positive integers with rows increasing left-to-right and columns increasing *weakly* bottom-to-top. Write $\mathcal{SSYT}(\lambda, r)$ for the set of semistandard Young tableaux of shape λ and maximum entry r.

A *multiset partition tableau* $\tilde{T}$ of shape λ is a filling of the Young diagram of λ with multisets along with at least λ_2 empty boxes in the first row (so that no nonempty box in the first row is adjacent to a box in the second row). The *content* of a tableau $\tilde{T}$ is the multiset partition consisting of the contents of non-empty boxes of $\tilde{T}$. For $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and

$\boldsymbol{b} \in \mathcal{W}_{s,m}$, write $\mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$ for the set of multiset partition tableaux of shape λ with content a multiset partition of $[k]^{\boldsymbol{a}} \uplus [\overline{m}]^{\boldsymbol{b}}$.

A *standard multiset partition tableau* is a multiset partition tableau T whose content is a set partition and whose rows increase left-to-right and columns increase bottom-to-top with respect to the last-letter order. Write $\mathcal{SMT}(\lambda, r)$ for the set of standard multiset partition tableaux of shape λ and content a set partition of $[r]$.

Finally, a *semistandard multiset partition tableau* $\hat{T}$ is a multiset partition tableau such that:

1. The rows and columns of $\hat{T}$ weakly increase in the last-letter order

2. For two identical blocks B and C of the content of $\hat{T}$,

   (a) If the blocks contain an even number of barred entries, then B and C cannot be placed in the same column.

   (b) If the blocks contain an odd number of barred entries, then B and C cannot be placed in the same row.

Write $\mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$ for the set of semistandard multiset partition tableaux of shape λ with content a multiset partition of $[k]^{\boldsymbol{a}} \uplus [\overline{m}]^{\boldsymbol{b}}$.

Example 7. Examples of these classes of tableaux of the same shape $\lambda = (5, 2, 1)$:

Given $T \in \mathcal{SMT}(\lambda, r + s)$ and $\boldsymbol{a} \in \mathcal{W}_{r,k}, \boldsymbol{b} \in \mathcal{W}_{s,m}$, write $\kappa_{\boldsymbol{a},\boldsymbol{b}}(T)$ for the result of applying $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ to the content of each box of T. For a multiset partition tableau $\tilde{T} \in \mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$, define its *standardization* to be the unique $T \in \kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\tilde{T})$ whose content is the standardization of the content of T such that for any two boxes with contents $A < B$ with $\kappa_{\boldsymbol{a},\boldsymbol{b}}(A) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(B)$, the box containing A is weakly to the right and weakly below the box containing B.

Example 8. Here we compute the standardization T of a semistandard multiset partition tableau $\tilde{T}$. At each step, we simultaneously replace all boxes with a particular content, proceeding in the last-letter order.

Definition 9. For multiset partition tableaux $\tilde{T}$ and $\tilde{S}$, let $\{\tilde{B}_1, \ldots, \tilde{B}_\ell\}$ be the set of distinct contents of the boxes of $\tilde{T}$ and $\tilde{S}$ in last-letter order. For $1 \leqslant i \leqslant \ell$, define the weak composition $\alpha^{\tilde{B}_i}(\tilde{T})$ by

$$\alpha^{\tilde{B}_i}(\tilde{T})_j = \#\text{boxes with content} \leqslant \tilde{B}_i \text{ in column } j.$$

We say that $\tilde{T}$ is larger than $\tilde{S}$ in the *column dominance order*, written $\tilde{S} \vartriangleleft \tilde{T}$, if $\alpha^{\tilde{B}_i}(\tilde{S}) \vartriangleleft \alpha^{\tilde{B}_i}(\tilde{T})$ for all $1 \leqslant i \leqslant \ell$.

Example 10. Two semistandard multiset partition tableaux compared in the column-dominance order:

$$
\begin{array}{ccc}
\begin{array}{|c|c|c|}
\hline
\bar{1} & 1\bar{2} & \\
\hline
\bar{1} & 1\bar{2} & \\
\hline
1 & 1 & 2 \\
\hline
\end{array}
& \vartriangleleft &
\begin{array}{|c|c|c|}
\hline
\bar{1} & 1\bar{2} & \\
\hline
2 & 1\bar{2} & \\
\hline
1 & 1 & \bar{1} \\
\hline
\end{array}
\end{array}
$$

α^1	$(1,1,0)$	$(1,1,0)$
α^2	$(1,1,1)$	$(2,1,0)$
$\alpha^{\bar{1}}$	$(3,1,1)$	$(3,1,1)$
$\alpha^{1\bar{2}}$	$(3,3,1)$	$(3,3,1)$

Remark 11. If the content of T is the standardization of a multiset partition as in Remark 4, then $\alpha^{\tilde{B}}(\kappa_{a,b}(T)) = \alpha^B(T)$ where B is the largest block of the content of T in last-letter order such that $\kappa_{a,b}(B) = \tilde{B}$. It's then clear to see that if T and S have this property and $T \vartriangleleft S$, then $\kappa_{a,b}(T) \vartriangleleft \kappa_{a,b}(S)$.

Example 12. Here we show the composition sequence for a semistandard multiset partition tableau along with that of its standardization:

$$
\begin{array}{ccc}
\begin{array}{|c|c|c|}
\hline
\bar{1} & 1\bar{2} & \\
\hline
2 & 1\bar{2} & \\
\hline
1 & 1 & \bar{1} \\
\hline
\end{array}
& &
\begin{array}{|c|c|c|}
\hline
7 & 49 & \\
\hline
5 & 38 & \\
\hline
1 & 2 & 6 \\
\hline
\end{array}
\end{array}
$$

		α^1	$(1,0,0)$	
α^1	$(1,1,0)$	α^2	$(1,1,0)$	
α^2	$(2,1,0)$	α^5	$(2,1,0)$	
		α^6	$(2,1,1)$	
$\alpha^{\bar{1}}$	$(3,1,1)$	α^7	$(3,1,1)$	
		α^{38}	$(3,2,1)$	
$\alpha^{1\bar{2}}$	$(3,3,1)$	α^{49}	$(3,3,1)$	

2.4 Symmetric Functions

Symmetric functions are a powerful tool for studying the representation theory of S_n and GL_n. The ring of symmetric functions consists of formal power series in an infinite number

of variables $X = \{x_1, x_2, \dots\}$ that are fixed under any permutation of a finite number of the variables. A special class of symmetric functions are the *Schur functions* s_λ for λ a partition. The Schur functions are defined by

$$s_\lambda = \sum_T x^T$$

where the sum is taken over all semistandard Young tableaux of shape λ and x^T is the monomial where the exponent of x_i is the number of times i occurs in the tableau T.

When all but the first n variables in the Schur function s_λ are set to zero, we obtain a polynomial $s_\lambda(X_n)$ in the variables $X_n = \{x_1, \dots, x_n\}$. Given a matrix $A \in GL_n$, evaluating $s_\lambda(X_n)$ at the eigenvalues of A gives the character value for the simple polynomial representation $W^\lambda_{GL_n}$. Using this fact, we will use symmetric function identities in Section 6.2 to describe the decomposition of a polynomial GL_n-module into simple modules. To that end, we introduce two special cases of Schur functions and the simple modules they compute the characters of: The *elementary symmetric function*

$$e_k = s_{(1^k)} = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

corresponds to the exterior power $\bigwedge^k \mathbb{C}^n$ and the *complete homogeneous symmetric function*

$$h_k = s_{(k)} = \sum_{i_1 \leqslant i_2 \leqslant \dots \leqslant i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

corresponds to the symmetric power $\mathrm{Sym}^k(\mathbb{C}^n)$.

2.5 Mutual Centralizers and the Partition Algebra

One natural place that decompositions like (1) appear is the setting of mutually centralizing actions. Let $V_n = \mathbb{C}^n$ be an n-dimensional vector space over $\mathbb{C}$. The symmetric group S_r acts on $V_n^{\otimes r}$ by permuting tensor factors. That is, for $\sigma \in S_r$,

$$\sigma.(v_1 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(r)}$$

where we apply σ^{-1}—rather than σ—to the subscripts to ensure that S_r acts on the left. The general linear group GL_n acts on $V_n^{\otimes r}$ diagonally where for $M \in GL_n$,

$$M.(v_1 \otimes \cdots \otimes v_r) = (Mv_1) \otimes \cdots \otimes (Mv_r).$$

According to classical Schur–Weyl duality, these actions are mutually centralizing. The consequences of such a pair of mutually centralizing actions is given generally in the following theorem called the double centralizer theorem. For a semisimple algebra A, write Λ^A for an indexing set of simple A-modules.

Theorem 13. *[24, Section 6.2.5][6, Section 4.2.1]Let A be a semisimple algebra acting faithfully on a module V and set $B = \mathrm{End}_A(V)$. Then B is semisimple and $\mathrm{End}_B(V) \cong A$. Furthermore, there is a set P (a subset of the indexing set of the simple A-modules) such that for each $x \in P$, W_A^x is a simple A-module occurring in the decomposition of V as an A-module. If we set $W_B^x = \mathrm{Hom}(W_A^x, V)$, then W_B^x is a simple B-module and the decomposition of V as an $A \times B$-module is*

$$V \cong \bigoplus_{x \in P} W_A^x \otimes W_B^x.$$

Moreover, the dimension of W_A^x is equal to the multiplicity of W_B^x in V as a B-module and the dimension of W_B^x is equal to the multiplicity of W_A^x in V as an A-module.

We are interested in applying the double centralizer theorem to the following situation. Let

$$e \in \mathrm{End}_{GL_n}({V_n}^{\otimes r}) \cong \mathbb{C}S_r$$

be an idempotent. Then $e{V_n}^{\otimes r}$ is a GL_n-module, and hence by restriction to the $n \times n$ permutation matrices an S_n-module. Then

$$B = \mathrm{End}_{S_n}(e{V_n}^{\otimes r}) \cong e\,\mathrm{End}_{S_n}({V_n}^{\otimes r})e$$

is a semisimple algebra. The following is a well-known fact (see [7, Section 6.2] or [14, Theorem 1.10.14]), which will help us study the simple B-modules.

Proposition 14. *Let A be an algebra and $e \in A$ an idempotent. Then eAe is an algebra with identity e, and the following can be said about its simple modules.*

1. *If S is a simple A-module, then eS is either zero or a simple eAe-module.*

2. *All simple eAe-modules arise in this way. That is, if T is a simple eAe-module, then there is a simple A-module S such that $eS \cong T$.*

When $n \geqslant 2r$, the centralizer algebra $\mathrm{End}_{S_n}({V_n}^{\otimes r})$ is isomorphic to the Partition algebra $P_r(n)$ introduced by Jones [10] and Martin [16] as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics, so the algebra B is given by $B \cong eP_r(n)e$. We devote the remainder of this section to describing the structure of $P_r(n)$ and its simple modules for $n \geqslant 2r$.

2.5.1 Diagram Basis of $P_r(x)$ and S_r actions

For r a positive integer and an indeterminate x, the partition algebra $P_r(x)$ is an associative algebra over $\mathbb{C}(x)$. When x is specialized to an integer $n \geqslant 2r$, the algebra $P_r(n)$ is isomorphic to the algebra of endomorphisms $\mathrm{End}_{S_n}({V_n}^{\otimes r})$ of ${V_n}^{\otimes r}$ as an S_n-module. The partition algebra has a basis called the *diagram basis* $\{\mathcal{L}_\pi : \pi \in \Pi_{2(r)}\}$ whose product has a combinatorial interpretation in terms of partition diagrams. To compute the product of $\mathcal{L}_\pi$ and $\mathcal{L}_\nu$, place a graph representing π above one representing ν and identify the vertices

on the bottom of π with the corresponding vertices of ν to create a three-tiered diagram. Let $\pi \circ \nu$ be the restriction of this diagram to the very top and very bottom, preserving which vertices are connected in the larger diagram and let $c(\pi, \nu)$ be the number of components entirely in the middle of the three-tiered diagram. Then $\mathcal{L}_\pi \mathcal{L}_\nu = x^{c(\pi, \nu)} \mathcal{L}_{\pi \circ \nu}$.

Example 15. Here we show the product of two diagram basis elements. Notice that two components are entirely in the middle, giving a coefficient of x^2.
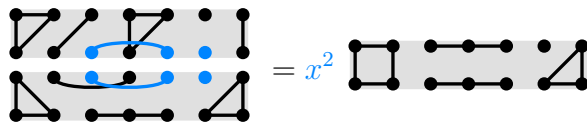
The symmetric group S_r sits naturally inside $P_r(x)$ where the permutation $\sigma \in S_r$ corresponds to the set partition $\{\{\sigma(1), \underline{1}\}, \ldots, \{\sigma(r), \underline{r}\}\}$. Writing $\mathcal{L}_\sigma$ for the corresponding diagram-basis element, the product $\mathcal{L}_{\sigma_1} \mathcal{L}_\pi \mathcal{L}_{\sigma_2}$ leads to an action of $S_r \times S_r$ on $\Pi_{2(r)}$ where we write $\sigma_1.\pi.\sigma_2$ for the resulting set partition so that

$$\mathcal{L}_{\sigma_1} \mathcal{L}_\pi \mathcal{L}_{\sigma_2} = \mathcal{L}_{\sigma_1.\pi.\sigma_2}.$$

The set partition $\sigma_1.\pi.\sigma_2$ is obtained from π by replacing each non-underlined element i with $\sigma_1(i)$ and each underlined element $\underline{i}$ with $\underline{\sigma_2^{-1}(i)}$.

The following straightforward fact will be useful later.

Lemma 16. *The subgroup* $(S_r \times S_r)^\pi$ *fixing a set partition* $\pi \in \Pi_{2(r)}$ *decomposes as a semidirect product*

$$(S_r \times S_r)^\pi = \left(\prod_{B \in \pi} K_B \right) \rtimes H$$

where K_B *consists of permutations within the block* B *and* H *consists of permutations of whole blocks of* π *with the same number of underlined and non-underlined elements.*

Example 17. For $\pi = \{\{1, 2, \underline{1}\}, \{3, 4, \underline{2}\}, \{5\}, \{\underline{3}, \underline{4}, \underline{5}\}\}$, the subgroup $(S_5 \times S_5)^\pi$ of permutations that fix π decomposes as above where

$$
\begin{aligned}
K_{\{1,2,\underline{1}\}} &= S_{\{1,2\}} \times S_{\{\underline{1}\}} \\
K_{\{3,4,\underline{2}\}} &= S_{\{3,4\}} \times S_{\{\underline{2}\}} \\
K_{\{5\}} &= S_{\{5\}} \\
K_{\{\underline{3},\underline{4},\underline{5}\}} &= S_{\{\underline{3},\underline{4},\underline{5}\}} \\
H &= \{\epsilon, (1\,3)(2\,4)(\underline{1}\,\underline{2})\}.
\end{aligned}
$$

2.5.2 Representations of $P_r(n)$

In [9], the authors construct the simple modules $W_{P_r}^\lambda$ of $P_r(n)$ for $n \geqslant 2r$. For $\lambda \in \Lambda^{P_r(n)}$, the module $W_{P_r}^\lambda$ consists of formal linear combinations of standard multiset partition tableaux

$$W_{P_r}^\lambda := \mathbb{C}\{v_T : T \in \mathcal{SMT}(\lambda, r)\}$$

with action given as follows. For a set partition $\pi \in \Pi_{2(r)}$ to act on a tableau T, first pull out the content of T, a set partition of $[r]$, into a single row. Then, put π on top of this row and identify the corresponding vertices. Form T' by replacing the content of each box in T with the set of vertices at the top of π that the box is connected to, and for each block entirely in the top of π, include it as the content of a box in the first row of T'. If two blocks above the first row are combined or the content of a box above the first row does not connect to the top of the partition diagram, the result is zero.

Example 18. Here we show the action of two different diagrams on the same tableau.
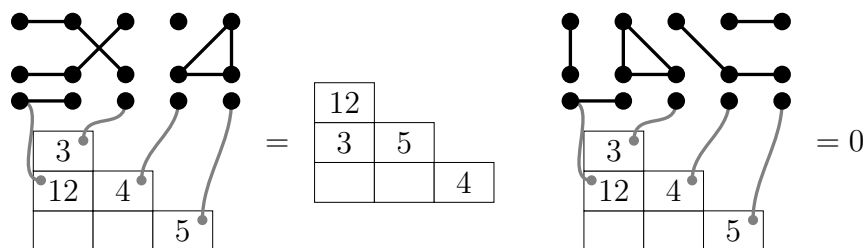
The result T of the above process may not have increasing rows and columns, so we need to make sense of what it means to write v_T for T nonstandard. The algorithm for writing v_T as a linear combination of standard tableaux is called the *straightening algorithm*. The straightening algorithm for $W_{P_r}^\lambda$ is the same as for the Specht modules of S_n applied to the rows above the first row of T, a fact which is clear from the construction in [9, Section 3.4]. A complete treatment of this algorithm for Specht modules can be found in [26, Section 2.6], but we summarize the process here.

Let $\lambda \in \Lambda^{P_r(n)}$. Because S_r sits inside $P_r(n)$, there is a natural S_r-action on $W_{P_r}^\lambda$ where σ acts by replacing each value i in $T \in \mathcal{SMT}(\lambda, r)$ with $\sigma(i)$. Setting $m = \lambda_2 + \cdots + \lambda_\ell$, there is also an action of S_m by permuting the boxes above the first row of T.

The simplest step of the straightening algorithm involves permuting boxes within the same column. Let T be a multiset partition tableau with content a set partition of $[n]$. If T' is obtained from T by swapping two boxes in the same column, then

$$v_{T'} = -v_T.$$

Applying this rule repeatedly allows us to assume that the tableau we want to straighten has increasing columns.

Now suppose T is a multiset partition tableau of shape λ with increasing columns. We say that T has a *decrease* at boxes a and b if the box b is immediately to the right of a in the same row and the content of a is larger than the content of b. Suppose T has a decrease at boxes a and b and let A be the set of boxes including a and each box above it and let B be the set of boxes including b and each box below it. The *Garnir transversal* $g(A, B)$ consists of all permutations of the boxes in $A \cup B$ so that their contents still increase up columns of T.

The following theorem is an immediate consequence of applying [26, Theorem 2.6.4] to the construction of $W_{P_r}^\lambda$ in [9].

Theorem 19. *Suppose T is a multiset partition tableau with content a set partition of $[r]$ that has a decrease along a row. Let $g(A, B)$ be the Garnir transversal for the corresponding subsets A and B of boxes of T as above. Then*

$$\sum_{\sigma \in g(A,B)} \operatorname{sgn}(\sigma) v_{\sigma.T} = 0$$

where $T \lhd \sigma.T$ for $\sigma \neq \epsilon$.

Using Theorem 19, we can write any nonstandard T as a linear combination of tableaux strictly larger in the column dominance order. Because the tableaux continue to increase in this order and there are only finitely many tableaux of a given content, this process eventually terminates, writing v_T as a linear combination of v_S for $S \in \mathcal{SMT}(\lambda, r)$.

Example 20. We apply one iteration of the straightening algorithm to the following tableau:

$$T = \begin{array}{c}\boxed{69}\\ \boxed{8}\,\boxed{35}\\ \boxed{2}\,\boxed{4}\\ \qquad\boxed{17}\end{array}$$

$$A = \{\{8\}, \{6, 9\}\}$$
$$B = \{\{4\}, \{3, 5\}\}$$

To conveniently write permutations of the boxes of T, we label the boxes above the first row by the following standard Young tableau:

$$\begin{array}{cc}\boxed{5}&\\ \boxed{3}&\boxed{4}\\ \boxed{1}&\boxed{2}\end{array} \quad .$$

Now the permutations of these boxes that leave the columns of T increasing are:

$$g(A, B) = \{\epsilon, (3\,4), (2\,4\,3), (4\,5\,3), (2\,4\,5\,3), (2\,3)(4\,5)\}.$$

By Theorem 19, we get the following equation (where we draw the tableau S in place of v_S):

Now we can solve for v_T on the left as a linear combination of tableaux with the decrease eliminated.

Remark 21. Note that in the above example, we are considering an action of S_5 on the boxes above the first row rather than the usual action of S_9 on the content of the tableau. In the case that the size of the contents of the boxes is preserved, we can embed this action of S_5 into S_9. For example, the permutation $(2\,3)(4\,5) \in S_5$ in the above example corresponds to $(4\,8)(3\,6)(5\,9) \in S_9$ so that

$$v_{(2\,3)(4\,5).T} = (4\,8)(3\,6)(5\,9)v_T.$$

This observation will be very helpful in developing an analogous straightening algorithm for semistandard multiset partition tableaux.

2.6 Super RSK

A *superalphabet* is a totally ordered set $\mathscr{A}$ with a map $\varepsilon : \mathscr{A} \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Recall that in $\mathbb{Z}/2\mathbb{Z}$, we have that $1 + 1 = 0$. If $\varepsilon(a) = 0$ we say that a is *even*, and if $\varepsilon(a) = 1$ we say that a is *odd*. For superalphabets $\mathscr{A}$ and $\mathscr{B}$, we call an element of $\mathscr{A} \times \mathscr{B}$ a *biletter* (a, b). We define a total order[1] on biletters where $(a_1, b_1) \prec (a_2, b_2)$ if either

1. $a_1 < a_2$

2. $a_1 = a_2$ and one of the following hold:

    (a) $\varepsilon(b_1) = 1$ and $\varepsilon(b_2) = 0$.
    (b) $\varepsilon(b_1) = \varepsilon(b_2) = 0$ and $b_1 < b_2$.
    (c) $\varepsilon(b_1) = \varepsilon(b_2) = 1$ and $b_1 > b_2$.

A biletter is *mixed* if $\varepsilon(a) + \varepsilon(b) = 1$. A *biword* of length m is a sequence $w = ((a_1, b_1), \dots, (a_m, b_m)) \in (\mathscr{A} \times \mathscr{B})^m$. A biword is called *ordered* if $(a_i, b_i) \preceq (a_j, b_j)$ whenever $i < j$ and called *restricted* if no mixed biletter is repeated. For $\tilde{A}$ a multiset of elements in $\mathscr{A}$ and $\tilde{B}$ a multiset of elements in $\mathscr{B}$ with $|\tilde{A}| = |\tilde{B}| = m$, write $\mathcal{ORBW}(\tilde{A}, \tilde{B})$ for the set of ordered restricted biwords $((a_1, b_1), \dots, (a_m, b_m))$ where $\{\!\{a_1, \dots, a_m\}\!\} = \tilde{A}$ and $\{\!\{b_1, \dots, b_m\}\!\} = \tilde{B}$.

Let $\tilde{A}$ be a multiset from a superalphabet $\mathscr{A}$. A *semistandard supertableau* of shape λ with content $\tilde{A}$ is a filling T of the Young diagram of λ with elements of A so that

1. The rows and columns of T are non-decreasing.

2. T is row-strict in the odd letters: if a appears twice in the same row, then $\varepsilon(a) = 0$.

3. T is column-strict in the even letters: if a appears twice in the same column, then $\varepsilon(a) = 1$.

Write $\mathcal{SSST}(\lambda, \tilde{A})$ for the set of semistandard supertableaux of shape λ with content $\tilde{A}$.

Following [17], we now define two operations on the set of semistandard supertableaux called 0-insertion and 1-insertion.

---

[1]In [17], the author denotes this order with a_i and b_i swapped as $\lhd$ and the order on the second coordinate given below as $\prec$.

Definition 22. The process of performing 0-insertion of a value a into a tableau depends on whether a is odd or even. We describe the process compactly here by writing the instructions for an even value first in orange and then the instructions for an odd value in blue.

Suppose a is (even/odd). To perform 0-insertion of a into a (row/column) of a tableau T,

(i) If a is larger than or equal to each element in the (row/column), place it at the end of the (row/column) and terminate.

(ii) Otherwise, consider the (left/bottom)-most box in the (row/column) with content larger than a and call it the bump site. Write b for the content of the bump site, and replace the content of the bump site with a.

(iii) If b is even, insert it into the row above the bump site. If b is odd, insert it into the column to the right of the bump site.

Write $T \xleftarrow{0} a$ for the result of inserting a into the first (row/column) of T.

Example 23. We give an example of 0-insertion where the alphabet is the positive integers with the usual map into $\mathbb{Z}/2\mathbb{Z}$. We color the odd numbers blue and even numbers orange to highlight which rules are being applied from Definition 22
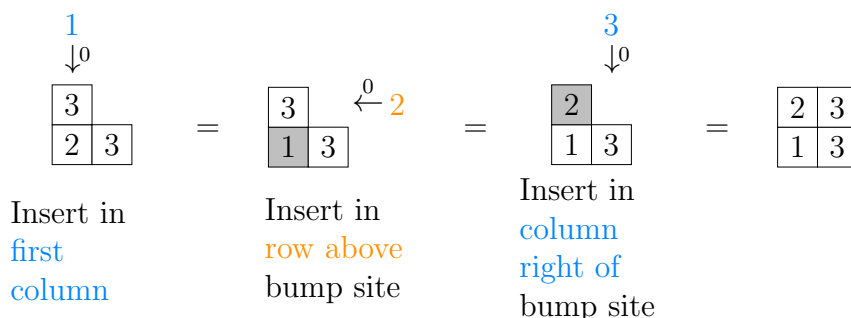
Similarly, define 1-insertion as in Definition 22, swapping "row" and "column," modifying step (i) to check if a is *strictly* larger, and modifying step (ii) to look for a box with content larger than or *equal* to a.

Example 24. We repeat the previous example, inserting 1 via 1-insertion instead.
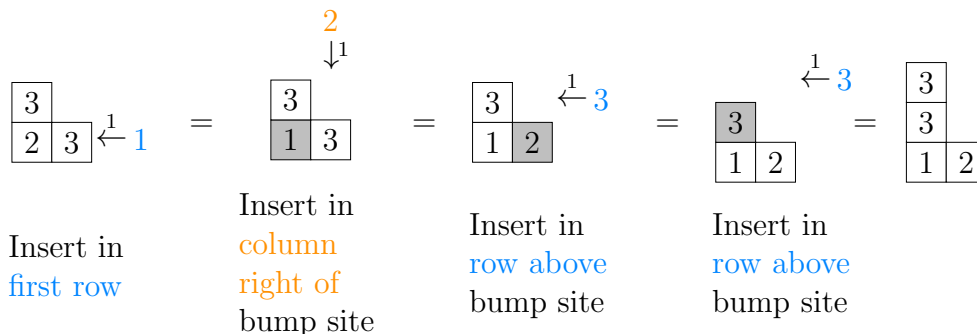
These dual notions of 0- and 1-insertion give rise to the following algorithm called *super RSK*.

Definition 25. For $w = ((a_1, b_1), \ldots, (a_m, b_m)) \in (\mathscr{A} \times \mathscr{B})^m$ an ordered, restricted biword, define $P_w^0 = \emptyset$ and $Q_w^0 = \emptyset$. Then inductively define

$$P_w^i = P_w^{i-1} \overset{\varepsilon(a_i)}{\longleftarrow} b_i$$

and define Q_w^i to be Q_w^{i-1} along with a_i in the box created in the insertion to P_w^{i-1}. Write $P_w = P_w^m$ and $Q_w = Q_w^m$ and

$$\mathrm{sRSK}(w) = (P_w, Q_w).$$

Theorem 26. *[17, Theorem 5.2] For any multiset $\tilde{A}$ from $\mathscr{A}$ and multiset $\tilde{B}$ from $\mathscr{B}$ with $|\tilde{A}| = |\tilde{B}| = m$, the map sRSK defines a bijection*

$$\mathcal{ORBW}(\tilde{A}, \tilde{B}, m) \overset{\sim}{\longleftrightarrow} \biguplus_{\lambda \vdash m} \mathcal{SSST}(\lambda, \tilde{A}) \times \mathcal{SSST}(\lambda, \tilde{B})$$

3 Super Multiset Partition Insertion

In this section, we present a modified version of the insertion algorithm on multiset partitions introduced in [4] built on the super RSK algorithm. Let $\mathscr{M}$ be the set of multisets with elements in $\{1, 2, \ldots\} \cup \{\bar{1}, \bar{2}, \ldots\}$ ordered by the last-letter order. This is a superalphabet with grading given by the parity of the number of barred elements:

$$\varepsilon(\tilde{S}) = \begin{cases} 0 & \tilde{S} \text{ has an even number of barred elements} \\ 1 & \tilde{S} \text{ has an odd number of barred elements} \end{cases}.$$

Definition 27. Let $\hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a}, \boldsymbol{b})}$. Let $\left\{\!\left\{ \tilde{S}_1, \ldots, \tilde{S}_m \right\}\!\right\}$ be the multiset of propagating blocks of of $\hat{\pi}$. Write $\tilde{S}_i^-$ for the multiset of underlined values in $\tilde{S}_i$ with the underlines removed and write $\tilde{S}_i^+$ for the remaining elements. Define the *biword associated to $\hat{\pi}$* to be the ordered biword

$$w(\hat{\pi}) = \begin{pmatrix} \tilde{S}_1^+ & \cdots & \tilde{S}_m^+ \\ \tilde{S}_1^- & \cdots & \tilde{S}_m^- \end{pmatrix} \in (\mathscr{M} \times \mathscr{M})^m.$$

By definition of the set $\hat{\Pi}_{2(\boldsymbol{a}, \boldsymbol{b})}$, the biword $w(\tilde{\pi})$ is restricted.

Definition 28. Let $\hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a}, \boldsymbol{b})}$ and $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$. Then $\mathrm{sRSK}(w(\hat{\pi})) = (P, Q)$ is a pair of semistandard supertableaux of the same shape μ with content equal to the restrictions of the propagating blocks of $\hat{\pi}$ to the underlined and non-underlined elements respectively. Note that n is at least twice the number of propagating blocks of $\hat{\pi}$, so in particular $n - |\mu| \geqslant |\mu|$. Hence, we can form a tableau P' of shape $\lambda = (n - |\mu|, \mu)$ from P by adding a first row consisting of a number of empty boxes followed by the blocks of

$\hat{\pi}$ with only non-underlined elements. Do the same for Q' adding the blocks with only underlined elements. Let

$$\mathrm{smRSK}(\hat{\pi}) = (P', Q').$$

Proposition 29. *For $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$, the map* smRSK *is a bijection:*

$$\hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})} \xleftrightarrow{\ \sim\ } \biguplus_{\substack{\lambda \vdash n \\ \mathcal{SSMT}(\lambda,\boldsymbol{a},\boldsymbol{b}) \neq \emptyset}} \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b}) \times \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b}) \qquad (3)$$

Proof. Note that $\hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$ cannot have two identical non-propagating blocks with an odd number of barred elements. From this observation and Theorem 26, it's clear that the image of $\hat{\pi}$ under smRSK is a pair of semistandard multiset partition tableaux in $\mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$.

Given such a pair, we can recover the multiset partition $\hat{\pi}$ by first removing the boxes in the first row to obtain the non-propagating blocks and then reversing super RSK on the remaining tableaux to form the propagating blocks. $\qquad\square$

Example 30. We give the result of smRSK on a restricted multiset partition $\hat{\pi}$.

$$w(\hat{\pi}) = \begin{pmatrix} 2 & 12 & \overline{1} & \overline{1} \\ \overline{12} & 2 & \overline{1} & 12 \end{pmatrix}$$

$$sRSK(w(\hat{\pi})) = \left(\begin{array}{|c|c|} \hline 12 & \overline{12} \\ \hline 2 & \overline{1} \\ \hline \end{array} \ , \ \begin{array}{|c|c|} \hline 12 & \overline{1} \\ \hline 2 & \overline{1} \\ \hline \end{array} \right)$$

$$smRSK(\hat{\pi}) = \left(\begin{array}{|c|c|c|c|} \hline 12 & \overline{12} & & \\ \hline 2 & \overline{1} & & \\ \hline & & \dots & 11 \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|c|c|} \hline 12 & \overline{1} & & & \\ \hline 2 & \overline{1} & & & \\ \hline & & \dots & 11 & \overline{2} \\ \hline \end{array} \right)$$

We will see in the next two sections that this bijection is an enumerative manifestation of the decomposition of an algebra $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ as a bimodule over itself.

4 The mixed multiset partition algebra $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$

Let $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m}$. The subgroup $S_{\boldsymbol{a}} \times S_{\boldsymbol{b}} \subseteq S_{r+s}$ consists of the permutations of $[r+s]$ that respect the coloring map $\kappa_{\boldsymbol{a},\boldsymbol{b}}$. Explicitly,

$$S_{\boldsymbol{a}} \times S_{\boldsymbol{b}} = \left(S_{\kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(1)} \times \cdots \times S_{\kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(k)} \right) \times \left(S_{\kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\overline{1})} \times \cdots \times S_{\kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\overline{m})} \right).$$

Example 31.

$$S_{(3,2)} \times S_{(2,1,1)} = S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6,7\}} \times S_{\{8\}} \times S_{\{9\}}.$$

We will usually write elements of $S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}$ as $\sigma\tau$ with $\sigma \in S_{\boldsymbol{a}}$ (which we identify with $S_{\boldsymbol{a}} \times 1 \subset S_{r+s}$) and $\tau \in S_{\boldsymbol{b}}$ (which we identify with $1 \times S_{\boldsymbol{b}} \subset S_{r+s}$).

Let $e_{\boldsymbol{a},\boldsymbol{b}}$ be the idempotent given by

$$e_{\boldsymbol{a},\boldsymbol{b}} = \frac{1}{|S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}|} \sum_{\sigma\tau \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}} \text{sgn}(\tau)\sigma\tau \in \mathbb{C}S_{r+s} \subseteq P_{r+s}(x).$$

Define the *mixed multiset partition algebra* over $\mathbb{C}(x)$ by

$$\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(x) := e_{\boldsymbol{a},\boldsymbol{b}}P_{r+s}(x)e_{\boldsymbol{a},\boldsymbol{b}}.$$

We begin this section by providing a basis for $\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$ indexed by elements of $\hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$.

Lemma 32. *Let G be a finite group acting on a finite set S, and let $f : G \to \{\pm 1\}$ be a group homomorphism. let $\mathbb{C}S$ be the formal $\mathbb{C}$-span of the elements of S. For $s \in S$,*

$$\sum_{g \in G} f(g)g.s = 0 \tag{4}$$

if and only if s is fixed by an element $g \in G$ with $f(g) = -1$.

Proof. First, suppose Equation (4) holds. Then in particular, the coefficient on s in the sum is zero. Because s occurs with positive coefficient (when g is the identity element), it must occur with a negative coefficient as well. Hence, there exists $g \in G$ with $g.s = s$ and $f(g) = -1$.

Conversely, suppose there exists $g \in G$ with $g.s = s$ and $f(g) = -1$. Let $H = \langle g \rangle \subseteq G$ be the subgroup generated by g. Let $t_1, \ldots, t_k$ be a transversal of H in G so that $G = \biguplus_{i=1}^{k} t_i H$. Then

$$\sum_{g \in G} f(g)g.s = \sum_{i=1}^{k} \sum_{j=1}^{|H|} f(t_i g^j) t_i g^j.s$$

$$= \sum_{i=1}^{k} f(t_i) \left(\sum_{j=1}^{|H|} f(g^j) \right) t_i.s$$

$$= \sum_{i=1}^{k} f(t_i) \left(\sum_{j=1}^{|H|} (-1)^j \right) t_i.s.$$

Because $f(g) = -1$, it must be that $|H|$ is even, so $\sum_{j=1}^{|H|} (-1)^j = 0$ and hence Equation (4) holds. $\square$

Lemma 33. *For $\pi \in \Pi_{2(r+s)}$,*

$$e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} = 0$$

if and only if $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) \notin \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$.

Proof. For ease of notation, let $G = (S_{\boldsymbol{a}} \times S_{\boldsymbol{b}})^2$. Applying Lemma 32 with $f : G \to \{\pm 1\}$ given by $f(\sigma_1 \tau_1, \sigma_2 \tau_2) = \operatorname{sgn}(\tau_1) \operatorname{sgn}(\tau_2)$ gives us that $e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} = 0$ if and only if π is fixed by some element

$$(\sigma_1 \tau_1, \sigma_2 \tau_2) \in (S_{\boldsymbol{a}} \times S_{\boldsymbol{b}})^2$$

with $\operatorname{sgn}(\tau_1) \operatorname{sgn}(\tau_2) = -1$.

By Lemma 16, the subgroup $(S_{r+s} \times S_{r+s})^\pi$ of $S_{r+s} \times S_{r+s}$ fixing π is given by

$$(S_{r+s} \times S_{r+s})^\pi = \left(\prod_{B \in \pi} K_B \right) \rtimes H.$$

To restrict to the subgroup G^π of $G \subseteq S_{r+s} \times S_{r+s}$ fixing π we take intersections, yielding

$$G^\pi = \left(\prod_{B \in \pi} K_B \cap G \right) \rtimes (H \cap G).$$

Then there exists $g \in G^\pi$ with $f(g) = -1$ if and only if one or both of the following hold:

(i) There exists $g \in K_B \cap G$ for some $B \in \pi$ with $f(g) = -1$.

(ii) There exists $g \in H \cap G$ with $f(g) = -1$.

In the first case, an element of $K_B \cap G$ is a permutation of elements within the block B that preserves which color the element is mapped to under $\kappa_{\boldsymbol{a},\boldsymbol{b}}$. There exists $g \in K_B \cap G$ with $f(g) = -1$ if and only if there are at least two elements of B that map to the same barred element under $\kappa_{\boldsymbol{a},\boldsymbol{b}}$.

In the latter case, an element of $H \cap G$ is a permutation of the blocks of π that only sends a block B to another block mapped to the same multiset under $\kappa_{\boldsymbol{a},\boldsymbol{b}}$. There exists $g \in H \cap G$ with $f(g) = -1$ if and only if there is a repeated block of $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$ with an odd number of barred elements.

Hence, $e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} = 0$ if and only if $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) \notin \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$. $\qquad \square$

Definition 34. Let $\hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$ and let π be its standardization. Define

$$\mathcal{D}_{\hat{\pi}} := e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}}.$$

Theorem 35. *The set $\{\mathcal{D}_{\hat{\pi}} : \hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}\}$ forms a basis for $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$.*

Proof. By Lemma 33, $e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} = 0$ unless $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$. Hence, the set

$$\{e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} : \pi \in \Pi_{2(r+s)}, \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}\}$$

spans $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$. Furthermore, if $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\nu)$, then $\pi = (\sigma_1 \tau_1, \sigma_2 \tau_2).\nu$ for some $(\sigma_1 \tau_1, \sigma_2 \tau_2) \in (S_{\boldsymbol{a}} \times S_{\boldsymbol{b}})^2$, so $e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}} = \operatorname{sgn}(\tau_1) \operatorname{sgn}(\tau_2) e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\nu e_{\boldsymbol{a},\boldsymbol{b}}$. We then need only one representative from each preimage $\kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\hat{\pi})$, and so the set

$$\{\mathcal{D}_{\hat{\pi}} : \hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}\}$$

spans $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$. Because the elements $\{\mathcal{D}_{\hat{\pi}} : \hat{\pi} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}\}$ are nonzero sums over distinct $(S_{\boldsymbol{a}} \times S_{\boldsymbol{b}})^2$-orbits, they are linearly independent and hence form a basis of $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$. $\quad \square$

Remark 36. This description of the basis leads immediately to a product formula. Given $\hat{\pi}, \hat{\nu} \in \hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}$ and π, ν their standardizations,

$$\begin{aligned}
\mathcal{D}_{\hat{\pi}}\mathcal{D}_{\hat{\nu}} &= (e_{\boldsymbol{a},\boldsymbol{b}}\mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}})(e_{\boldsymbol{a},\boldsymbol{b}}\mathcal{L}_\nu e_{\boldsymbol{a},\boldsymbol{b}}) \\
&= \frac{1}{|S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}|} \sum_{\sigma\tau \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}} \operatorname{sgn}(\tau) e_{\boldsymbol{a},\boldsymbol{b}}\mathcal{L}_\pi \sigma\tau\mathcal{L}_\nu e_{\boldsymbol{a},\boldsymbol{b}} \\
&= \frac{1}{|S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}|} \sum_{\sigma\tau \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}} \operatorname{sgn}(\tau) x^{c(\pi,\sigma\tau.\nu)} e_{\boldsymbol{a},\boldsymbol{b}}\mathcal{L}_{\pi \circ (\sigma\tau.\nu)} e_{\boldsymbol{a},\boldsymbol{b}}.
\end{aligned}$$

On diagrams, this amounts to the following process. First draw the diagrams for $\hat{\pi}$ and $\hat{\nu}$ in standard form and place the former on top of the latter. Then average over all permutations of identically colored vertices on top of the diagram for $\hat{\nu}$, taking their product as set partition diagrams and recording the sign of the permutation of the open circles.

Example 37. We compute the product of and in $\mathbb{MP}_{(2),(2)}(x)$:

Notice that the last diagram does not correspond to a restricted multiset partition, so it projects to zero by Lemma 33.

4.1 Realizing $\mathbb{MP}_{a,b}(x)$ as a centralizer algebra

Let $\boldsymbol{a} \in \mathcal{W}_{r,k}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m}$. When x is specialized to an integer $n \geqslant 2r + 2s$, we can realize $\mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ as a centralizer algebra of the diagonal S_n-action on $\operatorname{Sym}^{\boldsymbol{a}}(V_n) \otimes \bigwedge^{\boldsymbol{b}} V_n$ as follows. First, considering $(V_n)^{\otimes r+s}$ as an S_{r+s}-module via permutation of tensor factors, note that

$$\operatorname{Sym}^{\boldsymbol{a}}(V_n) \otimes \bigwedge\nolimits^{\boldsymbol{b}} V_n \cong e_{\boldsymbol{a},\boldsymbol{b}}(V_n)^{\otimes r+s}.$$

Hence,

$$\begin{aligned}
\operatorname{End}_{S_n}(\operatorname{Sym}^{\boldsymbol{a}}(V_n) \otimes \bigwedge\nolimits^{\boldsymbol{b}} V_n) &\cong e_{\boldsymbol{a},\boldsymbol{b}} \operatorname{End}_{S_n}(V_n^{\otimes r+s}) e_{\boldsymbol{a},\boldsymbol{b}} \\
&\cong e_{\boldsymbol{a},\boldsymbol{b}} P_{r+s}(n) e_{\boldsymbol{a},\boldsymbol{b}} \\
&\cong \mathbb{MP}_{\boldsymbol{a},\boldsymbol{b}}(n).
\end{aligned}$$

Remark 38. If $n < r + s$, then the centralizer algebra $\text{End}_{S_n}(V_n^{\otimes r+s})$ is the subalgebra of $P_r(n)$ consisting of diagram basis elements indexed by set partitions with at most n blocks. Hence, the requirement above that $n \geqslant 2r + 2s$ is necessary to achieve the full algebra $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ as a centralizer and for Theorem 13 to guarantee semisimplicity.

5   Representations of $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$

In order to construct the simple $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$-modules, we make the following definition inspired by Proposition 14.

Definition 39. Let $\boldsymbol{a} \in \mathcal{W}_{r,k}$, $\boldsymbol{b} \in \mathcal{W}_{s,m}$ and $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$. For $\lambda \in \Lambda^{P_{r+s}(n)}$, define

$$W_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}^{\lambda} := e_{\boldsymbol{a},\boldsymbol{b}} W_{P_{r+s}}^{\lambda}.$$

Let $\Lambda^{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}$ be the set of $\lambda \in \Lambda^{P_{r+s}(n)}$ for which $W_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}^{\lambda} \neq 0$. By Proposition 14 and the double centralizer theorem, we know that

$$\{W_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}^{\lambda} : \lambda \in \Lambda^{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}\}$$

forms a complete set of simple $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$-modules. This section is devoted to providing bases for these modules.

Definition 40. For $\tilde{T} \in \mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$, let T be its standardization and define

$$y_{\tilde{T}} := e_{\boldsymbol{a},\boldsymbol{b}} v_T.$$

For $S \in \kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\tilde{T})$, let $\sigma_S \tau_S \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}$ be any permutation so that $\sigma_S \tau_S v_S = v_T$. Then

$$e_{\boldsymbol{a},\boldsymbol{b}} v_S = \text{sgn}(\tau_S) y_{\tilde{T}}.$$

Lemma 41. *Recall that an even multiset is one with an even number of barred elements. If $\tilde{T} \in \mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$ has a repeated even entry in a column, then $y_{\tilde{T}} = 0$.*

Proof. If $\tilde{T}$ has a repeated even entry in a column, then any $T \in \kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\tilde{T})$ has two boxes with contents R and S in the same column with $\kappa_{\boldsymbol{a},\boldsymbol{b}}(R) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(S)$ having an even number of barred elements. For any permutation $\sigma\tau \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}$ with $\sigma\tau(R) = S$ and $\sigma\tau(S) = R$ that fixes every other element of $[r]$, τ must then be an even permutation. Furthermore, $\sigma v_T = v_{T'}$ where T' is the tableau obtained by swapping the two boxes in the same column. By the straightening algorithm for set partition tableaux, we have that $\sigma\tau v_T = -v_T$. Applying $e_{\boldsymbol{a},\boldsymbol{b}}$ to both sides yields the following:

$$e_{\boldsymbol{a},\boldsymbol{b}} \sigma\tau v_T = -e_{\boldsymbol{a},\boldsymbol{b}} v_T$$
$$\text{sgn}(\tau) e_{\boldsymbol{a},\boldsymbol{b}} v_T = -e_{\boldsymbol{a},\boldsymbol{b}} v_T$$
$$e_{\boldsymbol{a},\boldsymbol{b}} v_T = -e_{\boldsymbol{a},\boldsymbol{b}} v_T,$$

so $e_{\boldsymbol{a},\boldsymbol{b}} v_T = 0$ and hence $y_{\tilde{T}} = 0$. $\qquad\qquad\square$

We may then assume that our tableaux $\tilde{T}$ do not have repeated even entries in their columns.

Lemma 42. *Let $\tilde{T} \in \mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$ and suppose $\tilde{T}$ has a decrease (or a weak decrease of odd entries) along a row. Then*

$$y_{\tilde{T}} = \sum_{\tilde{S}} c_{\tilde{S}} y_{\tilde{S}}$$

where the sum is over $\tilde{S}$ where the decrease (or weak decrease) has been eliminated and $\tilde{S}$ is larger than $\tilde{T}$ in the column dominance order.

Proof. Suppose $\tilde{T}$ has a decrease (or weak decrease of odd entries) in a row. Then there is an $S \in \kappa_{\boldsymbol{a},\boldsymbol{b}}^{-1}(\tilde{T})$ that has the same content as the standardization of $\tilde{T}$ but has a decrease $a > b$ in the same position as the (weak) decrease in $\tilde{T}$. Let A be the set of boxes of S including a and above and B be the set of boxes including b and below. Let $g(A, B)$ be the Garnir transversal associated to these subsets. Then,

$$\sum_{\eta \in g(A,B)} \operatorname{sgn}(\eta) v_{\eta.S} = 0$$

where each $\eta.S$ has the decrease removed and is larger than S in the column-dominance order. Then multiplying by $e_{\boldsymbol{a},\boldsymbol{b}}$, we get

$$\sum_{\eta \in g(A,B)} \operatorname{sgn}(\eta) e_{\boldsymbol{a},\boldsymbol{b}} v_{\eta.S} = \sum_{\eta \in g(A,B)} \operatorname{sgn}(\eta) \operatorname{sgn}(\tau_{\eta.S}) y_{\kappa_{\boldsymbol{a},\boldsymbol{b}}(\eta.S)} = 0$$

where $\sigma_{\eta.S} \tau_{\eta.S} \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}$ is any element so that $\sigma_{\eta.S} \tau_{\eta.S} v_{\eta.S} = v_T$ where T is the standardization of $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\eta.S)$.

Each $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\eta.S) \neq \tilde{T}$ has the decrease (or weak decrease) eliminated and by Remark 11 is larger than $\tilde{T}$ in the column dominance order. We then need only show that the coefficient on $y_{\tilde{T}}$ in the above sum is nonzero.

If $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\eta.S) = \tilde{T}$, then η only permutes boxes of S that are identical under $\kappa_{\boldsymbol{a},\boldsymbol{b}}$. That is, there exists a permutation $\sigma\tau \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}$ such that $v_{\eta.S} = \sigma\tau v_S$ (see Remark 21). We now compute an element $\sigma_{\eta.S} \tau_{\eta.S}$ such that $\sigma_{\eta.S} \tau_{\eta.S} v_{\eta.S} = v_T$ starting from such an element for S.

$$\sigma_S \tau_S v_S = v_T$$
$$\sigma_S \tau_S (\sigma\tau)^{-1} \sigma\tau v_S = v_T$$
$$(\sigma_S \sigma^{-1})(\tau_S \tau^{-1}) v_{\eta.S} = v_T$$

Hence, we may choose $\sigma_{\eta.S} \tau_{\eta.S} = (\sigma_S \sigma^{-1})(\tau_S \tau^{-1})$. Then,

$$\operatorname{sgn}(\eta) \operatorname{sgn}(\tau_{\eta.S}) = \operatorname{sgn}(\eta) \operatorname{sgn}(\tau_S \tau^{-1})$$
$$= \operatorname{sgn}(\eta) \operatorname{sgn}(\tau) \operatorname{sgn}(\tau_S).$$

Note that because we can assume that A and B are strictly increasing in even entries and we are not considering the case of a weak even decrease, $A \cup B$ has no repeated even entries. Hence, η permutes only odd boxes, leaving the even boxes fixed, so $\text{sgn}(\eta) = \text{sgn}(\tau)$. Then,

$$\text{sgn}(\eta)\,\text{sgn}(\tau_{\eta.S}) = \text{sgn}(\tau_S).$$

Because each $y_{\tilde{T}}$ in the sum appears with the same nonzero coefficient $\text{sgn}(\tau_S)$, the coefficient on $y_{\tilde{T}}$ is nonzero, and we have written $y_{\tilde{T}}$ as the desired linear combination. $\square$

Theorem 43. *The set $\{y_{\hat{T}} : \hat{T} \in \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})\}$ forms a basis for $W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}$.*

Proof. Clearly, the set $\{e_{\boldsymbol{a},\boldsymbol{b}} v_T : T \in \mathcal{SMT}(\lambda, r+s)\}$ spans $W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}$, and so by Lemma 41 the vectors $y_{\tilde{T}}$ for $\tilde{T} \in \mathcal{MT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$ without a repeated even entry in any column span $W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}$. If $\tilde{T}$ has a repeated odd-sized entry within a row, we can apply Lemma 42 to write it as a linear combination of multiset partition tableaux that do not have this repeat. The resulting tableaux may have repeated odd entries or decreases in other locations, in which case we repeatedly apply the lemma. Because the tableaux are always larger in the column dominance order, this process eventually terminates, writing $y_{\tilde{T}}$ as a linear combination of $y_{\hat{S}}$ where $\hat{S} \in \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})$. Hence, the set $\{y_{\hat{T}} : \hat{T} \in \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})\}$ spans $W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}$.

Comparing dimensions in the decomposition of $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$ as a bimodule over itself and applying the super multiset RSK bijection (3), we obtain the following equation.

$$\sum_{\lambda \in \Lambda^{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}} (\dim(W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}))^2 = \dim(\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n))$$
$$= |\hat{\Pi}_{2(\boldsymbol{a},\boldsymbol{b})}|$$
$$= \sum_{\lambda \in \Lambda^{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}} |\mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})|^2$$

The spanning set above shows that $\dim(W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}) \leqslant |\mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})|$ for each λ, so the above equality implies that $\dim(W^\lambda_{\text{MP}_{\boldsymbol{a},\boldsymbol{b}}}) = |\mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})|$ and the spanning set $\{y_{\hat{T}} : \hat{T} \in \mathcal{SSMT}(\lambda, \boldsymbol{a}, \boldsymbol{b})\}$ is in fact a basis. $\square$

The construction of the simple $\text{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$-modules as projections of simple $P_{r+s}(n)$-modules leads to a straightforward formula for the action of a multiset partition on a semistandard multiset partition tableau. That is, for $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$,

$$e_{\boldsymbol{a},\boldsymbol{b}} \mathcal{L}_\pi e_{\boldsymbol{a},\boldsymbol{b}}.e_{\boldsymbol{a},\boldsymbol{b}} v_T = \sum_{(\sigma,\tau) \in S_{\boldsymbol{a}} \times S_{\boldsymbol{b}}} \text{sgn}(\tau) e_{\boldsymbol{a},\boldsymbol{b}} (\mathcal{L}_{\pi.\sigma\tau}.v_T).$$

In terms of diagrams, this formula amounts to first pulling out the content of the tableau as a one-row diagram, placing the diagram for $\tilde{\pi}$ on top. Then average over all permutations of identically-colored vertices at the bottom of $\tilde{\pi}$, noting the sign of the permutation of the open circles. The action is then computed exactly as for the partition algebra and the results summed with the appropriate sign.

Example 44. Here we show the action of a multiset partition on a tableau. Note that the permutation swapping the orange vertices stabilizes the diagram, so we include a factor of two to account for this multiplicity.

5.1 Cellularity

In this section, we briefly remark on the above results in the language of cellular algebras, which have distinguished bases encoding representation-theoretic information.

Definition 45. An algebra A over $\mathbb{C}$ is *cellular* with *cell datum* (Λ, i, M, C) if the following conditions are satisfied.

(C1) The finite set Λ is partially ordered, to each element $\lambda \in \Lambda$, there is an associated finite set $M(\lambda)$, and A has a $\mathbb{C}$-basis

$$\left\{ C_{S,T}^{\lambda} : (S,T) \in M(\lambda) \times M(\lambda), \lambda \in \Lambda \right\}.$$

(C2) The map i is a linear anti-automorphism of A with $i^2 = 1$ and $i(C_{S,T}^{\lambda}) = C_{T,S}^{\lambda}$.

(C3) For each $\lambda \in \Lambda$, each $S, T \in M(\lambda)$, and each $a \in A$,

$$aC_{S,T}^{\lambda} = \sum_{U \in M(\lambda)} r_a(U,S) C_{U,T}^{\lambda} + r'$$

where r' is a linear combination of basis elements with upper index μ strictly smaller than λ and the coefficients $r_a(U,S) \in \mathbb{C}$ do not depend on T.

The basis $C^\lambda_{S,T}$ is called a *cellular basis*. In [30, Theorem 4.1], the author proves that $P_r(x)$ is a cellular algebra. It has a cellular basis with associated combinatorial data:

- The set $\Lambda = \{\lambda \vdash k | k \in \{0, 1, \ldots, r\}\}$ is ordered under the graded dominance order.

- The map i acts by reflecting the partition diagram vertically.

- The set $M(\lambda) = \mathcal{SMT}((2r - |\lambda|, \lambda), r)$ consists of the set partition tableaux of shape $(2r - |\lambda|, \lambda)$.

Because the anti-involution i for $P_r(x)$ takes a diagram corresponding to a permutation in S_r to its inverse, we see that $i(e_{\boldsymbol{a},\boldsymbol{b}}) = e_{\boldsymbol{a},\boldsymbol{b}}$. It is then a consequence of [12, Proposition 4.3] that $\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$ is cellular with respect to the restriction of i. The following theorem is then a consequence of the construction of the irreducible representations of $\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)$.

Theorem 46. *The algebra* $\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(x)$ *has a cellular basis with associated combinatorial data:*

- *The set* $\Lambda = \{(\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)}) : \lambda \in \Lambda^{\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(2r)}\}$ *is ordered under the graded dominance order.*

- *The map i acts by reflecting the partition diagram vertically.*

- *The set* $M(\lambda) = \mathcal{SSMT}((2r - |\lambda|, \lambda), \boldsymbol{a}, \boldsymbol{b})$ *consists of the multiset partition tableaux of shape* $(2r - |\lambda|, \lambda)$.

6 Decompositions of Multivariate Polynomial Rings

In this final section, we apply the above results on super RSK and the mixed multiset partition algebra to describe how a certain multivariate polynomial ring decomposes as an S_n-module and GL_n-module. Let

$$\mathbb{C}[X_{n \times m}; \Theta_{n \times m'}] := \mathbb{C}[x_{i,j}, \theta_{i,j} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant j' \leqslant m']$$

where the variables $x_{i,j}$ commute with all variables and $\theta_{i,j}\theta_{a,b} = -\theta_{a,b}\theta_{i,j}$ if either $i \neq a$ or $j \neq b$ and $\theta_{i,j}^2 = 0$. For $r + s = n$, $\boldsymbol{a} \in \mathcal{W}_{r,m}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m'}$, write

$$\mathbb{C}[X_{n \times m}; \Theta_{n \times m'}]^{\boldsymbol{a},\boldsymbol{b}}$$

for the subspace spanned by monomials with degree $\boldsymbol{a}_t$ in the variables $x_{i,t}$ and degree $\boldsymbol{b}_t$ in the variables $\theta_{i,t}$.

Example 47.

$$x_{11}x_{11}x_{31}x_{22}x_{32}\theta_{11}\theta_{21}\theta_{42}\theta_{23} \in \mathbb{C}[X_{3 \times 5}; \Theta_{3 \times 4}]^{(3,2),(2,1,1)}.$$

6.1 As an S_n-module

The symmetric group S_n acts on the ring $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ by permuting the first index of the variables. That is, for $\sigma \in S_n$,

$$\sigma.x_{i,j} = x_{\sigma(i),j}$$
$$\sigma.\theta_{i,j} = \theta_{\sigma(i),j}.$$

To understand this ring as an S_n-module, we want to think of it as decomposing into projections by the idempotents $e_{\boldsymbol{a},\boldsymbol{b}}$. That is, we notice that

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]^{\boldsymbol{a},\boldsymbol{b}} \cong e_{\boldsymbol{a},\boldsymbol{b}} V_n^{\otimes m+m'}$$

as S_n-modules where the action on the latter space arises from the usual diagonal action on $V_n^{\otimes m+m'}$. The following theorem can then be recovered as a straightforward application of Theorem 13 and Theorem 43.

Theorem 48. [20, Theorem 3.1] Let $n \geqslant 2|\boldsymbol{a}| + 2|\boldsymbol{b}|$. Then as an S_n-module,

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]^{\boldsymbol{a},\boldsymbol{b}} \cong \bigoplus_{\lambda \in \Lambda^{\mathrm{MP}_{\boldsymbol{a},\boldsymbol{b}}(n)}} \left(W_{S_n}^\lambda\right)^{\oplus|\mathcal{SSMT}(\lambda,\boldsymbol{a},\boldsymbol{b})|}.$$

6.2 As a GL_n-module

If we identify $\mathbb{C}[X_{n\times m}]$ with the symmetric algebra $\mathrm{Sym}(V_n^{\otimes m})$ and $\mathbb{C}[\Theta_{n\times m'}]$ with the exterior algebra $\bigwedge(V_n^{\otimes m'})$, the standard diagonal action of GL_n on $V_n^{\otimes r}$ extends to a GL_n action on the ring $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ (see [6, Section 5.6.2] for details).

In this section we employ the super RSK correspondence to provide a combinatorial interpretation of the decomposition of this ring as a GL_n-module. To that end, we want to view the ring as decomposing into tensor products of symmetric and alternating powers. That is, we notice that

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]^{\boldsymbol{a},\boldsymbol{b}} \cong \mathrm{Sym}^{\boldsymbol{a}}(\mathbb{C}^n) \otimes \bigwedge^{\boldsymbol{b}}(\mathbb{C}^n)$$

where

$$\mathrm{Sym}^{\boldsymbol{a}}(\mathbb{C}^n) = \mathrm{Sym}^{\boldsymbol{a}_1}(\mathbb{C}^n) \otimes \cdots \otimes \mathrm{Sym}^{\boldsymbol{a}_m}(\mathbb{C}^n)$$

and

$$\bigwedge^{\boldsymbol{b}}(\mathbb{C}^n) = \bigwedge^{\boldsymbol{b}_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge^{\boldsymbol{b}_{m'}}(\mathbb{C}^n).$$

From this perspective, it's clear that the character of this subspace as a GL_n module is given by $h_{\boldsymbol{a}}(X_n) e_{\boldsymbol{b}}(X_n)$ where

$$h_{\boldsymbol{a}} = h_{\boldsymbol{a}_1} \cdots h_{\boldsymbol{a}_m} \text{ and } e_{\boldsymbol{b}} = e_{\boldsymbol{b}_1} \cdots e_{\boldsymbol{b}_{m'}}.$$

Let $\mathcal{SSMT}'(\lambda, \boldsymbol{a}, \boldsymbol{b})$ be the set of semistandard multiset partition tableaux whose content has all blocks of size one and whose unbarred and barred elements have multiplicities given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Theorem 49. *Let $\boldsymbol{a} \in \mathcal{W}_{r,m}$ and $\boldsymbol{b} \in \mathcal{W}_{s,m'}$. Then,*

$$h_{\boldsymbol{a}}e_{\boldsymbol{b}} = \sum_{\lambda \vdash r+s} |\mathcal{SSMT}'(\lambda, \boldsymbol{a}, \boldsymbol{b})| s_\lambda$$

Proof. Each monomial in $h_{\boldsymbol{a}}e_{\boldsymbol{b}} = h_{\boldsymbol{a}_1} \ldots h_{\boldsymbol{a}_m} e_{\boldsymbol{b}_1} \ldots e_{\boldsymbol{b}_{m'}}$ is of the form

$$\left(x_{i_1^{(1)}} \ldots x_{i_{\boldsymbol{a}_1}^{(1)}}\right) \ldots \left(x_{i_1^{(m)}} \ldots x_{i_{\boldsymbol{a}_m}^{(m)}}\right)\left(x_{j_1^{(1)}} \ldots x_{j_{\boldsymbol{b}_1}^{(1)}}\right) \ldots \left(x_{j_1^{(m')}} \ldots x_{j_{\boldsymbol{b}_{m'}}^{(m')}}\right)$$

where each factor $\left(x_{i_1^{(s)}} \ldots x_{i_{\boldsymbol{a}_s}^{(s)}}\right)$ is the contribution of $h_{\boldsymbol{a}_s}$ and $\left(x_{j_1^{(t)}} \ldots x_{j_{\boldsymbol{b}_t}^{(t)}}\right)$ is the contribution of $e_{\boldsymbol{b}_t}$. These monomials are in bijection with ordered restricted biwords where the monomial as written above corresponds to the biword obtained by putting the multiset

$$\left\{\!\!\left\{(s, i_1^{(s)}), \ldots, (s, i_{\boldsymbol{a}_s}^{(s)}) : 1 \leqslant s \leqslant m\right\}\!\!\right\} \uplus \left\{\!\!\left\{(\bar{t}, j_1^{(t)}), \ldots, (\bar{t}, j_{\boldsymbol{b}_t}^{(t)}) : 1 \leqslant t \leqslant m'\right\}\!\!\right\}$$

in order.

Under the super RSK correspondence, these biwords are in bijection with pairs (U, T) of the same shape $\lambda \vdash r + s$ where

1. U is a semistandard Young tableau whose content is precisely the multiplicities of the subscripts of the variables in the monomial.

2. T is a semistandard multiset partition tableau with entries all size one and multiplicities of unbarred and barred values given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Rearranging by partition shape, we can rewrite the sum

$$h_{\boldsymbol{a}}e_{\boldsymbol{b}} = \sum_{\lambda \vdash r+s} \sum_{(U,T)} x^U$$

$$= \sum_{\lambda \vdash r+s} |\mathcal{SSMT}'(\lambda, \boldsymbol{a}, \boldsymbol{b})| \sum_{U \in \mathcal{SSYT}(\lambda)} x^U$$

$$= \sum_{\lambda \vdash r+s} |\mathcal{SSMT}'(\lambda, \boldsymbol{a}, \boldsymbol{b})| s_\lambda.$$

$\square$

Example 50. Here we show the correspondence for a monomial in $h_3 h_2 e_2 e_2$.

$$(x_1 x_1 x_3)(x_2 x_3)(x_1 x_2)(x_2 x_4)$$

$$\updownarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\ 1 & 1 & 3 & 2 & 3 & 1 & 2 & 2 & 4 \end{pmatrix}$$

$$\updownarrow$$

Corollary 51. *As a GL_n-module,*

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]^{\boldsymbol{a},\boldsymbol{b}} \cong \bigoplus_{\ell(\lambda)\leqslant n} \left(W_{GL_n}^\lambda\right)^{\oplus|\mathcal{SSMT}'(\lambda,\boldsymbol{a},\boldsymbol{b})|} .$$

Acknowledgements

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