A Note on Graphs of k-Colourings

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Abstract

For a graph G, the k-colouring graph of G has vertices corresponding to proper k-colourings of G and edges between colourings that differ at a single vertex. The graph supports the Glauber dynamics Markov chain for k-colourings, and has been extensively studied from both extremal and probabilistic perspectives.

In this note, we show that for every graph G, there exists k such that G is uniquely determined by its k-colouring graph, confirming two conjectures of Asgarli, Krehbiel, Levinson and Russell. We further show that no finite family of generalised chromatic polynomials for G, which encode induced subgraph counts of its colouring graphs, uniquely determine G.

Mathematics Subject Classifications: 05C15, 05C31

1 Introduction

Let G be a graph on vertex set $V(G)$ and edge set $E(G)$. Throughout this paper, all colourings are proper, and a *k-colouring* is a proper colouring using at most k colours from a fixed palette, say $[k] \coloneqq \{1, \dots, k\}$. The *chromatic polynomial* $\pi_G(k)$ counts the number of k-colourings of G as a function of k. Chromatic polynomials were first considered for planar maps by Birkhoff [2] in 1912, and then for arbitrary graphs by Whitney [10] in 1932. Since then, they have been well-studied in the literature, with considerable interest in ways in which they can be computed, their algebraic properties, and generalisations (see [7] for a classical introduction).

A more detailed picture of the set of k-colourings of a graph G is given by the *k-colouring graph* $\mathcal{C}_k(G)$: this has vertex set the k-colourings of G, and edges between pairs of k-colourings that differ at precisely one vertex of G. Random walks on the k-colouring graph give the Glauber dynamics Markov chain, which has been extensively studied from the perspective of random sampling and approximate counting of k-colourings (see for example [4, 5, 9]). The k-colouring graph has also been investigated in the context of combinatorial reconfiguration (see, for example, the surveys in [6, Chapter 10] and [8]).

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The chromatic polynomial $\pi_G(k)$ counts the number of vertices in $\mathcal{C}_k(G)$. Asgarli, Krehbiel, Levinson and Russell [1] recently introduced a more general family of functions by replacing vertex counts with counts of instances of a fixed arbitrary graph: for graphs G and H, and $k \in \mathbb{N}$, the *generalised chromatic polynomial* $\pi_G^{(H)}(k)$ is the number of subsets of $V(\mathcal{C}_k(G))$ that induce a subgraph isomorphic to H as a function of k. Thus $\pi_G^{(K_1)}(k)$ is the chromatic polynomial of G, and $\pi_G^{(K_2)}(k)$ counts the number of edges in $\mathcal{C}_k(G)$ (see fig. 1 for an example of a structure that contributes to $\pi_{P_3}^{(C_4)}(4)$).

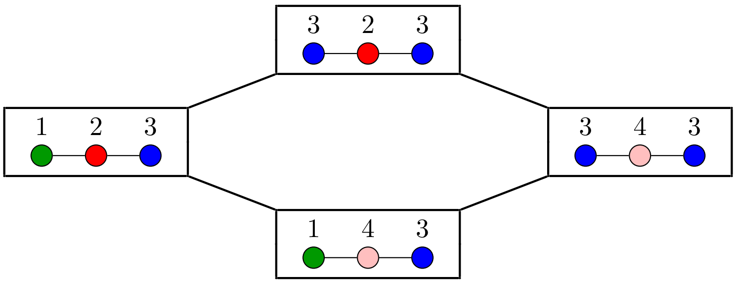

Figure 1. An induced C_4 in $\mathcal{C}_k(P_3)$ for $k \geqslant 4$.

For fixed graphs G and H, Asgarli et al. proved that $\pi_G^{(H)}(k)$ is a polynomial in k sufficiently large relative to the size of H. In Section 2, we strengthen this result to show that $\pi_G^{(H)}(k)$ is a polynomial without restriction.

Theorem 1. *For any fixed graphs G and H, the function $\pi_G^{(H)}(k)$ is a polynomial in k.*

Asgarli et al. also discuss the extent to which a graph G is determined by the invariants $\pi_G^{(H)}$. Letting $\mathcal{G}$ be the set of finite graphs, they conjecture that the collection of polynomials $\{\pi_G^{(H)}(k)\}_{H \in \mathcal{G}}$, or equivalently the collection of all colouring graphs $\{\mathcal{C}_k(G)\}_{k \in \mathbb{N}}$, is a complete graph invariant.

Conjecture 2 (Conjecture 6.1 [1])**.** For any graph G, the collection $\{\mathcal{C}_k(G)\}_{k \in \mathbb{N}}$ uniquely determines G.

They also make the stronger conjecture that finitely many colouring graphs suffice.

Conjecture 3 (Conjecture 6.2 [1])**.** There exists some function $f : \mathcal{G} \to \mathbb{N}$ such that for any graph G, the collection $\{\mathcal{C}_k(G)\}_{k=1}^{f(G)}$ uniquely determines G.

In Section 3, we confirm both conjectures by proving a stronger result.

Theorem 4. *Let G be a graph on n vertices. For any natural number $k > 5n^2$, the pair $(k,\ \mathcal{C}_k(G))$ uniquely determines the isomorphism class of G.*

Our proof of Theorem 4 provides an algorithm for reconstructing G. Note that the data used consists of the colouring graph $\mathcal{C}_k(G)$ together with the corresponding value k which we assume is known to satisfy $k > 5n^2$. The number of vertices in G is deduced from $\mathcal{C}_k(G)$ within the proof.

Since the collection of all colouring graphs $\{\mathcal{C}_k(G)\}_{k\in\mathbb{N}}$ holds the same information as the collection of generalised chromatic polynomials $\{\pi_G^{(H)}(k)\}_{H\in\mathcal{G}}$, another natural direction to investigate is whether a finite subcollection of generalised chromatic polynomials suffices to distinguish all non-isomorphic graphs. In Section 4 we give a negative answer.

Theorem 5. *No finite family of generalised chromatic polynomials is a complete graph invariant.*

We conclude with two open problems in Section 5.

2 Polynomiality

Let $\pi_G^{(H)}(k)$ denote the number of induced copies of H in $\mathcal{C}_k(G)$. We extend the standard proof of polynomiality for chromatic polynomials via partitions to generalised chromatic polynomials.

Theorem 1. *For any fixed graphs G and H, the function $\pi_G^{(H)}(k)$ is a polynomial in k.*

Proof. Let $h = |H|$ and $n = |G|$. We will say that a partition $P_1 \cup \cdots \cup P_s$ of $V(G) \times [h]$ is *valid* if $P_j \cap (V(G) \times \{i\})$ is an independent set in G for each $i \in [h]$ and each j. Fix an ordering $\prec$ on the vertices of $\mathcal{C}_k(G)$. Any collection $S = \{c_1 \prec \ldots \prec c_h\}$ of k-colourings of G defines a function $c\colon V(G) \times [h] \to [k]$ by $c(v, i) = c_i(v)$ for each $v \in V(G)$, $i \in [h]$, and c induces a valid partition $P_c = \{c^{-1}(i)\colon i \in [k]\}$ of $V(G) \times [h]$.

The graph induced by S in $\mathcal{C}_k(G)$ depends only on P_c, in the sense that if another collection of h colourings S' defines a partition $P_{c'}$ of $V(G) \times [h]$ then S and S' induce isomorphic subgraphs of $\mathcal{C}_k(G)$ if $P_c = P_{c'}$. Each partition P of $V(G) \times [h]$ thus corresponds to a fixed induced graph. Provided P is valid and consists of t non-empty parts, we can colour its parts in $(k)_t$ different ways, where $(k)_t := k(k-1)\cdots(k-t+1)$ denotes the falling factorial. Thus, each such partition that yields an induced copy of H contributes exactly $(k)_t$ induced copies of H to $\mathcal{C}_k(G)$. Writing $N_t^{(H)}$ for the number of valid partitions with exactly t parts yielding an induced copy of H, the generalised chromatic polynomial of H is given by the formula

$$\pi_G^{(H)}(k) = \sum_{t=1}^{n} N_t^{(H)} \binom{k}{t} t! = \sum_{t=1}^{n} N_t^{(H)} (k)_t \ .$$

Since each summand is a polynomial and n is fixed, $\pi_G^{(H)}$ is a polynomial as well. □

3 Complete invariance

We now prove that the collection of colouring graphs gives a complete graph invariant. Let G be a graph on n vertices. A vertex $c \in V(\mathcal{C}_k(G))$ is *rainbow* if it represents a colouring of G using n distinct colours; that is, $c(u) \neq c(v)$ for any distinct u and v in $V(G)$. Our strategy for reconstructing G is to use a $\mathcal{C}_k(G)$ with k large enough so that most vertices of

$C_k(G)$ correspond to a rainbow colouring; we will then be able to use the clique structure to reconstruct the graph.

Lemma 6. *Let G be a graph on n vertices. If c_1, c_2, c_3 are vertices of $\mathcal{C}_k(G)$ inducing a copy of K_3, then c_1, c_2, c_3 differ as colourings at a single vertex of G.*

Proof. Suppose that c_1 and c_2 differ at vertex u of G, while c_2 and c_3 differ at vertex v with $u \neq v$. Then c_1 and c_3 differ at both vertices u and v, so $c_1 c_3$ is not an edge of $\mathcal{C}_k(G)$, a contradiction. □

It follows that vertices in any clique in $\mathcal{C}_k(G)$ correspond to colourings which differ at a single vertex v of G. We say that such cliques are *generated* by v. For a colouring c in $\mathcal{C}_k(G)$, let $\mathcal{J}(c)$ be the collection of maximal cliques containing c in $\mathcal{C}_k(G)$. When $k \geqslant n+3$, $\mathcal{J}(c)$ consists of n cliques, each generated by a distinct vertex of G. Say that c is *typical* if for each $v \in G$, the clique generated by v in $\mathcal{J}(c)$ is of size $k - \deg(v)$. We note that every rainbow colouring is typical.

Lemma 7. *Let G be a graph on n vertices and take any natural number $k > 3n^2$. Then $\mathcal{C}_k(G)$ uniquely determines the degree sequence of G. Moreover, more than half of all vertices in $\mathcal{C}_k(G)$ are rainbow.*

Proof. The number of k-colourings of G is at most k^n, and there are $\binom{k}{n} \cdot n! \geqslant k^n(1 - \frac{n}{k})^n$ colourings of G using n colours. Since $k > 3n^2$, the proportion of vertices in $\mathcal{C}_k(G)$ that are rainbow is at least $(1 - \frac{n}{k})^n > 1/2$. For each vertex $c \in \mathcal{C}_k(G)$, we consider the sizes of maximal cliques containing c. The majority will be typical and therefore give the same collection of sizes. From any such typical vertex c, we can then deduce the degree sequence of G by subtracting the size of each maximal clique containing c from k. □

We are now ready to prove the main theorem.

Proof of Theorem 4. Consider a vertex c of $\mathcal{C}_k(G)$. Since we know $k \geqslant n+3$, the value of n is given by the number of cliques in $\mathcal{J}(c)$. Let $J_u, J_v \in \mathcal{J}(c)$ be two cliques generated by distinct vertices $u, v \in V(G)$ respectively. We will show that, by counting 4-cycles that contain c and intersect $J_u \setminus c$ and $J_v \setminus c$, we can determine whether u and v are adjacent in G (see fig. 2).

Claim. *Let c_0 be rainbow and $J_u, J_v \in \mathcal{J}(c_0)$ be distinct. Write t_{uv} for the number of 4-cycles containing c_0 with at least one vertex in each of $J_u \setminus c_0$ and $J_v \setminus c_0$, and $d_u = \deg(u)$, $d_v = \deg(v)$.*

- *If $uv \notin E(G)$ then $t_{uv} \geqslant k^2 - k(d_u + d_v + 2)$.*
- *If $uv \in E(G)$ then $t_{uv} \leqslant k^2 - k(d_u + d_v + 2) - k + 2n^2 + 3n$.*

Proof. Let $c_1 \in J_u \setminus c_0$ and $c_3 \in J_v \setminus c_0$. There is a 4-cycle (c_0, c_1, c_2, c_3) in $\mathcal{C}_k(G)$ precisely when there is some colouring c_2 that differs from c_0 at u and v only, and satisfies $c_2(v) = c_1(v)$ and $c_2(u) = c_3(u)$. An example of such a 4-cycle is given in fig. 1.

Suppose that $uv \notin E(G)$. Since $c_3(u)$ must be distinct from both $c_0(u)$ and the colours of any neighbour of u in c_0, there are $k - d_u - 1$ choices for c_3. Similarly, there are $k - d_v - 1$

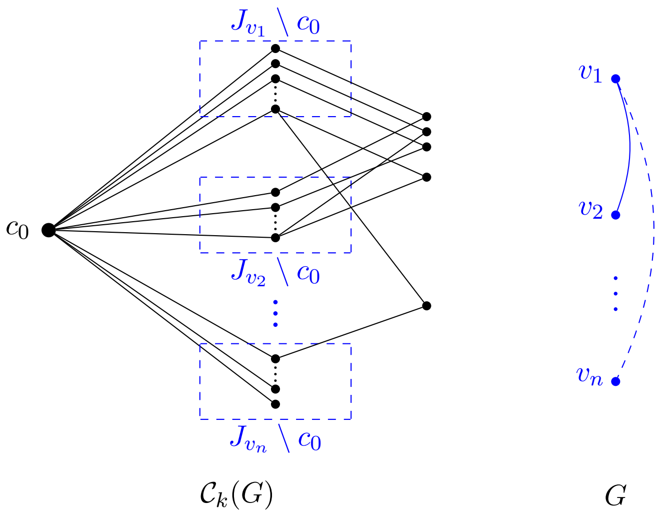

Figure 2. Detecting edges in G by counting 4-cycles containing c_0 in $\mathcal{C}_k(G)$.

choices for c_1, and hence the number of 4-cycles containing c_0 and one vertex from each of J_u and J_v is

$$k^2 - k(d_u + d_v + 2) + d_u d_v + d_u + d_v + 1 \geqslant k^2 - k(d_u + d_v + 2).$$

Next suppose that $uv \in E(G)$, and suppose first that we choose $c_3(u)$ to be a colour not used by c_0. Then there are $(k - n)$ choices for c_3, and since $c_2(v)$ must be distinct from each of $c_0(v)$, $c_3(u)$ and the colours of each neighbour of v in c_0, this leaves $(k - d_v - 2)$ choices for c_1 for a total of $(k-n)(k-d_v-2)$ pairs (c_1, c_3). Similarly, if we choose c_1 first, we count $(k-n)(k-d_u-2)$ colour pairs. Any colour pair in which both $c_1(v)$ and $c_3(u)$ are selected from the set of $k - n$ colours not used by c_0 is counted twice above, so the total number of colour pairs in which at least one of $c_1(v)$ and $c_3(u)$ is a colour not used by c_0 is

$$\begin{aligned} &(k-n)[(k-d_u-2)+(k-d_v-2)-(k-n-1)] \\ &= k^2 - k(d_u+d_v+3) + n(d_u+d_v-n+3) \\ &\leqslant k^2 - k(d_u+d_v+3) + n^2 + 3n. \end{aligned}$$

Since there are at most n^2 ways to choose $c_1(v)$ and $c_3(u)$ from colours used by c_0, we have the desired bound on t_{uv}. ■

We now build a candidate graph G_c from each vertex c of $\mathcal{C}_k(G)$ by considering pairs of cliques $J_u, J_v \in \mathcal{J}(c)$ and adding the edge uv in $E(G_c)$ whenever $t_{uv} \geqslant k^2 - k(k - |J_u| + k - |J_v| + 2)$. When $k > 5n^2$ we have $2n^2 + 3n < k$ for all positive n, so when c is rainbow, $t_{uv} \geqslant k^2 - k(d_u + d_v + 2)$ if and only if $uv \notin E(G)$. Furthermore, when c is rainbow, we have $d_u = k - |J_u|$ for each $u \in V(G)$. Substituting this term into our formula for t_{uv}, we see that our candidate graph G_c is isomorphic to G whenever c is rainbow. Since Lemma 7 guarantees that the majority of vertices in $\mathcal{C}_k(G)$ are rainbow, more than half of these candidates are isomorphic to G, and so G can be reconstructed by majority vote. □

4 Finite families of polynomials

We now work towards proving Theorem 5. First, we show that for any finite collection $\mathcal{F}$ of connected graphs, the polynomials $\{\pi_G^{(H)}(k) \colon H \in \mathcal{F}\}$ cannot distinguish all graphs. This is implied by the following result.

Lemma 8. *For each natural number m, there is a pair of non-isomorphic graphs G, G' such that for every connected graph H with at most m edges, $\pi_G^{(H)} = \pi_{G'}^{(H)}$.*

Proof. Let m be given, and choose any natural number $n > m + 1$. Consider the graph G_0 obtained from the path $v_1, \dots, v_{3n}$ by adding a new vertex v adjacent to v_{n-1} and v_n, and a new vertex v' adjacent to v_n and v_{n+1}. Define G and G' to be the subgraphs of G_0 induced by $\{v, v_1, \dots, v_{3n}\}$ and $\{v', v_1, \dots, v_{3n}\}$, respectively (see fig. 3).

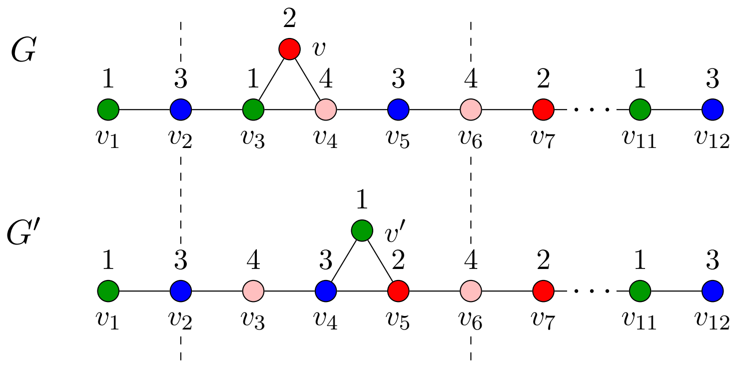

Figure 3. A k-colouring c of G for $n = 4$, and the corresponding k-colouring $f_X(c)$ of G'. In this example $v_{n-t_X} = v_2$ and $v_{n+t_X} = v_6$, so f recolours only vertices in the segment strictly between v_2 and v_6.

Fix any connected graph H with at most m edges, and some natural number k. We will show that $\pi_G^{(H)}(k) = \pi_{G'}^{(H)}(k)$ for each k by finding a bijection f between induced copies of H in $\mathcal{C}_k(G)$ and in $\mathcal{C}_k(G')$. For each copy X of H in $\mathcal{C}_k(G)$, we will define $f(X)$ in terms of an intermediate map f_X that transforms colourings of G into colourings of G'.

We first construct f_X. For $G^* \in \{G, G'\}$, let X^* be an induced copy of H in $\mathcal{C}_k(G^*)$. Say that a vertex $v \in V(G^*)$ corresponds to an edge c_1c_2 of X^* if it is the unique vertex at which the colourings c_1 and c_2 differ. Then, define t_{X^*} to be the smallest positive integer such that neither $v_{n-t_{X^*}}$ nor $v_{n+t_{X^*}}$ correspond to any edge of X^*. Such a value $t_{X^*} \in [n]$ exists because X^* only has $m < n - 1$ edges.

Working in G, let X be an induced copy of H in $\mathcal{C}_k(G)$. Fix $c \in V(X)$ and consider colours modulo k, taking $[k]$ as the set of representatives. We define $f_X(c)$ to colour each vertex u of G' by

$$(f_X(c))(u) = \begin{cases} c(v_{n-t_X}) + c(v_{n+t_X}) - c(v_{n-i}) & \text{if } u = v_{n+i} \text{ for } i \in (-t_X, t_X), \\ c(v_{n-t_X}) + c(v_{n+t_X}) - c(v) & \text{if } u = v', \\ c(u) & \text{otherwise.} \end{cases}$$

fig. 3 provides an example of a colouring c of G and a corresponding colouring $f_X(c)$ of G'.

Next, let X' be an induced copy of H in $\mathcal{C}_k(G')$, and fix $c' \in V(X')$. We similarly define $g_{X'}(c')$ to map colourings of G' to colourings of G by

$$(g_{X'}(c'))(u) = \begin{cases} c'(v_{n-t_{X'}}) + c'(v_{n+t_{X'}}) - c'(v_{n-i}) & \text{if } u = v_{n+i} \text{ for } i \in (-t_{X'}, t_{X'}), \\ c'(v_{n-t_{X'}}) + c'(v_{n+t_{X'}}) - c'(v') & \text{if } u = v, \\ c'(u) & \text{otherwise.} \end{cases}$$

Let $f(X)$ be the subgraph of $\mathcal{C}_k(G')$ induced by $f_X(V(X))$, and let $g(X')$ be the subgraph of $\mathcal{C}_k(G)$ induced by $g_{X'}(V(X'))$.

For each $c \in V(X)$, we observe that $f_X(c)$ is indeed a proper colouring of G'. We now show that X and $f(X)$ are isomorphic. Since H is connected and neither v_{n-t_X} nor v_{n+t_X} correspond to an edge of X, the colours of v_{n-t_X} and v_{n+t_X} are constant across all colourings in $V(X)$. Hence, the value $c(v_{n-t_X}) + c(v_{n+t_X})$ is the same across each colouring c in X. It is then straightforward to check that two colourings c_1 and c_2 are adjacent in X if and only if $f_X(c_1)$ and $f_X(c_2)$ are adjacent in $f(X)$. This implies that $t_X = t_{f(X)}$, and that $f(X)$ is an induced copy of H in $\mathcal{C}_k(G')$. Making symmetric observations about $g_{X'}$, we now observe that $g_{f(X)}$ is the inverse of f_X, and that therefore f is also invertible with inverse g. That is, f is a bijection between induced copies of H in $\mathcal{C}_k(G)$ and induced copies of H in $\mathcal{C}_k(G')$, and the result follows. □

We now extend this result to finite collections of disconnected graphs via a standard argument.

Lemma 9. *Let H be a graph with connected components $R_1, \ldots, R_t$. Then there is a finite collection $\mathcal{F}$ of connected graphs such that $\{\pi_G^{(F)} \colon F \in \mathcal{F}\}$ uniquely determines $\pi_G^{(H)}$ for every graph G.*

Proof. We proceed by induction on the number of connected components t of H, with the base case being when H is any connected graph. Suppose H has at least $t > 1$ components $R_1, \ldots, R_t$.

The product $\pi_G^{(R_1)}(k) \cdots \pi_G^{(R_t)}(k)$ counts the number of tuples $(\rho_1, \ldots, \rho_t)$ of injective maps $\rho_i \colon V(R_i) \to V(\mathcal{C}_k(G))$ such that for each $i \in [t]$, $\rho_i(V(R_i))$ induces a copy of R_i in $\mathcal{C}_k(G)$. Fix such a tuple of injective maps and let F be the subgraph of $\mathcal{C}_k(G)$ induced by the images of the maps, i.e. by the vertices in $\bigcup_i \rho_i(V(R_i))$. Notice that for a fixed graph H, there are finitely many possible isomorphism classes for the graph F (all such graphs have at most $|H|$ vertices), and that F is either isomorphic to H, or else has fewer connected components than H. If we fix such an isomorphism class F, its contribution to the count $\pi_G^{(R_1)}(k) \cdots \pi_G^{(R_t)}(k)$ is precisely $\pi_G^{(F)}(k)$ times the number $N(F, H)$ of tuples $(\rho_1, \ldots, \rho_t)$ which produce the vertices of the same copy of F in G. Hence, letting $\mathcal{F}^*$ be the family of non-isomorphic graphs other than H that can be obtained in this way, the following equality holds:

$$\pi_G^{(H)} = \pi_G^{(R_1)} \cdots \pi_G^{(R_t)} - \sum_{F \in \mathcal{F}^*} N(F, H) \cdot \pi_G^{(F)}. \tag{1}$$

By the induction hypothesis, each graph $F \in \mathcal{F}^*$ has a finite collection of connected graphs $\mathcal{F}_F$ (possibly $\mathcal{F}_F = \{F\}$) which determine $\pi_G^{(F)}$, and so the finite family $\mathcal{F} = \bigcup_{F\in\mathcal{F}^*} \mathcal{F}_F$ of connected graphs determines $\pi_G^{(H)}$. □

Remark 10. For connected H, the preceding proof can still be used to find a finite family of graphs $\mathcal{F}(H)$, not containing H or depending on G, such that the collection $\{\pi_G^{(F)} \colon F \in \mathcal{F}(H)\}$ determines $\pi_G^{(H)}$ (when H is disconnected this comes directly from the proof with formula given by eq. (1)). Namely, run the proof with a disconnected graph H^+ that contains H as one connected component. Then, isolating the term $\pi_G^{(H)}$ (which now occurs among the components and possibly in $\mathcal{F}^*$) in eq. (1) gives the relevant formula.

Theorem 5. *No finite family of generalised chromatic polynomials is a complete graph invariant.*

Proof. Let $H_1, \ldots, H_t$ be a finite family of graphs. By Lemma 9, there is a finite collection $\mathcal{F}$ of connected graphs such that for every graph G the generalised chromatic polynomials $\pi_G^{(H_1)}, \ldots, \pi_G^{(H_t)}$ only depend on $\{\pi_G^{(F)} \colon F \in \mathcal{F}\}$. The theorem now follows from choosing m in Lemma 8 to be larger than $\max\{|E(F)| \colon F \in \mathcal{F}\}$. □

5 Open problems

Asgarli et al. conjectured that $\pi_G^{(K_2)}$ determines the chromatic polynomial $\pi_G^{(K_1)}$, that is, if $\pi_{G_1}^{(K_2)} = \pi_{G_2}^{(K_2)}$ then $\pi_{G_1}^{(K_1)} = \pi_{G_2}^{(K_1)}$ [1, Conjecture 5.2]. This remains unverified, but in light of Remark 10 we ask a broader question.

Problem 11. For which graphs H does there exist an H' such that $\pi_G^{(H')}$ determines $\pi_G^{(H)}$ for every graph G?

Define the *graph product* of G_1, G_2 to be the graph $G_1 \square G_2$ on vertex set $V(G_1) \times V(G_2)$ with adjacencies between vertices (a_1, b_1), (a_2, b_2) if and only if $a_1 = a_2$ and $b_1 b_2 \in E(G)$ or $b_1 = b_2$ and $a_1 a_2 \in E(G)$. Notice that for any graph G on n vertices and any integer k, the graph $\mathcal{C}_k(G)$ is obtained from the product $Q(k, n) \coloneqq K_k \square \cdots \square K_k$ of n copies of K_k by removing vertices corresponding to k-colourings of vertices of G which are not proper in G. As such, every graph H whose polynomial $\pi_G^{(H)}$ is nonzero for some graph G must be an induced subgraph of $Q(k, n)$ for some k and n. Problem 11 is therefore only interesting when H is an induced subgraph of $Q(k, n)$.

It is well known that the chromatic polynomial does not distinguish all graphs, and we have shown in Theorem 5 that no finite family of generalised chromatic polynomials suffices to distinguish all graphs. But what about typical graphs? Bollobás, Pebody and Riordan [3, Conjecture 2] raised the intriguing conjecture that almost every graph is determined by its chromatic polynomial; they asked the same question for the Tutte polynomial (which is a stronger invariant). In a similar vein, we ask a weakening of their chromatic polynomial conjecture for finite families of generalised chromatic polynomials. Let $\mathcal{G}(n, p)$ be the random graph on n vertices obtained by sampling each edge independently with probability p.

Problem 12. Is there a finite set of graphs $H_1, \dots, H_t$ such that, for almost every $G \in \mathcal{G}(n, \frac{1}{2})$, if $\pi_{G'}^{(H_i)} = \pi_G^{(H_i)}$ for all i then $G' \cong G$?

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