Graph Product Structure for *h*-Framed Graphs

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Abstract

Graph product structure theory expresses certain graphs as subgraphs of the strong product of much simpler graphs. In particular, an elegant formulation states that every planar graph is a subgraph of the strong product of a path and of a bounded treewidth graph. Such formulations allow to lift combinatorial results for bounded treewidth graphs to graph classes for which the product structure holds, such as to planar graphs [Dujmović et al., J. ACM, 67(4), 22:1-38, 2020].

In this paper, we join the search for extensions of this powerful tool beyond planarity by considering the *h*-framed graphs, a graph class that includes 1-planar, optimal 2-planar, and *k*-map graphs (for appropriate values of *h*). We establish a graph product structure theorem for *h*-framed graphs stating that the graphs in this class are subgraphs of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique of size $3\lfloor\frac{h}{2}\rfloor + \lfloor\frac{h}{3}\rfloor - 1$. This allows us to improve over the previous structural theorems for 1-planar and *k*-map graphs. Our results lead to significant progress over the previous bounds on the queue number, non-repetitive chromatic number, and *p*-centered chromatic number of these graph classes, e.g., we lower the currently best upper bound on the queue number of 1-planar graphs and *k*-map graphs from 115 to 82 and from $\lfloor \frac{33}{2}(k+3\lfloor\frac{k}{2}\rfloor-3)\rfloor$ to $\lfloor \frac{33}{2}(3\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor-1)\rfloor$, respectively. We also employ the product structure machinery to improve the current upper bounds on the twin-width of 1-planar graphs from O(1) to 72. All our structural results are constructive and yield efficient algorithms to obtain the corresponding decompositions.

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1 Introduction

Graph product structure theory [19] was recently introduced and is receiving considerable attention, as it gives deep insights that allow a host of combinatorial tools to be applied [22]. Despite being a relatively new development, it is having significant impact [35]. Initially, it was introduced to settle a long-standing conjecture by Heath, Leighton and Rosenberg [26] related to the queue number of planar graphs [19]. Recently, it has been further exploited to solve several other combinatorial problems that were open for years, e.g., it was used to prove that planar graphs have bounded non-repetitive chromatic number [19], to improve the best known bounds for *p*-centered colorings of planar graphs and graphs excluding any fixed graph as a subdivision [13], to find shorter adjacency labelings of planar graphs [9, 23], and to find asymptotically optimal adjacency labelings of planar graphs [17, 25]. Moreover, a hereditary variant of product structure theory has been investigated in [30].

In its simplest form, the product structure theorem states that every planar graph is a subgraph of the strong product of a path and of a planar graph of treewidth at most 6 [19, 37]. The bound on the treewidth can be improved by allowing more than two graphs in the strong product, as it is known that every planar graph is a subgraph of the strong product of a path, of a 3-cycle, and of a planar graph of treewidth at most 3 [19] (see also [31]). These theorems are attractive since they describe planar graphs in terms of graphs of bounded treewidth, which are considered much simpler than arbitrary planar graphs. Furthermore, they enable combinatorial results that hold for graphs of bounded treewidth to be generalised for planar graphs and, more generally, for graphs where similar structural theorems can be obtained. On the algorithmic side, Bose, Morin, and Odak have recently shown that the graphs involved in the product structure theorem for planar graphs can be computed in linear time [12], improving upon [34]. More precisely, in [12], it is shown that the mapping from the input graph to a subgraph of the product can be computed in linear time. Observe that, the graphs involved in the product structure theorem for planar graphs are not prescribed a priori, but derived from the input (planar) graph; however, Biedl, Eppstein, and Ueckerdt [8] have recently shown that the problem of testing the embeddability of a graph in a host graph that is the strong product of a given path P and a given graph H is NP-complete, even when H has small pathwidth or tree-depth.

Analogous results are known for graphs of bounded Euler genus [19], apex-minor-free graphs [19], graphs with bounded degree in proper minor-closed classes [18], and graphs in non-minor closed classes [20]; see [22] for a survey. Related to our work are the structural theorems for k-planar and k-map graphs (the former ones are the graphs that can be drawn with at most k crossings per edge, whereas the latter ones are the contact-graphs of pairwise interior-disjoint regions homeomorphic to closed disks such that at most k regions may share the same point). In particular, it is known that every k-planar graph is a subgraph of the strong product of a path, of a graph of treewidth at most $\frac{1}{6}(k+4)(k+3)(k+2)-1$, and of a clique on $6(k+1)^2$ vertices [28], while every k-map graph is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $k+3\lfloor\frac{k}{2}\rfloor-3$ vertices [20].

Our contribution. In this work, our focus is on the class of *h*-framed graphs, which were recently introduced as a notable subclass of k-planar and a superclass of k-map graphs (for appropriate values of k) [4]; a graph is *h*-framed, if it admits a drawing on the Euclidean plane whose uncrossed edges induce a biconnected spanning plane graph with faces of size at most h. Since any simple h-framed graph is $O(h^2)$ -planar, it follows from the aforementioned product structure theorem for k-planar graphs that every simple h-framed graph is a subgraph of the strong product of a path, of a graph with treewidth $O(h^6)$, and of a clique of size $O(h^4)$. Dujmović, Morin and Wood [20] presented an improved product structure theorem for h-framed graphs: Every h-framed graph is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $h+3\lfloor\frac{h}{2}\rfloor-3$ vertices. In this work, we show a stronger structural result by showing that every h-framed graph is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1$ vertices, that is, our improvement is on the size of the clique¹. Recently, generalizing the result in [20], Distel et al. [15] showed that any multigraph that admits an h-framed drawing on a surface of Euler genus q is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $\max\{2g\lfloor \frac{h}{2} \rfloor, h+3\lfloor \frac{h}{2} \rfloor -3\}$ vertices. Since any planar graph is a subgraph of some triangulation (and thus of a 3-framed graph), for h = 3, Theorem 2 coincides with the product structure theorem for planar graphs proved in [19]. This is an indication that our theorem may be tight for the graphs in this family. Furthermore, we provide an alternative formulation to the one of Theorem 2, which will allow us to derive improved bounds on related problem as discuss below: Every h-framed graph is a subgraph of the strong product of the |h/2|-power of a path, a planar graph with treewidth at most 3 and a clique on $\max(3, h-2)$ vertices; see Theorem 5. All our structural results are constructive and yield efficient algorithms to obtain the corresponding decompositions. Our techniques provide improved upper bounds on the queue number, on the non-repetitive chromatic number, and on the *p*-centered chromatic number of *h*-framed graphs that are linear in h; see Theorem 13, Corollary 19, and Lemma 20, respectively. Finally, by extending the product structure machinery, we are able to give an efficient construction to obtain an explicit, linear in h, upper bound on the twin-width of h-framed graphs, while the currently best explicit upper bound derives from the one for k-planar graphs and it is hence exponential in $O(h^2)$ [10, 11]; see Theorem 22.

Consequences on related graph classes. Since 1-planar and optimal 2-planar graphs are subgraphs of 4- and 5-framed graphs, respectively [3, 7], and since k-map graphs are subgraphs of k-framed graphs [4], the product structure theorems mentioned above imply significant improvements on the currently best bounds for the following parameters (refer to Table 1). For definitions, see Section 4.

• Queue number: Using Theorem 2, we improve the best known upper bound on the queue number of k-map graphs from $\lfloor \frac{33}{2}(k+3\lfloor \frac{k}{2} \rfloor - 3) \rfloor$ [20] to $\lfloor \frac{33}{2}(3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1) \rfloor$ (Corollary 14), whereas, using Theorem 5, we lower the best known upper bounds on

¹Note that we have obtained this result completely independently and without being aware of the result in [20].

		$h ext{-framed}/h ext{-map}$	1-planar	opt 2-planar
queue num	old	$\lfloor \tfrac{33}{2}(h+3\lfloor \tfrac{h}{2} \rfloor -3) \rfloor$	115	132
	new	$\left\lfloor \frac{33}{2}(3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1) \right\rfloor$	82	82
non-repetitive chr. num	old	$4^4 \cdot (h+3\lfloor \frac{h}{2} \rfloor -3)$	1792	2048
	new	$4^4 \cdot \left(3 \lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1\right)$	1536	1536
p-centered chr. num	old	$(h+3\lfloor \frac{h}{2} \rfloor -3)(p+1)\chi_{\mathrm{p}}^{\mathrm{pl. 3-tr.}}$	$7(p+1) \chi_{\rm p}^{\rm pl. 3-tr.}$	$8(p+1) \chi_{\rm p}^{\rm pl. \ 3-tr.}$
	new	$(3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1)(p+1) \chi_{\mathrm{p}}^{\mathrm{pl. 3-tr.}}$	$6(p+1) \chi_{\rm p}^{\rm pl. \ 3-tr.}$	$6(p+1) \chi_{\rm p}^{\rm pl. \ 3-tr.}$
twin-width	old	_	O(1)	O(1)
	new	$11h + 51\sqrt{h} + 64$	72	75

Table 1: Previous [20] and new bounds on the queue number, non-repetitive and p-centered chromatic number, and twin-width for *h*-framed, 1-planar, optimal 2-planar, and *k*-map graphs. We denote by $\chi_p^{\text{pl. 3-tr.}}$ the *p*-centered chromatic number of planar 3-trees.

the queue number of 1-planar and optimal 2-planar graphs from 115 and 132 [20], respectively, both to 82 (Theorem 17).

- Non-repetitive chromatic number: Theorem 2 allows us to improve the best known upper bound on the non-repetitive chromatic number of k-map graphs from 4⁴ ⋅ (k + 3 \lfloor \frac{k}{2} \rfloor 3) [20] to 4⁴ ⋅ (3 \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor 1). In particular, for the class of 1-planar graphs our improvement is from 1792 to 1536 (Corollary 19). The latter is a bound that notably holds also for optimal 2-planar graphs, which forms an improvement over the one of 2048 that was previously known [20].
- **p**-centered chromatic number: Theorem 2 allows us to improve the best known upper bound on the non-repetitive chromatic number of k-map graphs from $(k + 3\lfloor \frac{k}{2} \rfloor 3)(p+1)\chi_{\rm p}^{\rm pl. 3-tr.}$ [20] to $(3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor 1)(p+1)\chi_{\rm p}^{\rm pl. 3-tr.}$, where $\chi_{\rm p}^{\rm pl. 3-tr.} \leq (p+1)(p\lceil \log(p+1)+2p+1)$ denotes the *p*-centered chromatic number of planar 3-trees [13]. In particular, we lower the best known upper bounds on the *p*-centered chromatic number of 1-planar and optimal 2-planar graphs from $7(p+1)\chi_{\rm p}^{\rm pl. 3-tr.}$ and $8(p+1)\chi_{\rm p}^{\rm pl. 3-tr.}$ [20], respectively, both to $6(p+1)\chi_{\rm p}^{\rm pl. 3-tr.}$ (Corollary 21).
- Twin-width: Theorem 22 provides the first explicit bound of $11h + 51\sqrt{h} + 64$ for the twin-width of (subgraphs of) *h*-framed graphs. This is best-possible up to a multiplicative constant since by [1] there exist graphs on *h* vertices having the twin-width at least (h 1)/2 for every *h*, and such graphs are trivially subgraphs of *h*-framed graphs. Specifically, consequent Corollary 29 improves the previous best upper bound on the twin-width of 1-planar and optimal 2-planar graphs, from O(1) [10] to 72 and 75, respectively (for more explicit bounds on the twin-width of these classes, refer to [11] which discusses the stronger parameter reduced bandwidth).



Figure 1: Illustration of: (a) a 4-framed topological graph whose skeleton edges are black and whose crossing edges are blue, and (b) the strong product $X \boxtimes Y$ of a planar graph X(red) and a path Y (blue).

Our improvement for k-map graphs is limited to certain value of k, as these graphs have bounded twin-width independently of k [10].

2 Preliminaries

For standard graph-theoretic terminology and notation we refer the reader, e.g., to [14].

Graphs. A graph is *simple* if it contains neither loops nor multi-edges. For a general graph G (not-necessarily simple), let si(G) denote the *simplification* of G, i.e., the simple graph obtained from G by removing all loops and replacing each set of parallel edges with a single edge. For any $i \ge 1$, the *i*-th power G^i of a graph G is the graph with the same vertex set as G, in which two vertices are adjacent if and only if they are at distance at most i in G. Clearly, $G \subseteq G^i$. A graph H is a *minor* of a graph G, if H can be obtained from a subgraph of G by contracting edges.

Topological graphs. A topological graph is a graph drawn on the plane. A plane graph is a topological graph with no crossing edges. A graph is k-planar if it is isomorphic to a topological graph in which each edge has at most k crossings. Furthermore, a k-planar graph with the maximum number of edges w.r.t. its number of vertices is called *optimal*. A k-map graph is one that admits a k-map, i.e., a representation where each vertex is a region homeomorphic to a closed disk, such that regions have pairwise disjoint interiors, no more than k regions share the same boundary point, and two regions touch if and only if the corresponding vertices are adjacent.

Given a topological graph G, the subgraph $\operatorname{sk}(G)$ of G consisting of all its vertices and uncrossed edges is the *skeleton* of G; refer to Figure 1a. A topological graph Gwhose skeleton $\operatorname{sk}(G)$ is biconnected is called *h*-framed [4], if all the faces of $\operatorname{sk}(G)$ have size at most h, and *internally h*-framed, if all the faces of $\operatorname{sk}(G)$, except for possibly one, have size at most h. The importance of this class lies in the following connections with k-planar and k-map graphs [4]. Optimal 1-planar and optimal 2-planar graphs are 4and 5-framed, respectively, while general 1-planar graphs can be augmented to 8-framed graphs, if multi-edges are forbidden, or to 4-framed graphs, if multi-edges are allowed. Finally, note that any k-map graph is a subgraph of a k-framed (multi-)graph [4] and of a 2k-framed simple graph [4, Thm. 32].

Treewidth. Let (\mathcal{X}, T) be a pair such that $\mathcal{X} = \{X_1, X_2, \ldots, X_\ell\}$ is a collection of subsets of vertices of a graph G, called *bags*, and T is a tree whose nodes are in one-to-one correspondence with the elements of \mathcal{X} . The pair (\mathcal{X}, T) is a *tree-decomposition* of G if it satisfies the following two conditions: (i) for every edge (u, v) of G, there exists a bag $X_i \in \mathcal{X}$ that contains both u and v, and (ii) for every vertex v of G, the set of nodes of T whose bags contain v induces a non-empty subtree of T. The *width* of a tree-decomposition (\mathcal{X}, T) of G is $\max_{i=1}^{\ell} |X_i| - 1$, while the *treewidth* tw(G) of G is the minimum width over all tree-decomposition of G.

Quotient graph. For a graph G and a partition \mathcal{P} of V(G), the quotient of G by \mathcal{P} , denoted by G/\mathcal{P} , is a graph containing a vertex v_P for each part P in \mathcal{P} (we say that v_P stems from P) and an edge $(v_{P'}, v_{P''})$ if and only if there exists a vertex in P' adjacent to a vertex in P'' in G. Note that, G/\mathcal{P} is a minor of G, if every part in \mathcal{P} induces a connected subgraph of G.

Strong product. The strong product of two graphs X and Y, denoted by $X \boxtimes Y$, is the graph whose vertex set $V(X \boxtimes Y)$ is the Cartesian product $V(X) \times V(Y)$, such that there exists an edge in $E(X \boxtimes Y)$ between the vertices $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in V(X \boxtimes Y)$ if and only if one of the following occurs: (a) $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$, (b) $y_1 = y_2$ and $(x_1, x_2) \in E(X)$, or (c) $(x_1, x_2) \in E(X)$ and $(y_1, y_2) \in E(Y)$; see Figure 1b. Dujmović et al. [19], Dujmović, Morin and Wood [20] and Ueckerdt, Wood, and Yi [37] showed the following main graph product structure results.

Theorem 1. Let G be a graph.

- a. If G is planar, then $G \subseteq P \boxtimes H$, for a path P and a planar graph H with $\operatorname{tw}(H) \leq 6$ [37].
- b. If G is planar, then $G \subseteq P \boxtimes H \boxtimes K_3$, for a path P and a planar graph H with $\operatorname{tw}(H) \leq 3$ [19].
- c. If G is 1-planar, then $G \subseteq P \boxtimes H \boxtimes K_7$, for a path P and a planar graph H with $\operatorname{tw}(H) \leq 3$ [20].
- d. If G is k-planar with k > 1, then $G \subseteq P \boxtimes H \boxtimes K_{18k^2+48k+30}$, for a path P and a graph H with tw $(H) \leq \frac{1}{6}(k+4)(k+3)(k+2) 1$ [20].
- e. If G is a k-map graph, then $G \subseteq P \boxtimes H \boxtimes K_{21k(k-3)}$, for a path P and a graph H with $tw(H) \leq 9$ [20].

Layering. Consider a graph G. A layering of G is an ordered partition $(V_0, V_1, ...)$ of V(G) such that, for every edge (v, w) of G with $v \in V_i$ and $w \in V_j$, it holds $|i - j| \leq 1$. If i = j, then (v, w) is an *intra-level edge*; otherwise, (v, w) is an *inter-level edge*. Each part V_i is called a *layer*. Let T be a BFS tree of G rooted at a vertex r. The BFS layering of *G* determined by *r* is the layering (V_0, V_1, \ldots) of *G* such that V_i contains all vertices of *G* at distance *i* from *r*. Given a partition \mathcal{P} of V(G) and a layering \mathcal{L} of *G*, the *layered* width of \mathcal{P} with respect to \mathcal{L} is the size of the largest set obtained by intersecting a part in \mathcal{P} and a layer in \mathcal{L} . The *layered* width of \mathcal{P} is the minimum layered width of \mathcal{P} over all layerings of *G*.

3 Computing the Product Structure

This section is devoted to the proof of a product structure theorem for h-framed graphs, summarized in the next theorem; several applications of this result are presented in Section 4.

Theorem 2 (Product Structure Theorem for *h*-Framed Graphs). Let *G* be a not-necessarily simple *h*-framed graph with $h \ge 3$. Then, $\operatorname{si}(G)$ is a subgraph of the strong product $H \boxtimes P \boxtimes K_{3|h/2|+|h/3|-1}$, where *H* is a planar graph with $\operatorname{tw}(H) \le 3$ and *P* is a path.

Note that, allowing multi-edges in the graphs supported by Theorem 2 may imply tighter bounds in the corresponding product structure for graphs that are *h*-framed but whose simplification is not. For instance, as already mentioned in Section 1, simple 1-planar graphs can always be augmented to simple 8-framed graphs and to 4-framed multigraphs. Then, Theorem 2 implies that every simple 1-planar graph is a subgraph of $H \boxtimes P \boxtimes K_{3\lfloor h/2 \rfloor + \lfloor h/3 \rfloor - 1}$, with h = 4.

The algorithm supporting Theorem 2 recursively decomposes G into parts with special properties, such that the resulting quotient graph will be H, and the additional properties of the constructed partition will imply the claimed product structure. We start with a technical setup followed by the core recursion in Lemma 4.

Layering G. Let T be a BFS tree of $\operatorname{sk}(G)$ rooted at an arbitrary vertex r incident to the unbounded face of $\operatorname{sk}(G)$. For an arbitrary G' and its implicitly fixed BFS tree T' (in our case, $G' = \operatorname{sk}(G)$ and T' = T), we call a path $P \subseteq G'$ vertical if P is a subpath of some root-to-leaf path of T'. Let $\mathcal{L} = (V_0, V_1, \ldots, V_b)$ be the BFS layering of $\operatorname{sk}(G)$ determined by r. Observe that, if P is a vertical path in $\operatorname{sk}(G)$, then P intersects every part of \mathcal{L} in at most one vertex. Given \mathcal{L} , we define a new ordered partition $\mathcal{W} = (W_0, W_1, \ldots, W_\ell)$ of the vertex set of G with $\ell = \lfloor b/\lfloor \frac{h}{2} \rfloor \rfloor - 1$, by merging consecutive $\lfloor \frac{h}{2} \rfloor$ -tuples of layers of \mathcal{L} . This is done as follows. For $i = 0, 1, \ldots, \ell$, we let $W_i := \bigcup_{j=0}^{\lfloor h/2 \rfloor - 1} V_{i \lfloor h/2 \rfloor + j}$ (assuming $V_x = \emptyset$ if x > b). Then, $\mathcal{W} := (W_0, W_1, \ldots, W_\ell)$ is a layering of G, as we prove below.

Property 3. $W := (W_0, W_1, \ldots, W_\ell)$ is a layering of G.

Proof. As for the edges of $\mathrm{sk}(G)$, we have that intra-level edges of \mathcal{L} are also intra-level edges of \mathcal{W} , whereas inter-level edges of \mathcal{L} are either intra-level edges or inter-level edges of \mathcal{W} . Next, we argue about the crossing edges of G. First, observe that each such an edge is a chord in some face of $\mathrm{sk}(G)$. Also, every chord of a face f of $\mathrm{sk}(G)$ has its ends at distance at most $\lfloor \frac{h}{2} \rfloor$ along f. This implies that G does not contain an edge with (u, v) with $u \in W_i$ and $v \in W_j$ with |i - j| > 1, which in turn implies that \mathcal{W} is a layering of G.

Partitioning G. The core of our algorithm is a construction of a special partition \mathcal{R} of V(G) such that $H = G/\mathcal{R}$ is a planar graph with $tw(H) \leq 3$, and the layered width of \mathcal{R} with respect to \mathcal{W} is not large. Our recursive decomposition of G is analogous to the one in [19] (as applied to planar graphs); however several non-trivial changes are needed to exploit the existence of the underlying (plane) skeleton of G. The algorithm starts from the unbounded face and recursively "dives" into gradually-shrinking areas of G.

Central in our approach is the following notion. For a cycle $C \subseteq \text{sk}(G)$, the subgraph of G bounded by C, denoted by G_C , is the subgraph of G formed by the vertices and edges of C and the vertices and edges of G drawn inside C. Consider a subset $U \subseteq V(G)$. For the partition \mathcal{L} (resp., the partition \mathcal{W}), the width of U with respect to \mathcal{L} (resp. to \mathcal{W}), denoted by $\lambda_{\mathcal{L}}(U)$ (resp. by $\lambda_{\mathcal{W}}(U)$), is the largest size of a set obtained by intersecting U and a part of \mathcal{L} (resp. of \mathcal{W}). We are now ready to present our main technical lemma.

Lemma 4. Let G be an n-vertex h-framed graph with $h \ge 3$ and let \mathcal{L} be a BFS layering of $\mathrm{sk}(G)$. Also, let C be a cycle in $\mathrm{sk}(G)$, and let G_C be the subgraph of G bounded by C. Further, for some $k \in \{1, 2, 3\}$, let P_1, \ldots, P_k be paths belonging to C such that $\mathcal{R}^0 = \{X_i : X_i := V(P_i), 1 \le i \le k\}$ is a partition of V(C). Then, it is possible to construct in $O(n^2)$ time a good partition \mathcal{R}' of $V(G_C)$, i.e., one that satisfies the following properties:

- 1. $\mathcal{R}' \supseteq \mathcal{R}^0$, and for every part $X \in \mathcal{R}' \setminus \mathcal{R}^0$, there exist $q \in \{1, 2, 3\}$ and $X' \subseteq X$ such that²
 - $X \setminus X'$ is a union of the vertex sets of at most q vertical paths of sk(G), and so, in particular, $\lambda_{\mathcal{L}}(X \setminus X') \leq q$, and
 - $|X'| \le h 3$ if q = 1, $|X'| \le \lfloor (h 1)/2 \rfloor 1$ if q = 2, and $|X'| \le \lfloor h/3 \rfloor 1$ if q = 3.
- 2. the quotient graph $H' = G_C/\mathcal{R}'$ is a planar graph with $tw(H') \leq 3$, and
- 3. the vertices of H' that stem from X_i , with $1 \le i \le k$, are incident to the same face of H' and induce a clique (i.e., either a vertex, or an edge, or a triangle).

Proof of Theorem 2. Let C denote the cycle bounding the unbounded face of sk(G), which, by a possible homeomorphism of the sphere, may be assumed to satisfy $|V(C)| \ge 3$. Based on the BFS tree T of sk(G) rooted in a vertex $r \in V(C)$, we define the following partition \mathcal{R}^0 of C: We split C into a path P_1 only consisting of the vertex r, and two paths P_2 and P_3 of lengths at most $\lfloor \frac{h-1}{2} \rfloor$ and $\lfloor \frac{h}{2} \rfloor$, respectively. Then, we set $\mathcal{R}^0 = \{V(P_1), V(P_2), V(P_3)\}$ and apply the algorithm given in the proof of Lemma 4. This way we obtain a good partition \mathcal{R}' of $V(G_C) = V(G)$ and graph $H' := G_C/\mathcal{R}'$ in $O(|V(G)|^2)$ time.

Note that, in general, $G_C \neq G$ as G may have edges drawn in the unbounded face (bounded by C) of sk(G). However, by setting H = H', we guarantee all edges of G in the unbounded face of sk(G) are "captured", since the quotient graph H' anyway contains a

²Property 1 of Lemma 4 will imply that $\lambda_{\mathcal{W}}(X) \leq 3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1$ in the proof of Theorem 2, but we will also make use of the stated more detailed treatment.

triangle on the vertices that stem from \mathcal{R}_0 . In fact, we have just obtained the graph H with the desired properties, i.e., H is planar and of tw $(H) \leq 3$.

What remains to prove is that G indeed is a subgraph of the strong product $H \boxtimes P \boxtimes K_{3\lfloor h/2 \rfloor + \lfloor h/3 \rfloor - 1}$ for some path P. Recall that the number of layers of the layering \mathcal{W} is $\ell + 1$, and that \mathcal{W} was obtained by merging consecutive $\lfloor \frac{h}{2} \rfloor$ -tuples of layers of \mathcal{L} . We set P to be the path on $\ell + 1$ vertices denoted in order by p_0, p_1, \ldots, p_ℓ . To a vertex $v \in V(G)$, we assign the pair (t, p_i) where $t \in V(H)$ if t stems from the part of \mathcal{R}' that v belongs to, and $v \in W_i \in \mathcal{W}$. This assignment is sound and unique.

If $vv' \in E(G)$ is any edge of G, and v and v' are assigned the pairs (t, p_i) and (t', p_j) as above, then $tt' \in E(H)$ or t = t' since $H = G/\mathcal{R}'$ is the quotient graph, and $p_i p_j \in E(P)$ or i = j since \mathcal{W} is a layering of G. Using Property 1 of Lemma 4, we furthermore estimate, for every part $X \in \mathcal{R}'$ and its $X' \subseteq X$ (cf. Property 1),

$$\begin{aligned} \lambda_{\mathcal{W}}(X) &\leq |X'| + \lambda_{\mathcal{L}}(X \setminus X') \cdot \lfloor h/2 \rfloor \\ &\leq \max\left(h - 3 + \lfloor h/2 \rfloor, \lfloor h/2 \rfloor - 1 + 2\lfloor h/2 \rfloor, \lfloor h/3 \rfloor - 1 + 3\lfloor h/2 \rfloor\right) \\ &\leq 3\lfloor h/2 \rfloor + \lfloor h/3 \rfloor - 1 \,, \end{aligned}$$

and hence at most $3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1$ vertices of G are assigned to the same pair (t, p_i) . This concludes the proof that $\operatorname{si}(G) \subseteq H \boxtimes P \boxtimes K_{3\lfloor h/2 \rfloor + \lfloor h/3 \rfloor - 1}$. \Box

We next present a variant of Theorem 2, which reduces the size of the clique in the product by replacing the path with a power of it. The variant does not immediately follow from the statement of Theorem 2. However, it can easily be derived by adopting in the proof of Theorem 2 the layering \mathcal{L} instead of the layering \mathcal{W} .

Theorem 5. Let G be an h-framed graph (where G is not necessarily simple). Then si(G) is a subgraph of the strong product of three graphs $H \boxtimes P^{\lfloor h/2 \rfloor} \boxtimes K_{\max(3,h-2)}$, where H is a planar graph with $tw(H) \leq 3$ and P is a path.

Proof. Recall that $\mathcal{L} = (V_0, V_1, \ldots, V_b)$ is a BFS layering of the skeleton $\mathrm{sk}(G)$, and thus every edge of G has ends in parts $V_i, V_j \in \mathcal{L}$ such that $|i - j| \leq \lfloor h/2 \rfloor$. Hence, we may choose P as the path on b + 1 vertices (p_0, p_1, \ldots, p_b) , use $P^{\lfloor h/2 \rfloor}$, and assign each vertex $v \in V(G)$ to the pair (t, p_i) where $t \in V(H)$ if t stems from the part of \mathcal{R}' that v belongs to, and $v \in V_i \in \mathcal{L}$. Now, the number of vertices of G assigned to the same pair (t, p_i) (where t stems from a part X) is at most $\lambda_{\mathcal{L}}(X) \leq |X'| + \lambda_{\mathcal{L}}(X \setminus X') \leq$ $\max(h - 3 + 1, \lfloor h/2 \rfloor - 1 + 2, \lfloor h/3 \rfloor - 1 + 3) = \max(3, h - 2)$. This concludes that $\mathrm{si}(G) \subseteq H \boxtimes P^{\lfloor h/2 \rfloor} \boxtimes K_{\max(3,h-2)}$.

3.1 Proof of Lemma 4

We prove Lemma 4 by providing a recursive procedure that we describe in the following. The base case of the recursion occurs when $V(G_C) = V(C)$ (i.e., there are no vertices in the interior of C and the edges in $E(G_C) \setminus E(C)$ are chords of C). In this case, the algorithm returns the partition $\mathcal{R}' = \mathcal{R}^0$, which is clearly good since the graph H' is a plane clique



Figure 2: Illustrations of graph $C \cup D$ for the separable case of G_C (Definition 6). The number a of τ -faces is 4 in (a) and 1 in (b). (c) A case when Property 2 of Definition 6 is not met (by τ_4).

of size k whose vertices stem from the parts of \mathcal{R}^0 . Note that, if $|E(G_C) \setminus E(C)| = 0$, then $V(G_C) = V(C)$, since G_C cannot have isolated vertices.

In the recursive step of the algorithm, we assume that there exist vertices and edges of G_C that lie in the interior of C. Our aim is to recurse on instances that contain fewer edges in the interior, but not on the boundary, of the cycle delimiting their unbounded face. We first need to handle a possible degenerate case³ of G_C . Recall that, since G is h-framed, all bounded faces of $G_C \cap \text{sk}(G)$ have length at most h.

Definition 6. We say that G_C is *separable* if the following conditions hold (see Figure 2):

- 1. The plane graph $G_C \cap \text{sk}(G)$ contains a bounded face σ_0 that intersects C in $a \ge 1$ disjoint maximal subpaths (some of the paths may consist of single vertices). Denoting by D the facial cycle of σ_0 , let τ_1, \ldots, τ_a be the bounded faces of the plane graph $C \cup D$ other than σ_0 .
- 2. For each τ_j , with j = 1, ..., a, the boundary of τ_j is a cycle C_j of sk(G), at most two parts of \mathcal{R}^0 intersect C_j , and every part of \mathcal{R}^0 intersecting C_j does so in a single subpath.⁴

Separable case. Suppose that G_C is separable and let G_j , j = 1, ..., a, denote the subgraph of G_C bounded by the facial cycle C_j of τ_j^5 . By definition, we have that $|E(G_j) \setminus E(C_j)| \subseteq |E(G_C) \setminus E(C)|$ (even when a = 1). Since $E(C_j) \setminus E(C) \neq \emptyset$, the latter implies $|E(G_j) \setminus E(C_j)| < |E(G_C) \setminus E(C)|$. Also, let Y denote the vertices of D that do not belong to C, i.e., $Y = V(D) \setminus V(C)$; refer to the hollow vertices of Figure 2. By the previous, we have $|Y| \leq |D| - 2 \leq h - 2$.

³Such a case does not explicitly occur in the original proof of [19], but the implicit case of a so-called [19] "tripod" with degenerate legs is analogous to what we are defining here.

⁴A part of \mathcal{R}^0 indeed may intersect the boundary C_j of τ_j in two subpaths (see Figure 2c). This, however, can happen only if $k \leq 2$.

⁵Note that, if a = 1, we have $|V(D) \cap V(C)| \ge 2$, since otherwise the face τ_1 would not be bounded by a cycle as required by Property 2 of Definition 6.

For j = 1, ..., a, let \mathcal{R}_j^0 be the partition of $V(C_j)$ consisting of the set $Y_j = Y \cap V(C_j)$, if it is not empty, and of the sets $X_i^j = X_i \cap V(C_j)$, i = 1, ..., k, if X_i^j is not empty; by Property 2 of Definition 6, \mathcal{R}_j^0 consists of at most three parts. Therefore, by a recursive application of our algorithm, each graph G_j , j = 1, ..., a, admits a good partition $\mathcal{R}_j \supseteq \mathcal{R}_j^0$ of $V(G_j)$.

We construct a partition \mathcal{R}' of $V(G_C)$ by putting into \mathcal{R}' the parts of \mathcal{R}_0 , the set Y (if non-empty), and the recursively obtained parts of each G_j that do not touch C_j ; formally, $\mathcal{R}' = \mathcal{R}_0 \cup \{Y\} \cup \bigcup_{j=1,\dots,a} (\mathcal{R}_j \setminus \mathcal{R}_j^0)$, or $\mathcal{R}' = \mathcal{R}_0 \cup \bigcup_{j=1,\dots,a} (\mathcal{R}_j \setminus \mathcal{R}_j^0)$ if $Y = \emptyset$. Note that \mathcal{R}' is indeed a partition of $V(G_C)$, since each vertex of G_C that lies in the interior of C must belong either to Y or to a part $X \in \mathcal{R}_j \setminus \mathcal{R}_j^0$, for some $j \in \{1, \dots, a\}$. In the following, we show that the constructed partition \mathcal{R}' is good. To this aim, we will exploit the next property.

Property 7. Under the conditions of Lemma 4, no vertex of any part from $\mathcal{R}' \setminus \mathcal{R}^0$ is adjacent to a vertex of $V(G) \setminus V(G_C)$.

Proof. Since \mathcal{R}^0 partitions V(C), and C is a cycle in sk(G), the drawing of C is an uncrossed simple closed curve in the topological graph G, and hence, by the Jordan curve theorem, no edge of G can have one end in $V(G_C) \setminus V(C)$ and the other end in $V(G) \setminus V(G_C)$.

Claim 8. The partition \mathcal{R}' constructed for the separable case of G_C is good.

Proof. For our proof, we additionally use Property 7 mentioned above, which we can further assume that it holds also for the recursive calls. First, we look at Property 1 of Lemma 4: this property holds true for every $X \in \mathcal{R}' \setminus (\mathcal{R}_0 \cup \{Y\})$ by the recursive construction, and for X = Y we pick any $y \in Y$ and set $Y' = Y \setminus \{y\}$, and by the previous we get $|Y'| \leq |Y| - 1 \leq h - 2 - 1 = h - 3$ and $Y \setminus Y' = \{y\}$ is a trivial vertical path.

It remains to analyze planarity and treewidth of the quotient graph H'. Recall that we have recursively obtained the planar quotient graphs $H_j := G_j/\mathcal{R}_j$, for $j \in \{1, \ldots, a\}$, and we may assume, from the recursive invocation of Property 3, that each H_j is a plane topological graph with the vertices stemming from the parts of \mathcal{R}_j^0 on the unbounded face. For further reference, we call these vertices stemming from \mathcal{R}_j^0 the *connectors* of H_j . By Property 7 following a recursive invocation of Lemma 4, no vertices of H_j other than the connectors will be adjacent to vertices of H' outside of H_j .

We start with drawing a plane (k + 1)-clique Q on the vertices stemming from the nonempty parts in $\mathcal{R}^0 \cup \{Y\}$. If k = 2, then at most two of the graphs G_j with $j \in \{1, \ldots, a\}$ intersect both parts of \mathcal{R}^0 , and for them we embed the corresponding (at most two) plane graphs H_j into the two triangular faces of Q, such that the vertices of Q are naturally identified with the connectors of H_j . The remaining quotient graphs H_j are then embedded into the drawing easily, since the connectors of each of them is identified with only one or two vertices of Q. Likewise, the desired plane drawing of H' is trivial if k = 1. If k = 3, then each pair of parts of \mathcal{R}^0 is intersected together by at most one of the graphs G_j , and then we embed the plane quotient graph H_j into the corresponding triangular face of Q,



Figure 3: Illustrations of graph $C \cup D$ for the general case of G_C , for (a) k = 1 and (b-c) k = 2.

with the appropriate identification of the connectors of H_j as in the case of k = 2. Again, the remaining quotient graphs H_j are embedded easily.

Altogether, we have obtained a plane drawing of H' such that the vertices stemming from the parts of \mathcal{R}^0 are on the same face; in particular, for k = 3, this is the triangular face of Q not containing the vertex which stems from the part Y. Moreover, the k parts of \mathcal{R}^0 are indeed pairwise adjacent in G_C already by edges of C. We have thus verified Property 3 of Lemma 4, and it remains to verify Property 2 in the aspect of treewidth of H'. We have recursively obtained, for each $j \in \{1, \ldots, a\}$, a tree-decomposition \mathcal{T}_j of the graph H_j , such that (by a folklore property of tree-decompositions) the clique of the vertices which stem from \mathcal{R}_j^0 is contained in a node ν_j of it. We create a new decomposition \mathcal{T} of H' from the disjoint union of \mathcal{T}_j over $j = 1, \ldots, a$ by adding a new node ν , holding the bag of vertices V(Q) and adjacent exactly to all ν_j . Further, for $i = 1, \ldots, k$, we rename each vertex of \mathcal{T}_j that stems from X_i^j , with $i = 1, \ldots, a$, as the vertex in Q that stems from X_i and, for $i = 1, \ldots, a$, we rename each vertex of \mathcal{T}_j that stems from Y_i as the vertex in Q that stems from Y. This is a valid tree-decomposition by Property 7, and it is of width at most 3 by Property 2 and the fact that $|V(Q)| - 1 = k \leq 3$.

General case. Now we move to the general case of G_C in Lemma 4. If $\mathbf{k} = \mathbf{1}$, we pick the bounded face σ_0 of $G_C \cap \text{sk}(G)$ incident to the single edge of $E(C) \setminus E(P_1)$; refer to Figure 3a. The face σ_0 then witnesses the separable case for G_C , by Definition 6, which is solved as above. If $\mathbf{k} = \mathbf{2}$, then we pick $e_0 \in E(C)$ as one of the edges joining P_1 and P_2 on C, and σ_0 as the bounded face of $G_C \cap \text{sk}(G)$ incident to e_0 ; see Figures 3b and 3c. Then we are back to the separable case for G_C with σ_0 , by Definition 6.

In the remainder, we assume $\mathbf{k} = \mathbf{3}$. First, we color every vertex v of G_C by the color $i \in \{1, 2, 3\}$ if the (unique) path in the BFS tree T from v to the root r hits $V(P_i)$ before possibly hitting other parts of \mathcal{R}^0 . In particular, the vertices of P_i are colored i. Our aim is to find, in the plane graph $F := G_C \cap \operatorname{sk}(G)$, a bounded face σ_1 containing vertices of all the three colors on its boundary. There our arguments divert from those used in [19]—since F is generally not a near-triangulation, and we additionally need that the face σ_1 intersects the boundary cycle C at most once (which requires additional care). We exploit the following.

The electronic journal of combinatorics 31(4) (2024), #P4.56



Figure 4: Illustrations for the proof of Claim 9. (a) Graph F and a zone F_1 . (b) The tri-colored triangle F_3 . (c) The desired face σ_1 in F.

Claim 9. In the setting above, there exists a cycle R bounding a bounded face σ_1 of F, such that V(R) contains all three of our colors, and R intersects C in at most one connected piece. Furthermore, the colors on R appear in three consecutive sections.

Proof. Our proof exploits the next definition.

Definition 10. A plane subgraph $F_1 \subseteq F$ is a *zone of* F if the following hold (see Figure 4a): (i) every vertex in $V(F_1) \cap V(C)$ is of degree at least 2 in F_1 , (ii) no vertex in $V(F) \setminus V(F_1)$ is adjacent to a vertex in $V(F_1) \setminus V(C)$, and (iii) in the plane drawing of F_1 inherited from G_C , every vertex from $V(F) \setminus V(F_1)$ is drawn in the unbounded face of F_1 .

Note that, similarly to Definition 6, if a facial cycle D in F intersects C in two or more places, then each of the remaining bounded faces of the plane subgraph $C \cup D$ bounds a zone of F. We say that the face of D divides F into these zones.

For our proof, we pick F_1 as an inclusion-minimal zone of F such that $V(F_1)$ intersects all three parts of \mathcal{R}^0 (i.e., F_1 meets all three colors on C). We consider the graph $F_2 := C \cup F_1$, and denote by $W_i \subseteq V(F_2)$, where $i \in \{1, 2, 3\}$, the set of vertices of F_2 colored i. By Definition 10, each set W_i induces a connected subgraph in F_2 , and so contracting each of W_1, W_2, W_3 into one vertex results in a tri-colored triangle F_3 which is a planar minor of F_2 ; see Figure 4b. By planarity of F_2 , there is a face σ_1 of F_2 such that the facial cycle $R \subseteq V(F)$ of σ_1 contracts down to F_3 (R is a cycle since the skeleton sk(G) is 2-connected); see Figure 4c. If R intersected C in two or more places, then Rwould divide F_2 into strictly smaller zones which, by the minimality of F_1 , each intersects at most two parts of \mathcal{R}^0 . However, the same then holds for the zones into which σ_1 divides the whole F, and so such σ_1 witnesses the case of separable G_C already solved, as can be easily checked from Definition 6.

Therefore, we have that V(R) contains all three of our colors, and R intersects C in at most one connected piece. Furthermore, the colors on R appear in three consecutive sections since the paths of the BFS tree T do not cross in the plane graph F. This concludes the proof.

Next, consider the set $V(R) \cap V(C)$. If this set contains all three colors, then all three colors occur on the path $R_0 := C \cap R$, and one of them, say color 1, occurs only on internal

vertices of R_0 (and nowhere else on C). In this case, the face σ_1 again witnesses the case of separable G_C with a = 1 which is solved as above. If, instead, the set $V(R) \cap V(C)$ does not contain all three colors, then we choose on R representatives – vertices $t_i \in V(R)$ of color i where i = 1, 2, 3, as in one of the following three possible cases of $V(R) \cap V(C)$ (refer to Figure 5):

- **C.1** If $V(R) \cap V(C)$ contains two colors, say 1 and 2, we choose $t_1, t_2 \in V(R) \cap V(C)$ as neighbors on C and $t_3 \in V(R) \setminus V(C)$ arbitrarily; refer to Figure 5a.
- **C.2** If $V(R) \cap V(C)$ contains one color, say 1, we choose $t_1 \in V(R) \cap V(C)$ arbitrarily and pick $t_2, t_3 \in V(R) \setminus V(C)$ such that $t_2t_3 \in E(R)$ (this is unique). Furthermore, up to symmetry between the colors 2 and 3, we may assume that the distance on Rbetween t_2 and V(C) is not smaller than the distance on R between t_3 and V(C); refer to Figure 5b.
- **C.3** If $V(R) \cap V(C) = \emptyset$, then, up to symmetry between the colors, we may assume that the color 3 occurs in V(R) no more times than each of the colors 1 and 2. We then choose $t_3 \in V(R)$ arbitrarily (of color 3), and set t_1 and t_2 to the two (unique)



Figure 5: Illustrations for the representatives t_1 , t_2 , and t_3 , and the vertical paths R_1 , R_2 , and R_3 . The vertices of R bound the gray shaded region. R' is depicted with black thick edges.

THE ELECTRONIC JOURNAL OF COMBINATORICS 31(4) (2024), #P4.56

vertices colored 1 and 2 on R that are neighbors of vertices of color 3 on R; refer to Figure 5c.

For i = 1, 2, 3, let R_i denote the unique vertical path in T from t_i to $V(P_i)$; see Figure 5. Note that some vertices t_i may lie on C, and then $R_i = t_i$ is a single-vertex path. Let Q be the subpath of R with the ends t_1 and t_2 and avoiding t_3 . We define $R' \subseteq R$ as the subpath or cycle (in the case R' = R) obtained from R by deleting all internal vertices of Q. Finally, we set $R^+ := R' \cup R_1 \cup R_2 \cup R_3$ which is a connected subgraph of $F(R^+$ will play the same role here as the so-called tripods in [19]).

Observe that $C \cup R^+$ is a 2-connected plane graph (in each of the three cases above). Moreover, it contains $a \in \{2, 3\}$ bounded faces τ_1, \ldots, τ_a , plus the bounded face σ_1 in the case of R' = R; for the latter see Figure 5c. We denote by C_j , $j \in \{1, \ldots, a\}$, the facial cycle of τ_j . It is now important to notice that each cycle C_j intersects at most two parts of \mathcal{R}^0 , which follows from our "multi-colored" choice of t_1, t_2, t_3 and R_1, R_2, R_3 in all three cases. Furthermore, every two parts of \mathcal{R}^0 are together intersected by at most one of C_j .

We next proceed similarly as in the separable case above. Let $G_j \subsetneq G_C$, $j \in \{1, \ldots, a\}$, be the strict subgraph of G_C bounded by C_j , and let \mathcal{R}_j^0 be the partition of $V(C_j)$ consisting of $V(C_j) \setminus V(C)$ and of the non-empty parts $X \cap V(C_j)$ over $X \in \mathcal{R}^0$. So, $|\mathcal{R}_j^0| \leq 3$. Therefore, by a recursive application of our algorithm, we may assume that each graph G_j admits a good partition $\mathcal{R}_j \supseteq \mathcal{R}_j^0$ of $V(G_j)$, with $j = 1, \ldots, a$.

We construct a partition $\mathcal{R}' \supseteq \mathcal{R}^0$ of $V(G_C)$ similarly as before; besides \mathcal{R}^0 we add the set $Z := V(R^+) \setminus V(C) \neq \emptyset$ as whole, and the recursively obtained parts of each G_j that do not touch C_j . Formally, $\mathcal{R}' = \mathcal{R}_0 \cup \{Z\} \cup \bigcup_{j=1,\dots,a} (\mathcal{R}_j \setminus \mathcal{R}_j^0)$. Note that \mathcal{R}' is a partition of $V(G_C)$ —in particular, each vertex of G_C which is not on C must belong either to Z or to a part $X \in \mathcal{R}_j \setminus \mathcal{R}_j^0$, for some $j \in \{1, \dots, a\}$, by induction. In the following, we show that the constructed partition \mathcal{R}' is good.

Claim 11. The partition \mathcal{R}' constructed for the general case of G_C is good.

Proof. Property 1 of Lemma 4 holds true for every $X \in \mathcal{R}' \setminus (\mathcal{R}_0 \cup \{Z\})$ by recursion. For X = Z we argue as follows. We choose $Z' := Z \setminus V(R_1 \cup R_2 \cup R_3) \subseteq R'$, and argue according to the Cases (C.1)–(C.3) that we distinguished for $V(R) \cap V(C)$. In the Case (C.1), i.e., $t_1, t_2 \in V(C)$, we have $V(R_1 \cup R_2) \cap Z = \emptyset$, and so $Z \setminus Z' = V(R_3)$ where R_3 is vertical in sk(G) and $\lambda_{\mathcal{L}}(Z \setminus Z') = 1$. In this case we also have $|Z'| \leq |R| - 3 \leq h - 3$, as desired. In the Cases (C.2) and (C.3), we similarly have that $Z \setminus Z'$ is made of 2 and 3 vertical paths, respectively, and the bounds on $\lambda_{\mathcal{L}}(Z \setminus Z')$ follow from that. In the Case (C.2) we also get $|Z'| \leq \lfloor \frac{1}{2}(|V(R)| - 1) \rfloor - 1 \leq \lfloor \frac{1}{2}(h - 1) \rfloor - 1$ as desired, since the distance from t_3 to V(C) on R is not more than $\frac{1}{2}(|V(R)| - 1)$. In the Case (C.3) we have that there are at most $\frac{1}{3}|V(R)|$ vertices of color 3 on R (which stay in Z' except the end of R_3), and so $|Z'| \leq \lfloor \frac{1}{3}|V(R)| \rfloor - 1 \leq \lfloor \frac{h}{3} \rfloor - 1$.

We now turn the attention to the quotient graph H := H'. Recall that we have recursively obtained the planar quotient graphs $H_j := G_j/\mathcal{R}_j$, for $j \in \{1, \ldots, a\}$, and we may assume, from the recursive invocation of Property 3, that each H_j is a plane topological graph with the vertices stemming from the parts of \mathcal{R}_j^0 on the unbounded face. We again call these vertices stemming from \mathcal{R}_j^0 the *connectors* of H_j , and we have from Property 7 that no vertices of H_j other than the connectors are adjacent to vertices of Houtside of H_j .

As previously, the graph H consists of a 4-clique Q on the vertices which stem from the four parts of $\mathcal{R}^0 \cup \{Z\}$, and of the union of the graphs H_j , with $j = 1, \ldots, a$, after identification of their connectors with the vertices of Q. Let z be the vertex of Q that stems from the part Z and w_1, w_2, w_3 be the vertices which stem from the parts X_1, X_2, X_3 of \mathcal{R}^0 . As already noted, none of the graphs H_j contains all three w_1, w_2, w_3 (as its connectors), and for every pair from $\{w_1, w_2, w_3\}$, say w_c and w_d , at most one of $H_j, j \in \{1, \ldots, a\}$, contains both w_c, w_d among its connectors. In such case, H_j is to be embedded in the triangular face $\{w_c, w_d, z\}$ of Q. Furthermore, if some $H_j, j \in \{1, \ldots, a\}$, contains only one of w_1, w_2, w_3 as its connector, say w_b , then H_j can be embedded in any of the two triangular faces of Q incident to the edge $\{w_b, z\}$.

Altogether, we have obtained a plane drawing of $H = G_C/\mathcal{R}'$ such that the vertices w_1, w_2, w_3 stemming from the parts of \mathcal{R}^0 are on the same triangular face. We have thus verified Property 3, and it remains to verify Property 2 in the aspect of treewidth of H. Again, we have recursively obtained a tree-decomposition \mathcal{T}_j of H_j for every $j \in \{1, \ldots, a\}$, such that the clique of the vertices which stem from \mathcal{R}_j^0 is contained in a node ν_j of it. We create a new decomposition \mathcal{T} of H from the disjoint union of \mathcal{T}_j over $j = 1, \ldots, a$ by adding a new node ν , holding the bag of vertices V(Q) and adjacent exactly to all ν_j . Further, for $i = 1, \ldots, k$, we rename each vertex of \mathcal{T}_j that stems from X_i^j , with $i = 1, \ldots, a$, as the vertex in Q that stems from X_i and, for $i = 1, \ldots, a$, we rename each vertex of \mathcal{T}_j that stems from Y. This is a valid tree-decomposition by Property 7, and it is of width 3 by Property 2 and the fact that |V(Q)| - 1 = 3.

We conclude the proof of Lemma 4 by discussing the time complexity of our algorithm, which follows the same ideas as the ones by Dujmović et al. [19] to compute the decomposition deriving from their product structure theorem for *n*-vertex planar graphs⁶. To show that a good partition of G_C can be obtained in $O(|V(G_C)|^2)$ time, it suffices to observe that the non recursive work needed to compute the graphs on which the recursive calls are applied can be easily implemented to run in $O(|V(G_C)|)$ time, by performing a visit of the planar skeleton of the input *h*-framed graph and of its BFS tree (provided that G_C is a topological *h*-framed graph). Since the total number of recursive calls is at most linear in $|V(G_C)|$, the total running time is thus quadratic in $|V(G_C)|$.

4 Consequences of the Product Structure

As mentioned in the introduction, Dujmović, Morin and Wood [20] have derived upper bounds on the queue number, on the non-repetitive chromatic number, and on the p-

⁶Note that subsequent improvements have brought the running time of this procedure first to $O(n \log n)$ [34] and finally to O(n) [12].

centered chromatic number of k-planar and k-map graphs exploiting Theorem 1. In the following, we present our improvements to each of these problems.

4.1 Queue number

A queue layout of a graph G is a linear order σ of the vertices of G together with an assignment of its edges to sets, called queues, such that no two edges in the same set nest. The queue number qn(G) of a graph G is the minimum number of queues over all queue layouts of G. Dujmović et al. [19] proved the following useful lemma concerning the queue number of graphs that can be expressed as subgraphs of the strong product of a path P, a graph H, and a clique K_{ℓ} on ℓ vertices.

Lemma 12 (Dujmović et al. [19]). If $G \subseteq P \boxtimes H \boxtimes K_{\ell}$ then $qn(G) \leq 3\ell qn(H) + \lfloor \frac{3}{2}\ell \rfloor$.

Combining Lemma 12 and Theorem 1(d), together with the fact that the queue number of planar 3-trees is at most 5 [2], Dujmović, Morin, and Wood showed the first constant upper bound on the queue number of k-planar graphs [20], thus resolving a long-standing open question. Analogously, by combining Lemma 12 and Theorem 2, we obtain the following.

Theorem 13. The queue number of h-framed graphs is at most

$$\lfloor \frac{33}{2} (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1) \rfloor.$$

Dujmović et al. [19] first showed the queue number of k-map graphs is at most $2(98(k+1))^3$. Later, by combining Theorem 1(e) and Lemma 12, Dujmović et al. [20] improved this bound to 32225k(k-3). More recently, Dujmović, Morin and Wood [20] have also observed that k-map graphs are k-framed and have exploited this observation to further improve this bound to $\lfloor \frac{33}{2}(k+3\lfloor \frac{k}{2} \rfloor - 3) \rfloor$. By Theorem 13, we can further improve these bounds by also leveraging the fact that these graphs are subgraphs of k-framed graphs [4].

Corollary 14. The queue number of k-map graphs is at most

$$\lfloor \frac{33}{2} \left(3 \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1 \right) \rfloor.$$

For $h \in \{4, 5\}$, Theorem 13 gives us an upper bound of 95. Since any 1-planar graph can be augmented to a (not-necessarily simple) 4-framed graph [3], Theorem 13 improves the currently best upper bound of 1-planar graphs from 115 [20] to 95. Since any optimal 2-planar graph is 5-framed, Theorem 13 improves the currently best upper bound on their queue number from 132 [20] to 95. Next, we show a generalization of Lemma 12 that allows further improvements.

Lemma 15. If $G \subseteq H \boxtimes P^i \boxtimes K_\ell$ then $qn(G) \leq i\ell + (2i+1)\ell qn(H) + \lfloor \frac{\ell}{2} \rfloor$.

The electronic journal of combinatorics 31(4) (2024), #P4.56

Proof. For convenience, let $\mathcal{X}_j = H \boxtimes P^j \boxtimes K_\ell$, with $1 \leq j \leq i$. Observe that the graphs \mathcal{X}_j , with $1 \leq j \leq i$, have the same vertex set and $\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$, with $1 \leq j < i$.

Let $P = (p_1, p_2, \ldots, p_z)$ and let $\langle x_1, x_2, \ldots, x_q \rangle$ be the vertex ordering of a qn(H)-queue layout of H. We set $V_{a,b} := \{v_{a,b}\} \times V(K_\ell)$, where $v_{a,b}$ denotes the vertex of $V(H) \times V(P)$ that stems from the vertex x_a of H and the vertex p_b of P. Note that, the sets $V_{a,b}$ form a partition of $V(\mathcal{X}_1)$. The following property follows from the proof of Lemma 12 given in [19].

Property 16 (Vertex order of Lemma 12). The queue layout of \mathcal{X}_1 in the proof of Lemma 12 is such that, for any two vertices $u \in V_{a,b}$ and $v \in V_{c,d}$, it holds that u precedes v in such a layout if and only if one of the following holds: Either b < d or b = d and a < c.

Our proof is by induction on *i*. In particular, we will show that \mathcal{X}_i has a queue layout whose vertex order σ satisfies Property 16 and uses at most $i\ell + (2i+1)\ell qn(H) + \lfloor \frac{\ell}{2} \rfloor$ queues. In the base case i = 1 and the result follows directly from Lemma 12.

Assume now that i > 1. Let Δ_i be the graph obtained by removing from P^i the edges that it shares with P^{i-1} , i.e., $\Delta_i = (V(P), E(P^i) \setminus E(P^{i-1}))$. Clearly, two vertices are adjacent in Δ_i if and only if they are at distance i in P.

Observe now that \mathcal{X}_i is the union of \mathcal{X}_{i-1} and $H \boxtimes \Delta_i \boxtimes K_\ell$. By induction, \mathcal{X}_{i-1} admits a queue layout Γ whose vertex order satisfies Property 16 and uses at most $(i-1)\ell + (2i-1)\ell \operatorname{qn}(H) + \lfloor \frac{\ell}{2} \rfloor$ queues. Therefore, in order to prove the statement, it suffices to show that the edges of $H \boxtimes \Delta_i \boxtimes K_\ell$ can be added in Γ by using at most $\ell + 2\operatorname{qn}(H)\ell$ queues. To this aim, we classify the edges of this graph into three sets $E_{|}$, E_n , and $E_{/}$. Namely, for each edge (u, v) with $u \in V_{a,b}$ and $v \in V_{c,d}$, we have that:

- $(u, v) \in E_{|}$, if b = d;
- $(u, v) \in E_n$, if b < d and a < c; and
- $(u, v) \in E_{/}$, if b < d and a > c.

First, we show that the edges in $E_{|}$ can be assigned to at most ℓ queues. For this, we recall that the number of queues in a queue layout coincides with the size of its largest rainbow [27], where a *rainbow* is a set of pairwise-nesting independent edges in a linear order of the vertices. Namely, let (u, v) and (u', v') be two edges in $E_{|}$ with $u \in V_{a_1,b}$, $v \in V_{a_2,b}, u' \in V_{a'_1,b'}$ and $v' \in V_{a'_2,b'}$. Assuming w.l.o.g. that $a_1 \leq a'_1$ holds, it follows that these two edges nest in σ , only if b = b', $a_1 = a'_1$, and $a_2 = a'_2$. Since each of the sets $V_{a_1,b}$ and $V_{a_2,b}$ contains at most ℓ vertices and since the vertices in $V_{a_1,b}$ precede the vertices of $V_{a_2,b}$ in σ , we have that the maximum rainbow formed by edges in $E_{|}$ has size at most ℓ in σ . Thus, ℓ queues suffice to embed all such edges in Γ [27].

Second, we that all the edges in E_n can be assigned to at most $\ell qn(H)$ queues. Since the proof that the edges in E_{ℓ} can be assigned to at most $\ell qn(H)$ queues is analogous, this concludes the proof of the lemma. Consider a partition of E_n into sets E_i , with $1 \leq i \leq qn(H)$, such that E_i contains all the edges (u, v) of E_n such $u \in V_{a,c}$, $v \in V_{c,d}$, and (a, c) belong to the *i*-th queue of H. Next, we show that the edges in each set E_i can be assigned to at most ℓ new queues in Γ . Consider any two nesting edges (u, v) and (u', v') in E_i with $u \in V_{a_1,b_1}$, $v \in V_{a_2,b_2}$, $u' \in V_{a'_1,b'_1}$ and $v' \in V_{a'_2,b'_2}$. By the definition of E_n , we have that $a_1 < a_2$, $a'_1 < a'_2$, $b_1 < b_2$, and $b'_1 < b'_2$. Since (u, v), $(u', v') \in E(\Delta_i)$ they have the same span and thus it follows that (u, v) and (u', v') nest in σ only if (i) $b_1 = b'_1$ and $b_2 = b'_2$ (that is, u and u' (resp. v and v') stem from the same vertex of Δ_i) and (ii) $a_1 = a'_1$ or $a_2 = a'_2$ (that is, at least one of the pairs u, u' and v, v' stem from the same vertex of H) [19]. Since each of the sets V_{a_1,b_1} and V_{a_2,b_2} contains at most ℓ vertices and since the vertices in V_{a_1,b_1} precede the vertices of V_{a_2,b_2} in σ , we have that the maximum rainbow formed by edges in E_i has size at most ℓ in σ . Thus, ℓ queues suffice to embed all such edges in Γ [27].

Lemma 15 in conjunction with Theorem 5 yields a quadratic (in h) upper bound on the queue number of h-framed graphs. However, for $h \leq 5$, it implies an improved bound on the queue number of 1-planar and optimal 2-planar graph, which we summarize in the following.

Theorem 17. The queue number of 1-planar and optimal 2-planar graphs is at most 82.

4.2 Non-repetitive chromatic number

An *r*-coloring of a graph G is a function $\phi : V(G) \to [r]$. A path $(v_1, v_2, \ldots, v_{2\tau})$ is repetitively colored by ϕ if $\phi(v_i) = \phi(v_{i+\tau})$ for $i = 1, 2, \ldots, \tau$. A coloring ϕ of G is nonrepetitive if no path of G is repetitively colored by ϕ . Clearly, non-repetitive colorings are proper, i.e., $\phi(u) \neq \phi(v)$ if u and v are adjacent in G. The non-repetitive chromatic number $\pi(G)$ of G is the minimum integer r such that G admits a non-repetitive r-coloring. In [21], Dujmović et al. developed the following lemma to upper-bound the non-repetitive chromatic number of graphs that can be expressed as subgraphs of the strong product of a path P, a graph H with tw(H), and a clique K_{ℓ} on ℓ vertices.

Lemma 18 (Dujmović et al. [21]). If $G \subseteq P \boxtimes H \boxtimes K_{\ell}$, then $\pi(G) \leq 4^{tw(H)+1} \cdot \ell$.

Using Lemma 18 and Theorem 1(c), Dujmović, Morin and Wood [20] provide an upper bound of 1792 and of 2048 on the non-repetitive chromatic number of 1-planar and optimal 2-planar graphs, respectively. Since 1-planar and optimal 2-planar graphs are 4-framed and 5-framed, respectively, we improve both bounds to 1536. By Lemma 18 and the product structure theorem for *h*-framed graph in [20], one can obtain an upper bound of $4^4 \cdot (h + 3\lfloor \frac{h}{2} \rfloor - 3)$ for the non-repetitive chromatic number of the graphs in this family. By Lemma 18 and Theorem 2, we can further improve this upper bound to $4^4 \cdot (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1)$. Also, since *k*-map graphs are subgraphs of *k*-framed graphs [4], their non-repetitive chromatic number is also improved from to $4^4 \cdot (k + 3\lfloor \frac{k}{2} \rfloor - 3)$ to $4^4 \cdot (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1)$. Specifically, we get the following.

Corollary 19. Let G be a graph.

- If G is 1-planar, then $\pi(G) \leq 4^4 \cdot 6$.
- If G is k-map, then $\pi(G) \leq 4^4 \cdot (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor 1).$
- If G is h-framed, then $\pi(G) \leq 4^4 \cdot (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor 1)$.

The electronic journal of combinatorics 31(4) (2024), #P4.56



Figure 6: Illustration of the operation of contracting two vertices u and v. The thick red edges connected w to vertices in $N_G(u)\Delta N_G(v)$.

4.3 *p*-centered chromatic number

For any $c, p \in \mathbb{N}$ with $c \ge p$, a *c*-coloring of a graph *G* is *p*-centered if, for every connected component *X* of *G*, at least one of the following holds: (i) the vertices of *X* are colored with more than *p* colors, or (ii) there exists a vertex *v* of *X* that is assigned a color different from the ones of the remaining vertices of *X*. For any graph *G*, the *p*-centered chromatic number $\chi_p(G)$ of *G* is the minimum integer *c* such that *G* admits a *p*-centered *c*-coloring. The following lemma is implied by combining results by Pilipczuk and Siebertz [36], Debski et al. [13] and Dujmović, Morin and Wood [20].

Lemma 20 ([13, 20, 36]). If $G \subseteq P \boxtimes H \boxtimes K_{\ell}$, where H is a planar graph with $\operatorname{tw}(H) \leq 3$, it holds that $\chi_p(G) \leq \ell(p+1)^2 (p \lceil \log(p+1) \rceil + 2p + 1)$.

By Lemma 20 and Theorem 1, Dujmović et, Morin and Wood [20] showed the following upper bounds: $\chi_p(G) \leq 7(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1)$ if G is a 1-planar graph, $\chi_p(G) \leq 8(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1)$ if G is an optimal 2-planar graph, $\chi_p(G) \leq (k+3\lfloor \frac{k}{2} \rfloor - 3)(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1)$ if G is a k-map graph, and $\chi_p(G) \leq (h+3\lfloor \frac{h}{2} \rfloor - 3)(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1)$ if G is an h-framed graph. By exploiting Theorem 2 and Lemma 20, we get the next.

Corollary 21. Let G be a graph.

- If G is 1-planar or optimal 2-planar, then $\chi_p(G) \leq 6(p+1)^2 (p \lceil \log(p+1) \rceil + 2p+1)$.
- If G is k-map, then $\chi_p(G) \leq (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor 1)(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1).$
- If G is h-framed, then $\chi_p(G) \leq (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor 1)(p+1)^2(p\lceil \log(p+1) \rceil + 2p+1).$

5 Bounding Twin-width

Besides the direct consequences of the product structure theorem(s) surveyed in Section 4, the construction presented in Section 3 has another strong implication described next.

Consider only simple graphs for the coming definition. A trigraph is a simple graph G in which some edges are marked as red. The edges of G which are not red are sometimes called (and depicted) black for distinction. With respect to the red edges only, we naturally speak about red neighbors and red degree in G. However, when speaking about edges, neighbors and/or subgraphs without further specification, we count both black and red edges together as one edge set. For the next definition refer to Figure 6. Given a pair of (possibly not adjacent) vertices $u, v \in V(G)$, we define a contraction of the pair u, v as the operation of creating a trigraph G' which is the same as G except that u, v are replaced with a new vertex w (said to stem from u, v) such that:

- the full neighborhood of w in G' (i.e., including the red neighbors), denoted by $N_{G'}(w)$, equals the union of the neighborhoods of u and v in G except u, v themselves, that is, $N_{G'}(w) = (N_G(u) \cup N_G(v)) \setminus \{u, v\}$, and
- the red neighbors of w in G', denoted by $N_{G'}^r(w)$, inherit all red neighbors of uand of v and add those in $N_G(u)\Delta N_G(v)$, that is, $N_{G'}^r(w) = (N_G^r(u) \cup N_G^r(v) \cup (N_G(u)\Delta N_G(v))) \setminus \{u,v\}$, where Δ denotes the symmetric set difference.

A partial contraction sequence of a trigraph G is a sequence of successive contractions $G = G_k, G_{k-1}, \ldots, G_0$ turning G into a trigraph G_0 , and its width is the maximum red degree of any vertex in any trigraph along the sequence. We speak about a contraction sequence of G if G_0 is a single vertex, in which case k = n - 1. The twin-width of G is the minimum width over all possible contraction sequences of G. To define the twin-width of a graph G, we consider G as a trigraph with no red edges.

The question about the maximum possible twin-width of planar graphs has recently attracted a lot of attention. After the first implicit (and astronomical) upper bounds on the twin-width of planar graphs, e.g. [10], we have seen a stream of improving explicit bounds [5, 11, 32], culminating with the current best upper bound of 8 by Hliněný and Jedelský [29]. This is complemented with a nearly matching lower bound of 7 by Král' and Lamaison [33], but determining the exact maximum value (7 or 8) is still an open question. We deem important to mention that the techniques used in the literature either exploit, more or less explicitly, the product structure machinery [5, 11, 32], or modifications thereof [29].

Beyond planarity, the twin-width of k-planar graphs is bounded for any fixed k by means of FO transductions [10]; however, such bound is not known as an explicit function. Explicit exponential asymptotic bounds for the twin-width of k-planar graphs (of order $2^{O(k)}$) are presented in the aforementioned [11] (with a generalization to higher surfaces). We give new explicit non-asymptotic bounds on the twin-width of (subgraphs of) h-framed and 1-planar graphs, where the bound for h-framed graphs is linear in h.

Theorem 22. Let G be an h-framed graph with $h \ge 4$. Then the twin-width of any simple spanning subgraph of G is at most $11h + 51\sqrt{h} + 64$.

The basic idea of the proof is to use the same recursive decomposition procedure that has been used to obtain the product structure in Lemma 4 to also obtain a good contraction sequence of small red degrees. We, however, need some preparatory work. To motivate it, we start with an informal outline of our proof:

- We recall the BFS layering $\mathcal{L} = (V_0, V_1, \ldots, V_b)$ of $\mathrm{sk}(G)$ as in Lemma 4. The crucial point of the proof is the following; if the constructed sequence contracts only vertices within a few (namely about \sqrt{h} , which is an optimum for the approach) consecutive layers of \mathcal{L} , then there will be only few other layers possibly containing red neighbors of each contracted vertex. Planarity of the skeleton $\mathrm{sk}(G)$, in a suitable setup, then restricts the number of possible red neighbors in each of the layers at every step of the sequence.
- The precise way of bounding the number of possible red neighbors in one layer of a graph along the contraction sequence closely follows the details of the decomposition recursively constructed in Lemma 4, and is handled by Claim 28 and its recursive proof. The definitions and statements preceding Claim 28 provide a technical background for the proof.
- There is one notable detail concerning the construction of our contraction sequence. While we are going to recursively contract parts of our graph (following the proof of Lemma 4), we inevitably create some red edges incident to the boundary of the recursively processed part(s). This is a problem for boundary vertices which also appear in the boundaries of two simultaneously processed smaller parts — every level of recursion then may add to the maximum red degree, eventually exceeding any fixed bounds. To resolve this problem, our proof introduces specially protected boundary vertices in Definition 25.

Consider some trigraph H. For a set $X \subseteq V(H)$, a (partial) contraction sequence of His *X*-restricted if every step contracts only pairs coming from X (including the vertices that stem from previous contractions in X). For a vertex partition \mathcal{P} of H, a (partial) contraction sequence is \mathcal{P} -respecting if every step contracts only pairs coming from some but one part $X \in \mathcal{P}$. For two vertices u and v belonging to the same part X of \mathcal{P} , the vertex w stemming from the contraction of u and v is assigned to X after the contraction. Naturally, if H_1 is a trigraph resulting from a \mathcal{P} -respecting partial contraction sequence of H, then by a "partition \mathcal{P} at H_1 " we mean the (unique) partition of $V(H_1)$ that stems from the parts of \mathcal{P} by the contractions. For an ordered vertex partition $\mathcal{P} = (X_0, X_1, X_2, \ldots)$ of H, we say that H is *m*-narrow in \mathcal{P} if for every edge $(v, w) \in H$ and suitable i and jwe have that $v \in X_i$, $w \in X_j$, and $|i - j| \leq m$.

The proof of Theorem 22 will be formulated within the following special setup which we outline already now for motivation. We assume fixed integer functions μ and ν (to be exactly specified later, namely of order $\Theta(\sqrt{h})$) such that the following holds for all $h \ge 4$:

$$1 \leq \mu(h), \nu(h) \leq \lfloor h/2 \rfloor \leq \mu(h) \cdot \nu(h)$$

Let $\mathcal{L}^{\nu(h)} = (L_0, L_1, L_2, \ldots)$ be the vertex partition of V(G) obtained from the above BFS layering \mathcal{L} of sk(G) such that $L_0 = V_0$ and the remaining parts in the partition are obtained by joining $\nu(h)$ -tuples of consecutive layers of \mathcal{L} , i.e., $L_i := \bigcup_{j:(i-1)\nu(h)+1 \leq j \leq i\nu(h)} V_j$ for $i = 1, 2, \ldots$ (note that $L_i = \emptyset$ for all sufficiently large *i*). Consider now any simple spanning sub(tri)graph $H \subseteq G$, i.e., V(H) = V(G) and so the partition $\mathcal{L}^{\nu(h)}$ applies to H as well as to G. We then have the following claim similar to Property 3:

Claim 23. The (tri)graph H is $\mu(h)$ -narrow in $\mathcal{L}^{\nu(h)}$.

Proof. Consider an edge e = (v, w) of H with $v \in L_i \in \mathcal{L}^{\nu(h)}$ and $w \in L_j \in \mathcal{L}^{\nu(h)}$. Edge e is at the same time an edge of G. If e belongs to $\mathrm{sk}(G)$, then we easily see that $|i - j| \leq 1 \leq \mu(h)$ since \mathcal{L} was a layering of $\mathrm{sk}(G)$. Otherwise, both ends v, w of e lie on the boundary of some face of $\mathrm{sk}(G)$, and so the distance from v to w in $\mathrm{sk}(G)$ is at most $\lfloor h/2 \rfloor$. For a contradiction, on the other hand, suppose (up to symmetry) that $i - j \geq \mu(h) + 1$. Then a shortest path from v to w in $\mathrm{sk}(G)$ must intersect each of the $\mu(h)$ parts $L_{j+1}, \ldots, L_{j+\mu(h)}$ in at least $\nu(h)$ vertices by the definition of $\mathcal{L}^{\nu(h)}$. This in turn means that the distance from v to w is more than $\mu(h) \cdot \nu(h) \geq \lfloor h/2 \rfloor$, which is a contradiction to the previous.

A direct consequence of Claim 23 and of the definition of a contraction is the following.

Corollary 24. A contraction of a pair of vertices from L_i in H may create red edges only to the vertices of $L_{i-\mu(h)} \cup \cdots \cup L_i \cup \cdots \cup L_{i+\mu(h)}$ (a union of $2\mu(h) + 1$ layers of $\mathcal{L}^{\nu(h)}$).

We say that a set $W \subseteq M$ is $[\alpha, \beta]$ -near vertical with respect to a partition \mathcal{P} of Mif the union of any $2\mu(h) + 1$ (consecutive or not) layers of \mathcal{P} intersects W in at most $\alpha \cdot \nu(h) \cdot (2\mu(h)+1) + \beta$ vertices. To illustrate this notion in our case of the BFS layering \mathcal{L} , observe that if $W = A \cup B \subseteq V(G)$ where the width of A with respect to \mathcal{L} is $\lambda_{\mathcal{L}}(A) \leq \alpha$ and $|B| \leq \beta$, then W is $[\alpha, \beta]$ -near vertical in $\mathcal{L}^{\nu(h)}$, but the full scope of this definition is broader.

We will exploit the following definition; refer to Figure 7.

Definition 25. Let H be a trigraph with a "boundary" set $B \subseteq V(H)$, a set $U := V(H) \setminus B$ of interior vertices, and vertices $p, p' \in B$ (p, p' will be called the *protected* vertices of G). Also, let $\mathcal{P} = (X_0, X_1, X_2, \ldots)$ be an ordered partition of V(H), such that H is $\mu(h)$ -narrow in \mathcal{P} .

We say that the triple (H, B, \mathcal{P}) is *marvelous* if the set B can be partitioned into four disjoint subsets $B_1, B_2, B_3, B_4 \subseteq B$, and there exists $Y \subseteq U$ and an index $\ell > 0$ such that

- (I) each of the sets B_1 , B_2 and B_3 is $[2, \kappa(h)]$ -near vertical with respect to \mathcal{P} , with $\kappa(h) := \lfloor \frac{h-1}{2} \rfloor 1$,
- (II) the set $B_4 \cup (U \setminus Y)$ is $[3, \sigma(h)]$ -near vertical with respect to \mathcal{P} , with $\sigma(h) := \lfloor \frac{h}{3} \rfloor 1$,
- (III) *H* has no red edge with both ends in $B \cup (U \setminus Y)$, and no red edge with an end p or p',
- (IV) $|Y \cap X_i| \leq 8$ for all $0 \leq i \leq \ell 2$, $|Y \cap X_{\ell-1}| \leq 7$, and $|Y \cap X_i| \leq 4$ for all $i \geq \ell$, and
- (V) for all $0 \leq i, j \leq \ell 2$, every vertex of $Y \cap X_i$ has at most 4 red neighbors in $Y \cap X_j$.



Figure 7: Illustration of Definition 25: A trigraph H with a distinguished vertex set B (the "boundary"), further partitioned into B_1, B_2, B_3, B_4 , and the special protected vertices p and p' with no incident red edges. The layers of the associated partition \mathcal{P} of V(H) are pictured horizontally. This informal illustration, in particular, features; vertices $y_1, y_2 \in Y \cap X_{\ell-2}$ which have each 4 red neighbors in $Y \cap X_j$ for $j \leq \ell - 2$, a vertex $u \in (U \setminus Y) \cap X_{\ell-2}$ which has 8 red neighbors in $Y \cap X_j$ for any $j \leq \ell - 2$, and vertices $b_1 \in B_1$ and $b_3 \in B_3$ having each potentially up to $16\mu(h) + 8$ red neighbors in $2\mu(h) + 1$ consecutive layers of $\mathcal{P} \cap Y$.

Now we give two core technical statements regarding Definition 25.

Lemma 26. Given a marvelous triple (H, B, \mathcal{P}) , the maximum red degree of a vertex in $V(H) \setminus B$ is at most

$$18\nu(h)\mu(h) + 3\kappa(h) + \sigma(h) + 9\nu(h) + 12\mu(h) + 6, \tag{1}$$

and the maximum red degree of a vertex in B is at most $16\mu(h) + 8$.

Proof. Let $\mathcal{P} = (X_0, X_1, X_2, \ldots)$ and Y be defined as in Definition 25. By the conditions (I) and (II) and the definition of near vertical, the number of potential red neighbors of a vertex x in $B \cup (U \setminus Y) = V(H) \setminus Y$ is at most $3 \cdot [2\nu(h) \cdot (2\mu(h) + 1) + \kappa(h)] + [3\nu(h) \cdot (2\mu(h) + 1) + \sigma(h)] = 9\nu(h) \cdot (2\mu(h) + 1) + 3\kappa(h) + \sigma(h)$. The number of potential red neighbors of x in Y can be estimated as follows. By (IV), there are at most 7 such neighbors in $Y \cap X_{\ell-1}$ (≤ 6 if $j = \ell - 1$), and at most 8 in each $Y \cap X_i$ where $j - \mu(h) \leq i \leq j$ since H is $\mu(h)$ -narrow. By (V), if $j \leq \ell - 2$, we moreover have at most 4 red neighbors of x in each $Y \cap X_i$ where $j - \mu(h) \leq i \leq \min(\ell - 2, j + \mu(h))$. Again by (IV), we have at most 4 red neighbors of x in each $Y \cap X_i$ where $\ell \leq i \leq j + \mu(h)$. Note that these subcases altogether cover all values of i between $j - \mu(h)$ and $j + \mu(h)$. By a simple case analysis, the sum of these upper estimates is maximized when $i = j = \ell - 1$, which gives an upper bound of $8\mu(h) + 6 + 4\mu(h) = 12\mu(h) + 6$ potential red neighbors of x in Y in the trigraph H. Together with the previous, it confirms (1).

Finally, for any vertex $y \in B$, we immediately get from (III), (IV) and H being $\mu(h)$ -narrow, that the number of potential red neighbors of y is at most $8(2\mu(h)+1) = 16\mu(h)+8$ in H.

Lemma 27. Let H be a trigraph with a "boundary" set B and a partition \mathcal{P} such that (H, B, \mathcal{P}) is marvelous. Then there exists a U-restricted \mathcal{P} -respecting partial contraction sequence of H, with $U := V(H) \setminus B$, resulting in a trigraph H_1 with a boundary set B satisfying the following properties:

- (a) H_1 has no red edge with both ends in B, and no red edge with an end p or p', and
- (b) $|U_0 \cap X_i^0| \leq 4$, for all $i \geq 0$, where $U_0 = V(H_1) \setminus B$ and $X_i^0 \subseteq V(H_1)$ stems from the part $X_i \in \mathcal{P}$ by contractions.

Furthermore, each triple (H', B, \mathcal{P}') , where H' is a trigraph occuring along the partial contraction sequence and \mathcal{P}' is the partition \mathcal{P} at H', is marvelous.

Proof. We do the proof by induction on |U|. We pick the largest index k such that $|U \cap X_k| > 4$, and if k does not exist, we immediately stop with the empty sequence and $H_1 = H$ satisfying the conclusion. Note that we may have $k > \ell$ since (IV) requires only $|Y \cap X_k| \leq 4$.

Now we contract any pair of distinct vertices $u, w \in U \cap X_k$ with the same adjacency to the protected vertices p, p' in H, i.e., that $N_H(u) \cap \{p, p'\} = N_H(w) \cap \{p, p'\}$, and that $u, w \in Y$ if possible. This is always sound since there can be at most four different adjacencies to p, p'. Denote by H' the trigraph resulting from the previous contraction in H. Let the new vertex that stems from the contraction of u, w be x, and set U' := $U \setminus \{u, w\} \cup \{x\}$ and $Y' := Y \cup \{x\}$. Clearly, this contraction does not create any red edge incident to p or p' by our choice of u, w, and also no red edge with both ends in $B \cup (U' \setminus Y')$ since this set is not affected.

Let $\ell' := k + 1$ and $\mathcal{P}' = (X'_0, X'_1, \dots)$ be the partition \mathcal{P} at H'. Consider the following properties of the triple (H', B, \mathcal{P}') , which are derived from those of Definition 25:

- (I') each of the sets B_1 , B_2 and B_3 is $[2, \kappa(h)]$ -near vertical with respect to \mathcal{P}' ,
- (II') the set $B_4 \cup (U' \setminus Y')$ is $[3, \sigma(h)]$ -near vertical with respect to \mathcal{P}' ,
- (III') H' has no red edge with both ends in $B \cup (U' \setminus Y')$, and no red edge with an end p or p',
- (IV') $|Y' \cap X'_i| \leq 8$ for all $1 \leq i \leq \ell' 2$, $|Y' \cap X'_{\ell'-1}| \leq 7$, and $|Y' \cap X'_i| \leq 4$ for all $i \geq \ell'$, and
- (V') for all $1 \leq i, j \leq \ell' 2$, every vertex of $Y' \cap X'_i$ has at most 4 red neighbors in $Y' \cap X'_i$.

Conditions (I') and (II') are trivially inherited by H' (and U', Y') from H (and from U, Y), and validity of (III') for H' has just been confirmed. (IV') is valid by our choice of k – in particular, $|Y' \cap X'_k| \leq 7$ since we have contracted in this part. (V') remains valid since we have not contracted among the relevant parts $X'_i = X_i$ for $i \in \{0, \ldots, k-1\}$. Therefore, the triple (H', B, \mathcal{P}') is marvelous, and we may apply the lemma inductively since |U'| < |U|.

We now return to the proof of the main result of this section.

Proof of Theorem 22. We start from the BFS layering $\mathcal{L} = (V_0, V_1, V_2, ...)$ of the skeleton $\mathrm{sk}(G)$, as in Section 3, i.e., we choose a root r from the unbounded face of $\mathrm{sk}(G)$ and define $V_i \subseteq V(G)$ as the set of those vertices at distance $i \ge 0$ from r in $\mathrm{sk}(G)$. Then, we define the ordered partition $\mathcal{L}^{\nu(h)} = (L_0 = V_0, L_1, L_2, ...)$ of V(G) by setting $L_i := \bigcup_{j:(i-1)\nu(h)+1 \le j \le i\nu(h)} V_j$ for $i \ge 1$, as used above in Claim 23. Let \overline{H} be a simple spanning subgraph of G.

In order to apply induction, which will in this case follow the recursive-decomposition steps shown in the proof of Lemma 4, we strengthen the statement of Theorem 22 as follows.

Claim 28. Let C be a cycle in sk(G) which has been encountered in the inductive proof of Lemma 4, and let G_C be the subgraph of G bounded by C and H_C the subgraph of \overline{H} induced on $V(G_C)$. Let $p, p' \in V(C)$ be arbitrary. Let $U := V(G_C) \setminus V(C)$. Let $\mathcal{L}_C^{\nu(h)}$ denote $\mathcal{L}^{\nu(h)}$ restricted to $V(H_C) = V(G_C)$. Then there exists a U-restricted $\mathcal{L}_C^{\nu(h)}$ -respecting partial contraction sequence of H_C , resulting in the trigraph H_C^0 and satisfying the following:

- a) The maximum red degree in the sequence is as (1) in Lemma 26, and the maximum red degree of vertices of V(C) is at most $16\mu(h) + 8$. There is no red edge in the sequence incident to the protected vertices p, p'.
- b) $|(V(H_C^0) \setminus V(C)) \cap L'_i| \leq 4$, where L'_i stems from L_i in the sequence, holds for $i \geq 0$.

In this setting of Claim 28, from the inductive proof of Lemma 4, we get a subgraph $R \subseteq \operatorname{sk}(G) \cap G_C$ such that $X := V(R) \setminus V(C)$ satisfies Property 1 of Lemma 4. Considering the $a \ge 2$ bounded faces of $C \cup R$ (which is 2-connected), we denote their bounding cycles by D_1, \ldots, D_a , and say that the cycle D_i is *nonempty* if the interior of D_i contains a vertex of G_C . Recall, from the separable case of Lemma 4, that a may be higher than 3. We furthermore have for free (from the proof of Lemma 4) that for each nonempty D_i we may apply Claim 28 inductively, and we only need to appropriately choose the protected vertices $p_i, p'_i \in V(D_i)$.

The purpose of choosing the pair of protected vertices in Claim 28 is to prevent "recursive accumulation" of red degree in the boundary vertices. If $p, p' \in V(D_i)$, then we choose $p_i = p$ and $p'_i = p'$. If $V(D_i) \cap \{p, p'\} = \emptyset$, then we choose p_i and p'_i as the ends of the path $D_i \cap C$, and we take one or both of p_i, p'_i arbitrarily if $D_i \cap C$ is one-vertex or empty. If, up to symmetry, $p \in V(D_i)$ and $p' \notin V(D_i)$, then we set $p_i = p$ and p'_i as one of the ends of $D_i \cap C$ – we take an arbitrary of the two ends except the special case described next; if $p \in V(D_i), p' \in V(D_j)$ and one end (or both ends) of $D_i \cap C$ is an end



Figure 8: Illustration of the proof of Theorem 22: This is the first step after a recursive application of Claim 28 to D_1 and D_2 in $H_C \subseteq G$, and outlining the setup required by Lemma 27.

(ends) of $D_j \cap C$, then we set p'_i to the end shared with $D_j \cap C$ (and the possible other shared end is symmetrically set to p'_j). This way we ensure that every vertex of C is *not* protected in at most one of the nonempty D_i 's.

We continue in the proof of Claim 28 for H_C as follows. Apply Claim 28 to (assumed nonempty, up to symmetry) D_1 , and to D_2 . Then concatenate the partial contraction sequence of H_{D_2} after the one of H_{D_1} . There is no confusion in this concatenation since the sequences do not touch the vertices of $V(D_1) \cup V(D_2)$, and there is no edge from $V(H_{D_1}) \setminus V(D_1)$ to $V(H_{D_2}) \setminus V(D_2)$ due to the plane skeleton $\mathrm{sk}(G)$. We denote the respective trigraphs resulting in the sequences by $H_{D_1}^1$ and $H_{D_2}^1$. In the concatented partial contraction sequence, every interior vertex of H_{D_1} and H_{D_2} , and also every vertex of V(C) thanks to the choice of protected vertices p_1, p'_1 and p_2, p'_2 , stay with red degrees as in Claim 28 a). For a vertex $y \in (V(D_1) \cap V(D_2)) \setminus V(C)$, the same holds since y has at most $16\mu(h) + 8$ incident red edges from the current one of the two sequences, plus possibly up to $4(2\mu(h) + 1) = 8\mu(h) + 4$ red edges from $H_{D_1}^1$ by Claim 28 b), together $24\mu(h) + 12 \leq 12\nu(h)\mu(h) + 12\mu(h) + 6\nu(h) + 6$ compared to (1) in Lemma 26. See Figure 8 for an illustration of the intermediate outcome.

Next, we are going to apply Lemma 27 to the resulting graph $H := H_{D_1}^1 \cup H_{D_2}^1 \cup R \cup C$ and the sets $B := V(C) \cup V(D_3) \cup \ldots \cup V(D_a), U = V(H) \setminus B, Y := V(H_{D_1}^1 \cup H_{D_2}^1) \setminus V(D_1 \cup D_2)$. For that we verify the conditions of Definition 25: The protected vertices p, p' are the same as in Claim 28. The trigraph H is $\mu(h)$ -narrow in the partition that stems from $\mathcal{L}_C^{\nu(h)}$ by

Claim 23. (I) and (II) follow from the course of induction in the claim and from Property 1



Figure 9: Illustration of the proof of Theorem 22: This is the (possible) second step after the first step leading to a subtrigraph H^2 and a subsequent recursive application of Claim 28 to D_3 in $H_C \subseteq G$. Again, we have the setup required by Lemma 27 for the next step, including a corresponding new definition of U and Y. Note that here we have $B_4 = \emptyset$ since a = 3 (meaning we are at the last step).

of Lemma 4, where the partition $B = B_1 \cup B_2 \cup B_3 \cup B_4$ is given by the paths P_1, P_2, P_3 from Lemma 4 and by $B_4 = V(R) \cap B$. (III) is true since the inductively obtained partial contraction sequences do not touch $C \cup R$ and since the protected vertices p, p' of C have been respected. (IV) is true for sufficiently large ℓ by Claim 28 b), and (V) follows the same way. Let H^2 denote the contracted trigraph resulting from this application of Lemma 27.

Subsequently, we apply Claim 28 to D_3 (if $a \ge 3$ and nonempty, otherwise we skip), and append the obtained partial contraction sequence from H_{D_3} to $H_{D_3}^1$ after the previous sequence ending with H^2 . Refer to Figure 9. We repeat (even for empty D_3 if $a \ge 3$) an analogous application of Lemma 27 to resulting $H := H^2 \cup H_{D_3}^1 \cup R^2 \cup C$ where R^2 is the restriction of R to H^2 . In this we again, as above, satisfy the conditions of Lemma 27, since the relevant conclusions for H^2 are the same as we had for $H_{D_1}^1$. We repeat possibly again, until we exhaust all cycles up to D_a , and satisfy the conclusions of Claim 28 by Lemma 27.

Since we have finished the proof of Claim 28, we can now apply it to the unbounded facial cycle C of sk(G), exactly as we have started with Lemma 4 previously. In the resulting trigraph H_C^0 , we can then greedily contract the remaining vertices, and by Claim 28 b) and Corollary 24, we in this final contraction sequence never exceed red degree of $|V(C)| + 4(2\mu(h) + 1) \leq 2\mu(h)\nu(h) + 4\mu(h) + 5$ which conforms to (1) in Lemma 27.

At last, we precisely choose $\mu(h) := \left\lceil \sqrt{\lfloor h/2 \rfloor/2} \right\rceil$ and $\nu(h) := 2\mu(h)$ in accordance with the assumptions stated above. Based on (1) we estimate the sought twin-width of

 $\overline{H} \subseteq G$ as follows:

$$18\nu(h)\mu(h) + 3\kappa(h) + \kappa'(h) + 9\nu(h) + 12\mu(h) + 6$$

$$= 36\mu(h)^{2} + 3(\lfloor (h-1)/2 \rfloor - 1) + \lfloor h/3 \rfloor - 1 + 30\mu(h) + 6$$

$$\leq 36(\lfloor h/2 \rfloor/2 + 2\sqrt{\lfloor h/2 \rfloor/2} + 1) + 3\lfloor (h-1)/2 \rfloor + \lfloor h/3 \rfloor - 4 + 30\sqrt{\lfloor h/2 \rfloor/2} + 36$$

$$= 15\lfloor h/2 \rfloor + 3(h-1) + \lfloor h/3 \rfloor + 102\sqrt{\lfloor h/2 \rfloor/2} + 68 \leq 11h + 51\sqrt{h} + 64.$$

$$\Box$$

Corollary 29. The twin-width of simple 1-planar graphs is at most 72, and that of simple optimal 2-planar graphs is at most 75.

Proof. We treat these graphs as spanning subgraphs of *h*-framed graphs with h = 4 and h = 5, respectively. From (2) we get a more precise estimate based on setting $\mu(h) = 1$, $\nu(h) = 2$, $\kappa'(h) = 0$, and $\kappa(h) = 0$ for h = 4 and $\kappa(h) = 1$ for h = 5, leading to the respectively claimed bounds of 72 and 75.

We point out that Theorem 22 implies an improvement on the twin-width of kmap graphs only up to a certain k, as these graphs have bounded twin-width independently of k [10].

6 Conclusions

In this paper we have provided a product structure theorem for h-framed graphs. Our approach is constructive and can easily be implemented to run in quadratic time to obtain the corresponding decomposition, provided that the input graph is a topological h-framed graph.

A major open question is to obtain a speed-up in the construction; the recent algorithmic advances in [12, 34] have the potential to lead to improvements in the running time. The conference version [6] of this paper asked the question whether each k-planar graph is a subgraph of the strong product of a path, a (planar) graph of constant treewidth H, and a clique whose size is a function f of k. In a recent development, the question was answered affirmatively if H is not required to be planar [16] and $f(k) \in 2^{O(\lfloor k/2 \rfloor!)}$. This leaves open the questions of whether H can be assumed to be planar and/or f can be bounded by a polynomial function of k.

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