A Note on Hamilton Decompositions of Even-Regular Multigraphs

Vincent Pfenninger[a]

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Abstract

In this note, we prove that every even regular multigraph on n vertices with multiplicity at most r and minimum degree at least $rn/2 + o(n)$ has a Hamilton decomposition. This generalises a result of Vaughan who proved an asymptotic version of the multigraph 1-factorisation conjecture. We derive our result by proving a more general result which states that dense regular multidigraphs that are robust outexpanders have a Hamilton decomposition. This in turn is derived from the corresponding result of Kühn and Osthus about simple digraphs.

Mathematics Subject Classifications: 05C45, 05C35, 05C70, 05C20, 05C38

1 Introduction

A *multidigraph* (or *directed multigraph*) D is a pair $(V(D), E(D))$ where $V(D)$ is a finite set and $E(D)$ is a multiset with elements from the set $(V(D)\times V(D))\setminus\{(v,v)\colon v\in V(D)\}$ (so loops are not allowed). We call $V(D)$ the *set of vertices of* D and $E(D)$ the *multiset of edges of* D. The *multiplicity of a multidigraph* D is the maximum number of times an edge appears in $E(D)$. For a vertex $v \in V(D)$, we denote by $d^+_D(v)$ the number of *outedges* at v in D, that is, edges of the form (v, x) for some $x \in V(D)$ (counting multiplicities). Similarly, $d^-_D(v)$ is the number of *inedges* at v in D. We say that D is s-regular, if $d^+_D(v) = d^-_D(v) = s$ for every $v \in V(D)$. A *Hamilton cycle* in a multidigraph is a directed cycle covering all the vertices. A *Hamilton decomposition* of a multidigraph D is a partition of $E(D)$ such that every part is the edge set of a Hamilton cycle of D. We also use analogous definitions for multigraphs (the undirected analogue to multidigraphs).

Vaughan proved that if n is even and G is a regular multigraph on n vertices with multiplicity at most r and minimum degree at least $rn/2+o(n)$, then G has a 1-factorisation[1] [9]. This is an approximate version of the multigraph 1-factorisation conjecture of Plantholt and Tipnis [8] which is a generalisation of the 1-factorisation conjecture [1]. The 1-factorisation conjecture states that if G is an s-regular graph on n vertices where n is an

[a]Institute of Discrete Mathematics, Graz University of Technology (pfenninger@math.tugraz.at).

[1]A 1-*factorisation* is a decomposition of the edge set into perfect matchings.

even integer and $s \geqslant n/2$, then G has a 1-factorisation. The multigraph 1-factorisation conjecture states that if G is an s-regular multigraph on n vertices with multiplicity at most r where n is an even integer and $s \geqslant rn/2$, then G has a 1-factorisation. The 1-factorisation conjecture was proved for all large graphs by Csaba, Kühn, Lo, Osthus, and Treglown [2]. Our aim in this note is to generalise the result of Vaughan by showing that if in addition the multigraph is even-regular, then it has a Hamilton decomposition.

Theorem 1. *For every $\varepsilon \in (0,1)$ and $r \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$. Let G be an s-regular n-vertex multigraph with multiplicity at most r where $s \in 2\mathbb{N}$ with $s \geqslant rn/2 + \varepsilon n$. Then G has a Hamilton decomposition.*

Observe that this does indeed imply the theorem of Vaughan since if G is an odd-regular graph on an even number of vertices we can first remove a perfect matching and then apply our result to the remaining graph.

We also prove a directed analogue of Theorem 1.

Theorem 2. *For every $\varepsilon \in (0,1)$ and $r \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$. Let D be an s-regular n-vertex multidigraph with multiplicity at most r where $s \geqslant rn/2 + \varepsilon n$. Then D has a Hamilton decomposition.*

We derive these results from a more general result about multidigraphs that are robust outexpanders. To state this result we need the following definitions.

For an n-vertex digraph D, a set $S \subseteq V(D)$, and $\nu \in (0,1)$, we define the *ν-robust outneighbourhood of S in D* to be $RN^+_{\nu,D}(S) \coloneqq \{v \in V(D)\colon |S \cap N^-_D(v)| \geqslant \nu n\}$. For $\nu, \tau \in (0,1)$, a simple[2] n-vertex digraph D is said to be a *robust (ν,τ)-outexpander* if for each set $S \subseteq V(D)$ with $\tau n \leqslant |S| \leqslant (1-\tau)n$, we have $|RN^+_{\nu,D}(S)| \geqslant |S| + \nu n$. For a multidigraph D, the *underlying simple digraph of D* is the digraph $\widetilde{D}$ obtained from D by dropping any multiple edges (that is an ordered pair of vertices is an edge in $\widetilde{D}$ if and only if it appears (at least once) in $E(D)$). We analogously define the *underlying simple graph* of a multigraph.

The *hierarchy* notation $0 < a \ll b < 1$ is a short form of stating that there exists a nondecreasing function $a_0\colon (0,1) \to (0,1)$ such that the statement that follows after holds for all $a, b \in (0,1)$ with $a \leqslant a_0(b)$ (that is the statement holds as long as a is sufficiently small in terms of b). Hierarchies with more variables are defined similarly and whenever $1/a$ appears in a hierarchy we implicitly assume that a is a positive integer.

The following theorem is our main result from which we derive all other results in this note.

Theorem 3. *Let $1/n \ll \nu \ll \tau \ll 1/r, \alpha \leqslant 1$. Let D be a n-vertex multidigraph with multiplicity at most r such that the following hold.*

(a) *D is s-regular for some $s \geqslant \alpha n$.*

(b) *The underlying simple digraph of D is a robust (ν,τ)-outexpander.*

[2] A multidigraph is *simple* if its multiplicity is 1 (such a multidigraph is also called a *digraph* or a *simple digraph*).

Then D has a Hamilton decomposition.

We also prove an analogous result for (undirected) multigraphs. For this we need the corresponding undirected version of the above definition of robust outexpansion. For an n-vertex graph G, a set $S \subseteq V(G)$, and $\nu \in (0,1)$, we define the *ν-robust neighbourhood of S in G* to be $RN_{\nu,G}(S) := \{v \in V(G)\colon |S \cap N_G(v)| \geqslant \nu n\}$. For $\nu, \tau \in (0,1)$, a simple n-vertex (undirected) graph G is said to be a *robust (ν,τ)-expander* if for each set $S \subseteq V(G)$ with $\tau n \leqslant |S| \leqslant (1-\tau)n$, we have $|RN_{\nu,G}(S)| \geqslant |S| + \nu n$. As discussed in [6], robust expansion is a very weak notion of quasirandomness, and thus, for example, random graphs of constant density are robust expanders with high probability.

Theorem 4. *Let $1/n \ll \nu \ll \tau \ll 1/r, \alpha \leqslant 1$. Let G be a n-vertex multigraph with multiplicity at most r such that the following hold.*

(a) *G is s-regular for some $s \in 2\mathbb{N}$ with $s \geqslant \alpha n$.*

(b) *The underlying simple graph of G is a robust (ν,τ)-expander.*

Then G has a Hamilton decomposition.

The case $r = 1$ of all these results was proved by Kühn and Osthus in [5]. Throughout, we ignore floors and ceilings whenever doing so does not affect the argument.

2 Proofs of secondary results

In this section, we prove that Theorem 3 implies Theorems 1, 2 and 4.

2.1 Proof of Theorem 4

To derive Theorem 4 from Theorem 3 we need the following two lemmas from [6]. The first of these lemmas allows us to orient the edges of a robust expander in such a way that the resulting digraph is a robust outexpander and the in-degree and out-degree of every vertex are not too far apart.

Lemma 5 ([6, Lemma 3.1])**.** *Suppose that $1/n \ll \eta \ll \nu, \tau, \alpha < 1$. Suppose that G is a robust (ν,τ)-expander on n vertices with $\delta(G) \geqslant \alpha n$. Then one can orient the edges of G in such a way that the resulting oriented graph D satisfies the following*[3]*:*

(i) *D is a robust $(\nu/4,\tau)$-outexpander.*

(ii) *$d^+_D(x) = (1 \pm \eta)\frac{d_G(x)}{2}$ and $d^-_D(x) = (1 \pm \eta)\frac{d_G(x)}{2}$.*

For a (simple) digraph D, we define its *minimum semi-degree* $\delta^0(D)$ to be

$$\delta^0(D) := \min_{\substack{\circ \in \{+,-\} \\ v \in V(D)}} d^\circ_D(v).$$

The second lemma allows us to find a regular spanning subdigraph of a robust outexpander with linear minimum semi-degree that is still a robust outexpander.

[3]For positive reals a, b, c, d, we write $a = b(c \pm d)$ for $b(c-d) \leqslant a \leqslant b(c+d)$.

Lemma 6 ([6, Lemma 3.4])**.** *Suppose that* $1/n \ll \nu' \ll \xi \ll \nu \leqslant \tau \ll \alpha < 1$. *Let* D *be a robust* (ν, τ)*-outexpander on* n *vertices with* $\delta^0(D) \geqslant \alpha n$. *Then* D *contains a* ξn*-factor*[4] *which is still a robust* (ν', τ)*-outexpander.*

We now show that Theorem 3 implies its undirected analogue Theorem 4.

Proof of Theorem 4 assuming Theorem 3. We choose new constants ν' and ξ such that $1/n \ll \nu' \ll \xi \ll \nu \ll \tau \ll 1/r, \alpha$. Note that we can write $G = G' \dot\cup H$ where G' is the underlying simple graph of G and H is a multigraph that is edge-disjoint from G'. By assumption, G' is a robust (ν, τ)-expander with $\delta(G') \geqslant \alpha n/r$. By Theorem 5, there exists an orientation D of G' such that D is a robust $(\nu/4, \tau)$-outexpander with $\delta^0(D) \geqslant \alpha n/3r$. By Theorem 6, D contains a ξn-factor F that is a robust (ν', τ)-outexpander. Let $\widetilde{G}$ be the undirected multigraph obtained from G by deleting the edges contained in F. Note that $\widetilde{G}$ is $(s - 2\xi n)$-regular. Since s is even, there exists a decomposition of $\widetilde{G}$ into cycles (we allow 2-cycles, that is, two parallel edges). Orienting each of these cycles consistently and then combining this with F gives an orientation $\widetilde{D}$ of G that is $s/2$-regular. Moreover, the underlying simple digraph of $\widetilde{D}$ contains F and thus is a robust (ν', τ)-outexpander. By applying Theorem 3 to $\widetilde{D}$, we get a Hamilton decomposition of $\widetilde{D}$ which, by dropping the orientations, gives the desired Hamilton decomposition of G. □

2.2 Proof of Theorem 2 and Theorem 1

Theorem 2 follows from Theorem 3 by the following lemma which states that the minimum semi-degree condition on D implies that D is a robust outexpander.

Lemma 7 ([6, Lemma 3.7])**.** *Let* $1/n \ll \nu \ll \tau \ll \varepsilon < 1$. *Let* D *be a digraph on* n *vertices with minimum semi-degree* $\delta^0(D) \geqslant (1/2 + \varepsilon)n$. *Then* D *is a robust* (ν, τ)*-outexpander.*

Proof of Theorem 2 assuming Theorem 3. Let ν and τ be new constants such that $1/n \ll \nu \ll \tau \ll \varepsilon, 1/r$. Note that the underlying simple digraph D' of D has minimum semi-degree $\delta^0(D') \geqslant (1/2 + \varepsilon/r)n$. By Theorem 7, D' is a robust (ν, τ)-outexpander. Hence we are done by Theorem 2. □

Analogously, Theorem 1 follows from Theorem 4 by the following lemma (the undirected version of Theorem 7).

Lemma 8 ([6, Lemma 3.8])**.** *Let* $1/n \ll \nu \ll \tau \ll \varepsilon < 1$. *Let* G *be a graph on* n *vertices with minimum degree* $\delta(G) \geqslant (1/2 + \varepsilon)n$. *Then* G *is a robust* (ν, τ)*-expander.*

Proof of Theorem 1 assuming Theorem 4. Let ν and τ be new constants such that $1/n \ll \nu \ll \tau \ll \varepsilon, 1/r$. Note that the underlying simple graph G' of G has minimum degree $\delta(G') \geqslant (1/2 + \varepsilon/r)n$. By Theorem 8, G' is a robust (ν, τ)-expander. Hence we are done by Theorem 4. □

[4] A ξn-factor is a ξn-regular spanning subdigraph.

3 Proof of Theorem 3

In this section we prove Theorem 3, the main theorem of this note. The main ingredients for our proof are the following two results from [5] and [7]. They state, respectively, that a robust outexpander with linear minimum semi-degree has a Hamilton decomposition if it is regular and can be almost decomposed into Hamilton cycles if it is almost regular.

Theorem 9 ([5, Theorem 1.2]). *Let $1/n \ll \nu \ll \tau \ll \alpha \leqslant 1$. Let D be a s-regular simple digraph on n vertices that is a robust (ν, τ)-outexpander where $s \geqslant \alpha n$. Then D has a Hamilton decomposition.*

Theorem 10 ([7, Corollary 1.2]). *Let $1/n \ll \xi \ll \delta \ll \nu \ll \tau \ll \alpha \leqslant 1$. Let D be an n-vertex digraph such that*

(i) $d_D^+(v) = (\alpha \pm \xi)n$, $d_D^-(v) = (\alpha \pm \xi)n$ *for all* $v \in V(D)$ *and*

(ii) D *is a robust* (ν, τ)*-outexpander.*

Then D contains at least $(\alpha - \delta)n$ edge-disjoint Hamilton cycles.

To prove Theorem 3, we also need the following four facts about robust outexpanders. The first is the fact that robust outexpanders with linear minimum semi-degree are Hamilton-connected.

Corollary 11 ([4, special case $\boldsymbol{p = 1}$ of Corollary 6.9]). *Let $1/n \ll \nu \ll \tau \ll \alpha \leqslant 1$. Let D be a simple digraph on n vertices that is a robust (ν, τ)-outexpander with $\delta^0(D) \geqslant \alpha n$. Then for any $x, y \in V(D)$, D contains a Hamilton path from x to y.*

The next fact is that a random subgraph of a robust outexpander is very likely to still be a robust outexpander.

Lemma 12 ([3, Lemma 4.12], [6, Lemma 3.2(ii)]). *Let $1/n \ll \nu \ll \tau, p \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices. Suppose that Γ is the spanning random subgraph of D obtained by taking each edge independently with probability p. Then, with probability at least $1 - \exp(-\nu^3 n^2)$, Γ is a robust $(p\nu/2, \tau)$-outexpander.*

Finally, we need the following two simple facts which follow easily from the definition of robust outexpanders.

Lemma 13 ([3, Lemma 4.2]). *Let $1/n \ll \varepsilon \ll \nu \ll \tau \leqslant 1$ and let D be digraph that is a robust (ν, τ)-outexpander on n vertices. Then any digraph obtained from D by deleting at most εn inedges and at most εn outedges at each vertex as well as deleting at most εn vertices is a robust $(\nu - 2\varepsilon, 2\tau)$-outexpander.*

Lemma 14 ([4, Lemma 6.6]). *Let $1/n \ll \nu \ll \tau \leqslant 1$. Let D be a simple digraph on n vertices that is a robust (ν, τ)-outexpander with $\delta^0(D) \geqslant 2\tau n$. Then for any $x, y \in V(D)$, there exists a (directed) path in D from x to y of length at most $1/\nu$.*

We are now ready to prove Theorem 3. The main idea of the proof is to randomly split the multidigraph D into r simple digraphs $D_1, \dots, D_r$. We then cover most edges in $D_2, \dots, D_r$ with Hamilton cycles using Theorem 10. Using a small number of edges of D_1, we then find Hamilton cycles covering the remaining edges of $D_2, \dots, D_r$. Finally, we show that the remaining subgraph of D_1 has a Hamilton decomposition using Theorem 9.

Proof of Theorem 3. Let $\alpha' := s/n \geqslant \alpha$. Let $\widetilde{D}$ be the underlying simple digraph corresponding to D. For each $e \in E(\widetilde{D})$, we denote by m_e the multiplicity with which e appears in D. For each $e \in E(\widetilde{D})$ independently, we let X_e be a subset of $[r]$ of size m_e chosen uniformly at random. For each $i \in [r]$, we let D_i be the digraph with $V(D_i) = V(D)$ and $E(D_i) = \{e \in E(\widetilde{D}) \colon i \in X_e\}$. Note that D_i is a simple digraph for each $i \in [r]$ and that $\dot{\bigcup}_{i\in[r]} D_i = D$ as a multidigraph.

Let ξ and δ be new constants with $1/n \ll \xi \ll \delta \ll \nu$. Let $u \in V(\widetilde{D})$ and $i \in [r]$. We have

$$\mathbb{E}[d^+_{D_i}(u)] = \sum_{\substack{v \in V(D) \\ e=(u,v)\in E(\widetilde{D})}} \binom{r-1}{m_e-1} \Big/ \binom{r}{m_e} = \sum_{\substack{v \in V(D) \\ e=(u,v)\in E(\widetilde{D})}} m_e/r = d^+_D(u)/r = s/r.$$

And similarly, $\mathbb{E}[d^-_{D_i}(u)] = s/r$. Thus by the Chernoff bound, we have

$$\mathbb{P}[d^+_{D_i}(v) \neq (1 \pm \xi)s/r],\ \mathbb{P}[d^-_{D_i}(v) \neq (1 \pm \xi)s/r] \leqslant \exp(-\Omega(n)).$$

Note that there is a coupling of D_i and the digraph $\widetilde{D}_{1/r}$ obtained from $\widetilde{D}$ by taking each edge independently with probability $1/r$ such that $\widetilde{D}_{1/r} \subseteq D_i$. Hence, since $\widetilde{D}$ is a robust (ν, τ)-outexpander, we have by Theorem 12, with probability at least $1-\exp(\nu^3 n^2)$, D_i is a robust $(\nu/2r, \tau)$-outexpander. Hence by a union bound, we have that with probability $1 - o(1)$ for every $i \in [r]$,

(i) $d^+_{D_i}(v) = (1 \pm \xi)s/r$, $d^-_{D_i}(v) = (1 \pm \xi)s/r$ for all $v \in V(D)$ and

(ii) D_i is a robust $(\nu/2r, \tau)$-outexpander.

Fix such a choice of $(D_1, \dots, D_r)$. Note that $(\alpha'/r - \xi\alpha'/r)n = (1-\xi)s/r \leqslant (1+\xi)s/r = (\alpha'/r + \xi\alpha'/r)n$. Hence by Theorem 10, for each $i \in [r]$, there is a set $\mathcal{H}_i$ of at least $(\alpha'/r - \delta)n$ edge-disjoint Hamilton cycles in D_i. For each $i \in [r]$, let $D'_i = D_i - \bigcup_{H\in\mathcal{H}_i} E(H)$ and note that $\Delta^0(D'_i) := \max_{\circ\in\{+,-\}, v\in V(D'_i)} d^\circ_{D'_i}(v) \leqslant \delta n + \xi\alpha'/rn \leqslant 2\delta n - 1$.

Let $D' = D_1 \cup \bigcup_{2\leqslant i\leqslant r} D'_i$. Observe that it suffices to show that D' has a Hamilton decomposition. Note that D' is s'-regular for some $s' \geqslant (1-2\xi)\alpha' n/r$ and since D_1 is a robust $(\nu/2r, \tau)$-outexpander, the underlying simple digraph of D' is also a robust $(\nu/2r, \tau)$-outexpander.

We now show that, for each $i \in \{2, \dots, r\}$, there exists a decomposition of D'_i into a set $\mathcal{M}_i$ of at most $16\delta^{1/2}n$ matchings each of size at most $\delta^{1/2}n$. Indeed, for each

$i \in \{2, \dots, r\}$, first decompose D'_i into two oriented graphs[5] $D^{\text{or}}_{i,1}$ and $D^{\text{or}}_{i,2}$. Since $\Delta^0(D'_i) \leqslant 2\delta n - 1$, the underlying graphs of $D^{\text{or}}_{i,1}$ and $D^{\text{or}}_{i,2}$ have maximum degree at most $4\delta n - 2$. Hence, by applying Vizing's theorem to the underlying graphs of $D^{\text{or}}_{i,1}$ and $D^{\text{or}}_{i,2}$, we obtain a decomposition of the edges of each of $D^{\text{or}}_{i,1}$ and $D^{\text{or}}_{i,2}$ into a set of at most $4\delta n$ matchings. These two decompositions together give a decomposition of the edges of D'_i into a set of at most $8\delta n$ matchings. From this the desired set of matchings is obtained by splitting each matching of size larger than $\delta^{1/2}n$ into $2/\delta^{1/2}$ matchings of as equal size as possible.

Let $\mathcal{M} = \bigcup_{2 \leqslant i \leqslant r} \mathcal{M}_i = \{M_1, \dots, M_t\}$, where $t \leqslant 16r\delta^{1/2}n \leqslant \delta^{1/4}n$. We construct a set of edge-disjoint paths $\mathcal{P} = \{P_1, \dots, P_t\}$ in D' such that $M_i \subseteq E(P_i)$ and $|V(P_i)| \leqslant \delta^{1/4}n$ for each $i \in [t]$ as follows. Let $i \in [t]$ and suppose that $P_1, \dots, P_{i-1}$ have already been constructed. Let $M_i = \{(u_j, v_j) \colon j \in [m]\}$ where $m \leqslant \delta^{1/2}n$ and let $D^* = D' - \bigcup_{1 \leqslant j \leqslant i-1} E(P_j) - \bigcup_{i+1 \leqslant j \leqslant t} M_i$. In order to show that D^* contains a path P_i such that $M_i \subseteq E(P_i)$ and $|V(P_i)| \leqslant \delta^{1/4}n$, we show by induction on $j \in [m]$ that there exists a path Q_j from u_1 to v_j in $D^* - \bigcup_{j+1 \leqslant j' \leqslant m}\{u_{j'}, v_{j'}\}$ of length at most $2j\nu^{-2}$ such that Q_j contains the edges $(u_{j'}, v_{j'})$ for all $j' \in [j]$ (then Q_m is the desired path P_i). To that end, let $j \in [m]$ and suppose that Q_{j-1} is a path from u_1 to v_{j-1} in $D^* - \bigcup_{j \leqslant j' \leqslant m}\{u_{j'}, v_{j'}\}$ of length at most $2(j-1)\nu^{-2}$ such that Q_{j-1} contains the edges $(u_{j'}, v_{j'})$ for all $j' \in [j-1]$. Let D^{**} be the underlying simple digraph of $D^* - \{v_j\} - \bigcup_{j+1 \leqslant j' \leqslant m}\{u_{j'}, v_{j'}\} - (V(Q_{j-1}) \setminus \{v_{j-1}\})$. Observe that D^{**} can be obtained from the underlying simple digraph of D' by deleting at most $\delta^{1/4}n$ inedges and at most $\delta^{1/4}n$ outedges at every vertex and then deleting at most $2\delta^{1/2}n + 2(j-1)\nu^{-2} \leqslant \delta^{1/4}n$ vertices. Hence by Theorem 13 (with $\delta^{1/4}$ and $\nu/2r$ playing the roles of ε and ν, respectively), D^{**} is a robust $(\nu/2r - 2\delta^{1/4}, 2\tau)$-outexpander. Since $\delta^0(D^{**}) \geqslant 4\tau n$, by Theorem 14 (with $\nu/2r - 2\delta^{1/4}$ and 2τ playing the roles of ν and τ, respectively), we have that there exists a path P^* in D^{**} from v_{j-1} to u_j of length at most $1/(\nu/2r - 2\delta^{1/4}) \leqslant \nu^{-2}$. Prepending Q_{j-1} and appending the edge (u_j, v_j) to P^* gives the desired path Q_j.

We now construct a set $\mathcal{H} = \{H_1, \dots, H_t\}$ of edge-disjoint Hamilton cycles in D' such that, for each $i \in [t]$, $E(P_i) \subseteq E(H_i)$. Let $i \in [t]$ and suppose that $H_1, \dots, H_{i-1}$ have already been constructed. Let x and y be the first and last vertex of P_i, respectively. Let $D^\diamond$ be the digraph obtained from D' by deleting the edges in $\bigcup_{1 \leqslant j \leqslant i-1} E(H_j) \cup \bigcup_{i+1 \leqslant j \leqslant t} E(P_j)$ and deleting all vertices of P_i except x and y. Observe that $D^\diamond$ is obtained from D' by deleting at most $\delta^{1/4}n$ inedges and at most $\delta^{1/4}n$ outedges at every vertex and then deleting at most $\delta^{1/4}n$ vertices. Hence by Theorem 13 (with $\delta^{1/4}$ and $\nu/2r$ playing the roles of ε and ν, respectively), $D^\diamond$ is a robust $(\nu/2r - 2\delta^{1/4}, 2\tau)$-outexpander. Since $\delta^0(D^\diamond) \geqslant \alpha' n/2r$, by Theorem 11 (with $\alpha'/2r$, 2τ, and $\nu/2r - 2\delta^{1/4}$ playing the roles of α, τ, and ν, respectively), $D^\diamond$ contains a Hamilton path from y to x which together with P_i forms the desired Hamilton cycle H_i in D'.

Let $D'' = D' - \bigcup_{H \in \mathcal{H}} E(H)$. It now suffices to show that D'' has a Hamilton decomposition. Note that D'' is a simple digraph as $D'' \subseteq D_1$. Since D'' is obtained from D' by deleting at most $\delta^{1/4}n$ inedges and $\delta^{1/4}n$ outedges at every vertex, we have by Theorem 13 (with $\delta^{1/4}$ and $\nu/2r$ playing the roles of ε and ν, respectively), that D''

[5] An *oriented graph* is a simple digraph such that for any vertices x and y at most one of (x, y) and (y, x) is an edge.

is a robust $(\nu/2r - 2\delta^{1/4}, 2\tau)$-outexpander. Moreover, since D' is s'-regular for some $s' \geqslant (1-2\xi)\alpha' n/r$, D'' is s''-regular for some $s'' \geqslant \alpha' n/2r$. Finally, by Theorem 9 (with $\alpha'/2r$, 2τ, and $\nu/2r - 2\delta^{1/4}$, playing the roles of α, τ, and ν, respectively), D'' has a Hamilton decomposition. □

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