Extremal Graphs for the Suspension of Edge-Critical Graphs

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Abstract

The Turán number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. For a vertex v and a multi-set $\mathcal{F}$ of graphs, the suspension $\mathcal{F} + v$ of $\mathcal{F}$ is the graph obtained by connecting the vertex v to all vertices of F for each $F \in \mathcal{F}$. For two integers $k \geqslant 1$ and $r \geqslant 2$, let H_i be a graph containing a critical edge with chromatic number r for any $i \in \{1, \ldots, k\}$, and let $H = \{H_1, \ldots, H_k\} + v$. In this paper, we determine $\mathrm{ex}(n, H)$ and characterize all the extremal graphs for sufficiently large n. This generalizes a result of Chen, Gould, Pfender and Wei on intersecting cliques.

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1 Introduction

Given a graph H, a graph G is called *H-free* if it contains no copy of H as a subgraph. The *Turán number* $\mathrm{ex}(n, H)$ of H is the maximum number of edges in an H-free graph on n vertices. Determining $\mathrm{ex}(n, H)$ is one of most important problems in extremal graph theory and the *Turán graph* plays a key role. For two integers n and r with $n \geqslant r \geqslant 2$, the Turán graph $T_r(n)$ is an n-vertex complete r-partite graph with parts of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Let $t_r(n)$ denote the number of edges in $T_r(n)$. The classical Turán's theorem [17] shows that $\mathrm{ex}(n, K_{r+1}) = t_r(n) = (1 - \frac{1}{r} + o(1))\binom{n}{2}$ and the only extremal graph is $T_r(n)$.

Let $\chi(H)$ denote the chromatic number of H. If there is an edge e of H such that $\chi(H - e) = \chi(H) - 1$, then we say that H is *edge-critical* and e is a *critical edge*. The celebrated Erdős-Stone-Simonovits Theorem [6, 7] states that $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}$. For an edge-critical graph H with $\chi(H) = r + 1$, Simonovits [16] proved that $T_r(n)$ is also the unique extremal graph for sufficiently large n.

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Theorem 1 (Simonovits [16]). *Let H be an edge-critical graph with $\chi(H) = r + 1 \geqslant 3$. Then there exists some n_0 such that $\mathrm{ex}(n, H) = t_r(n)$ for all $n \geqslant n_0$, and the unique extremal graph is $T_r(n)$.*

Although the Turán numbers of non-bipartite graphs are asymptotically determined by Erdős-Stone-Simonovits theorem, it is still a challenge to determine the exact Turán numbers for many non-bipartite graphs. There are only a few graphs whose Turán numbers are determined exactly, including edge-critical graphs and some other specific graphs (e.g. see [8, 10, 12, 13, 19, 20, 21, 22]). Among all the existing results, the Turán number of the graph consisting of some specific graphs that intersect in exactly one common vertex is widely studied (e.g. see [4, 9, 14, 11, 18]).

In this paper, we mainly consider edge-critical graphs intersecting in a special vertex. For a vertex v and a multi-set $\mathcal{F}$ of graphs, the suspension $\mathcal{F} + v$ of $\mathcal{F}$ is the graph obtained by connecting the vertex v to all vertices of F for each $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$, then we simply write $F + v$ instead of $\mathcal{F} + v$. We call the vertex v the *center vertex* of $\mathcal{F} + v$. If $\mathcal{F}$ is a multi-set consisting of k copies of K_{r-1}, then the graph $\mathcal{F} + v$ is known as a (k, r)-*fan*, denoted by $F_{k,r}$. Erdős, Füredi, Gould and Gunderson [5] first considered the Turán number of $F_{k,3}$ (also known as the friendship graph), and established the following result.

Theorem 2 (Erdős, Füredi, Gould and Gunderson [5]). *For every $k \geqslant 1$, and for every $n \geqslant 50k^2$,*

$$\mathrm{ex}(n, F_{k,3}) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

For general r, Chen, Gould, Pfender and Wei [2] determined $\mathrm{ex}(n, F_{k,r})$ for sufficiently large n.

Theorem 3 (Chen, Gould, Pfender and Wei [2]). *For every $k \geqslant 1$ and $r \geqslant 2$, and for every $n \geqslant 16k^3r^8$,*

$$\mathrm{ex}(n, F_{k,r}) = t_{r-1}(n) + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

We further extend this result and determine $\mathrm{ex}(n, H)$ for $H := \mathcal{F} + v$, where $\mathcal{F}$ consists of k edge-critical graphs $H_1, H_2, \ldots, H_k$ with $\chi(H_i) = r$ for each $1 \leqslant i \leqslant k$. Let $\mathcal{G}_{n,k,r}$ be a family of graphs, each of which is obtained from Turán graph $T_r(n)$ by embedding two vertex disjoint copies of K_k in one partite set if k is odd and embedding a graph with $2k - 1$ vertices, $k^2 - 3k/2$ edges with maximum degree $k - 1$ in one partite set if k is even. Our main result is as follows.

Theorem 4. *Suppose that $k \geqslant 1$ and $r \geqslant 2$ are integers. Let H_i be an edge-critical graph with $\chi(H_i) = r$ for each $i \in \{1, \ldots, k\}$, and let $H := \{H_1, H_2, \cdots, H_k\} + v$. Then, for sufficiently large n,*

$$\mathrm{ex}(n, H) = t_r(n) + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

Moreover, the $\mathcal{G}_{n,k,r}$ is the family of extremal graphs for H.

This paper is organized as follows. In the remainder of this section, we describe notations and terminologies used in our proofs. In Section 2, we make a reduction of Theorem 4, and prove it assuming Theorem 8. We prove Theorem 8 in Section 3.

Natation. Let G be a graph. We use $e(G)$, $\delta(G)$ and $\Delta(G)$ to denote the number of edges, minimum degree and maximum degree in G, respectively. For $S, T \subseteq V(G)$, we use $G[S]$ denote the graph induced by S. For $v \in V(G)$, let $N_S(v)$ denote the set of vertices in S adjacent to v and $d_S(v) = |N_S(v)|$. Let $N_S(T) = \bigcap_{v\in T} N_S(v)$. Let $V - S = \{v \in V : v \notin S\}$. In particular, if $S = V(G)$, then we substitute $N_G(v)$ and $d_G(v)$ for $N_{V(G)}(v)$ and $d_{V(G)}(v)$, respectively. A *matching* in G is a set of edges from $E(G)$, no two of which share a common vertex. The *matching number* of G, denoted by $\nu(G)$, is the maximum number of edges in a matching in G. An *r-partition* of G is a partition of $V(G)$ into r pairwise disjoint nonempty subsets $V_1, V_2, \dots, V_r$. For an integer t, let $[t] = \{1, 2, \dots, t\}$.

2 Reduction to H-free graphs with large minimum degree

In this section, we make a reduction in preparation for the proof of Theorem 4. We first introduce a function related to the number of edges in a graph with bounded matching number and maximum degree. Let G be a graph with its matching number $\nu(G)$ and maximum degree $\Delta(G)$. Define

$$f(\nu,\Delta) = \max\left\{e(G) \mid \nu(G) \leqslant \nu, \Delta(G) \leqslant \Delta\right\}.$$

Abbott, Hanson and Sauer [1] studied this function for $\nu = \Delta = k-1$, and proved that

$$f(k-1,k-1) = \begin{cases} k^2-k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

The extremal graphs are graphs with $2k-1$ vertices, $k^2 - 3k/2$ edges with maximum degree $k-1$ if k is even, or two vertex disjoint copies of K_k if k is odd. For general ν and Δ, Chvátal and Hanson [3] established the following theorem.

Theorem 5 (Chvátal and Hanson [3])**.** *For every $\nu \geqslant 1$ and $\Delta \geqslant 1$,*

$$f(\nu,\Delta) = \nu\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leqslant \nu\Delta + \nu. \tag{1}$$

Definition 6 (Good partition)**.** For two integers $k, r \geqslant 2$, call a partition $V_1, V_2, \dots, V_r$ of G k-good, if the following properties hold for each $i \in [r]$:

(i) $\Delta(G[V_i]) \leqslant k-1$,

(ii) $\sum_{j\in[r]\setminus\{i\}} \nu(G[V_j]) \leqslant k-1$, and

(iii) $d_{V_i}(u) + \sum_{j\in[r]\setminus\{i\}} \nu\left(G[N_G(u)\cap V_j]\right) \leqslant k-1$ for each $u \in V_i$.

Chen, Gould, Pfender and Wei [2] characterised the properties of a k-good partition of graphs by showing the following lemma.

Lemma 7 (Chen, Gould, Pfender and Wei [2])**.** *Suppose that G has a k-good partition $V_1, V_2, \ldots, V_r$. Let G' be the minimal induced subgraph of G such that $e(G') - \sum_{1\leqslant i<j\leqslant r} |V_i'||V_j'|$ is maximal, where $V_i' = V(G') \cap V_i$ for each $i \in [r]$. Then the following properties hold:*

(i) $e(G') - \sum_{1\leqslant i<j\leqslant r} |V_i'||V_j'| \leqslant f(k-1, k-1)$;

(ii) *For each $i \in [r]$ and $x \in V_i'$, we have $0 < d_{G'}(x) - |V(G')\setminus V_i'| \leqslant k-1-\sum_{j\in[r]\setminus\{i\}} \nu(G'[V_j'])$;*

(iii) *If $\nu(G'[V_i']) \geqslant 2$ for each $i \in [r]$, then $e(G') - \sum_{1\leqslant i<j\leqslant r} |V_i'||V_j'| < f(k-1, k-1)$.*

Now, we begin with a reduction of our main theorem via the existence of a k-good partition, and prove Theorem 4 assuming Theorem 8. We leave the proof of Theorem 8 in the next section.

Theorem 8. *For two integers $k, r \geqslant 2$, let H_i be an edge-critical graph with $\chi(H_i) = r$ for each $i \in [k]$, and let $H := \{H_1, H_2, \ldots, H_k\} + u$. If G is an H-free graph with n vertices and $\delta(G) \geqslant \frac{r-1}{r}n - k$, then G contains a k-good partition for sufficiently large n.*

Proof of Theorem 4 given Theorem 8. Let $\mathcal{G}_{n,k,r}$ be the family of graphs defined in Section 1. We first show that G is H-free for each $G \in \mathcal{G}_{n,k,r}$. Otherwise, we consider an embedding of H with the center vertex v into G. Without loss of generality, we may assume that $e(G[V_1]) = f(k-1, k-1)$. Note that $E(H_j+v)\cap E(G[V_1]) \neq \emptyset$ for each $j \in [k]$ in view of $\chi(H_j) = r$. It follows that $v \notin V_1$ as $\Delta(G[V_1]) < k$ by the construction of $\mathcal{G}_{n,k,r}$. Suppose that $v \in V_s$ for some $s \in [r]\setminus\{1\}$. In this situation, we have $E(H_j+v)\cap E(G[V_1])$ are pairwise disjoint for any $j \in [k]$. This means that $\nu(G[V_1]) \geqslant k$, a contradiction. Thus, G is H-free and $e(G) = t_r(n) + f(k-1, k-1)$, implying the lower bound.

In what follows, we prove that $e(G) \leqslant t_r(n) + f(k-1, k-1)$ for any H-free graph G on n vertices. We first show that this is true if $\delta(G) \geqslant \frac{r-1}{r}n - k$. By Theorem 8 and Lemma 7, there is a k-good partition $V_1, V_2, \ldots, V_r$ of G such that

$$e(G) \leqslant \sum_{1\leqslant i<j\leqslant r} |V_i||V_j| + f(k-1, k-1) \leqslant t_r(n) + f(k-1, k-1),$$

as desired. Next, we aim to deal with small vertices. For a graph F and $f \in V(F)$, we say f is a *small vertex* of F if $d(f) < \frac{r-1}{r}|V(F)| - k$. We first delete a small vertex in G. As long as there is a small vertex in the resulting graph, we delete it. We keep doing this until the remaining graph G^* (G^* maybe empty) has no small vertices. If

$n^* := |V(G^*)| < \sqrt{kn/4}$, then

$$\begin{aligned} e(G) &< e(G^*) + \sum_{i=n^*+1}^{n} \left(\frac{r-1}{r}i - k\right) \\ &< \frac{kn}{8} + \frac{r-1}{r}\frac{(n+\sqrt{kn/4})(n-\sqrt{kn/4})}{2} - \left(n - \sqrt{\frac{kn}{4}}\right)k \\ &< \frac{r-1}{r}\frac{n(n-1)}{2} \leqslant t_r(n), \end{aligned}$$

as required. Thus, we may assume that n^* is sufficiently large and $\delta(G^*) \geqslant \frac{r-1}{r}n^* - k$. This implies that $e(G^*) \leqslant t_r(n^*) + f(k-1,k-1)$ as G^* is also H-free. It follows that

$$\begin{aligned} e(G) &< e(G^*) + \sum_{i=n^*+1}^{n} \left(\frac{r-1}{r}i - k\right) \\ &\leqslant t_r(n^*) + f(k-1,k-1) + \sum_{i=n^*+1}^{n} \left(\frac{r-1}{r}i - k\right) \\ &\leqslant t_r(n) + f(k-1,k-1), \end{aligned} \tag{2}$$

where the last inequality holds as $t_r(s-1) + \frac{r-1}{r}(s-1) \leqslant t_r(s)$. Thus, $e(G) < t_r(n) + f(k-1,k-1)$.

Now, we prove the uniqueness of the extremal graph. Let G be an H-free graph with

$$e(G) = t_r(n) + f(k-1,k-1). \tag{3}$$

Then $\delta(G) \geqslant \frac{r-1}{r}n - k$ by the above argument. It follows from Theorem 8 that G has a k-good partition $V_1, V_2, \ldots, V_r$. By Lemma 7, there exists a minimal induced subgraph G' of G such that

$$e(G) - \sum_{1\leqslant i<j\leqslant r} |V_i||V_j| \leqslant e(G') - \sum_{1\leqslant i<j\leqslant r} |V_i'||V_j'| \leqslant f(k-1,k-1). \tag{4}$$

Without loss of generality, suppose that $|V_i'| > 0$ for $1 \leqslant i \leqslant s$ and $|V_i'| = 0$ for $s+1 \leqslant i \leqslant r$. Then, for each $i \in [s]$ and $x \in V_i'$

$$0 < d_{G'}(x) - |V(G')\backslash V_i'| \leqslant k - 1 - \sum_{j\in[s]\backslash\{i\}} \nu(G'[V_j']),$$

implying that $\nu(G'[V_i']) \geqslant 1$ and $\sum_{j\in[s]\backslash\{i\}} \nu(G'[V_j']) \leqslant k-2$. In addition, by (3) and (4), we have

$$\sum_{i=1}^{r} e(G[V_i]) = \sum_{i=1}^{s} e(G'[V_i']) = f(k-1,k-1). \tag{5}$$

Case 1. $\sum_{j\in[s]\setminus\{i_0\}} \nu(G'[V_j']) = 0$ for some $i_0 \in [s]$. This implies that $V_j' = \emptyset$ for each $j \in [r] \setminus \{i_0\}$ and $G' = G[V_{i_0}']$. Thus, $e(G[V_{i_0}]) \geqslant e(G') = f(k-1, k-1)$. It follows from (5), the definition of good partition and the result of Abbott, Hanson and Sauer [1] that $G \cong G_{n,k,r} \in \mathcal{G}_{n,k,r}$.

Case 2. $1 \leqslant \sum_{j\in[s]\setminus\{i\}} \nu(G'[V_j']) \leqslant k-2$ for each $i \in [s]$. Clearly, there exists an $i_0 \in [r]$ such that $\nu(G'[V_{i_0}']) = 1$; otherwise, we get a contradiction by Lemma 7 (iii) and (4). Without loss of generality, suppose that $\nu(G'[V_1']) = 1$. Then, we have

$$
\begin{aligned}
\sum_{i=1}^{s} e(G'[V_i']) &\leqslant \sum_{2\leqslant i\leqslant s} f\left(\nu(G'[V_i']), k-1\right) + f(1, k-1) \\
&\leqslant f\left(\sum_{2\leqslant i\leqslant s} \nu(G'[V_i']), k-1\right) + f(1, k-1) \\
&\leqslant f(k-2, k-1) + f(1, k-1) \\
&\leqslant f(k-1, k-1),
\end{aligned}
$$

where the last inequality is strictly true for $k \geqslant 5$. This leads to a contradiction in view of (5). It remains to consider the situation for $k \leqslant 4$.

If $k = 3$, then $s = 2$ and $G[V_1'] \cong G[V_2'] \cong K_3$. This together with (5) yields that G is a graph formed by the complete r-partite graph with classes $V_1, \ldots, V_r$ embedding two triangles $x_1y_1z_1$ and $x_2y_2z_2$ in V_1 and V_2, respectively. Recall that $V_1, V_2, \ldots, V_r$ is a k-good partition of G. But $d_{G[V_1]}(x_1) + \nu\left(G[N_{V_2}(x_1)]\right) = 3 > k-1$, a contradiction to Definition 6 (iii). If $k = 4$, then $G[V_1'] \cong K_{1,3}$ and $\sum_{i=2}^{s} \nu(G[V_2']) = 2$. As the same argument of the case $k = 3$, we can find a vertex $x \in V_1$ such that $d_{G[V_1]}(x) + \sum_{i=2}^{s} \nu\left(G[N_{V_i}(x)]\right) = 5 > k-1$, a contradiction to Definition 6 (iii). Thus, we complete the proof of Theorem 4. $\square$

3 H-free graphs with large minimum degree

In this section, we give a proof of Theorem 8. We first present the following useful lemmas given by Roberts and Scott [15].

Lemma 9 (Roberts and Scott [15]). *Let F be a graph with a critical edge and $\chi(F) = r+1 \geqslant 3$, and let $f(n) = o(n^2)$ be a function. If G is an F-free graph with n vertices and $e(G) \geqslant t_r(n) - f(n)$, then G can be made r-partite by deleting $O(n^{-1}f(n)^{3/2})$ edges.*

Lemma 10 (Roberts and Scott [15]). *Let $r \geqslant 2$ and $t \geqslant 1$ be integers. Suppose that the graph $G \subseteq T_r(rn)$ is $T_r(rt)$-free. Then $e(G) \leqslant t_r(rn) - n^2/2$ for sufficiently large n.*

We also need another easy lemma about edge-critical graphs.

Lemma 11. *Let G be an edge-critical graph with $\chi(G) = r \geqslant 2$, and let $G^* = G + u$. If v_1v_2 is a critical edge of G, then both uv_1 and uv_2 are critical edges of G^*.*

Proof. By symmetry, it suffices to show that uv_1 is a critical edge of G^*. Since $\chi(G) = r$ and v_1v_2 is a critical edge of G, there is a partition $(V_1, \ldots, V_r)$ of G such that V_i is an independent set of G for each $i \in [r-1]$ and $V_r = \{v_1\}$. Let G' be the graph obtained from G^* by deleting the edge uv_1. Clearly, $(V_1, \ldots, V_{r-1}, V_r \cup \{u\})$ is an r-coloring of G'. This means that uv_1 is a critical edge of G^* in view of $\chi(G^*) = r+1$. □

Now, we are in a position to prove Theorem 8.

Proof of Theorem 8. Suppose that H_j is an edge-critical graph with a critical edge u_jv_j and $\chi(H_j) = r$ for each $j \in [k]$. Let $H := (H_1, H_2, \ldots, H_k) + u$ and $H_j^* = H_j + u$ for $j \in [k]$. By Lemma 11, we have uv_j is a critical edge of H_j^*. Then, by Theorem 1, there exists a constant p_j (or p_j^*) such that H_j can be embedded in $T_{r-1}((r-1)p_j) + e$ with $u_jv_j = e$ (or H_j^* can be embedded in $T_r(rp_j^*) + e$ with $uv_j = e$) for each $j \in [k]$, where e is any edge inside a vertex class of $T_{r-1}((r-1)p_j)$ (or $T_r(rp_j^*)$).

Let G be an H-free graph with maximum number of edges. This means that $G + e$ contains a copy of H for any $e \notin E(G)$. It follows that G contains a subgraph D which is a copy of $H - e_0$ for some $e_0 \in H$. Without loss of generality, we may assume that $e_0 \in H_k^*$ and v_0 is the center vertex of D. Let D' denote the subgraph which is a copy of $(H_1, \ldots, H_{k-1}) + u$ in D with the center vertex v_0. Note that $\delta(G) \geqslant \frac{r-1}{r}n - k$. Let $\ell = |V(D')|$. Choose a subset $S \subseteq N_G(v_0) - V(D')$ such that

$$|S| = \frac{r-1}{r}n - k - \ell. \tag{6}$$

Clearly, $G[S]$ is H_k-free as G is H-free. We show that $G[S]$ is close to $T_{r-1}(|S|)$. Note that

$$\delta(G[S]) \geqslant \delta(G) - (n - |S|) \geqslant \frac{r-2}{r}n - 2k - \ell = |S| - \frac{n}{r} - k. \tag{7}$$

This implies that

$$\begin{aligned} e(G[S]) &\geqslant \frac{|S|\left(|S| - \frac{n}{r} - k\right)}{2} \geqslant \frac{|S|(|S|+1)}{2} - \frac{|S|\left(\frac{n}{r} + k + 1\right)}{2} \\ &= \frac{r-2}{r-1}\frac{|S|(|S|+1)}{2} - \frac{(k+\ell-1)+(r-1)(k+1)}{2(r-1)}|S|. \end{aligned}$$

For simplicity, let $C_{k,\ell} = \frac{(k+\ell-1)+(r-1)(k+1)}{2(r-1)}$. Since $(1 - \frac{1}{r-1})\frac{|S|(|S|+1)}{2} \geqslant t_{r-1}(|S|)$, we have

$$e(G[S]) \geqslant t_{r-1}(|S|) - C_{k,\ell}|S|. \tag{8}$$

For a partition $(S_1, S_2, \ldots, S_{r-1})$ of $G[S]$, we define an $(r-1)$-partite graph

$$G_S[S_1, S_2, \ldots, S_{r-1}] = \left(S, \{v_iv_{i'} \in E(G) \,:\, v_i \in S_i, v_{i'} \in S_{i'}, 1 \leqslant i < i' \leqslant r-1\}\right).$$

Now we partition S into $(S_1, S_2, \ldots, S_{r-1})$ such that $e(G_S[S_1, S_2, \ldots, S_{r-1}])$ is maximum. By Lemma 9, for some constant c

$$\sum_{1 \leqslant i \leqslant r-1} e(G[S_i]) \leqslant c|S|^{1/2} \leqslant cn^{1/2}. \tag{9}$$

This together with (8) implies that

$$\left| |S_i| - \frac{n}{r} \right| \leqslant \varepsilon_1 \frac{n}{r} \tag{10}$$

for some $\varepsilon_1 \in (0, 10^{-6})$. Fix $i \in [r-1]$. For $x_i \in S_i$ and $i' \in [r-1]$ with $i' \neq i$, we have

$$d_{S_{i'}}(x_i) \geqslant \delta(G[S]) - \sum_{q \in [r-1] \setminus \{i,i'\}} |S_q| - cn^{\frac{1}{2}} \geqslant \frac{1}{r}n - 2k - \ell - \frac{r-3}{r}\varepsilon_1 n - cn^{\frac{1}{2}} \geqslant (1-\varepsilon_2)\frac{n}{r} \tag{11}$$

for some $\varepsilon_2 \in (2\varepsilon_1, 10^{-5})$.

Now, we show that $E(G[S_i])$ is empty for each $i \in [r-1]$. Suppose that there exists an edge $uv \in E(G[S_1])$. Pick $B_1 \subseteq (S_1 - \{u,v\})$ with $|B_1| = \frac{n}{2r}$. By (10) and (11), we have

$$|N_{S_i}(u) \cap N_{S_i}(v)| \geqslant (1 - 2(\varepsilon_1 + \varepsilon_2))\frac{n}{r} \geqslant \frac{n}{2r}$$

for each $2 \leqslant i \leqslant r-1$. We can pick a subset $B_i \subseteq (N_{S_i}(u) \cap N_{S_i}(v))$ with $|B_i| = \frac{n}{2r}$ for each $2 \leqslant i \leqslant r-1$. Let $B = \bigcup_{i \in [r-1]} B_i$. Recall that $G[S]$ is H_k-free. It follows from Theorem 1 that $G_B[B_1, B_2, \ldots, B_{r-1}]$ is $T_{r-1}((r-1)p_k)$-free for some constant $p_k > 0$. By Lemma 10, we have

$$e(G_B[B_1, B_2, \ldots, B_{r-1}]) \leqslant t_{r-1}\left(\frac{(r-1)n}{2r}\right) - \frac{n^2}{8r^2},$$

implying that there are at least $\frac{n^2}{8r^2}$ edges missing between vertex classes S_i for $i \in [r-1]$. This together with (9) shows that

$$e(G[S]) \leqslant t_{r-1}(|S|) + cn^{\frac{1}{2}} - \frac{n^2}{8r^2},$$

a contradiction to (8). Therefore $E(G[S_i]) = \emptyset$ for each $i \in [r-1]$.

Since $e(G[S_i]) = 0$, we can further improve $|S_i|$ for each $i \in [r-1]$ by showing that

$$|S_i| \leqslant |S| - \delta(G[S]) \leqslant \frac{n}{r} + k \tag{12}$$

in view of (6) and (7). This further implies that

$$|S_i| = |S| - \sum_{i' \in [r-1] \setminus \{i\}} |S_{i'}| \geqslant \frac{n}{r} - k(r-1) - \ell. \tag{13}$$

Moreover, for $x \in S_i$ and $i' \in [r-1] \setminus \{i\}$, it follows from (7) and (12) that

$$d_{S_{i'}}(x) \geqslant \delta(G[S]) - \sum_{q \in [r-1] \setminus \{i,i'\}} |S_q| \geqslant \frac{n}{r} - k(r-1) - \ell. \tag{14}$$

Recall that D' denote the subgraph which is a copy of $(H_1, \ldots, H_{k-1}) + u$ in D. We consider the vertices not in $S \cup V(D')$. Let $S_0 = V(G) - S - V(D')$. Then

$$|S_0| = n - \ell - |S| = \frac{n}{r} + k. \tag{15}$$

For $x_i \in S_i$ with $i \in [r-1]$, by $\delta(G) \geqslant (r-1/r)n - k$

$$d_{S_0}(x_i) \geqslant d_G(x_i) - (|S \cup V(D')| - |S_i|) \geqslant |S_i| \geqslant \frac{n}{r} - k(r-1) - \ell = |S_0| - kr - \ell. \quad (16)$$

Let

$$a = kr + \ell, \;\; p^* = \sum_{i \in [k]} p_i^*, \quad (17)$$

and

$$S_0^* = \{x \in S_0 : d_{S_0}(x) \geqslant p^*(r-1)a + k\}.$$

Claim 12. $|S_0^*| \leqslant a(r-1)$.

Proof. Suppose that $|S_0^*| \geqslant a(r-1)+1$. For each $i \in [r-1]$, let

$$S_0^i = \left\{ v \in S_0^* : d_{S_i}(v) \geqslant \frac{|S_i|}{a+1} \right\}.$$

Notice that $d_{S_0}(x_i) \geqslant |S_0| - a$ for $x_i \in S_i$ by (16) and (17). In other words, there are at most a vertices in S_0 are not adjacent to x_i. If $X \subseteq S_0^*$ with $|X| = a+1$, then $S_i \subseteq \bigcup_{x \in X} N(x)$ for $i \in [r-1]$. This implies that $|S_0^i| \geqslant |S_0^*| - a$. Thus, $\left|\bigcap_{i \in [r-1]} S_0^i\right| \geqslant |S_0^*| - a(r-1) \geqslant 1$. We can choose a vertex $v \in S_0^*$ such that for each $i \in [r-1]$

$$d_{S_i}(v) \geqslant \frac{|S_i|}{a+1}. \quad (18)$$

In the following, we aim to find a copy of H with a center vertex v.

For $i \neq i' \in [r-1]$ and $x \in S_i$, recall that $d_{S_{i'}}(x) \geqslant n/r - k(r-1) - \ell \geqslant |S_i| - a$ by (12), (14) and (17). This together with (18) shows that

$$N_{S_i}(Y \cup \{v\}) \geqslant N_{S_i}(v) - a|Y| \geqslant \frac{|S_i|}{a+1} - aC_Y, \quad (19)$$

where $Y \subseteq S - S_i$ and $|Y| = C_Y$ is a constant. For sufficiently large n, by (19), we can pick $Y_1^1 \subseteq N_{S_1}(v)$ with $|Y_1^1| = p_1^*$, $Y_i^1 \subseteq N_{S_i}(v) \cap \left(\bigcap_{i' \in [i-1]} N_{S_i}(Y_{i'}^1)\right)$ satisfying $|Y_i^1| = p_1^*$ successively for $2 \leqslant i \leqslant r-1$. Note that the size of $|Y_i^1|$ is to ensure that H_1^* can be embedded in $G_{Y_1 \cup Y_0^j}[Y_0^1, Y_1^1, \ldots, Y_{r-1}^1] + vx_1$. Fix i, we choose

$$Y_i^j \subseteq \left(N_{S_i}(v) \cap \left(\bigcap_{i' \in [i-1]} N_{S_i}(Y_{i'}^j) \right) \right) - \bigcap_{j' \in [j-1]} Y_i^{j'}$$

satisfying $|Y_i^j| = p_j^*$ successively for $2 \leqslant j \leqslant k$. For $j \in [k]$, let $Y_j = \bigcup_{i \in [r-1]} Y_i^j$. Then $|Y_j| = (r-1)p_j^*$. Now, we find a copy of H_j^* in $G[Y_j \cup S_0]$. It follows from (16) that

$$\begin{aligned} |N(Y_j) \cap N_{S_0}(v)| &\geqslant |N_{S_0}(v)| - \sum_{u \in Y_j} (|S_0| - d_{s_0}(u)) \\ &\geqslant p^*(r-1)a + k - p_j^*(r-1)a \\ &\geqslant (p^* - p_j^*)(r-1)a + k. \end{aligned}$$

Thus, for $j \in [k]$, we can pick $x_j \in N(Y_j) \cap N_{S_0}(v)$ such that $x_1, x_2, \ldots, x_k$ are pairwise distinct. Again by (16), for $j \in [k]$, we have

$$|N_{S_0}(Y_j)| \geqslant |S_0| - p_j^*(r-1)a = \frac{n}{r} + k - p_j^*(r-1)a.$$

This means that we can pick $Y_0^1 \subseteq N_{S_0}(Y_1) - \{v, x_1, x_2, \ldots, x_k\}$, with $|Y_0^1| = p_1^*$, and pick

$$Y_0^j \subseteq N_{S_0}(Y_j) - \left(\bigcup_{j' \in [j-1]} Y_0^j \cup \{v, x_1, x_2, \ldots, x_k\} \right)$$

with $|Y_0^j| = p_j^*$ for $j = 2, 3, \ldots, k$ successively.

For $j \in [k]$, we have chosen $x_j \in N_{S_0}(v)$ and subsets Y_j, Y_0^j. It is easy to see that $G[Y_j \cup Y_0^j]$ contains a copy of $T_r(rp_j^*)$. This together with the choice of the edge vx_j and Theorem 1 shows that $H_j^* = H_j + u$ can be embedded in $G_{Y_j \cup Y_0^j}[Y_0^j, Y_1^j, \ldots, Y_{r-1}^j] + vx_j$ such that $u = v$. Thus, we can get a copy of H in G, a contradiction. □

Now, we consider the vertices in S_0 with small degree in $G[S_0]$. Let $Z_0 = S_0 - S_0^*$, $Z_i = S_i$ for $i \in [r-1]$ and $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_{r-1}$. It follows from Claim 12 that

$$\frac{n}{r} + k = |S_0| \geqslant |Z_0| \geqslant |S_0| - a(r-1) \geqslant \frac{n}{r} + k - a(r-1). \tag{20}$$

By (12), (13) and (17), we have

$$\frac{n}{r} + k \geqslant |Z_i| = |S_i| \geqslant \frac{n}{r} - a + k \tag{21}$$

for $i \in [r-1]$, and then

$$|V(G) - Z| \leqslant \left| V(G) - \bigcup_{i=0}^{r-1} S_i \right| + a(r-1) \leqslant \ell + a(r-1). \tag{22}$$

Recall that $\delta(G) \geqslant \frac{r-1}{r}n - k$ and $d_{Z_0}(x) \leqslant p^*(r-1)a + k - 1$ for $x \in Z_0$. This together with (21) and (22) shows that for $x \in Z_0$ and $i \in [r-1]$,

$$\begin{aligned} d_{Z_i}(x) &\geqslant d_G(x) - |V(G) - Z| - d_{Z_0}(x) - \sum_{i' \in [r-1] \setminus \{i\}} |Z_{i'}| \\ &\geqslant \frac{r-1}{r}n - k - \ell - a(r-1) - (p^*(r-1)a + k - 1) - (r-2)\left(\frac{n}{r} + k\right) \\ &\geqslant \frac{n}{r} - ((p^*+1)(r-1)+1)\,a - p^*(r-1)k \\ &\geqslant \frac{n}{r} - 2\,((p^*+1)(r-1)+1)\,a. \end{aligned} \tag{23}$$

For every $i \in [r-1]$ and every $x \in Z_i$, by Claim 12 and (16), we have

$$d_{Z_0}(x) \geqslant d_{S_0}(x) - |S_0^*| \geqslant \frac{n}{r} - a + k - a(r-1) = \frac{n}{r} - ar + k. \tag{24}$$

Claim 13. *For every $x \in V(G) - Z$, there exists an $i = i(x)$ such that $d_{Z_i}(x) < k$. Moreover, such an i is unique.*

Proof. Suppose that there exists a vertex $v \in V(G) - Z$ such that $d_{Z_i}(v) \geqslant k$ for each $i \in \{0\} \cup [r-1]$. Let $d_{Z_{\hat{i}}}(v) = \min\{d_{Z_i}(v) : 0 \leqslant i \leqslant r-1\}$ for some $\hat{i} \in \{0\} \cup [r-1]$. In the following, we consider the case $\hat{i} = 0$ (for other cases, i.e., $\hat{i} \in [r-1]$, we can derive a similar conclusion through analogous reasoning). By the pigeonhole principle, we have $d_{Z_0}(v) \leqslant d_G(v)/r$. Thus, for $i \in [r-1]$, we have

$$\begin{aligned} d_{Z_i}(v) &\geqslant d_G(v) - |V(G) - Z| - d_{Z_0}(v) - \sum_{i' \in [r-1] \setminus \{i\}} d_{Z_{i'}}(v) \\ &\geqslant d_G(v) - (\ell + a(r-1)) - \frac{d_G(v)}{r} - \sum_{i' \in [r-1] \setminus \{i\}} |Z_{i'}| \\ &\geqslant \frac{r-1}{r} d_G(v) - \frac{r-2}{r} n - ar \geqslant \frac{n}{2r^2}. \end{aligned} \tag{25}$$

Now, we construct k r-partite graphs. Recall that $|N_{Z_0}(v)| \geqslant k$. We can pick k distinct vertices $x_1, x_2, \ldots, x_k$ in $N_{Z_0}(v)$ and choose k pairwise disjoint subsets Y_0^1, Y_0^2, $\ldots$, Y_0^k in $Z_0 - \{x_1, x_2, \ldots, x_k\}$ with $|Y_0^j| = \frac{n}{4kr^2}$ for $j \in [k]$. By (21), (23) and (25), we have

$$|N_{Z_i}(v) \cap N_{Z_i}(x_j)| \geqslant d_{Z_i}(v) - (|Z_i| - d_{Z_i}(x_j)) \geqslant \frac{n}{4r^2}. \tag{26}$$

For $i \in [r-1]$, we can choose k pairwise disjoint Y_i^1, Y_i^2, $\ldots$, Y_i^k such that $Y_i^j \subseteq N_{Z_i}(v) \cap N_{Z_i}(x_j)$ and $|Y_i^j| = \frac{n}{4kr^2}$ for $j \in [k]$. This is possible, since we can choose $Y_i^1 \subseteq N_{Z_i}(v) \cap N_{Z_i}(x_1)$. Suppose that $Y_i^1, \ldots, Y_i^{j'-1}$ have been chosen. Due to (26), we choose

$$Y_i^{j'} \subseteq (N_{Z_i}(v) \cap N_{Z_i}(x'_j)) \setminus \bigcup_{j \in [j'-1]} Y_i^j.$$

Let $Y^j = \bigcup_{i=0}^{r-1} Y_i^j$ for $j \in [k]$. Then, we obtain k r-partite graphs $G_{Y^j}[Y_0^j, Y_1^j, \ldots, Y_{r-1}^j]$.

Since G is H-free, there exists $j_0 \in [k]$ such that $G_{Y^{j_0}}[Y_0^{j_0}, Y_1^{j_0}, \ldots, Y_{r-1}^{j_0}]$ is $T_r(rp^*_{j_0})$-free by Theorem 1. Thus, by Lemma 10, we have

$$e(G_{Y^{j_0}}[Y_0^{j_0}, Y_1^{j_0}, \ldots, Y_{r-1}^{j_0}]) \leqslant t_r\left(\frac{n}{4kr}\right) - \frac{n^2}{32k^2r^4}.$$

This means that

$$e\left(G_{Z \cup \{v\}}\left[Z_0 \cup \{v\}, Z_1, \ldots, Z_{r-1}\right]\right) \leqslant t_r(n) - \frac{n^2}{32k^2r^4}. \tag{27}$$

On the other hand, by (14), (21) and (24), we have

$$
\begin{aligned}
e\left(G_{Z\cup\{v\}}\left[Z_0\cup\{v\}, Z_1,\ldots,Z_{r-1}\right]\right) &\geqslant e\left(G_Z\left[Z_0, Z_1,\ldots,Z_{r-1}\right]\right)\\
&= \sum_{x\in S} d_{Z_0}(x) + \sum_{i=1}^{r-2} \sum_{\substack{i<i'\leqslant r-1\\ x\in S_{i'}}} d_{S_i}(x)\\
&\geqslant \left(\frac{n}{r} - ar\right)\sum_{i=1}^{r-1} i|S_i| = t_r(n) - \frac{2(a(r+1)-k)}{r}n,
\end{aligned}
$$

a contradiction to (27).

Now, we prove the uniqueness of $i = i(x)$ for $x \in V(G) - Z$. Suppose that there exists $x \in V(G) - Z$ and $i, i' \in \{0\} \cup [r-1]$ such that both $d_{Z_{i_1}}(x)$ and $d_{Z_{i_2}}(x)$ are less than k. This means $Z_{i_1} \cup Z_{i_2}$ has at least $|Z_{i_1}| + |Z_{i_2}| - 2k + 2$ vertices that are not adjacent to x. Thus $d_G(x) \leqslant n - (|Z_{i_1}| + |Z_{i_2}| - 2k) < (1-1/r)n - k$ in view of (21), a contradiction. □

By Claim 13, for each $x \in V(G) - Z$, there is a unique $i = i(x)$ such that $d_{Z_i}(x) < k$. We can put x in $Z_{i(x)}$. Then, we get an r-partition $(V_0, V_1, \ldots, V_{r-1})$ of G with $Z_i \subseteq V_i$ for $i \in \{0\} \cup [r-1]$. For $x \in V(G)$, we consider the degree of x in V_i with $x \notin V_i$. Suppose first that $x \in V_{i(x)}$ for $x \in V(G) - Z$. For $0 \leqslant i \leqslant r-1$ with $i \neq i(x)$, by (20), (21) and (22),

$$
d_{V_i}(x) \geqslant d_G(x) - d_{Z_{i(x)}}(x) - |V(G) - Z| - \sum_{\substack{0\leqslant i'\leqslant r-1\\ i'\notin\{i, i(x)\}}} |Z_{i'}| \geqslant \frac{n}{r} - ar - k. \tag{28}
$$

Then, we bound $d_{V_{i'}}(x)$ for $x \in V_i$ and $i \neq i'$. For $i \in \{0\} \cup [r-1]$, it follows from (20) and (21) that

$$
\frac{n}{r} + k + \ell + a(r-1) \geqslant |Z_i| + |V(G) - Z| \geqslant |V_i| \geqslant |Z_i| \geqslant \frac{n}{r} + k - a(r-1). \tag{29}
$$

Let $x \in V_i$ and $i' \neq i$, $0 \leqslant i' \leqslant r-1$. Combining (23), (24) and (28),

$$
d_{V_{i'}}(x) \geqslant d_{Z_{i'}}(x) \geqslant \frac{n}{r} - 2\left((p^*+1)(r-1)+1\right)a. \tag{30}
$$

Let $b_1 = k + \ell + a(r-1)$ and $b_2 = 2\left((p^*+1)(r-1)+1\right)a$. Fixing $i \in \{0\} \cup [r-1]$, (29) and (30) can be reduced to

$$
\frac{n}{r} + b_1 \geqslant |V_i| \geqslant \frac{n}{r} + k - a(r-1) \tag{31}
$$

and

$$
d_{V_{i'}}(x) \geqslant \frac{n}{r} - b_2 \geqslant |V_{i'}| - (b_1 + b_2) \tag{32}
$$

for $x \in V_i$ and $i' \neq i$, $0 \leqslant i' \leqslant r-1$. By (31) and (32), we have

$$
\begin{aligned}
e\left(G\left[V_0, V_1, \ldots, V_{r-1}\right]\right) &= \sum_{i=0}^{r-2} \sum_{\substack{i<i'\leqslant r-1\\ x\in V_{i'}}} d_{V_i}(x) \geqslant \left(\frac{n}{r} - b_2\right)\sum_{i=1}^{r-1} i|V_i|\\
&= t_r(n) - \frac{2(a(r-1)-k+b_2)}{r}n.
\end{aligned}\tag{33}
$$

In what follows, we prove that $(V_0, V_1, \dots, V_{r-1})$ is a k-good partition of G.

First, we show that $(V_0, V_1, \dots, V_{r-1})$ satisfies (1) of Definition 6. Note that $V_i \neq \emptyset$ for $0 \leqslant i \leqslant r-1$ by (29). If $\Delta(G[V_i]) \geqslant k$ for some $i \in \{0, 1, \dots, r-1\}$, then we can choose $x \in V_i$ and $x_1, x_2, \dots, x_k$ in $N_{V_i}(x)$. As in the proof of Claim 13, we can find k r-partite graphs and one of them is $T_r(rp_j^*)$-free for some $j \in [k]$ by Theorem 1. Thus, by Lemma 10, we have

$$e\left(G\left[V_0, V_1, \dots, V_{r-1}\right]\right) \leqslant t_r(n) - \varepsilon n^2 \tag{34}$$

for some $\varepsilon > 0$, a contradiction to (33).

Then, we show that $(V_0,\ V_1, \dots, V_{r-1})$ satisfies (2) of Definition 6. Otherwise, by symmetry, suppose that $\sum_{i\in[r-1]} \nu(G[V_i]) \geqslant k$. Let $x_1y_1, x_2y_2, \dots, x_ky_k$ be the matching M of G with $x_jy_j \in V_{i(x_jy_j)}$ for some $i(x_jy_j) \in [r-1]$. We use M to find k $(r-1)$-partite graphs. By (31) and (32), we have

$$\left|\bigcap_{j\in[k]} \left(N_{V_0}(x_j) \cap N_{V_0}(y_j)\right)\right| \geqslant |V_0| - 2k(b_1 + b_2) \geqslant 0.$$

Choose a vertex $v \in \bigcap_{j\in[k]}(N_{V_0}(x_j) \cap N_{V_0}(y_j))$. For $j \in [k]$ and $q \in [r-1]$ with $q \neq i(x_jy_j)$, let $X_q^j = N_{V_q}(v) \cap N_{V_q}(x_j) \cap N_{V_q}(y_j)$. Clearly, $|X_q^j| \geqslant n/r - 3(b_1 + b_2) \geqslant n/(2r) + 2k$ by (32). We first choose $r-2$ subsets Y_q^1 with $q \in [r-1]$ and $q \neq i(x_1y_1)$ such that $Y_q^1 \subseteq X_q^1$ and $|Y_q^1| = \frac{n}{2kr}$. Then, for $j \in \{2, 3, \dots, k\}$, we choose $r-2$ subsets Y_q^j with $q \in [r-1]$ and $q \neq i(x_jy_j)$ such that $Y_q^j \subseteq X_q^j \setminus \bigcup\limits_{s\in[j-1]} Y_q^s$ and $\left|Y_q^j\right| = \frac{n}{2kr}$. Finally, we choose

$$Y_{i(x_jy_j)}^j \subseteq N_{V_{i(x_jy_j)}}(v) \setminus \left(\{x_j, y_j\} \cup \left(\bigcup_{s\in[k]\setminus\{j\}} Y_{i(x_jy_j)}^s\right)\right)$$

with $|Y_{i(x_jy_j)}^j| = \frac{n}{2kr}$ for $j \in [k]$. Let $Y^j = \bigcup_{i\in[r-1]} Y_i^j$ for $j \in [k]$. Then, we obtain k $(r-1)$-partite graphs $G_j = G_{Y^j}[Y_1^j, Y_2^j, \dots, Y_{r-1}^j]$. Since G is H-free, there exists some $j_0 \in [k]$ such that G_{j_0} is $T_r(rp_{j_0})$-free by Theorem 1. Thus, by Lemma 10, we have

$$e(G_{j_0}) \leqslant t_r\left(\frac{n}{2k}\right) - \frac{n^2}{8k^2r^2}.$$

This means that

$$e\left(G[V_0, V_1, \dots, V_{r-1}]\right) \leqslant t_r(n) - \frac{n^2}{8k^2r^2},$$

a contradiction to (33).

In the end, we show that $(V_0, V_1, \dots, V_{r-1})$ satisfies (3) of Definition 6. Otherwise, by symmetry, suppose that there exists $v \in V_0$ such that $d_{V_0}(v) + \sum_{i\in[r-1]} \nu\left(G[N_{V_i}(v)]\right) \geqslant k$. Thus, we can pick $z_1, z_2, \dots, z_s$ in $N_{V_0}(v)$ and $(k-s)$-matching $x_{s+1}y_{s+1},\ x_{s+2}y_{s+2},\ \dots,$ x_ky_k in $\bigcup_{i\in[r-1]} G[N_{V_i}(v)]$ such that x_j and y_j are in the same vertex class $V_{i(x_jy_j)}$ for $s+1 \leqslant j \leqslant k$. As the same methods used to verify (1) and (2) of Definition 6, we can show that $e(G[V_0, V_1, \ldots, V_{r-1}]) \leqslant t_r(n) - \varepsilon n^2$ for some $\varepsilon > 0$ by finding s k-partite graphs $Y_1, Y_2, \ldots, Y_s$ with $v_j \in V(Y_j)$, and $(k-s)$ $(r-1)$-partite graphs $Y_{k-s+1}, \ldots, Y_k$ with $x_j, y_j \in V(Y_j)$, a contradiction to (33). Thus, we conclude that $(V_0, V_1, \ldots, V_{r-1})$ is a k-good partition of G, completing the proof of Theorem 8. □

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