On Density Conditions for Transversal Trees in Multipartite Graphs

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Abstract

Let G be an r-partite graph such that the edge density between any two parts is at least α. How large does α need to be to guarantee that G contains a connected transversal, that is, a tree on r vertices meeting each part in one vertex? And what if instead we want to guarantee the existence of a Hamiltonian transversal?

In this paper we initiate the study of such extremal multipartite graph problems, obtaining a number of results and providing many new constructions, conjectures and further questions.

Mathematics Subject Classifications: 05C35, 05C80

1 Introduction

1.1 Background

Given an r-partite graph G with r-partition $\sqcup_{i=1}^{r} V_i$, denote its *r-partite density* $d_r(G)$ by

$$d_r(G) := \min_{1\leqslant i<j\leqslant r} \frac{|E(G[V_i, V_j])|}{|V_i|\cdot|V_j|}.$$

(Here we assume $V_i \neq \emptyset$ for every i.)

A *transversal* G' of G is a subgraph of G induced by a set of vertices U meeting each part V_i in exactly $|U \cap V_i| = 1$ vertex. Clearly G' can be viewed a subgraph of K_r, the complete graph on r-vertices. Given a family $\mathcal{F}$ of non-empty subgraphs of K_r, we say that G has *$\mathcal{F}$-free transversals* if every transversal G' of G is $\mathcal{F}$-free. In this paper we are concerned with a generalisation of the following Turán-type problem:

Problem 1.1 (Density Turán problem for multipartite graphs)**.** Determine the supremum $\pi_r(\mathcal{F})$ of the $\alpha \leqslant 1$ such that there exists an r-partite graph G with $d_r(G) = \alpha$ and $\mathcal{F}$-free transversals.

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Problem 1.1 was introduced by Bondy, Shen, Thomassé and Thomassen [8], who considered the case when $\mathcal{F}$ consists of a single graph, the triangle K_3. Bondy, Shen, Thomassé and Thomassen gave a beautiful proof that any tripartite graph G with $d_3(G) > \frac{-1+\sqrt{5}}{2}$ (the reciprocal of the golden ratio) must contain a triangle, and that this result is best possible. They also fully resolved a more general inhomogeneous version of the problem. Further, they showed than in any *infinite*-partite graph, a partite density strictly greater than $\frac{1}{2}$ suffices to guarantee the existence of a triangle.

This latter result was improved by Pfender [18], who showed that for $r \geqslant 12$, any r-partite graph with triangle-free transversals can have r-partite density at most $\frac{1}{2}$ (the cases $4 \leqslant r \leqslant 11$ remain open to this day — it is known from [8] that $\pi_4(\{K_3\}) > 1/2$). More generally, Pfender showed that for any $t \in \mathbb{Z}_{\geqslant 2}$ there exists $r \in \mathbb{Z}_{\geqslant t}$ such that any r-partite graph with K_t-free transversals can have r-partite density at most $\frac{t-2}{t-1}$, which is easily seen to be best possible by taking a suitable intersection of a $(t-1)$-partite Turán graph with an $(r-1)$-partite graph. Thus, in our notation, Pfender's result is a multigraph analogue of Turán's theorem stating that for every integer $t \geqslant 2$, $\pi_r(\{K_t\}) = \frac{t-2}{t-1}$ for all r sufficiently large.

Pfender's result was generalised by Narins and Tran [17], who obtained a multipartite analogue of the Erdős–Stone theorem, with a surprising twist. Given a graph F with chromatic number $\chi(F)$, they showed that if F is *almost colour-critical* (see [17] for a definition of this term), then for all r sufficiently large,

$$\pi_r(\{F\}) = \frac{\chi(F) - 2}{\chi(F) - 1},$$

as one would expect; however, if F is not almost-colour critical, then Narins and Tran showed that $\pi_r(\{F\}) \geqslant \frac{\chi(F)-2}{\chi(F)-1} + \frac{1}{(\chi(F)-1)^2(r-1)^2} > \frac{\chi(F)-2}{\chi(F)-1}$ for all $r \geqslant |F|$.

Bondy, Shen, Thomassé and Thomassen's results were extended in three further directions. First of all Baber, Johnson and Talbot [5] considered the problem of minimising the density of triangles in tripartite graphs with 3-partite density above $\frac{-1+\sqrt{5}}{2}$ (which can be viewed as a multipartite version of the Rademacher–Turán problem). Secondly, Markström and Thomassen [15] determined the partite density needed to guarantee a copy of $K_{r+1}^{(r)}$, the complete r-uniform hypergraph on $r+1$ vertices, in an $(r+1)$-partite r-uniform hypergraph.

Finally, in a direction related to the one we pursue in the present paper, Nagy [16] studied the partite density needed in a subgraph of a blow-up of a graph H to guarantee the existence of H as a transversal subgraph. In particular, Nagy determined this critical density in the case when H is a tree or a cycle, relating it to the spectral properties of H's adjacency matrix. Nagy's work was generalised by Csikváry and Nagy [9], who obtained inhomogeneous versions and extensions of Nagy's results.

Beyond the problems considered in this paper, multipartite graphs are widely-studied objects in extremal graph theory. Multipartite graphs appear in applications of the Szemerédi regularity lemma, and are the subject of an influential family of problems posed by Bollobás, Erdős and Szemerédi [7], on which research is still ongoing [13]. Finally, there is a connection between multipartite graphs (and especially the more general H-partite graphs we shall focus on) and 1-dependent random graph models, which we expound on in Section 2.2. Given the applications of such 1-dependent models in percolation theory [3, 19], this gives strong motivation for studying the connectivity-related questions we shall consider.

1.2 Setting of this paper

In this paper we study a slightly more general form of Problem 1.1. Following Nagy [16], given a host graph H we will consider *H-partite graphs*, which we define below.

Definition 1.2 (***H***-partite graphs)**.** A *weighted H-partite graph* (henceforth abbreviated to H-partite graph) is a graph G together with a *canonical H-partition* $V(G) := \bigsqcup_{x\in V(H)} V_x$ and a weight function $w:\ V(G) \to [0,1]$ satisfying the following:

1. for each $x \in V(H)$, $\sum_{v\in V_x} w(v) = 1$ (i.e. for each $x \in V(H)$, w gives rise to a probability distribution over V_x);
2. $E(G) = \bigcup_{xx'\in E(H)} E(G[V_x, V_y])$ (i.e. G is a subgraph of the blow-up of the host graph H associated with the canonical partition $\bigsqcup_{x\in V(H)} V_x$).

Remark 1.3. Any subgraph of a blow-up of the host graph H can be viewed as a weighted H-partite graph by letting the weight function correspond to the uniform distribution over each part V_x, $x \in V(H)$ (i.e. setting $w(v) = 1/|V_x|$ for every $v \in V_x$).

Conversely, given any weighted H-partite graph G where the weight function w takes rational values only, one can construct a subgraph G' of a blow-up of H in a natural way: let $N \in \mathbb{N}$ be chosen so that $Nw(v) \in \mathbb{N}$ for all $v \in V(G)$ with $w(v) > 0$. Then replace each $v \in V(G)$ by a set B_v of $Nw(v)$ vertices, and put a complete bipartite graph between B_u and B_v whenever $uv \in E(G)$. Since any weight function on a finite graph can be arbitrarily well-approximated by a rational weight function, it follows that the study of subgraphs of blow-ups of H and weighted H-partite graphs are (asymptotically) equivalent.

We thus choose in the rest of this paper to refer to the slightly more general class of weighted H-partite graphs as H-partite graphs — the latter setting makes for much cleaner results and proofs, and for the Turán-type problems we consider the two settings are fully equivalent by a result of Bondy, Shen and Thomassé, Proposition 2.1 given in the next section.

The notion of r-partite density generalises to the H-partite setting in a natural way:

Definition 1.4 (***H***-partite density)**.** Given an H-partite graph G, we define the *H-partite density profile* of G to be

$$\alpha(G) := \left(\alpha_{xy}\right)_{xy\in E(H)},$$

where $\alpha_{xy} := \sum_{u\in V_x, v\in V_y} w(u)w(v)\mathbb{1}_{uv\in E(G)}$ is the edge density of the (weighted) bipartite subgraph of G induced by $V_x \sqcup V_y$. Further, the *H-partite density* of G is $d_H(G) := \min\{\alpha_{xy} :\ xy \in E(H)\}$.

Similarly to the r-partite setting, a *transversal* of an H-partite graph G is a subgraph of G induced by a set of vertices U meeting each part V_x, $x \in V(H)$ of the canonical partition of G in exactly one vertex. Our interest in this paper is then the following generalisation of Problem 1.1:

Problem 1.5. [Density Turán problem for H-partite graphs] Given a non-empty family $\mathcal{F}$ of non-empty subgraphs of a host graph H, determine

$$\pi_H(\mathcal{F}) := \sup\Big\{d_H(G): \ G \text{ is an } H\text{-partite graph with } \mathcal{F}\text{-free transversals}\Big\}.$$

As alluded to above, a result of Bondy, Shen, Thomassé and Thomassen implies that for finite host graphs H the supremum in the definition of $\pi_H(\mathcal{F})$ is in fact a maximum (see Proposition 2.1), and in particular for $H = K_r$ we have $\pi_r(\mathcal{F}) = \pi_{K_r}(\mathcal{F})$ for all non-empty families $\mathcal{F}$ of non-empty graphs on at most r vertices.

Even in the case $H = K_r$ of Problem 1.5, which is equivalent to Problem 1.1 as we remarked above, and whose origins can be traced back to remarks of Bollobás [6, page 324] in the late 1960s, many interesting questions remain open, notably when it comes to spanning structures — indeed the whole extent of the literature can be found in the previous subsection.

However a need to consider the more general setting of host graphs H other than K_r stems from applications to percolation theory. As we explain in Section 2.2, weighted H-partite graphs correspond to an important family of locally dependent random graph models (referred to as "colouring models" in the recent work of Lengler, Martinsson, Petrova, Schnider, Steiner, Weber and Welzl [12]); in particular it would be highly desirable to better understand such models when the host graph H is an $N \times N$ square grid, a d-dimensional hypercube or the (infinite) integer square lattice. This motivates the study of the more general H-partite setting of Problem 1.5.

1.3 Results

We investigate the H-partite density threshold for connected transversals in H-partite graphs. Given a positive integer r, let $\mathcal{T}_r$ denote the family of all trees on r vertices. Our first result is as follows: for every graph H obtained from K_n by deleting a non-empty matching, we determine the smallest H-partite density forcing the existence of a connected transversal in an H-partite graph.

Theorem 1.6. *Let $r \geqslant 3$ and let M be a non-empty matching in K_r. Then for the graph $H = K_r - M$ obtained from K_r by deleting the edges of the matching M, we have*

$$\pi_H(\mathcal{T}_r) = \frac{1}{2}.$$

We show however that when $H = K_r$, the connectivity threshold dips strictly below $1/2$:

Theorem 1.7. *For all $r \geqslant 3$,*

$$\frac{r-2}{2(r-1)^2}\left(3r - 4 - \sqrt{5r^2 - 16r + 12}\right) \leqslant \pi_{K_r}(\mathcal{T}_r) \leqslant \frac{1}{2} - \frac{1}{4r-6}.$$

Remark 1.8. It is easily checked that for $r \geqslant 4$, the sequence given by

$$u_r := \frac{r-2}{2(r-1)^2}\left(3r - 4 - \sqrt{5r^2 - 16r + 12}\right)$$

is strictly increasing and satisfies $u_4 = \frac{8-2\sqrt{7}}{9} = 0.300944\ldots$ and $\lim_{r\to\infty} u_r = \frac{3-\sqrt{5}}{2} = 0.381966\ldots$.

Remark 1.9. In the case $r = 3$ of Theorem 1.7, it is a fairly trivial exercise to improve the upper bound to 1/4, matching the lower bound and establishing that $\pi_{K_3}(\mathcal{T}_3) = \frac{1}{4}$; this result also follows from [12, Theorem 7, Colouring model with two colours].

We conjecture that the lower bound in Theorem 1.7 is always tight, and in particular that a K_r-partite density of $\frac{3-\sqrt{5}}{2}$ guarantees the existence of a connected transversal:

Conjecture 1.10. The lower bound in Theorem 1.7 is tight. In particular, for any $r \geqslant 4$, every r-partite graph with r-partite density at least $\frac{3-\sqrt{5}}{2}$ contains a transversal tree on r vertices.

Update added in revision: *Conjecture 1.10 was recently proved in an asymptotic form by Lengler, Martinsson Petrova, Schnider, Steiner, Weber and Welzl [12]. More precisely, they proved in [12, Theorem 7, colouring model] that for any $\varepsilon > 0$ there exists r_0 such that for every $r \geqslant r_0$ any weighted r-partite graph with r-partite density at least $\frac{3-\sqrt{5}}{2} + \varepsilon$ must contain a connected transversal. The proof uses an ingenious and highly non-trivial 13-pages long probabilistic argument, that may well have further application to extremal problems for r-partite graphs.*

Remark 1.11. An interesting question is why Conjecture 1.10 appears to be much harder to prove than Theorem 1.6. We speculate this may be due to the simpler structure of the extremal constructions in Theorem 1.6: these consist of weighted r-partite graphs in which each part contains at most two vertices. On the other hand, the conjectured extremal configuration for Conjecture 1.10, Construction 3.4, features one part containing $r-1$ different vertices, requiring any argument to take a 'global' approach in order to capture its more complex structure.

In addition to the results above, we prove some general upper bounds on the threshold for the existence of connected transversals.

Proposition 1.12. *Let H be any connected graph on r vertices. Then $\pi_H(\mathcal{T}_r) \leqslant \frac{r-2}{r-1}$.*

This elementary bound is sharp in general, as can be seen by considering stars.

Proposition 1.13. *For the star $K_{1,r-1}$, we have $\pi_{K_{1,r-1}}(\mathcal{T}_r) = \frac{r-2}{r-1}$*

Before stating the next result, recall that given two graphs G_1 and G_2, their *Cartesian product* $G_1 \times G_2$ is the graph on the set $V(G_1) \times V(G_2)$ in which vertices (u_1, u_2) and (v_1, v_2) are joined by an edge if either $u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. For r even, the ladder on r vertices is the Cartesian product $K_2 \times P_{r/2}$ of a single edge with a path of length $r/2$. As a consequence of works of Nagy [16] and Day, Falgas–Ravry and Hancock [10], $\pi_H(\mathcal{T}_r)$ is known when H is a tree or a ladder on r vertices (see Section 2.2). Since clearly $\pi_H(\mathcal{T}_r) \leqslant \pi_{H'}(\mathcal{T}_r)$ whenever H' is a supergraph of H, this yields the following:

Corollary 1.14. *Let $r \in \mathbb{Z}_{\geqslant 2}$, and let H be a graph on r vertices. If H contains a Hamiltonian path P_r, then $\pi_H(\mathcal{T}_r) < \frac{3}{4}$, while if H contains a spanning ladder $K_2 \times P_{r/2}$, then $\pi_H(\mathcal{T}_r) < \frac{2}{3}$.*

We also investigate small cases of Problem 1.5. Up to isomorphism, the connected graphs on 4-vertices consist of K_4, K_4^- (K_4 with an edge removed), $K_{2,2} = C_4$ (the cycle on 4 vertices), $K_{1,3}$ (the star on 4 vertices), P_4 (the path on 4 vertices) and $K_4 - P_3$ (K_4 with a path on three vertices deleted). We summarise our bounds for $\pi_H(\mathcal{T}_4)$ for these graphs H as well as for the 5-cycle C_5 in the table below.

Graph H	Connected transversal threshold	Result
K_4	$\pi_{K_4}(\mathcal{T}_4) \stackrel{?}{=} \frac{8-2\sqrt{7}}{9} = 0.3009\ldots$	Theorem 1.7/Conjecture 1.10
	$\pi_{K_4}(\mathcal{T}_4) < 2 - 2\sqrt{\frac{2}{3}} = 0.36701\ldots$	Theorem 3.3
K_4^-	$\pi_{K_4^-}(\mathcal{T}_4) = \frac{1}{2}$	Theorem 1.6
$K_{2,2} = C_4$	$\pi_{K_{2,2}}(\mathcal{T}_4) = \frac{1}{2}$	Theorem 1.6
$K_4 - P_3$	$\pi_{K4-P_3}(\mathcal{T}4) = 4 - 2\sqrt{3} = 0.5358\ldots$	Proposition A.1
P_4	$\pi_{P_4}(\mathcal{T}_4) = \frac{-1+\sqrt{5}}{2} = 0.6180\ldots$	Nagy [16, Corollary 3.13]
$K_{1,3}$	$\pi_{K_{1,3}}(\mathcal{T}_4) = \frac{2}{3}$	Proposition 1.13
C_5	$\pi_{C_5}(\mathcal{T}_5) = \frac{1}{2}$	Proposition A.2

Finally, we consider multipartite versions of Dirac's theorem. Let $\mathcal{C}_r$ denote the collection of Hamilton cycles on r labelled vertices. We prove (perhaps surprisingly) that the r-partite density threshold for Hamiltonicity in r-partite graphs is strictly greater than 1/2.

Theorem 1.15. *Fix $r \in \mathbb{Z}_{\geqslant 4}$. Let $p_\star = p_\star(r)$ be the unique real solution in $(\frac{1}{2}, 1)$ to the cubic equation*

$$(r-2) - (4r-10)p + (6r-14)p^2 - (4r-8)p^3 = 0. \tag{1.1}$$

Then $\pi_{K_r}(\mathcal{C}_r) \geqslant (p_\star)^2 + (1-p_\star)^2 > \frac{1}{2}$.

Remark 1.16. Asymptotic analysis of (1.1) yields that $p_\star = \frac{1}{2} + \frac{1}{2r} + O\left(r^{-2}\right)$ and hence that ${p_\star}^2 + (1-p_\star)^2 = \frac{1}{2} + \frac{1}{2r^2} + O\left(r^{-3}\right)$.

We conjecture, however, that the Hamiltonicity threshold should converge to 1/2 as $r \to \infty$.

Conjecture 1.17 (Multipartite Dirac Conjecture)**.** $\lim_{r\to\infty} \pi_{K_r}(C_r) = \frac{1}{2}$.

Here again, we prove some lower bounds for the $r = 4$ case.

Proposition 1.18. $0.5707\ldots \leqslant \pi_{K_4}(\mathcal{C}_4) \leqslant \frac{1}{\sqrt{3}} = 0.5773\ldots$.

1.4 Organisation of the paper

We gather some preliminary remarks in Section 2. Our main results on connected transversals, Theorems 1.6 and 1.7 are proved in Section 3, while our results on Hamiltonian transversals can be found in Section 4. We end the paper in Section 5 with some remarks and discussion regarding a long list of related problems. The results on the connectivity thresholds for the small graphs $K_4 - P_3$ and C_5 are included together with their proofs in Appendix A.

1.5 Notions and notation

We use $[r]$ to denote the set $\{1, 2, \ldots, r\}$ and, given a set S, we write $S^{(t)}$ for the collection of all subsets of S of size t. Throughout the paper, we use standard graph theoretic notions and notation, which we recall below for the reader's convenience.

A graph is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(2)}$ forms the edges of the graph. A subgraph of G is a graph G' with $V(G') \subseteq V(G')$ and $E(G') \subseteq E(G)$. A spanning subgraph of G is a subgraph G' with $V(G') = V(G)$. Given a set $U \subseteq V(G)$, the subgraph of G induced by U is the graph $G[U] := (U, E(G) \cap U^{(2)})$.

The neighbourhood $N_G(x)$ of a vertex x in a graph G is the collection of vertices $y \in V(G)$ such that $\{x, y\}$ is an edge of G; the degree $\deg(x) = \deg_G(x)$ of x is the size of its neighbourhood. The adjacency matrix of a graph G, $A = A(G)$ is a matrix with rows and columns indexed by vertices of V, with $A_{u,v} = \mathbb{1}_{\{u,v\} \in E(G)}$.

A path of length $\ell - 1$ in a graph G is a sequence of ℓ distinct vertices $\{v_1, v_2, \ldots v_\ell\}$ with $\{v_i, v_{i+1}\} \in E(G)$ for all $i \in [\ell - 1]$. Two vertices in a graph are connected if they are joined by a finite path; this is an equivalence relation on $V(G)$, whose equivalence classes are the connected components of G. A graph is connected if it consists of a single connected component. A tree is a minimally connected graph; a vertex of degree 1 in a tree is called a leaf.

We denote by K_r the complete graph on r vertices $K_r = ([r], [r]^{(2)})$, and $K_{r,s}$ the complete bipartite graph whose vertex-set is the disjoint union of an r-set A and an s-set B, and whose edges include all pairs $\{a, b\}$ with $a \in A, b \in B$. The graph $K_{1,r-1}$ is known as the star on r vertices. We also denote by C_r the cycle on r vertices, $C_r = ([r], \{\{i, i+1\} : \ i \in [r-1]\} \cup \{\{r, 1\}\})$. A matching in a graph G is a collection of vertex-disjoint edges.

Whenever there is no ambiguity, we write uv for a pair $\{u, v\}$. Similarly we write F for the subgraph family $\{F\}$. Finally, when we consider H-partite graphs with $H = K_r$, the complete graph on r-vertices, we write 'r-partite' rather than 'K_r-partite', and similarly '$d_r(G)$' and '$\pi_r(\mathcal{F})$ rather than '$d_{K_r}(G)$' and '$\pi_{K_r}(\mathcal{F})$', so as to avoid notational clutter (note that it is an easy corollary to Proposition 2.1 that the notions of $\pi_r(\mathcal{F})$ and $\pi_{K_r}(\mathcal{F})$ we introduced do coincide, and thus can be conflated). In the context of (weighted) H-partite graphs, given a set A of vertices we write $w(A) := \sum_{a \in A} w(a)$ for the sum of the weights of the vertices contained therein.

2 Preliminaries

2.1 Compactness and computability

Using a simple convexity argument, Bondy, Shen, Thomassé and Thomassen [8] proved the useful fact that if $H = K_3$, then for the problems we are considering, there always exists a finite extremal example in which all the parts have bounded sizes. However, their argument easily generalises to the statement of the next Proposition, as noted by several authors, see e.g. Nagy [16, Lemma 2.1] for a formal proof.

Proposition 2.1 (Bondy, Shen, Thomassé, Thomassen)**.** *For any non-empty host graph H and any family $\mathcal{F}$ of non-empty subgraphs of H, there exists an H-partite graph G such that*

(i) (extremality) G has $\mathcal{F}$-free transversals and H-partite density $\pi_H(\mathcal{F})$;

(ii) (finiteness) for every vertex $x \in V(H)$, the corresponding part V_x in the canonical partition of G contains at most $\deg_H(x)$ vertices.

So in principle, determining $\pi_H(\mathcal{F})$ is a finitely computable problem — albeit one in which the complexity rises very quickly as the average degree in H increases.

2.2 Random transversals and connection to 1-dependent random graphs

A useful tool when studying H-partite graphs is to consider random transversals. More specifically, given an H-partite graph G, a *random transversal* of G is obtained by first selecting a representative $\mathbf{v}_x \in V_x$ independently at random from each of the parts $(V_x)_{x \in V(H)}$ of the canonical H-partition of G, with $\mathbb{P}(\mathbf{v}_x = u) = w(u)$ for every $x \in V(H)$ and $u \in V_x$. The random transversal T is then the subgraph of G induced by the randomly chosen representatives, $T := G[\{\mathbf{v}_x : \ x \in V(H)\}]$ (which may be viewed as a random spanning subgraph of H in the natural way).

Random transversals are a special class of 1-dependent random graphs, whose definition we recall below. Given a host graph $H = (V(H), E(H))$ and a probability measure μ on the collection of subsets of $E(H)$, we can build a random graph $\mathbf{H}$ from H by setting $V(\mathbf{H}) = V(H)$ and letting $E(\mathbf{H})$ be a μ-random subset of $E(H)$, i.e. a subset of $E(H)$ chosen at random according to the probability distribution given by μ. Thus $\mathbf{H}$ is a random variable taking values in the collection of spanning subgraphs of H, and we refer to it as a *random graph model on H*.

Definition 2.2 (**1**-dependent random graph models)**.** Let H be a host graph. A random graph model $\mathbf{H}$ on H is said to be 1-*dependent* if whenever A and B are disjoint subsets of $V(H)$, the random induced subgraphs $\mathbf{H}[A]$ and $\mathbf{H}[B]$ are mutually independent random variables.

Informally, a random graph model is 1-dependent if events 'living' (defined by what happens) on disjoint vertex-subsets are mutually independent. Denote by $\mathcal{M}_{1,\geqslant p}(H)$ the collection of 1-dependent random graph models $\mathbf{H}$ on H such that for each edge $xy \in E(H)$, $\mathbb{P}(xy \in E(\mathbf{H})) \geqslant p$.

It is clear that given an H-partite graph G with $d_H(G) \geqslant p$, taking a random transversal of G gives us a 1-dependent random graph model over H from $\mathcal{M}_{1,\geqslant p}(H)$. While in general there exist 1-dependent random graphs that cannot be constructed from H-partite graphs, H-partite graphs are arguably the most important class of 1-dependent random graph models from the point of view of applications and constructions. They have notably been recently studied in [12] under the name "colouring models".

As noted by Balister and Bollobás [2], "1-dependent percolation models have become a key tool in establishing bounds on critical probabilities" in percolation theory. Despite their usefulness, however, many basic questions about the behaviour of 1-dependent models are open.

With regards to the problems studied in this paper, set

$$p_{\mathrm{conn}}(H) := \sup\{p \in [0,1] : \ \exists \mathbf{H} \in \mathcal{M}_{1,\geqslant p}(H) \text{ such that } \mathbb{P}\left(\mathbf{H} \text{ is connected}\right) = 0\}$$

be the 1-dependent critical probability for connectivity over H. By the observation above that H-partite graphs correspond to a special class of 1-dependent models, it follows that for any connected host graph H on r vertices,

$$\pi_H(\mathcal{T}_r) \leqslant p_{\mathrm{conn}}(H). \tag{2.1}$$

Day, Falgas–Ravry and Hancock showed in [10, Theorems 15, 16] that $p_{\mathrm{conn}}(P_r) = \frac{1}{4}\left(3 - \tan^2\left(\frac{\pi}{2r}\right)\right)$, which is exactly the value of $\pi_{P_r}(P_r)$ determined by Nagy [16] (and implies his result), and that $p_{\mathrm{conn}}(K_r) = \frac{1}{2}\left(1 - \tan^2\left(\frac{\pi}{2r}\right)\right)$ (in which case we do not believe we have equality in (2.1)).

Further in [10, Theorem 26], Day, Falgas–Ravry and Hancock showed that $p_{\mathrm{conn}}(P_r \times K_2) < \frac{2}{3}$ for every $r \geqslant 1$; here $P_r \times K_2$ denotes the ladder graph on $2r$ vertices (the Cartesian product of the path P_r with the edge K_2, obtained by taking two disjoint copies of P_r and joining each vertex in one copy by an edge to the corresponding vertex in the other copy). Corollary 1.14 follows as an immediate corollary of (2.1) and our discussion so far. Finally, we note the proofs of [10, Theorems 30 and 26] implicitly determine $\pi_{P_r \times K_2}(\mathcal{T}_{2r}) = p_{\mathrm{conn}}(P_r \times K_2)$, though the common value of these two quantities is only given as the optimal solution to a (complicated) set of recursive inequalities.

2.3 Paths, stars and trees

As mentioned in the introduction, Nagy [16, Theorem 3.9] determined $\pi_T(T)$ for every tree T on r vertices in terms of the largest eigenvalue of T's adjacency matrix $A(T)$.

Theorem 2.3 (Nagy)**.** *Let T be a tree on r vertices, and let $\lambda = \lambda_{\max}(T)$ be the largest eigenvalue of $A(T)$. Then $\pi_T(T) = 1 - \frac{1}{\lambda^2}$.*

By results of Lovász and Pelikán [14, Theorem 2], among all trees T on r vertices the star $K_{1,r-1}$ is the one maximising the value of $\lambda_{\max}(T)$, and satisfies $\lambda_{\max}(K_{1,r}) = \sqrt{r-1}$ [14, Theorem 3]. Together with Theorem 2.3 this immediately implies Propositions 1.13 and 1.12 — for the latter, note that if H is any connected graph on r vertices and T is any spanning tree of H we have

$$\pi_H(\mathcal{T}_r) \leqslant \pi_T(T) = 1 - \frac{1}{\lambda_{\max}(T)} \leqslant 1 - \frac{1}{\lambda_{\max}(K_{1,r-1})} = \frac{r-2}{r-1}.$$

For future reference, let us also note here that the results of Lovász and Pelikán [14, Theorems 2,3] taken together with Nagy's Theorem 2.3 imply that the minimum of $\pi_T(T)$ over all r-vertex trees T is attained when T is the path P_r on r vertices, for which we have

$$\pi_{P_r}(P_r) = 1 - \frac{1}{4\cos^2(\frac{\pi}{r+1})}; \tag{2.2}$$

see [16, Corollary 3.13] or [10, Theorem 16] for an elementary proof of a stronger result using a variant of the Lovász local lemma.

For the sake of making this paper self-contained, we give here short, direct proofs of Propositions 1.12 and 1.13 that do no rely on the work of Lovász, Pelikán and Nagy, as well as of an illustrative 'star absorption' lemma.

Lemma 2.4 (Star absorption lemma). *Consider a $K_{1,N}$-partite graph G with partite density $p > 1/2$. Suppose that there is a subset A of the part corresponding to the centre of the star $K_{1,N}$ such that $w(A) = \alpha > 1 - p$. Then there exists a vertex $a \in A$ such that all but at most $\left(\frac{1-p}{\alpha}\right)N$ of the leaf-parts of G contain a vertex joined to a by an edge of G.*

Proof. Suppose no vertex $a \in A$ sends an edge to more than θN of the leaf-parts of G. Then by the partite density condition, we have:

$$pN \leqslant (1-\alpha)N + \alpha\theta N.$$

Rearranging yields $1 - \theta \leqslant \frac{1-p}{\alpha}$. In particular, there exists a vertex $a \in A$ such that all but at most a $\frac{1-p}{\alpha}$ proportion of the leaf-parts of G send an edge to a. □

Let us now sketch how this lemma may be applied to find transversals with large or connected components. Consider an arbitrary part V_0 in an $(r+1)$-partite graph G with $(r+1)$-partite density $p \geqslant 1/2$. By averaging and relabelling, there exists a vertex v_0 in V_0 sending edges to an α_i proportion of part V_i (meaning a subset of V_i with weight α_i) for $1 \leqslant i \leqslant r$, where $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots \geqslant \alpha_r$ and $\sum_{i=1}^r \alpha_i \geqslant pr$.

Observe that if $\alpha_{r/2} > 1/2$ then by the pigeon-hole principle and our partite density condition, for each $i \in \{1, 2, \ldots, r/2\}$ there exists a vertex in $V_{r/2+i}$ sending an edge of G to $N(v_0) \cap V_i$. We can thus find a transversal copy of a 2-subdivision of a star with $r/2$ leaves and v_0 as its centre, and hence a connected transversal of G. Also if $\alpha_r > 0$, then G contains a transversal copy of $K_{1,r}$, i.e. a connected transversal, since for every $i \in [r]$ there exists $v_i \in V_i$ with $v_0 v_i \in E(G)$.

On the other hand if $\alpha_{r/2} \leqslant 1/2$ and $\alpha_r = 0$, then there exists a least $t < r/2$ such that $\alpha_{t+1} \leqslant 1/2$, and a greatest $s \geqslant 1$ such that $\alpha_{r+1-s} = 0$. The α_i then satisfy

$$pr \leqslant \frac{r-t-s}{2} + \sum_{i=1}^{t} \alpha_i. \tag{2.3}$$

Applying Lemma 2.4 successively to $V_1, V_2, \ldots, V_t$ to 'absorb' the bad parts $V_{r+1-s}, \ldots, V_r$, we see that provided

$$s \prod_{i=1}^{t} \left(\frac{1-p}{\alpha_i} \right) < 1, \tag{2.4}$$

there exists a transversal connected subgraph of G, consisting of a central vertex joined by an edge to the centres of a 'star-forest' (a collection of vertex-disjoint stars on r vertices). This simple idea underlies the proofs of Theorems 1.6 and 1.7 and may have further applications to the study of H-partite graphs.

Proof of Proposition 1.12. Recall that H is an r-vertex graph. Let G be an H-partite graph with H-partite density $p > (r-2)/(r-1)$. Let $\mathbf{T}$ be a random transversal of G, and let S be any spanning tree of H. Then

$$\mathbb{E}|E(\mathbf{T}) \cap E(S)| \geqslant p(r-1) > r-2.$$

It follows that with strictly positive probability $|E(\mathbf{T}) \cap E(S)| = r-1$, and thus that H contains a connected transversal. $\square$

Proof of Proposition 1.13. The upper bound was proved in Proposition 1.12, and also follows directly from an application of Lemma 2.4 with $\alpha = 1$, $N = r-1$ and $1-p < \frac{1}{r-1}$. For the lower bound, consider a $K_{1,r-1}$-partite graph obtained by setting the centre part V_0 of the star to be $[r-1]$ with the uniform weighting $w(i) = 1/(r-1)$, letting each of the $r-1$ leaf parts V_i, $1 \leqslant i \leqslant r-1$ consist of a single vertex v_i, and adding all edges iv_j, $1 \leqslant i, j \leqslant r-1$ with $i \neq j$. This is easily seen to have no spanning connected transversal, and $K_{1,r}$-partite density $\frac{r-2}{r-1}$. $\square$

3 Transversal trees in H-partite graphs

3.1 K_r-partite graphs

Construction 3.1. Fix $\alpha \in [0,1]$. For $r \geqslant 3$, we construct a weighted r-partite graph $G^b = G^b(\alpha)$ as follows: for $i \in [r-1]$, we set $V_i := \{(i,i),(i,r)\}$ with weight function $w((i,i)) = \alpha$ and $w((i,r)) = 1-\alpha$, while we set $V_r := \{(r,i) : \ i \in [r-1]\}$ with the uniform weight function $w((r,i)) = 1/(r-1)$ for all $i \in [r-1]$. A partite edge uv is present in G^b if and only if $u_2 = v_2 = r$ (note this implies $u_1, v_1 \neq r$) or $u_1 = r$ and $u_2 \neq v_2$.

Proposition 3.2. *For $\alpha = \frac{1}{2(r-1)}\left(3r-4-\sqrt{5r^2-16r+12}\right)$, the graph G^b has r-partite density*
$$\rho^b(r) := \frac{r-2}{2(r-1)^2}\left(3r-4-\sqrt{5r^2-16r+12}\right),$$
and contains no connected transversal.

Proof. Consider any transversal T of G^b, with t_i the vertex in $T \cap V_i$. Then $t_r = (r, j)$ for some $j \in [r-1]$, and is joined by an edge to all vertices t_i, $i \in [r-1] \setminus \{j\}$ with $t_i = (i, i)$. Further, the restriction of G^b to $T \setminus \{t_r\}$ consists of the disjoint union of a (possibly empty) clique, corresponding to those vertices t_i, $i \in [r-1]$, with $t_i = (i, r)$, and of a (possibly empty) independent set, corresponding to those t_i with $t_i = (i, i)$.

The transversal subgraph of G^b induced by T is thus one of the following: the disjoint union of a star centred at t_r and a non-empty clique containing t_j (this occurs if $t_j = (j, r)$), or the disjoint union of a star centred at t_r, a (possibly empty) clique and an isolated vertex t_j (this occurs if $t_j = (j, j)$). In either case, the spanning transversal is not connected, establishing half of the proposition.

For the other half, observe that the edge density between parts V_i and V_j for $1 \leqslant i < j \leqslant r$ is $(1-\alpha)^2$ if $j < r$ and $\frac{r-2}{r-1}\alpha$ if $j = r$. The function $\min\left(\frac{r-2}{r-1}\alpha, (1-\alpha)^2\right)$ is maximised for $\alpha \in [0, 1]$ when
$$\alpha^2 - \frac{3r-4}{r-1}\alpha + 1 = 0$$
i.e. when $\alpha = \frac{1}{2(r-1)}\left(3r-4-\sqrt{5r^2-16r+12}\right)$, at which point it attains the value $\rho^b(r)$. □

Proof of Theorem 1.7. The lower bound on $\pi_{K_r}(\mathcal{T}_r)$ follows from Proposition 3.2. For the upper bound, consider an r-partite graph G with edge density $p > \frac{r-2}{2r-3}$. Suppose for contradiction that G contains no connected transversal. By averaging, there exists a vertex $v_r \in V_r$ such that
$$\sum_{i \in [r-1]} w(N_G(v_r) \cap V_i) \geqslant (r-1)p. \tag{3.1}$$

We may assume without loss of generality that $\alpha_i := w(N(v_r) \cap V_i)$ is a decreasing sequence. If $\alpha_{r-1} > 0$, then for every $i \in [r-1]$ there exists $v_i \in V_i$ with $v_i v_r \in E(G)$, and G contains a transversal star on r vertices, a contradiction. Let therefore $s > 0$ be the maximal integer such that $\alpha_{r-s+1} = 0$. By averaging, we have $\alpha_1 \geqslant \frac{r-1}{r-1-s}p > \frac{r-1}{2r-3}$. Let $t \geqslant 1$ be the maximal integer such that $\alpha_t \geqslant \frac{r-1}{2r-3}$.

Observe that for any $i \leqslant t$ and $j > r-s$, there exist vertices $v_i \in V_i$ and $v_j \in V_j$ with $v_i v_j \in E(G)$. Indeed, by our choice of p, we have $\alpha_i \geqslant \frac{r-1}{2r-3} > 1-p$. Thus if $t \geqslant s$, we have that G contains a subdivision of a star on a total of r vertices as a transversal, a contradiction. We may therefore assume $s > t$.

Now by (3.1) we have

$$\frac{(r-1)(r-2)}{2r-3} < (r-1)p \leqslant \sum_{i=1}^{r-1} \alpha_i \leqslant (r-1-s-t)\frac{r-1}{2r-3} + t,$$

which after rearranging terms and using our assumption $s > t$ implies

$$\frac{r-1}{2r-3} > \frac{s(r-1)-t(r-2)}{2r-3} \geqslant \frac{(r-1)+t}{2r-3},$$

a contradiction. $\square$

For the special case $r = 4$, we can prove a slightly better upper bound on the threshold for connected transversals.

Theorem 3.3. $\pi_4(\mathcal{T}_4) \leqslant 2 - 2\sqrt{\frac{2}{3}} = 0.36701\ldots$.

Proof. Consider a 4-partite graph G with edge density p with $1/3 \leqslant p < 2/3$ and no transversal tree on 4 vertices. Then each vertex of G sends edges to at most two parts of the canonical partition of G. By averaging and relabelling parts if necessary, it follows that there exists a vertex $v_1 \in V_1$ whose neighbourhood has maximum weight over all vertices of G, sending no edge to V_4, and with $\alpha := w(N(v_1) \cap V_2)$, $\beta := w(N(v_1) \cap V_3)$ satisfying $\alpha \geqslant \beta$ and

$$\alpha + \beta \geqslant 3p. \tag{3.2}$$

Since $p > 1/3$, this implies that α and β are both non-zero. Given a vertex $u \in V_4$, set

$$(x_u, y_u, z_u) := (w(N(u) \cap V_1), w(N(u) \cap V_2), w(N(u) \cap V_3)).$$

Since G contains no transversal tree on 4 vertices, (x_u, y_u, z_u) has at most two non-zero coordinates, and sends no edge to v or $N(v)$. Thus one of the following holds:

(1) $x_u = 0$, $y_u \leqslant 1 - \alpha$, $z_u \leqslant 1 - \beta$;

(2) $0 < x_u \leqslant 1$ and $y_u = 0$, and $z_u \leqslant \min(1 - \beta, \alpha + \beta - x_u)$;

(3) $0 < x_u \leqslant 1$, $0 < y_u \leqslant \min(1 - \alpha, \alpha + \beta - x_u)$ and $z_u = 0$.

Let $\theta_i := w\{u \in V_4 : (x_u, y_u, z_u) \text{ satisfies } (i)\}$. Then by the density condition on G between the parts V_4 and V_i, $i \in [3]$, we have:

$$(1 - \theta_1) \geqslant p \tag{3.3}$$
$$(1 - \alpha)(1 - \theta_2) \geqslant p \tag{3.4}$$
$$(1 - \beta)(\theta_1 + \theta_2) \geqslant p. \tag{3.5}$$

Now adding $(\theta_1+\theta_2)$ times (3.4) to $(1-\theta_2)$ times (3.5) and combining it with our lower bound (3.2) on $\alpha+\beta$, we obtain

$$(1+\theta_1)p \leqslant (\theta_1+\theta_2)(1-\theta_2)(2-\alpha-\beta) \leqslant (\theta_1+\theta_2)(1-\theta_2)(2-3p).$$

Rearranging terms, we get

$$\frac{p}{2-3p} \leqslant \frac{(\theta_1+\theta_2)(1-\theta_2)}{1+\theta_1} := f(\theta_1,\theta_2). \tag{3.6}$$

Now the partial derivative of $f(x,y)$ with respect to x is $(1-y)^2/(1+x)^2$, which is strictly positive for $x \geqslant 0$. In particular, since $\theta_1 \leqslant 1-p$ by (3.3), it follows that the right hand-side of (3.6) is at most

$$f(1-p,\theta_2) = 1-\theta_2 - \frac{(1-\theta_2)^2}{2-p}.$$

Substituting this into (3.6) and rearranging terms, we get the following quadratic inequality for θ_2:

$$(\theta_2)^2 - p\theta_2 + \frac{7p-2-4p^2}{2-3p} \leqslant 0.$$

In particular, the discriminant

$$p^2 - 4\left(\frac{7p-2-4p^2}{2-3p}\right) = \frac{2-p}{2-3p}\left(3p^2-12p+4\right)$$

must be non-negative. Solving $3p^2-12p+4 \geqslant 0$ for $p \leqslant 1$ yields $p \leqslant 2-2\sqrt{\frac{2}{3}}$, concluding the proof of the proposition. □

3.2 $(K_r - M)$-partite graphs

Construction 3.4. Let 12 be the missing edge in K_r^-. We construct a K_r^--partite graph G^b as follows: let $V_1 = \{1\}$ and $V_2 = \{2\}$; for every $i \in [r] \setminus [2]$, set $V_i = \{1,2\}$. Place a uniform weight function on each of the parts, and include an edge $xy \in V_i \times V_j$ (with $i \neq j$) in G^b if and only if $x = y$.

Proof of Theorem 1.6. For the lower bound, it is easily checked that the K_r^- partite graph G^b given in Construction 3.4 has K_r^-partite density exactly equal to $1/2$ and contains no connected transversal, since every transversal component can meet at most one of V_1 and V_2.

For the upper bound, let H be a graph on $[r]$ in which each vertex has degree at least $r-2$, i.e. a graph obtained from K_r by deleting a matching. Assume without loss of generality that $[r-2] \subseteq N_H(r)$.

Consider an H-partite graph G with H-partite density $p > 1/2$. Let $V_1, V_2, \ldots, V_r$ denote the parts from the H-partition of $V(G)$. If $r = 3$, then without loss of generality we have that both 13 and 23 are edges of H and by averaging there is a vertex $v_3 \in V_3$ sending edges to both V_1 and V_2, and hence a connected transversal. We may thus assume in what follows that $r \geqslant 4$.

By averaging and our assumption on $N_H(r)$, there exists a vertex $v_r \in V_r$ such that for every $i \in [r-2]$, there are $\alpha_i|V_i|$ edges of G from v_r to V_i, where the α_i are reals from $[0,1]$ satisfying:

$$\frac{r-2}{2} < p(r-2) \leqslant \sum_{i=1}^{r-1} \alpha_i. \tag{3.7}$$

Let t be the number of i for which $\alpha_i \geqslant 1/2$, and let s be the number of i for which $\alpha_i = 0$. Then

$$\sum_{i=1}^{r-2} \alpha_i \leqslant \frac{r-2-s-t}{2} + t = \frac{r-2+t-s}{2}.$$

Combining this with (3.7), we have that $t \geqslant s+1$. Since every vertex of H is incident with at most one non-edge, it follows from this inequality that there exists an injection f from the s-set $B := \{i \in [r-2] : \ \alpha_i = 0\} \cup \{r-1\}$ into the t-set $A := \{i \in [r-2] : \ \alpha_i > 1/2\}$ such that $if(i) \in E(H)$ for all $\in B$. We shall use this to find a subdivision of a star as a subgraph of a transversal of G.

Pick for each $i \in [r-1]$ a vertex $v_i \in V_i$ such that (i) if $i \notin B$ then $v_i v_r \in E(G)$, while (ii) if $i \in B$ then $v_i v_{f(i)} \in E(G)$. Clearly if we can do this then we have found our desired subdivision of a star centred at v_r as a subgraph of a transversal of G. Let us therefore verify we can find good choices of the vertices $v_i \in V_i$, $i \in [r-1]$. For $i \in [r-1] \setminus (B \cup f(B))$, we just pick v_i to be an arbitrary vertex in $N_G(v_e) \cap V_i$ and (i) is trivially satisfied.

Next consider $i \in B$, and the associated index $f(i) \in A$. By the edge density condition, at least a $p > 1/2$ proportion of the edges between V_i and $V_{f(i)}$ are present in G (since $if(i) \in E(H)$ by construction of the injection f). In particular, since $|N_G(v_r) \cap V_{f(i)}| = \alpha_{f(i)}|V_{f(i)}| \geqslant \frac{1}{2}|V_{f(i)}|$, there exist $v_i \in V_i$ and $v_{f(i)} \in N_G(v_r) \cap V_{f(i)}$ such that $v_i v_{f(i)}$ is an edge of G. Clearly for these choices of v_i and $v_{f(i)}$ we have that (ii) is satisfied for i. Further, by construction, $v_r v_{f(i)} \in E(G)$, and (i) is satisfied for the index $f(i) \notin B$. Thus there exists good choices of the vertices $v_i \in V_i$, $i \in [r-1]$, and we are done. $\square$

4 A multipartite Dirac density problem

4.1 Multipartite graphs with no Hamiltonian transversals

We begin by giving two easy constructions that give the lower bound of $\pi_r(C_r) \geqslant 1/2$ for the multipartite Dirac and odd cycle problems.

Construction 4.1 (Two colour construction). For $r \geqslant 4$, we construct a weighted r-partite graph G_1 as follows. We let the parts V_i, $i \in [r-2]$ consist of $r-2$ disjoint copies of

$\{0,1\}$. Further we let $V_{r-1} = \{0\}$ and $V_r = \{1\}$. We let the weight function w be constant and equal to $1/2$ on $\sqcup_{i\in[r-2]} V_i$, and constant and equal to 1 on $V_{r-1} \cup V_r$. Given $x \in V_i$ and $y \in V_j$ with $i \neq j$, we let xy be an edge of G_1 if either $x = y$ or $\{i, j\} = \{r-1, r\}$.

It is easily checked that Construction 4.1 has r-partite density $1/2$ and contains no transversal Hamilton cycle, so that $\pi_r(C_r) \geqslant \frac{1}{2}$ for all $r \geqslant 3$.

Construction 4.2 (Parity construction). For $r \geqslant 5$, we construct a weighted r-partite graph G_2 as follows. We let the parts V_i, $i \in [r]$ consist of r disjoint copies of $\{0,1\}$, and the weight function to be the constant function $w : V(G_2) \to \{1/2\}$. Given $x \in V_i$ and $y \in V_j$ with $1 \leqslant i < j \leqslant r$, we let xy be an edge of G_2 if and only if $x + y$ is odd.

It is easily checked that Construction 4.2 has r-partite density $1/2$ and contains no transversal cycle of odd length. In particular, for $r \geqslant 3$ and odd this gives an alternative proof that $\pi_r(C_r) \geqslant \frac{1}{2}$. We now give a refined version of Construction 4.1, which gives an improved upper bound for $\pi_r(C_r)$.

Construction 4.3. Let $p_1, p_2, p_3 \in (0,1)$. For $r \geqslant 4$, we construct a weighted r-partite graph $G_3 := G_3(p_1, p_2, p_3)$ as follows. We let the parts V_i, $i \in [r-2]$ consist of $r-2$ disjoint copies of $\{0, r-1\}$. Further we let $V_{r-1} = \{0, r\}$ and $V_r = \{1, 2, \ldots r-2, r-1\}$, with all parts pairwise disjoint.

We now define a weight function w in the following way. If $v \in V_i$ for some $i \in [r-2]$, then $w(v) = p_1$ if $v = 0$ and $w(v) = 1 - p_1$ otherwise. If $v \in V_{r-1}$, then $w(v) = p_2$ if $v = 0$ and $w(v) = 1 - p_2$ otherwise. If $v \in V_r$, then $w(v) = p_3$ if $v = r-1$, and $w(v) = (1-p_3)/(r-2)$ otherwise.

Finally, we specify the edges of G. Given $x \in V_i$ and $y \in V_j$ with $1 \leqslant i < j \leqslant r-1$, we let xy be an edge of G_3 if and only if $x = y = 0$ or $x = y = r-1$. For $x \in V_i$ with $i \leqslant r-2$ and $y \in V_r$, we let xy be an edge of G_3 if either $x = r-1$ or $y = i$. Finally if $x \in V_{r-1}$ and $y \in V_r$, we let xy be an edge if either $x = 0$ and $y = r-1$ or $x = r$.

Proof of Theorem 1.15. We prove first of all that the graph G_3 given in Construction 4.3 above does not contain a transversal C_r. Indeed, this can be seen via a simple analysis: suppose for a contradiction that we had chosen $v_i \in V_i$ for $i \in [r]$, and that $G := G_3[\{v_1, v_2, \ldots, v_r\}]$ contains a copy C of C_r.

Note that, for all $i \in [r]$, the vertex v_i is connected to at least two other parts in G. Otherwise, v_i cannot be part of a transversal cycle. This, together with the fact that vertex $r \in V_{r-1}$ is only connected to V_r, implies $v_{r-1} = 0$. Further, since a Hamilton cycle is 2-connected, $G_3[\{v_1, v_2, \ldots, v_{r-1}\}]$ must be connected, which implies that $v_i = 0$ for all $i \in [r-2]$(since $v_{r-1} = 0$ and edges between parts V_i and V_j for $i < j < r$ exist only between vertices with the same labels).

To complete the cycle C, we then need distinct $i, j \in [r-1]$ such that v_i and v_j have a common neighbour in V_r. However, for all $i \in [r-1]$, the vertex $0 \in V_i$ is only connected to the vertex $i \in V_r$, a contradiction.

We have thus shown that G_3 has C_r-free transversals. It now remains to show that we can achieve $d_{K_r}(G_3) > 1/2$ for judicious choices of p_1, p_2, p_3. This will be achieved by considering a simple optimisation problem. For $i < j$, the edge density between V_i and V_j in G_3 is equal to

- $(p_1)^2 + (1-p_1)^2$ if $1 \leqslant i < j \leqslant r-2$;
- p_1p_2 if $1 \leqslant i \leqslant r-2$ and $j = r-1$;
- $(1-p_1) + (1-p_3)/(r-2)$ if $1 \leqslant i \leqslant r-2$ and $j = r$;
- $(1-p_2) + p_2p_3$ if $i = r-1$ and $j = r$.

We must pick p_1, p_2, p_3 to ensure the minimum of these four quantities is strictly greater than $1/2$. This is a simpler task than solving the optimisation problem of maximising the minimum of these quantities. Fix ε with $0 < \varepsilon < \frac{1}{2(r-1)}$. Pick $p_1 = \frac{1}{2} + \varepsilon$, $p_2 = 1 - \varepsilon$ and $p_3 = 1 - (r-1)\varepsilon$ (note all three of these quantities are in $(0,1)$ by our choice of ε). Then for these choices of the parameters p_1, p_2, p_3, we have

$$\begin{aligned} d_{K_r}(G_3) &= \min\left((p_1)^2 + (1-p_1)^2, p_1p_2, (1-p_1) + \frac{(1-p_3)}{r-2}, (1-p_2) + p_2p_3\right) \\ &= \min\Big(\frac{1}{2} + \varepsilon^2, \frac{1}{2} + \frac{\varepsilon}{2}(1-2\varepsilon), \frac{1}{2} + \frac{\varepsilon}{r-2}, \frac{1}{2} + \left(\frac{1}{2} - (r-1)\varepsilon\right) + (r-1)\varepsilon^2\Big), \end{aligned}$$

which is strictly greater than $\frac{1}{2}$, as required. This shows that $\pi_r(C_r) > \frac{1}{2}$, as claimed.

In fact, we can explicitly solve the optimisation problem needed to work out the best lower bound on $\pi_{K_r}(C_r)$ we can get from Construction 4.3. Set $p_1 = p_\star$, where $p_\star$ is the unique real solution in $(\frac{1}{2}, 1)$ to the cubic equation

$$(r-2) - (4r-10)p + (6r-14)p^2 - (4r-8)p^3 = 0, \tag{4.1}$$

and let

$$p_2 = \frac{(p_\star)^2 + (1-p_\star)^2}{p_\star} \qquad\qquad p_3 = 1 - (r-2)p_\star(2p_\star - 1).$$

Then for these choices of parameters, we have $d_{K_r}(G_3) = (p_\star)^2 + (1-p_\star)^2$. □

4.2 Transversal squares in 4-partite graphs

Proof of Proposition 1.18. The lower bound follows from the case $r = 4$ Theorem 1.15, solving the cubic equation (1.1) explicitly. For the upper bound, suppose G is a weighted 4-partite-graph with $d_{K_4}(G) = p > 1/\sqrt{3}$. Let $\sqcup_{i=1}^4 V_i$ be the canonical partition of G. Select vertices $\mathbf{v_i} \in V_i$ for $i \in [4]$ independently at random, with $v_i = v$ with probability $w(v)$ for every $v \in V_i$.

Observe that the edges of K_4 may be decomposed into 3 perfect matchings, M_1, M_2 and M_3, the union of any two of which gives a copy of C_4. Now the expected number of M_j, $1 \leqslant j \leqslant 3$, such that both of the edges in M_j are present in $G[\{v_1, v_2, v_3, v_4\}]$ is $3p^2 > 1$. It follows from Markov's inequality that with probability at least $(3p^2 - 1)/2 > 0$, $G[\{v_1, v_2, v_3, v_4\}]$ contains a C_4. Thus G fails to have C_4-free transversals, as required. □

5 Further open problems

The most obvious directions for future research are, of course, to prove Conjectures 1.10 and Conjecture 1.17 on the (asymptotic) r-partite thresholds for connectivity and Hamiltonicity in r-partite graphs. As in [8] and [9], it may be useful for inductive approaches to these conjectures to consider *inhomogeneous* versions of Problem 1.1, and to determine the set of K_r-partite density profiles $\alpha = (\alpha_e)_{e\in E(K_r)}$ needed to guarantee the existence of a desirable transversal subgraphs. Finally Lengler, Martinsson, Petrova, Schnider, Steiner, Weber and Welzl's recent and himpressive probabilistic proof of an asymptotic version of Conjecture 1.10 in [12] provides a new set of techniques for attacking these problems, which should be investigated further.

Beyond Conjectures 1.10 and 1.17, there are a number of other natural problems we would like to highlight.

5.1 Component evolution, long paths and large treess

In this paper, we focus on the problem of finding connected transversals in multipartite graphs. What if instead we looked for transversals containing a connected component of order at least t?

Problem 5.1 (Extremal component evolution)**.** For each t with $3 \leqslant t \leqslant r$, determine $\pi_{K_r}(\mathcal{T}_t)$.

We provide below a family of constructions giving lower bounds for this problem for various values of t.

Construction 5.2 (Intersecting palette construction). Fix $t \in \mathbb{Z}_{\geqslant 2}$. For $r \geqslant \binom{2t-1}{t}$, we construct a weighted r-partite graph G^t with r-partition $V(G^t) := \sqcup_{i=1}^r V_i$ as follows.

Let $\sqcup_{X\in[2t-1]^{(t)}} S_X$ be a balanced partition of $[r]$ into $\binom{2t-1}{t}$ sets S_X indexed by the t-elements subsets $X \in [2t-1]^{(t)}$. For each part V_j, $j \in [r]$ with $j \in S_X$, we let V_j consist of a copy of X, and we let the weight function be constant and equal to $1/t$ over V_j. We then add an edge between $x \in V_i$ and $y \in V_j$ $(i \neq j)$ if and only if $x = y$. See Figure 5.1 for the case $r = 6$, $t = 2$.

Proposition 5.3. *For every $t \geqslant 2$ fixed and every $r \geqslant \binom{2t-1}{t}$, the graph G^t has r-partite density $1/t^2$ and contains no transversal connected component containing more than $\left\lceil \binom{2t-2}{t-1}/\binom{2t-1}{t} r\right\rceil = \left\lceil \frac{t}{2t-1} r\right\rceil$ vertices.*

Proof. Straightforward analysis of Construction 5.2. □

Proposition 5.3 suggests the following sub-problem of Problem 5.1 may be particularly fruitful to study.

Problem 5.4. Given $\alpha \in (0, 1)$, let $f_r(\alpha)$ denote the maximum r-partite density of an r-partite graph whose transversal components have size at most αr. Determine the asymptotic behaviour of $f_r(\alpha)$ as $r \to \infty$.

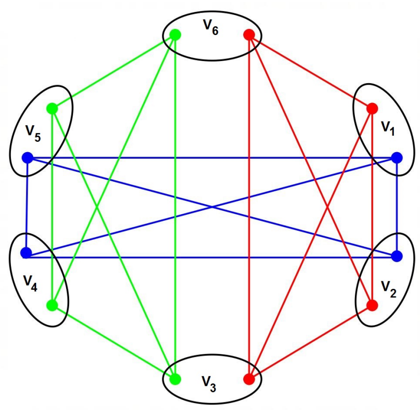

Figure 1: Construction 6.2 for $r = 6$ and $t = 2$.

In a different direction, Erdős and Gallai determined the largest size of a graph on n vertices containing no path of length ℓ. It is natural to consider the analogous problem in our setting, and in particular to determine whether long transversal paths are easier to avoid than large transversal trees (i.e. whether the answer to the problem below is greater than the answer to Problem 5.1 with $t = \ell$).

Problem 5.5. Given ℓ with $3 \leqslant \ell \leqslant r$, determine $\pi_{K_r}(P_\ell)$.

5.2 Bipartite graphs

Another potentially interesting direction to consider is the density Turán H-partite problem when $H = K_{r,r}$. Clearly, the connectivity threshold in that setting is at least $\pi_{K_{2r}^-} = 1/2$ (by Theorem 1.6 and monotonicity). We ask whether this lower bound might be tight (which would imply a significant strengthening of Theorem 1.6):

Question 5.6. Is $\pi_{K_{r,r}}(\mathcal{T}_{2r}) = \frac{1}{2}$?

Similarly to Pfender, we could also look for transversal copies of smaller complete bipartite subgraphs.

Problem 5.7 (Multipartite Zarankiewicz problem)**.** Given intergers $2 \leqslant s \leqslant r$, determine the quantity $\pi_{K_{r,r}}(K_{s,s})$.

This problem is of particular interest when $s = 2$. Note that by the Erdős–Stone–Simonovits theorem, $\pi_{K_{r,r}}(K_{2,2}) \to 0$ as $r \to \infty$, so the question is about the order of the rate of decay with respect to r.

Given the connection to locally dependent percolation theory outlined in Section 2.2, it would also be of interest to study the connectivity problem for Q_d-partite graphs, where Q_d denotes the d-dimensional hypercube graph. In particular, we have the following questions:

Question 5.8 (Appearance of a transversal giant)**.** Fix $\varepsilon > 0$. For d sufficiently large, what is $\pi_{Q_d}(\mathcal{T}_{\lfloor \varepsilon 2^d \rfloor})$?

Question 5.9 (Connectivity)**.** What is $\pi_{Q_d}(\mathcal{T}_{2^d})$?

We note Questions 5.8 and 5.9 are Q_d-partite versions of questions of Falgas–Ravry and Pfenninger and of Balister, Johnston, Savery and Scott on 1-dependent random graphs. In particular, [11, Conjecture 1.18] would imply the answer to Question 5.8 is at most $\frac{1}{2}$. We give below a simple matching lower bound.

Proposition 5.10. *Fix $d \in \mathbb{N}$ and let $s = 1 + \max_{1 \leqslant r \leqslant d-1} \left(\binom{d}{r-1} + \binom{d}{r} + \binom{d}{r+1}\right)$. Then $\pi_{Q_d}(\mathcal{T}_s) \geqslant \frac{1}{2}$. In particular, for all $\varepsilon > 0$ fixed and d sufficiently large, the answer to Question 5.9 is at least* $1/2$.

Proof. For each $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \{0,1\}^d$, we let $V_{\mathbf{x}}$ be given by

$$V_{\mathbf{x}} = \begin{cases} \{0\} & \text{with weight function } w(\{0\}) = 1 & \text{if } \sum_{i=1}^d x_i \equiv 0 \mod 4 \\ \{0,1\} & \text{with } w(\{0\}) = w(\{1\}) = \frac{1}{2} & \text{if } \sum_{i=1}^d x_i \equiv 1 \mod 2 \\ \{1\} & \text{with } w(\{1\}) = 1 & \text{if } \sum_{i=1}^d x_i \equiv 2 \mod 4, \end{cases}$$

with the sets $V_{\mathbf{x}}$ chosen vertex-disjoint. Given an edge $\mathbf{xy} \in E(Q_d)$, we place an edge between $u \in V_{\mathbf{x}}$ and $v \in V_{\mathbf{y}}$ if and only $u = v$. This is easily seen to give rise to a Q_d-partite graph with Q_d-partite density $\frac{1}{2}$ and such that every connected component in a transversal subgraph must be contained within the union of at most three consecutive layers $L_t := \{\mathbf{x} \in Q_d : \sum_{i=1}^d x_i = t\}$. Since the size of three consecutive layers is at most $s - 1 = O(2^d/\sqrt{d}) = o(d)$, the proposition follows immediately. □

5.3 Cycles

Using results of Nagy, we can obtain the following bounds on the threshold for connectivity in C_r-partite graphs:

Proposition 5.11. *For any $r \in \mathbb{Z}_{\geqslant 4}$, we have:*

$$\frac{3 - \tan^2\left(\frac{\pi}{\lfloor \frac{r}{2} \rfloor + 2}\right)}{4} \leqslant \pi_{C_r}(\mathcal{T}_r) = \pi_{C_r}(P_r) \leqslant \frac{3 - \tan^2\left(\frac{\pi}{r+2}\right)}{4}.$$

Proof. For the upper bound, we have

$$\pi_{C_r}(\mathcal{T}_r) \leqslant \pi_{C_r}(C_r) = \pi_{P_{r+1}}(P_{r+1}) = \frac{3 - \tan^2\left(\frac{\pi}{r+2}\right)}{4},$$

where the last two equalities follow from [16, Theorem 4.6]. For the lower bound, view C_r as the union of paths P^1 and P^2 on $\lceil r/2 \rceil + 1$ and $\lfloor r/2 \rfloor + 1$ vertices respectively, with the left and right endpoints of P^1 identified with the left and right endpoints of P_2. Now consider a P^2-partite graph G^2 with no transversal copy of P^2 and such that $d_{P^2}(G^2) = \pi_{P_{\lfloor r/2 \rfloor+1}}(P_{\lfloor r/2 \rfloor+1})$. Since the length of P^2 is less or equal to the length of G^1, we can clearly use G^2 to make a P^1-partite graph G^1 with no transversal copy of P^1 and the same partite density as G^2. Then the union G of G^1 and G^2 (identifying the parts corresponding to the endpoints of P^1 and P^2 as appropriate) is a C_r-partite graph in which every transversal has at least two edges missing, and in particular is not connected. Thus $\pi_{C_r}(\mathcal{T}_r) \geqslant \pi_{P^2}(P^2)$, and the lower bound then follows from [16, Corollary 3.13]. □

By Theorem 1.6 (in the case $r = 4$) and Proposition A.2 (in the case $r = 5$), the lower bound in Proposition 5.11 is tight for $r \in \{4, 5\}$.

Question 5.12. Is the lower bound in Proposition 5.11 tight for all $r \geqslant 4$?

In a different direction, one could ask for the K_r-partite density needed to force the existence of a cycle of a given length ℓ (or of length at least/at most ℓ) as a subgraph of a transversal. This gives rise to the following family of problems

Problem 5.13 (Extremal girth and circumference)**.** For every ℓ with $3 \leqslant \ell \leqslant r$, determine $\pi_{Kr}(C_\ell)$, $\pi_{K_r}(\{C_t : t \leqslant \ell\})$ and $\pi_{K_r}(\{C_t : t \geqslant \ell\})$.

One could also ask about the appearance of odd cycles. As noted in the introduction, Pfender showed [18, Corollary 5] that $\pi_{K_r}(K_3) = 1/2$ for $r \geqslant 12$. By a simple result of Bondy, Shen, Thomassé and Thomassen [8, Theorem 1], there exist r-partite graphs with r-partite density $\frac{1}{2}$ and no odd cycles in transversals. This gives the following:

Corollary 5.14. *For all $r \geqslant 4$, $\pi_{K_r}(\bigcup\{C_t : t \text{ odd}\}) \geqslant \frac{1}{2}$, with equality for all $r \geqslant 12$.*

Question 5.15. For which r between 4 and 12 does equality hold in Corollary 5.14?

5.4 Spectrum of the connectivity threshold

For any r, let $\mathcal{D}_r := \{\pi_H(\mathcal{T}_r) : H \text{ is a connected } r\text{-vertex graph}\}$ denote the collection of thresholds for the existence of connected spanning transversals for connected r-vertex graphs.

Problem 5.16. Characterize $\mathcal{D}_r$.

We note that by Proposition 2.1 for every r and every connected graph H on r-vertices the value of $\pi_H(\mathcal{T}_r)$ can in principle be reduced to the solution of a quadratic optimization problem. As shown by our result for $K_4 - P_3$ (Proposition A.1) or by the earlier result of Nagy [16, Corollary 3.13] for paths, even for small r the set $\mathcal{D}_r$ may contain irrational numbers.

What is more interesting than the structure of $\mathcal{D}_r$ for specific small r is of course the asymptotic behaviour. We know that for all connected non-complete graphs H on r vertices, we have

$$\frac{1}{2} = \pi_{K_r^-}(\mathcal{T}_r) \leqslant \pi_H(\mathcal{T}_r) \leqslant \pi_{K_{1,r-1}}(\mathcal{T}_r) = \frac{r-2}{r-1}.$$

From Nagy's results, Theorem 2.3 and (2.2), we have that $\lim_{r\to\infty} \pi_{P_r}(\mathcal{T}_r) = \frac{3}{4}$. Further, it follows from [10, Theorem 26] that for the ladder $P_r \times K_2$, we have $\lim_{r\to\infty} \pi_{P_r \times K_2}(\mathcal{T}_{2r}) = \frac{2}{3}$. Finally the recent result of Lengler, Martinsson, Petrova, Schnider, Steiner, Weber and Welzl [12] shows that $\frac{3-\sqrt{5}}{2} = \lim_{r\to\infty} \pi_{K_r}(\mathcal{T}_r)$. Thus $\frac{3-\sqrt{5}}{2}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ and 1 are accumulation points for the sequence $\mathcal{D}_r$ in the sense that we can find sequences of r-vertex connected graphs H_r with $\pi_{H_r}(\mathcal{T}_r)$ tending to these numbers.

Question 5.17. What are the other accumulation points for the sequence of finite sets $\mathcal{D}_r$?

In particular, it would be nice to know whether or not one should expect the collection of accumulation points of $(\mathcal{D}_r)_{r\in\mathbb{Z}_{\geqslant 2}}$ to form a well-ordered set, or to contain only algebraic numbers of degree at most 2.

5.5 Factors

Besides spanning trees and Hamilton cycles, another widely-studied class of spanning subgraphs is that of F*-factors*: given a graph F and a graph H such that $|V(H)| \equiv 0 \mod |V(F)|$, an F-factor in H is a collection of vertex-disjoint copies of F that together cover all the vertices of H. We denote by mF such a collection of m vertex-disjoint copies of F. What is the rs-partite threshold for a K_s-factor?

Problem 5.18 (Multipartite Hajnal–Szemerédi)**.** For $r \in \mathbb{Z}_{\geqslant 2}$ and $s \in \mathbb{Z}_{\geqslant 3}$, determine $\pi_{K_{rs}}(rK_s)$.

Remark 5.19. Problem 5.18 is very different from the analogous problem for graphs, where one asks for the minimum degree condition for the existence of a K_s-factor. Indeed, by the Hajnal–Szemerédi theorem, a minimum degree of $2n$ is necessary to force the existence of a K_3-factor in a graph on $3n$ vertices. On the other hand, it follows from the Bondy–Shen–Thomassé–Thomassen Theorem [8, Corollary 3] and partitioning a $3n$-partite graph into n tripartite graphs that for any n,

$$\pi_{K_{3n}}(nK_3) \leqslant \pi_{K_3}(K_3) = \frac{-1+\sqrt{5}}{2} < \frac{2}{3}.$$

Similarly, one could ask for cycle factors.

Problem 5.20 (Multipartite Abbasi)**.** Given $\ell \in \mathbb{Z}_{\geqslant 3}$, determine $\pi_{K_{r\ell}}(rC_\ell)$.

Partitioning an $r\ell$-partite graphs into r disjoint ℓ-partite graphs, we immediately have that $\pi_{K_{r\ell}}(rC_\ell) \leqslant \pi_{K_\ell}(C_\ell)$, which is strictly greater than $\frac{1}{2}$ for all $\ell \geqslant 4$ (by Theorem 1.15) and which we conjecture tends to $1/2$ as $\ell \to \infty$ (Conjecture 1.17). This should be compared with Abbasi's result [1, Chapter 6] that any $r\ell$-vertex graph with minimum degree at least $\frac{r}{\ell}\left\lfloor\frac{\ell}{2}\right\rfloor$ contains a C_ℓ-factor.

5.6 Universality

The Erdős–Sós conjecture states that every n-vertex graph with strictly more than $\frac{n}{2}(k-1)$ edges contains every tree on at most k vertices as a subgraph. Such universality questions are natural in the multipartite setting also:

Question 5.21. Let $3 \leqslant s \leqslant r$ be fixed. What is the threshold

$$\begin{aligned}\alpha_{\text{tree-universal}}(r,s) := \inf\big\{\alpha > 0 : d_{K_r}(G) \geqslant \alpha \ \Rightarrow \\ \forall T \in \mathcal{T}_s,\ T \text{ is a subgraph of a transversal of } G\big\} \quad ?\end{aligned}$$

By Proposition 1.12, it is immediate that $\alpha_{\text{tree-universal}}(r,s) \leqslant \frac{s-2}{s-1}$. However, we suspect that the actual threshold may be significantly lower when r is larger than s. Given Nagy's result, Theorem 2.3, together with Lovász and Pelikán's results on the largest eigenvalues in the adjacency matrix of trees, the star $K_{1,s-1}$ could well be the hardest tree to find as a transversal. This would suggest that $\alpha_{\text{tree-universal}}(r,s)$ should be no larger than $\frac{s-2}{r-1}$: by averaging, any r-partite graph with larger density has a spanning transversal with average degree greater than $s-2$.

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A Appendix: Connectivity threshold for $H = K_4 - P_3$ and $H = C_5$

Proposition A.1. $\pi_{K_4-P_3}(\mathcal{T}_4) = 4 - 2\sqrt{3}$.

Proof. For the lower bound, consider the following construction. We view $H = K_4 - P_3$ as a triangle on a vertex set $\{1, 2, 3\}$ with a pendant edge $\{0, 1\}$ attached to the vertex 1. We construct an H-partite graph by letting $V_0 = \{v_0\}$, $V_1 = \{v_2, v_3, v_X\}$ with the weighting $w(v_2) = w(v_3) = 2 - \sqrt{3}$ and $w(v_X) = 2\sqrt{3} - 3$, and let V_2, V_3 be two disjoint copies of $\{v_1, v_Y\}$ with the weighting $w(v_1) = 2 - \sqrt{3}$ and $w(v_Y) = \sqrt{3} - 1$. We then add the edges v_0v_2, v_0v_3 between V_0 and V_1, the edges v_1v_i, v_1v_X and v_Yv_X between V_i and V_1 ($i \in \{2, 3\}$) and the edge v_Yv_Y between V_2 and V_3. It is easily checked that the resulting H-partite graph has H-partite density $4 - 2\sqrt{3}$ and has no connected transversal.

For the upper bound, it follows from Proposition 2.1 (and possibly replacing some vertices by two clones, each assigned half the weight) that it is enough to consider H-partite graphs with $|V_0| = 1$, $|V_1| = 3$ and $|V_2| = |V_3| \leqslant 2$. Consider such an H-partite graph G with $d_H(G) = p > 0$ and no connected transversal. By Proposition 1.13, we may assume $p \leqslant 2/3$.

Set $V_1 = \{x_1, x_2, x_3\}$, $V_2 = \{y_1, y_2\}$ and $V_3 = \{z_1, z_2\}$. We know there exists at least one edge from V_2 to V_3, so without loss of generality we may assume that $y_2z_2 \in E(G)$. Let $U_2 \subseteq V_2$ and $U_3 \subseteq V_3$ be the set of vertices in $V_2 \cup V_3$ incident with an edge from V_2 to V_3. Further set $W_1 \subseteq V_1$ be the (non-empty) set of vertices sending an edge to V_0. Since G contains no connected transversal, there is no edge of G from W_1 to $U_2 \cup U_3$.

Given a set $S \subseteq V(G)$, let $w(S) := \sum_{v \in S} w(v)$ be the total weight of the vertices in S. By the partite density condition between V_0 and V_1 we have $w(W_1) \geqslant p$, and similarly by the partite density condition between V_2 and V_3 we have $w(U_2)w(U_3) \geqslant p$. Further by the partite density conditions between V_1 and V_2 we have $1 - p \geqslant w(W_1)w(U_2)$, and similarly we have $1 - p \geqslant w(W_1)w(U_3)$. We deal with two special cases to show we can restrict our attention to a graph G with a similar structure to our lower bound construction (albeit with potentially different weights).

Case 1: $W_1 = V_1$. Then our inequalities tell us $1 - p \geqslant \max(w(U_2), w(U_3))$, and $w(U_2)w(U_3) \geqslant p$, so that $(1 - p)^2 \geqslant p$ and hence $p \leqslant \frac{\sqrt{3}-5}{2} < \frac{1}{2} < 4 - 2\sqrt{3}$. Thus moving forward, we may assume that $|W_1| < |V_1|$.

Case 2: W_1 sends edges to at most one of V_2, V_3. Relabelling parts, we may assume without loss of generality that W_1 sends no edges to V_3. Then looking at the partite densities between V_1 and V_3, we have $w(W_1) \leqslant 1 - p$. Now $w(W_1) \geqslant p$, as observed above, because of the partite density between V_0 and V_1. This immediately implies $p \leqslant \frac{1}{2} < 4 - 2\sqrt{3}$. Thus moving forward, we may assume that W_1 sends edges to both of V_2 and V_3.

The final case. Since G does not contain a connected transversal, each vertex of W_1 can send an edge to at most one of V_2 and V_3. Thanks to the previous case, we already know $|W_1| < |V_1| = 3$. Thus in the remainder of the proof, we may assume without loss of generality all of the following hold: $W_1 = \{x_2, x_3\}$, $U_2 = \{y_2\}$, $U_3 = \{z_2\}$ and

$\{x_2y_1, x_3z_1\} \subseteq E(G)$. By definition of W_1, U_2 and U_3, we then have the following inequalities:

$$
\begin{aligned}
&p \leqslant w(x_2) + w(x_3) && \text{(density between } V_0 \text{ and } V_1\text{)},\\
&p \leqslant w(x_2)(1 - w(y_2)) + (1 - w(x_2) - w(x_3)) && \text{(density between } V_1 \text{ and } V_2\text{)},\\
&p \leqslant w(x_3)(1 - w(z_2)) + (1 - w(x_2) - w(x_3)) && \text{(density between } V_1 \text{ and } V_3\text{)},\\
&p \leqslant w(y_2)w(z_2) && \text{(density between } V_2 \text{ and } V_3\text{)},
\end{aligned}
$$

with all weights $w(v)$ taking values in $[0, 1]$ and $w(x_2) + w(x_3) \leqslant 1$. We now analyse this system of inequalities and deduce that $p \leqslant 4 - 2\sqrt{3}$.

Noting that the right hand-side of the second and third inequalities above are decreasing in $w(y_2), w(z_2)$ while the right-hand side of the fourth inequality is increasing in $w(y_2), w(z_2)$, we may assume that, reducing the weight of y_2 or z_2 if necessary, we have $w(y_2)w(z_2) = p$. Further, note that the right hand-side of the second and third inequalities are decreasing in $w(x_2)$ and $w(x_3)$ respectively, while the right hand-side of the first inequality is increasing in $w(x_2)$ and $w(x_3)$. Reducing the weight of x_2 and x_3 until the first inequality is tight, we see that we may assume that $w(x_2)+w(x_3) = p$. Thus we may eliminate two of our variables and, rearranging terms, rewrite our system of inequalities as:

$$
2p - 1 \leqslant \min \left\{ w(x_2)\left(1 - w(y_2)\right),\ (p - w(x_2))\left(1 - \frac{p}{w(y_2)}\right) \right\},
$$

with $w(x_2) \in [0, p]$, $w(y_2) \in [p, 1]$. In particular, we have

$$
(2p - 1)^2 \leqslant w(x_2)(p - w(x_2))\left(1 - w(y_2)\right)\left(1 - \frac{p}{w(y_2)}\right) \leqslant \left(\frac{p}{2}\right)^2 (1 - \sqrt{p})^2 .
$$

Solving the inequality $2p - 1 \leqslant \frac{p}{2}(1 - \sqrt{p})$ for $p \in [0, \frac{2}{3}]$, we deduce that $d_H(G) = p \leqslant 4 - 2\sqrt{3}$, as required. □

Proposition A.2. $\pi_{C_5}(\mathcal{T}_5) = \frac{1}{2}$.

Proof. For the lower bound, we have $\pi_{C_5}(\mathcal{T}_5) \geqslant \pi_{K_5^-}(\mathcal{T}_5) = \frac{1}{2}$ by monotonicity and Theorem 1.6, where K_5^- denotes K_5 with one edge removed.

For the upper bound, it follows from Proposition 2.1 (and possibly replacing some vertices by two clones, each assigned half the weight) that it is enough to consider C_5-partite graphs G with canonical partitions $\sqcup_{i=1}^5 V_i$ satisfying $|V_i| = \{x_i, y_i\}$ for each $i \in [5]$. Let G be such a graph with no connected transversal and C_5-partite density p, and suppose for a contradiction that $p > \frac{1}{2}$.

If there exists some i such that neither x_i nor y_i sends an edge into both V_{i-1} and V_{i+1} (winding round modulo 5), then $p = d(G) \leqslant \min\{w(x_i), w(y_i)\} \leqslant \frac{1}{2}$, contradicting our assumption on p. Thus, without loss of generality, we may assume that for every $i \in [5]$ the vertex x_i sends an edge into both V_{i-1} and V_{i+1}. If some x_i is adjacent to both x_{i-1} and x_{i+1}, then we have a transversal tree in G. Further if there is no i such that x_ix_{i+1}

is an edge of G, then x_1, y_2, x_3, y_4, x_5 induces a transversal path on 5 vertices, i.e. a connected transversal.

We may therefore assume without loss of generality that x_1x_2, y_5x_1 and x_2y_3 are all edges of G, and further that at least one of y_5x_4, y_3x_4 is an edge of G. Thus G contains a path on 5 vertices, i.e. a connected transversal, a contradiction. This concludes the proof that $\pi_{C_5}(\mathcal{T}_5) \leqslant \frac{1}{2}$. $\square$