The Number of Spanning Trees in 4-Regular Simple Graphs

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Submitted: Nov 23, 2023; Accepted: Oct 24, 2024; Published: Nov 29, 2024

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Abstract

Extending an earlier work by Kostochka for subcubic graphs, we show that a connected graph G with minimum degree 2 and maximum degree 4 has at least $75^{n_4/5+n_3/10+1/5}$ spanning trees, where n_i is the number of vertices of degree i in G, unless G is the complete graph on 5 vertices or obtained from the complete graph on 6 vertices by deleting the edges of a perfect matching. This, in particular, allows us to determine the value of the inferior limit of the normalised number of spanning trees (introduced by Alon) over the class of connected 4-regular graphs to be $75^{1/5}$.

Mathematics Subject Classifications: 05C05, 05C35, 05C07

1 Introduction

In 1983, McKay [4] managed to determine precisely the asymptotics of the maximum number of spanning trees a d-regular simple graph can have, by proving that

$$\limsup_{n\to\infty} \left\{\tau(G)^{1/n} \,:\, G \in \mathcal{C}(n,d)\right\} = \frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}} = d - \frac{1}{2} + o(1),$$

where $\tau(G)$ is the number of spanning trees of the graph G, and $\mathcal{C}(n,d)$ is the class of all connected d-regular n-vertex simple graphs. It is remarkable that the exact value could be pinpointed. By contrast, the situation is less understood for the analogous inferior limit. Let us set $c(d) = \liminf_{n\to\infty} \left\{\tau(G)^{1/n} \,:\, G \in \mathcal{C}(n,d)\right\}$. In 1990, Alon [1] proved the following inequalities:

$$d - \Theta\left(d(\log\log d)^2/\log d\right) \leqslant c(d) \leqslant \left[(d+1)^{d-2}(d-1)\right]^{1/(d+1)} = d - \Theta(\log d), \qquad (1)$$

where $d \geqslant 3$ for the second inequality. It is worth mentioning that Alon proved many other properties of $c(d)$ in his work, notably observing that the inferior limit defining $c(d)$ is actually an infinum, and also establishing the general lower bound $\sqrt{2} \leqslant c(d)$.

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The asymptotic range for $c(d)$ was reduced five years later: subtly modifying Alon's approach, Kostochka [3] showed that $d - \Theta((\log d)^2) \leqslant c(d)$. In addition, this stronger lower bound is obtained as a corollary of a bound for connected graphs with given degree sequences, rather than connected regular graphs. In this work, Kostochka also addressed one of Alon's question: can we determine exact values of $c(d)$? Alon [1] predicted this should be generally difficult. Kostochka [3] was able to provide an almost exact lower bound for the number of spanning trees of a connected n-vertex graph with maximum degree 3 and minimum degree at least 2. Specifically, he proved the following.

Theorem 1. *Let G be a graph with minimum degree at least 2 and maximum degree at most 3. Either G is isomorphic to K_4 or $\tau(G) \geqslant 2^{3(n_3+2)/4}$, where n_3 is the number of vertices of degree 3 in G.*

As a direct corollary of Theorem 1, one obtains that $c(3) \geqslant 2^{3/4}$, which is sufficient to prove equality because the upper bound in (1) is precisely $2^{3/4}$ when $d = 3$.

The purpose of this work is to extend Theorem 1 to graphs with maximum degree at most 4, and as a by-product determine the exact value of $c(4)$. Kostochka's argument follows an inductive structural analysis. We proceed similarly, but are required to involve various new tools and techniques, such as a lift operation on vertices. We obtain the following statement in which, generalising the notation used in Theorem 1, for a graph G with minimum degree at least 2 we define n_i to be the number of vertices of degree i, for each positive integer i; and we also set

$$f(G) = 75^{\frac{1}{5}\cdot(n_4+n_3/2+1)}.$$

For any integer $n \geqslant 3$, we let K_n be the complete graph with n vertices. We define K_6^- to be the graph obtained from K_6 by deleting the edges of an arbitrary perfect matching of K_6. We observe that $\tau(K_5) = 5^3 < 177 < f(K_5)$ and $\tau(K_6^-) = 384 < 421 < f(K_6^-)$. However, as we shall demonstrate, these two graphs are the only two connected graphs G with minimum degree 2 and maximum degree at most 4 such that $\tau(G) < f(G)$. We let $\mathcal{E} = \{K_5, K_6^-\}$.

Theorem 2. *If G is a connected graph with minimum degree 2 and maximum degree (at most) 4, then*

$$\tau(G) \geqslant f(G) = 75^{n_4/5+n_3/10+1/5},$$

unless $G \in \mathcal{E} = \{K_5, K_6^-\}$.

A direct consequence of Theorem 2 is that $c(4) \geqslant 75^{1/5}$. This is sufficient to obtain equality again thanks to the upper bound in (1), which is thus proved to be tight also for 4-regular graphs.

Corollary 3. *We have $c(4) = 75^{1/5}$.*

We follow an inductive approach to establish Theorem 2. The fact that there are two exceptions to the stated lower bound thus obliges us, before applying the induction hypothesis, to check if the smaller graph could be one of these exceptions, and if so then provide an argument not using the induction hypothesis. A remark is that, when this might happen in our proof, the considered graph must then be small: it has at most 9 vertices. It is thus possible, and we have done it, to use a computer to verify the stated lower bound on all connected graphs with at most 9 vertices, minimum degree at least 2 and maximum degree (at most) 4. However, since we did not find these verifications to hinder too much the general flow of the proof, we decided to explicitly include these exceptional cases along with the number of their spanning trees, showing they do satisfy the announced lower bound. The reader can, nonetheless, safely ignore these situations, provided a computer verification would be satisfying enough.

Before ending this introduction, we mention that, surprisingly, the analogous questions for connected regular (loopless) multigraphs could be completely solved. First, Bogdanowicz [2] provided an exact lower bound, along with the extremal graphs, for connected cubic multigraphs; next, Pekárek and the current authors [6] provided a full solution, for all values of n, the number of vertices, and of d, the degree.

1.1 Notation

We use standard graph theory notation. In particular, given a graph G, we write $V(G)$ for its vertex set, and $E(G)$ for its edge set. Given a subset V' of $V(G)$, we let $G[V']$ be the subgraph of G induced by V'. We write $G - V'$ to mean $G[V(G) \setminus V']$, and abbreviate $G - \{v\}$ to $G - v$. Similarly, if $e \in E(G)$ then $G - e$ is the graph obtained from G by deleting the edge e, so that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. For a vertex $v \in V(G)$, we let $\deg_G(v)$ be the degree of v in G, and we set $\delta(G) = \min\{\deg_G(v) : v \in V(G)\}$ while $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$.

Spanning trees and forests of a graph will be seen as subset of edges. We will contract subsets of vertices in a graph G: when performing contractions, we always delete multiple edges and loops that could occur, thus obtaining again a simple graph G'. We canonically identify edges in G' and in G when there is no ambiguity. It will be useful to keep in mind that if T' is a spanning tree of a graph G' obtained from a graph G by contracting the set $\{x, y\} \subseteq V(G)$, then (using the canonical identification of edges) T' is a spanning forest of G composed of 2 trees, separating x and y.

1.2 Preliminaries

We occasionally perform a "lifting" operation. Although everything in this section works also for multigraphs, we here purposely restrict the setting to simple graphs. The lifting operation is thus defined as follows. Let x, y_1 and y_2 be three distinct vertices in a graph H such that y_1 and y_2 are not adjacent, and suppose that f_i is an edge in H between x and y_i, for $i \in \{1, 2\}$. (In other words, $\{x, y_1, y_2\}$ induces in H a star centred at x.) *Lifting* f_1 and f_2 means deleting the two edges f_1 and f_2 and adding an edge between y_1 and y_2.

If x is a vertex of degree $2m$ in H, a *complete lift* of x is the process of first performing a sequence of m lifts of pairs of edges incident with x and then deleting the vertex x (which is, by then, isolated), thereby producing a graph H_x. We underline that if a complete lift at x is possible, then H_x is necessarily connected if x is not a cut-vertex of H. Moreover, with our definition, it is possible to perform a complete lift at x such that the degree of every vertex in H_x is the same as its degree in H only if there exists a matching of size m in the complement of the subgraph of H induced by $N_H(x)$.

We shall use the following relation, which was discovered and demonstrated by Ok and Thomassen [5].

Theorem 4 (Ok and Thomassen [5]). *Let H be a graph with a vertex x of degree $2m$. Let H_x be a graph obtained from H by a complete lift of x. Then*

$$\tau(H) \geqslant c_m \, \tau(H_x),$$

with

$$c_m = \min_{d_1,d_2,\dots,d_k} \min_X \frac{\prod_{i=1}^k d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers $d_1,\dots,d_k$ with varying length k such that $\sum_{i=1}^k d_i = 2m$, and over all connected k-vertex multigraphs X with degree sequence $d'_1, d'_2, \dots, d'_k$ such that $d'_i \leqslant d_i$ for each $i \in \{1,\dots,k\}$.

In particular, Ok and Thomassen [5] showed that $c_3 = \frac{8}{3}$. (As noticed earlier [6], the set of cases to be considered in the definition of c_m can actually be reduced to the multigraphs with degree sequence exactly $d_1, d_2, \dots, d_k$.)

In the sequel, we will sometimes want to estimate the number of spanning forests of a (small) graph that separates two given vertices. More precisely, given a graph G and 2 of its vertices v_1 and v_2, what is the number of spanning forests of G formed by the disjoint union of 2 trees T_1 and T_2 such that T_i contains v_i for $i \in \{1,2\}$? A general way to answer is to notice that this number is precisely $\tau(G+uv) - \tau(G)$ if $uv \notin E(G)$; and $\tau(G) - \tau(G-uv)$ if $uv \in E(G)$. Indeed, in the former case the sought forests bijectively correspond to the spanning trees of $G+uv$ that contain the edge uv, which is equal to $\tau(G+uv) - \tau(G)$. The latter case is similar.

Finally, we introduce three small graphs that will often occur as subgraphs in our forthcoming analysis. The graph X_5 is obtained from K_4 by adding a vertex w with degree 2, the graph X'_5 is obtained from K_4 by adding a vertex w with degree 3, and the graph X_6 is obtained from X_5 by adding a vertex w' with degree 2 and neighbourhood disjoint from that of w, as depicted in Figure 1. We note that $\tau(X_5) = 40$, $\tau(X'_5) = 75$ and $\tau(X_6) = 100$.

2 Proof of Theorem 2

For each $n \geqslant 3$, we define $\mathcal{A}_n$ be the class of connected n-vertex graphs with minimum degree at least 2 and maximum degree at most 4. We assume the existence of a smallest integer $n \geqslant 3$ such that there is $G \in \mathcal{A}_n \setminus \mathcal{E}$ satisfying $\tau(G) < f(G)$. Note that $|V(G)| \geqslant 4$. Our aim is to derive a contradiction, which will establish Theorem 2.

(A). *The graph G has no cut edge.*

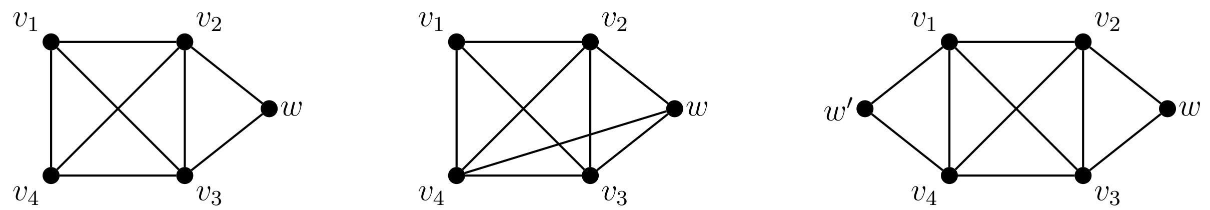

Figure 1: The graphs X_5 (left), X_5' (middle) and X_6 (right). We have $\tau(X_5) = 40$, $\tau(X_5') = 75$ and $\tau(X_6) = 100$.

Proof. Suppose, on the contrary, that e is a cut edge of G. There exists a path P in G that contains e, has internal vertices of degree 2 only and end-vertices — which we call u and v — each of degree at least 3. Deleting the path P forms two connected graphs G_1 and G_2, each in some $\mathcal{A}_s$ where $s \geqslant 3$. Moreover, $\delta(G_1), \delta(G_2) < 4$ so neither G_1 nor G_2 belongs to $\mathcal{E}$. Since adding the edges of the path P to the (disjoint) union of any spanning tree of G_1 and any spanning tree of G_2 produces a spanning tree of G, we deduce that

$$\tau(G) = \tau(G_1)\tau(G_2) \geqslant f(G_1)f(G_2) = 75^{1/5-2\cdot 1/10}f(G) = f(G).$$

This contradiction concludes the proof. ⌟

(B). *The graph G has no cut vertex.*

Proof. Suppose, on the contrary, that u is a cut vertex of G. Since G has no cut edge by (A), we necessarily have $\deg_G(u) = 4$, and $G - u$ is composed of exactly 2 components C_1 and C_2, each containing precisely 2 neighbours of u. For $i \in \{1, 2\}$, let $G_i = C_i + u$, so $\deg_{G_i}(u) = 2$. Then, G_1 and G_2 are connected graphs, each in some $\mathcal{A}_s$ where $s \geqslant 3$, and neither of them belongs to $\mathcal{E}$. In addition, the union of any spanning tree of G_1 and any spanning tree of G_2 produces a spanning tree of G. It follows that $\tau(G) = \tau(G_1)\tau(G_2) \geqslant f(G_1)f(G_2)$. Moreover, $f(G_1)f(G_2) = 75^{1/5-1/5}f(G) = f(G)$. This contradiction concludes the proof. ⌟

We continue with a straightforward property, that will be useful for us later on.

(C). *If v is a vertex of degree 2 in G, then v is contained in a triangle of G.*

Proof. Suppose, on the contrary, that v is a vertex of degree 2 in G, with neighbours x and y that are not adjacent in G. Let G' be the graph obtained from G by deleting the vertex v and adding the edge xy. Then $\tau(G) \geqslant \tau(G')$ and $f(G') = f(G)$. Moreover, $G' \in \mathcal{A}_{n-1}$ and hence if $G' \notin \mathcal{E}$ then $\tau(G) \geqslant f(G)$. If $G' \in \mathcal{E}$, then G is obtained from either K_5 or K_6^- by subdividing an arbitrary edge (all edges being isomorphic). However, this contradicts that G violates Theorem 2, as then we have $\tau(G) = 200 > 178 = \lceil f(G) \rceil$ in the former case, and $\tau(G) = 608 > 422 = \lceil f(G) \rceil$ in the latter case. □

(D). *The graph G has no subgraph isomorphic to K_4.*

Proof. Suppose that $V' = \{v_1, v_2, v_3, v_4\}$ induces a clique in G. For $i \in \{1, 2, 3, 4\}$, let u_i be the neighbour of v_i not in V', if it exists. We can assume that u_1 exists because otherwise G is K_4, but $\tau(K_4) = 16 > 14 = \lceil f(K_4) \rceil$. By (B), either G is obtained from K_5 by deleting at most 2 edges incident with u_1, or we can assume that u_2 exists and is different from u_1. The former case cannot happen, because first G is not K_5; second if G is obtained from K_5 by deleting one edge, then $\tau(G) = 75 = f(G)$; and third if G is obtained from K_5 by deleting two adjacent edges, then $\tau(G) = 40 > 32 = \lceil f(G) \rceil$.

It thus remains to deal with the latter case. We discriminate with respect to the maximum number D of neighbours in V' a vertex in $V(G) \setminus V'$ has. Since $G \notin \mathcal{E}$, we have $D \in \{1, 2, 3\}$.

- If $D = 1$, then all (existing) vertices u_i are pairwise distinct, and we form G' by contracting V' into a new vertex v'. Because $D = 1$, and u_1 and u_2 exist and are distinct, we have $G' \in \mathcal{A}_{n-3}$. Moreover, notice that $f(G') = 75^{-3/5} f(G)$ (whatever the existence of u_3 or u_4). We observe that $G' \notin \mathcal{E}$, for otherwise G is one of the two graphs in Figure 2. However, this is impossible since we would then have $\tau(G) > f(G)$. It follows that $\tau(G) \geqslant \tau(K_4) \cdot \tau(G') = 16\tau(G') \geqslant 16 \cdot 75^{-\frac{3}{5}} f(G) > f(G)$.

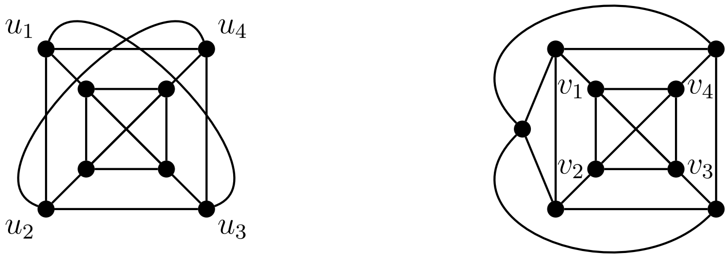

Figure 2: The graph H_1 (left) satisfies $\tau(H_1) = 3456 > 2372 = \lceil f(H_1) \rceil$, and the graph H_2 (right) satisfies $\tau(H_2) = 10469 > 5625 = f(H_2)$.

- If $D = 3$, then assume without loss of generality that $u_2 = u_3 = u_4$, and call this vertex w. Because u_1 has degree at least 2 and is not a cut vertex by (B), we observe that $\deg_G(w) = 4$. Moreover w and u_1 are not adjacent, for otherwise G is obtained from K_5 by subdividing an edge (yielding the vertex u_1), which as was pointed out in the proof of (C) would imply that $\tau(G) = 200 > 178 = f(G)$.

  Now, we form G' by contracting $V' \cup \{w\}$ (which in G induces K_5 minus an edge, so that $\tau(G[V' \cup \{w\}]) = 75$) into a new vertex v', which must have degree 2. So we infer that $G' \in \mathcal{A}_{n-4} \setminus \mathcal{E}$. Moreover, $f(G') = 75^{-5 \cdot \frac{1}{5}} f(G) = 75^{-1} f(G)$. It follows that $\tau(G) \geqslant \tau(G[V' \cup \{w\}]) \cdot \tau(G') \geqslant 75 \cdot 75^{-1} f(G) = f(G)$.

- If $D = 2$, then we can assume without loss of generality that $u_2 = u_3$, and we call this vertex w. If u_4 does not exist, then $\deg_G(w) \geqslant 3$, for otherwise u_1 would be a cut-vertex of G, thereby contradicting (B). We form G' by contracting V' into a single vertex v', which thus has degree 2 in G'. So $G' \in \mathcal{A}_{n-3} \setminus \mathcal{E}$ and hence $\tau(G') \geqslant f(G') = 75^{-4/5} f(G)$.

We now consider an arbitrary spanning forest F' of $G' - v' = G - V'$ that can be extended into a spanning tree of G' by adding edges in $\{v'u_1, v'w\}$. We next prove that the ratio between the number of spanning trees of G that contain F' and the number of spanning trees of G' that contain F' is at least $\frac{75}{2}$.

If u_1 and w are in different components of F', then there is exactly 1 way to extend F' into a spanning tree of G', specifically by adding to F' both edges incident to v'. Moreover, there are $\tau(X_5) = 40$ spanning trees of G that contain F', all obtained by adding to F' the edge u_1v_1, and a spanning tree of $G[V' \cup \{w\}] \sim X_5$.

If u_1 and w are in a same component of F', that is, if F' is a spanning tree of $G' - v'$, then there are $\deg_{G'}(v') = 2$ ways to extend F' into a spanning tree of G'. Moreover, there are $\tau(X_5') = 75$ spanning trees of G that contains F', all obtained by adding to F' a spanning tree of the graph constructed from $G[V' \cup \{u_1, w\}]$ by identifying u_1 and w, thereby producing X_5'.

Consequently, we infer that $\tau(G) \geqslant \frac{75}{2} \cdot \tau(G') \geqslant \frac{75}{2} \cdot 75^{-4/5} f(G) > f(G)$.

If u_4 exists, then the situation can be best treated under two cases, depending on whether or not u_1 and u_4 are the same vertex.

- If $u_1 \neq u_4$, then assume first that w has degree 2 in G. In particular, w is adjacent to neither of u_1 and u_4. So if we form G' by contracting the vertices in $V' \cup \{w\}$ into a single vertex v', it follows that $G' \in \mathcal{A}_{n-4} \setminus \mathcal{E}$, since $\deg_{G'}(v') = 2$, and also that $f(G') = 75^{-4/5} f(G)$. Moreover, $\tau(G) \geqslant \tau(G[V' \cup \{w\}]) \cdot \tau(G') = 40 \cdot \tau(G')$, as $G[V' \cup \{w\}] \sim X_5$. Consequently, $\tau(G) \geqslant 40 f(G') = 40 \cdot 75^{-4/5} f(G) > f(G)$.

  Suppose now that w has degree at least 3 in G. We then contract the vertices in V' into a single vertex v'. Note that $G' \in \mathcal{A}_{n-3} \setminus \mathcal{E}$, as $\deg_{G'}(v') = 3$, and hence $\tau(G') \geqslant f(G') = 75^{-4/5} f(G)$. We consider an arbitrary spanning forest F' of $G' - v' = G - V'$ that can be extended into a spanning tree of G' by adding edges in $\{v'u_1, v'u_4, v'w\}$. Again, we prove that the ratio between the number of spanning trees of G that contain F' and the number of spanning trees of G' that contain F' is at least $\frac{75}{2}$.

  * If u_1, u_4 and w are in different components of F', then there exists exactly 1 spanning tree of G' that contains F': it is obtained by adding to F' all three edges incident to v' in G'. On the other hand, there are $\tau(X_5) = 40$ spanning trees of G that contain F', all obtained by adding to F' the edges u_iv_i for $i \in \{1, 4\}$, and a spanning tree of $G[V'] \sim X_5$.
  * If u_1 and u_4 are in the same component of F', and w is in a different one, then there are exactly 2 spanning trees of G' that contain F': they are obtained by adding to F' the edge wv' and one of the 2 edges in $\{u_1v', u_4v'\}$.

On the other hand, there are $\tau(X_6) = 100$ spanning trees of G that contain F', all obtained by adding to F' a spanning tree of the graph constructed from $G[V' \cup \{w, u_1, u_4\}]$ by identifying u_1 and u_4, thereby producing X_6.

  * If w and u are in the same component of F', and u' is in a different one, where $\{u, u'\} = \{u_1, u_4\}$, then there are exactly 2 spanning trees of G' that contain F': they are obtained by adding the edge between u' and V' and one of the 2 edges between $\{u, w\}$ and V'. On the other hand, we obtain $\tau(X_5') = 75$ different spanning trees of G containing F' by adding to F' the edge between u' and V', and a spanning tree of the graph constructed from $G[V' \cup \{w, u\}]$ by identifying w and u, thereby producing X_5'.
  * If all of u_1, u_4 and w are in the same component of F', then there are exactly 3 spanning trees of G' that contain F': they are obtained by adding to F' any of the 3 edges incident to v' in G'. On the other hand, there are $\tau(K_5) = 125$ different spanning trees of G that contain F', obtained by adding to F' a spanning tree of the graph constructed from $G[V' \cup \{u_1, u_4, w\}]$ by identifying the vertices u_1, u_4 and w, thereby producing K_5.

We conclude that $\tau(G) \geqslant \frac{75}{2}\tau(G') \geqslant \frac{75}{2} \cdot 75^{-4/5} f(G) > f(G)$.

– Suppose now that $u_1 = u_4$, and let us call w' this vertex.

Let H be the subgraph of G induced by $V' \cup \{w, w'\}$. Then H has either 10 or 11 edges, depending on whether or not w and w' are adjacent in G. We first observe that H cannot be the whole graph G, since H does not contradict the statement of Theorem 2. Indeed, if w and w' are not adjacent in G then $H \sim X_6$ so $\tau(H) = 100 > 75 = f(H)$; while otherwise $\tau(H) = 225 > 178 = \lceil f(H) \rceil$. By (B), we thus deduce that each of w and w' has degree at least 3.

If w and w' are adjacent, then (B) further implies that both vertices have degree 4 in G. Furthermore, if w and w' have a common neighbour, then G is the graph H_4 depicted in Figure 3. However, this is not possible as $\tau(H_4) = 575 > 422 = \lceil f(H_4) \rceil$. Now, contracting $V' \cup \{w, w'\}$ into a single vertex v' yields a graph $G' \in \mathcal{A}_{n-5} \setminus \mathcal{E}$, because $\deg_{G'}(v') = 2$. Moreover, $f(G') = 75^{-6/5} f(G)$, and since in this case $\tau(H) = 225$, we infer that $\tau(G) \geqslant 225\tau(G') \geqslant 225 \cdot 75^{-6/5} f(G) > f(G)$.

Similarly, if w and w' are not adjacent in G, then we form G' from $G - V'$ by adding an edge between w and w'. Notice that $\delta(G') \in \{2, 3\}$ so $G' \in \mathcal{A}_{n-4} \setminus \mathcal{E}$. Observe also that $f(G') = 75^{-5/5} f(G) = 75^{-1} f(G)$, whatever the degrees of w and w' are in $\{3, 4\}$. We assert that every spanning tree of G' yields at least 100 spanning trees of G, in a way that all of them are distinct. Indeed, consider a spanning tree T' of G'. If T contains the edge ww', then $T' - ww'$ is a spanning forest of $G - V'$ into 2 trees (such that w and w' belong to different trees). Therefore, adding to $T' - ww'$ any of the 100 spanning trees of $H \sim X_6$ yields a spanning tree of G. If now T' does not use the edge ww', then it can be extended into a spanning tree of G by adding to it any spanning forest of $H = G[V' \cup \{w, w'\}]$ composed of 2 trees separating w and w'. There are $\tau(H + ww') - \tau(H) = 225 - 100 = 125$ such forests. All spanning trees of G thus created are pairwise distinct, since they either differ in $G - V'$ (that is, on $T' - ww'$), or in $G[V' \cup \{w, w'\}]$. Consequently, we infer that $\tau(G) \geqslant 100\tau(G') \geqslant 100 \cdot 75^{-1} f(G) > f(G)$.

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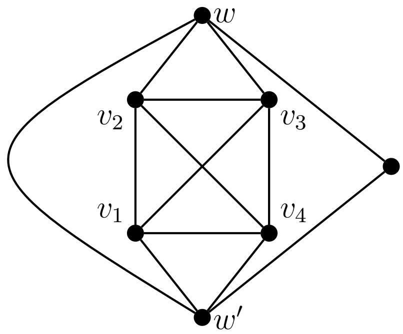

Figure 3: The graph H_4 satisfies $\tau(H_4) = 575 > 422 = \lceil f(H_4) \rceil$.

Our next aim is to prove that G has no subgraph isomorphic to the diamond D_4. We proceed in two steps. Let D_5 be the graph obtained from the diamond D_4 with edges $\{ux, uy, vx, vy, xy\}$ by adding a vertex w adjacent only to x and to y.

(E). *The graph G has no subgraph isomorphic to D_5.*

Proof. First note that, by (D), none of the edges uv, uw and vw belong to $E(G)$. We form G' from G by deleting the vertices x and y, and adding the edges uv, uw and vw. We observe that $G' \in \mathcal{A}_{n-2}$, and we note that $f(G') = 75^{-2/5} f(G)$. Consider any spanning forest F' of $G' - \{uv, uw, vw\} = G - \{x, y\}$ that can be extended to a spanning tree of G' by adding (possibly 0) edges in $\{uv, uw, vw\}$. We show that the ratio between the number of spanning trees of G that contain F' and the number of spanning trees of G' that contain F' is at least $\frac{20}{3}$.

- If u, v and w belong to 3 different components of F', then there are exactly 3 spanning trees of G' that contains F': they are obtained by adding any 2 edges from the triangle uvw. On the other hand, there are 20 spanning trees of G than contain F', obtained by adding to F' a spanning tree of $G[u, v, w, x, y] \sim D_5$.

- If exactly 2 vertices among u, v, w are in a same component of F', then F' has exactly 2 components and we assume, (by symmetry), that u, v are in the same component and w is in a different component. Then there are exactly 2 spanning trees of G' that contain F': they are obtained by adding to F' either of the 2 edges uw and vw. On the other hand, adding to F' any spanning forest of $G[u, v, w, x, y] \sim D_5$

composed of 2 trees T_u and T_v separating u and v yields a spanning tree of G that contains F'. As reported earlier, since u and v are not adjacent in $G[u, v, w, x, y]$, the number of such forests is $\tau(G[u, v, w, x, y] + uv) - \tau(G[u, v, w, x, y]) = 40 - 20 = 20$.

- If all of u, v, w are in one component of F', then necessarily F' is a spanning tree of G'. Moreover, there are 15 spanning trees of G that contain F', obtained by adding to F' either the edge xy and one of the 6 edges between $\{x, y\}$ and $\{u, v, w\}$; or one of the 3 edges in $\{xu, xv, xw\}$ and one of the 3 edges in $\{yu, yv, yw\}$.

We deduce that $\tau(G) \geqslant \frac{20}{3} \cdot \tau(G')$. If $G' \notin \mathcal{E}$, then $\tau(G') \geqslant f(G') = 75^{-2/5} f(G)$, and we hence conclude that $\tau(G) \geqslant \frac{20}{3} \cdot 75^{-2/5} f(G) > f(G)$.

If $G' \in \mathcal{E}$, then G is one of the two graphs of Figure 4, which however is impossible as then $\tau(G) > f(G)$. ⌟

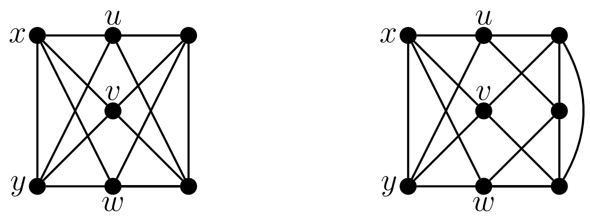

Figure 4: The graph H_5 (left) satisfies $\tau(H_5) = 1200 > 1001 = \lceil f(H_5) \rceil$; and the graph H_6 (right) satisfies $\tau(H_6) = 3610 > 2372 = \lceil f(H_6) \rceil$.

(F). *The graph G has no subgraph isomorphic to the diamond D_4.*

Proof. Suppose, on the contrary, that the subset of vertices $\{u, v, x, y\}$ induces a diamond in G, where u and v are not adjacent. By (E), we know that $N_G(x) \cap N_G(y) = \{u, v\}$. Let G' be obtained from G by adding an edge between u and v, and next contracting the edge xy to produce a new vertex z. So $\{u, v, z\}$ induces a triangle in G', the degrees of u and v in G' are the same as those in G, and $N_{G'}(z) = (N_G(x) \cup N_G(y)) \setminus \{x, y\}$. We have $G' \in \mathcal{A}_{n-1}$ and we observe that $f(G') = 75^{-1/5} f(G)$, whatever the degrees of x and y in G are — each of them being either 3 or 4.

We now consider any spanning forest F' of $G' - \{uv, uz, vz\}$ that is contained in a spanning tree of G'. We recall that if x has in G a neighbour x' not in $\{u, v, y\}$, then in the sequel we identify the edge xx' of G with the edge zx' of G'; and we do similarly for y. This allows us to consider F' as a forest contained in G, which necessarily separates x and y. We next prove that the ratio between the number of spanning trees of G that contain F' and the number of spanning trees of G' that contain F' is at least $\frac{5}{2}$.

- If u, v and z are all in different components of F', then F' is contained in exactly $\tau(K_3) = 3$ spanning trees of G'. On the other hand, there are 8 spanning trees of G that contain F', all obtained by adding to F' one of the 8 spanning trees of $G[u, v, x, y] \sim D_4$.

- If u and v are in a same component of F', different from that of z, then F' is contained in exactly 2 spanning trees of G': they are obtained by adding to F' either the edge zu or the edge zv. On the other hand, F' is contained in 8 spanning trees of G, obtained by adding to F' a spanning forest of $G[V'] \sim D_4$ composed of 2 trees separating u and v. There are $\tau(G[V'] + uv) - \tau(G[V']) = 16 - 8 = 8$ such forests.

- If z is in the same component of F' as exactly one of u and v, say u by symmetry, then F' is again contained in exactly 2 spanning trees of G'. On the other hand, F' is contained in 5 spanning trees of G, obtained as follows: assuming by symmetry that F' (now viewed as a subgraph of G) contains a path between u and x, and thus not between u and y, we obtain a spanning tree of G by adding to F' any spanning forest of $G[V']$ composed of 2 trees separating u and x. There are $\tau(G[V']) - \tau(G[V'] - ux) = 8 - 3 = 5$ such forests.

- If all of u, v, z are in a same component of F', then F' is a spanning tree of G'. On the other hand, F' is contained in 3 spanning trees of G, obtained as follows. We assume by symmetry that F', now viewed as a subgraph of G, has a path between u and x, and thus not between u and y. If y and v are in different components of F', then we add to F' any of the 3 edges between y and $\{u, v, x\}$; while if F' contains also a path between v and y, then we add to F' any of the 3 edges in $\{xv, xy, yu\}$.

We deduce that $\tau(G) \geqslant \frac{5}{2} \cdot \tau(G')$. If $G' \notin \mathcal{E}$, then $\tau(G') \geqslant f(G') = 75^{-1/5} f(G)$, and we hence conclude that $\tau(G) \geqslant \frac{5}{2} \cdot 75^{-1/5} f(G) > f(G)$.

If $G' \in \mathcal{E}$, then observe that G' must be isomorphic to K_6^-. Indeed, otherwise G' is K_5, and then G must be K_6^-, and thus belongs to $\mathcal{E}$, which is not the case. If now G' is K_6^-, then G must be isomorphic to the graph depicted in Figure 5, which contradicts that G is a counter-example. ⌟

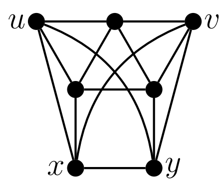

Figure 5: The graph H_7 satisfies $\tau(H_7) = 1183 > 1001 = \lceil f(H_7) \rceil$.

We now show that G has clique number 2.

(G). *The graph G does not contain a triangle.*

Proof. Suppose, on the contrary, that $V' = \{x, y, z\}$ induces a triangle in G. Observe that (F) implies that every vertex of G not in V' has at most 1 neighbour in V'. The analysis is best treated under three cases, regarding the degrees of x, y and z.

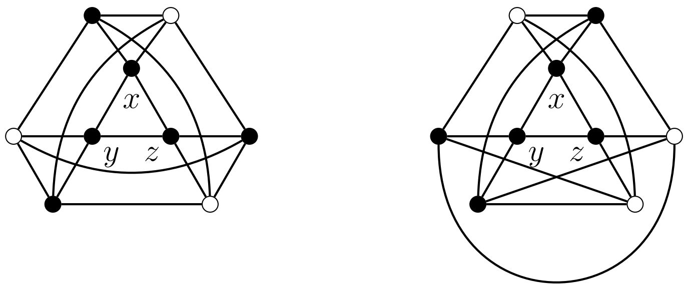

Figure 6: The graph H_8 (left) satisfies $\tau(H_8) = 12096 > 5625 = f(H_8)$; and the graph H_9 (right) satisfies $\tau(H_9) = 12480 > 5625 = f(H_9)$.

- If $\deg_G(x) + \deg_G(y) + \deg_G(z) \leqslant 10$, then $|N_G(x) \cup N_G(y) \cup N_G(z) \setminus V'| \leqslant 4$. Consequently, if G' is obtained from G by contracting V' into a single vertex v', then $G' \in \mathcal{A}_{n-2}$. Moreover, $\tau(G) = \tau(K_3) \cdot \tau(G') = 3\tau(G')$. Observe now that $f(G') = 75^{-1/5} f(G)$, as can be seen by considering $\deg_{G'}(v')$ and recalling that at most 2 vertices of V' have degree 4 in G, the third one then having degree 2. Therefore, if $G' \notin \mathcal{E}$ then $\tau(G) \geqslant 3 \cdot 75^{-1/5} f(G) > f(G)$. We deduce that $G' \in \mathcal{E}$. However, this contradicts (D) or (F). Indeed, if $G' = K_5$, then letting a, b, c, d be the 4 vertices of G outside V', necessarily $G[\{a, b, c, d\}] \sim K_4$. If $G' = K_6^-$, then let a, b, c, d, e be the 5 vertices of G outside V', such that e is adjacent to no vertex in V'. Necessarily, a vertex in V' has at least 2 neighbours among a, b, c, d. These 3 vertices thus induce a triangle in G, and the vertex e is adjacent to exactly 2 of them, thereby forming a copy of D_4 in G.

- If 2 vertices in V' have degree 4, say x and y, and z has degree 3, then let $G_0 = G - z$. Call w the unique neighbour of z not in V'. Notice that $G_0 \in \mathcal{A}_{n-1}$ because G has no cut-vertex by (B) and, if w has degree 2 in G, then (C) implies that the other neighbour of w in G is either x or y, which would contradict (F). By the deletion-contraction formula, $\tau(G_0) = \tau(G_0') + \tau(G_0'')$ where $G_0' = G_0 - xy$ and G_0'' is obtained from G_0 by contracting the edge xy into a new vertex x''. We observe that $G_0' \in \mathcal{A}_{n-1}$ and $G_0'' \in \mathcal{A}_{n-2}$ because none of x, y and z is a cut-vertex in G, by (B), and z has degree 3 while x and y have degree more than 2 in G. Furthermore, $f(G_0') = 75^{-3/5} f(G)$ and $f(G_0'') = 75^{-2/5} f(G)$. Last, we also note that none of G_0' and G_0'' belongs to $\mathcal{E}$, as both contain a vertex of degree 2 or 3, namely w.

  Consider an arbitrary spanning tree T' of G_0'. Then T' is a spanning tree of G_0 that does not use the edge xy. Adding to T' any of the 3 edges of G incident with z yields a spanning tree of G that contains T', and precisely 1 edge in $\{xy, zx, zy, zw\}$. As $G_0' \notin \mathcal{E}$, we thus obtain in total at least $3 \cdot 75^{-3/5} f(G)$ such spanning trees of G.

  Consider an arbitrary spanning tree T'' of G_0''. The edge set of T'' forms a spanning forest of G with exactly 3 components, one composed only of the vertex z, one containing both w and either x or y, say x by symmetry, and another one containing y. Each such forest extends into 5 different spanning trees of G, by adding to it any pair of edges in $\{xy, xz, yz, zw\}$ but the pair $\{xz, zw\}$. As $G_0'' \notin \mathcal{E}$, we thus obtain in total at least $5 \cdot 75^{-2/5}$ spanning trees of G, all containing precisely 2 edges in $\{xy, zx, zy, zw\}$.

In total, we have thus obtained at least $(5 \cdot 75^{-2/5} + 3 \cdot 75^{-3/5})f(G)$ spanning trees of G, which is more than $f(G)$.

- It remains to deal with the case where each of x, y and z has degree 4 in G. First consider the graph G_{xy} obtained from G by deleting the edges zx and zy, and contracting the edge xy into a new vertex. Then $G_{xy} \in \mathcal{A}_{n-1}$ by (B), and since z has degree 2 in G_{xy}, we further know that $G_{xy} \notin \mathcal{E}$. Consequently, $\tau(G_{xy}) \geqslant f(G') = 75^{-2/5}f(G)$. Adding the edge xy to any spanning tree of G_{xy} yields a spanning tree of G that contains the edge xy, but none of the edges zx and zy. We thus obtain at least $75^{-2/5}f(G)$ such spanning trees of G. Using an analogous argument where we contract xz or yz instead of xy, we in total obtain $3 \cdot 75^{-2/5}$ different spanning trees of G, each containing precisely 1 edge in $\{xy, xz, yz\}$.

  Second, consider the graph G_1 obtained by contracting the whole triangle xyz into a single new vertex u, which thus has degree 6. We observe that it is possible to completely lift the vertex u, unless G has precisely 9 vertices and the 6 vertices not in $\{x, y, z\}$ induce a copy of $K_{3,3}$. Indeed, let $N = N_G(\{x, y, z\}) \setminus \{x, y, z\}$. The 6-vertex subgraph of G induced by N has maximum degree at most 3, and at least 2 of its vertices have degree at most 2 unless G has precisely 9 vertices, because G has no cut-vertex by (B). From there, the facts that G has no K_4 and no D_4, by (D) and (F), yield the statement quickly. We deduce that if u cannot be completely lifted, then G is either the graph H_8 or the graph H_9, both depicted in Figure 6. However, none of these graphs contradicts Theorem 2, so that u can indeed be completely lifted in G_1, yielding a graph $G' \in \mathcal{A}_{n-3}$. Theorem 4 then guarantees that $\tau(G_1) \geqslant \frac{8}{3}\tau(G')$.

  Now, if the graph G' thus obtained belongs to $\mathcal{E}$, then $G' = K_6^-$, since $|V(G')| \geqslant 6$. It follows that the 6 vertices not in V' induce a 4-vertex path P with two additional vertices, both joined to the end-vertices of P, and to one of the inner vertices of P, in a way that every vertex has degree 3. Using that G contains no diamond by (F), we infer that G is one of the 4 graphs depicted in Figure 7. However, none of them is a counter example to Theorem 2, and therefore $G' \notin \mathcal{E}$. Consequently, $\tau(G') \geqslant f(G') = 75^{-3/5}f(G)$. We deduce that $\tau(G_1) \geqslant \frac{8}{3} \cdot 75^{-3/5}f(G)$. Observe now that every spanning tree of G_1 yields $\tau(K_3) = 3$ different spanning trees of G, thus yielding at least $8 \cdot 75^{-3/5}f(G)$ different spanning trees of G, each containing precisely 2 edges in $\{xy, xz, yz\}$.

  In total, we have constructed $(3 \cdot 75^{-2/5} + 8 \cdot 75^{-3/5})f(G)$ different spanning trees of G, which is more than $f(G)$. This concludes the proof.

$\lrcorner$

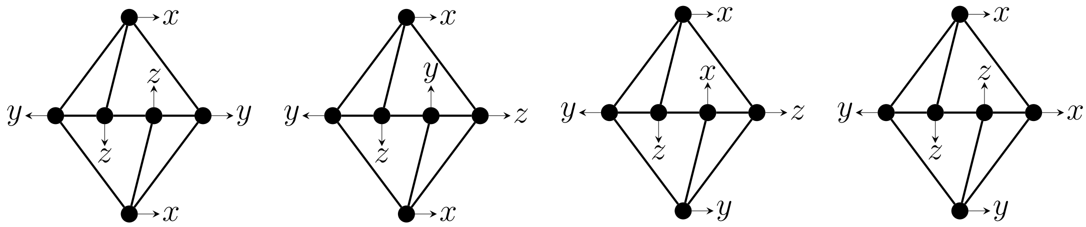

Figure 7: In these pictures, the triangle xyz is not shown, but from each vertex starts an arrow pointing to its unique neighbour in the triangle xyz. The graphs are named $H_{10}, \ldots, H_{13}$ from left to right. We note that $f(H_i) = 5625$ for each $i \in \{10, \ldots, 13\}$. We have $\tau(H_{10}) = 12096$; $\tau(H_{11}) = 12321$; $\tau(H_{12}) = 11729$ and $\tau(H_{13}) = 11520$.

We can now prove our last property of the graph G, that the neighbours of any vertex of degree 4 must induce a subgraph with at least 1 edge.

(H). *If x is a vertex of degree 4 in G, then $G[N_G(x)]$ is not empty.*

Proof. Suppose, on the contrary, that $N_G(x) = \{a, b, c, d\}$ induces no edge in G. We consider the 3 graphs that may be formed by doing a complete lift at x: we can indeed completely lift the vertex x, and it amounts to deleting x and adding the edges of a perfect matching between its neighbours. Thus, from $G - x$, we obtain G_1 by adding the edges in $M_1 = \{ab, cd\}$; G_2 by adding the edges in $M_2 = \{ac, bd\}$; and G_3 by adding the edges in $M_3 = \{ad, bc\}$. As x is not a cut-vertex of G by (B), we have $G_1, G_2, G_3 \in \mathcal{A}_{n-1}$. Moreover, since $G[N_G(x)]$ is empty, we see that $G_1, G_2, G_3 \notin \mathcal{E}$. Therefore $\tau(G_j) \geqslant f(G_j) = 75^{-1/5} f(G)$ for each $j \in \{1, 2, 3\}$.

Now we consider all spanning forests of $G - x$ that are contained in a spanning tree of G. Note that they include all spanning forests of $G_i - M_i$ that are contained in a spanning tree of G_i. We classify in Table 1 all these forests of $G - x$ into 15 types, regarding which vertices among a, b, c, d belong to the same component — really, only 5 types up to the symmetry of the roles played by the neighbours of x. For each such type, we calculate the number of ways each corresponding forest may be extended into a spanning tree of G, G_1, G_2 and G_3 — this number indeed depends only on the type of the forest.

For each $i \in \{1, \ldots, 15\}$, let f_i be the number of forests of type i, let λ_i be the number of ways to extend a forest of type i to a tree of G, and let $\lambda_{i,j}$ be the number of ways to extend a forest of type i to a tree of G_j.

We have

$$\tau(G) = \sum_{i=1}^{15} \lambda_i f_i \qquad \text{and} \qquad \forall j \in \{1, 2, 3\}, \quad \tau(G_j) = \sum_{i=1}^{15} \lambda_{i,j} f_i. \tag{2}$$

We further read off from Table 1 that

$$\forall i \in \{1, \ldots, 15\}, \quad \lambda_i \geqslant \lambda_{i,1} + \lambda_{i,2} + \lambda_{i,3}.$$

Type	Forest type	G	G_1	G_2	G_3
1	a, b, c, d	1	0	0	0
2	ab, c, d	2	0	1	1
3	ac, b, d	2	1	0	1
4	ad, b, c	2	1	1	0
5	bc, a, d	2	1	1	0
6	bd, a, c	2	1	0	1
7	cd, a, b	2	0	1	1
8	ab, cd	4	0	2	2
9	ac, bd	4	2	0	2
10	ad, bc	4	2	2	0
11	a, bcd	3	1	1	1
12	b, acd	3	1	1	1
13	c, abd	3	1	1	1
14	d, abc	3	1	1	1
15	$abcd$	4	1	1	1

Table 1: All the possible types of spanning forests of $G - x$, along with the number of ways to extend them into spanning trees of G, G_1, G_2 and G_3.

Consequently, (2) now implies that $\tau(G) \geqslant \tau(G_1) + \tau(G_2) + \tau(G_3)$. As noted earlier, $\tau(G_j) \geqslant f(G_j) = 75^{-1/5} f(G)$ for each $j \in \{1, 2, 3\}$, and hence $\tau(G) \geqslant 3{\cdot}75^{-1/5} f(G) > f(G)$. This contradiction concludes the proof. ⌟

We are now ready to demonstrate Theorem 2.

Proof of Theorem 2. Suppose the existence of a smallest positive integer n such that there exists $G \in \mathcal{A}_n \setminus \mathcal{E}$ with $\tau(G) < f(G)$. Clearly, $n \geqslant 4$. Moreover, $G \neq K_4$ as $\tau(K_4) = 16 > 13 = \lceil f(K_4) \rceil$. If G has maximum degree less than 4, then Theorem 1 implies that $G \geqslant 2^{3(n_3+2)/4} > 75^{n_3/10}$ as $2^{3/4} > 75^{1/10}$. Therefore, G contains a vertex v of degree 4. By (H), there are 2 neighbours u and w of v that are adjacent in G. Consequently, $\{u, v, w\}$ induces a triangle in G, which contradicts (G). □

Acknowledgements

Zelealem B. Yilma received support from the Qatar Foundation through Carnegie Mellon University in Qatar's Seed Research program.

References

[1] Alon, N. The number of spanning trees in regular graphs. *Random Structures & Algorithms 1*, 2 (1990), 175–181.

[2] Bogdanowicz, Z. R. On the minimum number of spanning trees in cubic multigraphs. *Disc. Math. Graph Theory 40* (2020), 149–159.

[3] Kostochka, A. V. The number of spanning trees in graphs with a given degree sequence. *Random Structures & Algorithms 6*, 2–3 (1995), 269–274.

[4] McKay, B. D. Spanning trees in regular graphs. *European Journal of Combinatorics 4*, 2 (1983), 149–160.

[5] Ok, S., and Thomassen, C. On the minimum number of spanning trees in k-edge-connected graphs. *Journal of Graph Theory 84*, 3 (2017), 286–296.

[6] Pekárek, J., Sereni, J.-S., and Yilma, Z. B. The minimum number of spanning trees in regular multigraphs. *Electron. J. Combin. 29*, 4 (2022), #P4.29.